HEIGHT 1 GROUP SCHEMES AND PRISMATIC F-GAUGES

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Abstract. We describe the prismatic F-gauge associated to a finite flat height one group scheme over a smooth variety of positive characteristic. As applications, we derive the description of the crystalline Dieudonné module of Berthelot-Breen-Messing in this case and recover results of Bragg-Olsson describing flat cohomology using a Hoobler-type sequence.

1. Introduction

Motivations. Let X be a smooth scheme over a perfect field k of characteristic $p > 0$, and let G/X be a finite flat abelian group scheme over X. The general theory of Anschütz-Le Bras [1] and Mondal [13, 14] associates to G a prismatic F-gauge $\mathscr{M}(G) \in \mathscr{D}_{\mathrm{qcoh}}(X^{\mathrm{syn}})$; that is, a quasi-coherent complex on the syntomification of X as defined in [4, 5]. The purpose of this article is to analyze the structure of $\mathscr{M}(G)$ in the case when G has coheight 1 (meaning the Cartier dual G^* of G has height 1). In addition to describing $\mathscr{M}(G)$ in simpler terms in this case, we explain how the structure of $\mathscr{M}(G)$ gives rise to several previously obtained results.

(1) The crystalline realization of $\mathscr{M}(G)$ is the crystalline Dieudonné module $M_{\mathrm{crys}}(G)$ of Berthelot-Breen-Messing [2], which in the case of a coheight 1 group scheme has a particularly simple structure [2, 4.3.6]. Namely, it is the pushforward along the canonical morphism of topoi $(X/k)_{\mathrm{crys}} \to (X/W(k))_{\mathrm{crys}}$ of the crystal on X/k defined by the vector bundle $\mathscr{L}ie_{G^*}$ on X. More precisely, if $U \hookrightarrow T$ is a PD-thickening over k of an étale X-scheme U then because of the divided powers on the ideal of U in T the Frobenius morphism $F_T : T \to T$ factors through a map $\Phi_T : T \to U$ and $M_{\mathrm{crys}}(G)_T = \Phi_T^* \mathscr{L}ie_{G^*}|_U$.

(2) By [11, Theorem B] the flat cohomology $R\Gamma(X, G)$ is isomorphic to $R\Gamma(X^{\mathrm{syn}}, \mathscr{M}(G^*)\{1\})$, where $(-)\{1\}$ denotes a Breuil-Kisin twist. In the height 1 case this flat cohomology can also be computed using a Hoobler-type sequence [7, 1.19]

$$R\Gamma(X, G)[1] \simeq R\Gamma(X,\ \mathscr{E} \otimes Z\Omega^1_X \xrightarrow{\rho - C} \mathscr{E} \otimes \Omega^1_X\ ),$$

where $(\mathscr{E}, \rho)$ is the vector bundle with semilinear map $\rho : F^*\mathscr{E} \to \mathscr{E}$ corresponding to G by [10, Exposé VIIA, Remarque 7.5]. An explanation for this comparison in the case of $G = \mu_p$ was given in [4, 4.4.7].

We will derive both (1) and (2) directly from the structure of the F-gauge $\mathscr{M}(G^*)$.

Statement of the main result.

1.1. The Nygaard-filtered prismatization $k^{\mathscr{N}}$ of k can be described as the stack quotient

$$k^{\mathscr{N}} = [\mathrm{Spf}(W(k)[u, t]/(ut - p))/\mathbf{G}_m],$$

where u (resp. t) has weight 1 (resp. -1). Let $j_t : X^{\mathscr{N}}_t \to X^{\mathscr{N}}$ (resp. $j_u : X^{\mathscr{N}}_u \to X^{\mathscr{N}}$) be the pullback along $X^{\mathscr{N}} \to k^{\mathscr{N}}$ of the closed substack $[\mathrm{Spec}(k[t])/\mathbf{G}_m] \subset k^{\mathscr{N}}$ (resp. $[\mathrm{Spec}(k[u])/\mathbf{G}_m]$). We also consider the stack $j_0 : X^{\mathscr{N}}_0 \to X^{\mathscr{N}}$ defined to be the base change of $B\mathbf{G}_{m,k} \subset k^{\mathscr{N}}$, so we have morphisms $i_t : X_0^{\mathcal{N}} \to X_t^{\mathcal{N}}$ and $i_u : X_0^{\mathcal{N}} \to X_u^{\mathcal{N}}$ for which $j_t \circ i_t = j_u \circ i_u$. As we discuss in section 2, there are natural maps of stacks

$$\pi_t : X_t^{\mathcal{N}} \to X, \quad \pi_u : X_u^{\mathcal{N}} \to X$$

which have the property that

$$\pi_t \circ i_t = F_X \circ \pi_i \circ i_u. \tag{1.1.1}$$

1.2. Consider a triple $(\mathscr{E}, \mathscr{E}', \rho)$, consisting of two vector bundles $\mathscr{E}$ and $\mathscr{E}'$ on X, and a map $\rho : F_X^*\mathscr{E} \to \mathscr{E}'$. Note that then we have a map

$$i_t^*\pi_t^*\mathscr{E} \xrightarrow{\simeq} i_u^*\pi_u^*F_X^*\mathscr{E} \xrightarrow{i_u^*\pi_u^*\rho} i_u^*\pi_u^*\mathscr{E}',$$

which we denote simply by $\rho|_{X_0{}^{\mathcal{N}}}$. Let $\mathscr{M}^{\mathcal{N}}_{(\mathscr{E},\mathscr{E}',\rho)}$ be the complex on $X^{\mathcal{N}}$ given by

$$\mathrm{cocone}\left(j_{t*}\pi_t^*\mathscr{E} \oplus j_{u*}\pi_u^*\mathscr{E}' \xrightarrow{\rho|_{X_0^{\mathcal{N}}} - i_u^*} j_{0*}i_u^*\pi_u^*\mathscr{E}' \right), \tag{1.2.1}$$

where the first term is placed in degree 0.

1.3. Consider the standard inclusions $j_{0HT} : X^{\Delta}_{p=0} \hookrightarrow X^{\mathcal{N}}_{p=0}$ and $j_{0\mathrm{dR}} : X^{\Delta}_{p=0} \hookrightarrow X^{\mathcal{N}}_{p=0}$. Then j_{0HT} (resp. $j_{0\mathrm{dR}}$) factors through a map $j'_{HT} : X^{\Delta}_{p=0} \hookrightarrow X^{\mathcal{N}}_u$ (resp. $j'_{dR} : X^{\Delta}_{p=0} \hookrightarrow X^{\mathcal{N}}_t$) and we have $\pi_u \circ j_{0HT} = \pi_t \circ j_{0\mathrm{dR}}$, and both maps are the standard morphism $\pi : X^{\Delta}_{p=0} \to X$ (induced by transmutation from the map of ring stacks $R : W/p \to \mathbf{G}_a$ defined in characteristic p). It follows that if $j^{\Delta} : X^{\Delta}_{p=0} \hookrightarrow X^{\Delta}$ is the inclusion then

$$j_{HT}^*\mathscr{M}^{\mathcal{N}}_{(\mathscr{E},\mathscr{E}',\rho)} \simeq j_*^{\Delta}\pi^*\mathscr{E}', \quad j_{\mathrm{dR}}^*\mathscr{M}^{\mathcal{N}}_{(\mathscr{E},\mathscr{E}',\rho)} \simeq j_*^{\Delta}\pi^*\mathscr{E}.$$

It follows that in order to descend $\mathscr{M}^{\mathcal{N}}_{(\mathscr{E},\mathscr{E}',\rho)}$ to a prismatic F-gauge one has to specify an isomorphism $\pi^*\mathscr{E}' \simeq \pi^*\mathscr{E}$. Therefore starting with a pair $(\mathscr{E}, \rho)$ consisting of a vector bundle $\mathscr{E}$ on X and a map $\rho : F_X^*\mathscr{E} \to \mathscr{E}$ we get a prismatic F-gauge $\mathscr{M}^{\mathrm{syn}}_{(\mathscr{E},\rho)} \in \mathscr{D}(X^{\mathrm{syn}})$ whose pullback to $X^{\mathcal{N}}$ is $\mathscr{M}^{\mathcal{N}}_{(\mathscr{E},\mathscr{E},\rho)}$.

The main result of this article is the following.

Theorem 1.4. *Let G/X be a finite flat abelian group scheme of height ≤ 1, let G^* denote the Cartier dual of G, and let $(\mathscr{E}, \rho)$ be the vector bundle on X with map $\rho : F_X^*\mathscr{E} \to \mathscr{E}$ associated to G by* [10, Exposé VIIA, Remarque 7.5]*. Then*

$$\mathscr{M}(G^*) \simeq \mathscr{M}^{\mathrm{syn}}_{(\mathscr{E},\rho)}.$$

In addition to proving this theorem we discuss how this yields a different perspective on points (1) and (2) above.

1.5. **Comparison with the crystalline theory.** We can also consider the de Rham stack $X^{\mathrm{dR}} = \phi^*X^{\Delta}$. Let $\tilde{j}_{\mathrm{dR}} : X^{\mathrm{dR}} \to X^{\mathcal{N}}$ be the twist of j_{dR} be the semilinear isomorphism $X^{\mathrm{dR}} \to X^{\Delta}$. For any object $U \hookrightarrow T$ of the crystalline site of $X/W(k)$ there is a natural map $g_T : T \to X^{\mathrm{dR}}$ (see section 4 for more discussion), and therefore we can consider the crystal defined by sending T to $g_T^*\mathscr{M}(G^*)$. By [14, 2.2.6] this crystal agrees with crystalline Dieudonné module $M_{\mathrm{crys}}(G^*)$ defined in [2].

Theorem 1.6. *The crystal defined by $(U \hookrightarrow T) \mapsto g_T^*\mathscr{M}_{(\mathscr{E},\rho)}$ is canonically isomorphic to $\Phi^*\mathscr{E}$, where $\Phi : (X/k)_{\mathrm{crys}} \to X_{\mathrm{\acute{e}t}}$ is the canonical morphism of topoi defined in* [2, 4.3.4]*, and the induced isomorphism $M_{\mathrm{crys}}(G^*) \simeq \Phi^*\mathscr{E}$ agrees with the one in* [2, 4.3.6].

1.7. **The Hoobler sequence.** By [11, Theorem B] we have an isomorphism

$$R\Gamma(X_{\mathrm{fppf}}, G) \simeq R\Gamma(X^{\mathrm{syn}}, \mathscr{M}(G^*)\{1\}),$$

and therefore by 1.4 also an isomorphism

$$R\Gamma(X_{\mathrm{fppf}}, G) \simeq R\Gamma(X^{\mathrm{syn}}, \mathscr{M}^{\mathrm{syn}}_{(\mathscr{E},\rho)}\{1\}).$$

Theorem 1.8. *For $m \geq 0$ there is a canonical isomorphism*

$$R\Gamma(X^{\mathrm{syn}}, \mathscr{M}^{\mathrm{syn}}_{(\mathscr{E},\rho)}\{m\})[m] \simeq R\Gamma(X,\ \mathscr{E} \otimes F_{X*}Z\Omega^m_X \xrightarrow{\rho - C} \mathscr{E} \otimes \Omega^m_X\ ),$$

which for $m = 1$ recovers the isomorphism in [7, 1.19] *using* [11, Theorem B]*. Here C denotes the map defined by the Cartier isomorphism, and we somewhat abusively write simply ρ for the semilinear map induced by the inclusion of forms and the map ρ.*

Remark 1.9. As discussed in [7, 3.17], one gets from 1.8 also a derived version for singular schemes by Kan extension.

Prerequisites. We assume that the reader is familiar with the basic theory of geometrization of prismatic cohomology as discussed in [4].

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2. The maps π_t and π_u

2.1. We discuss some of the necessary constructions that will be used later. Let X be a k-scheme. Recall that $X^{\mathscr{N}}$ is defined by transmutation from a ring stack $\mathbf{G}_a^{\mathscr{N}}$ over $k^{\mathscr{N}} = [(\mathrm{Spf}(W[u,t]/(ut - p))/\mathbf{G}_m]$, where u has weight 1 and t has weight -1. For a p-nilpotent ring R the groupoid $k^{\mathscr{N}}(R)$ is the groupoid of triples $(\mathscr{L}, u_{\mathscr{L}}, t_{\mathscr{L}})$, where $\mathscr{L}$ is a line bundle on $\mathrm{Spec}(R)$, and $u : \mathscr{O}_{\mathrm{Spec}(R)} \to \mathscr{L}$ and $t : \mathscr{L} \to \mathscr{O}_{\mathrm{Spec}(R)}$ are maps of line bundles such that $ut = p$ on $\mathscr{O}_{\mathrm{Spec}(R)}$. Geometrically, we can view these as maps of functors $u_{\mathscr{L}} : \mathbf{G}_a \to \mathbf{V}(\mathscr{L})$ and $t_{\mathscr{L}} : \mathbf{V}(\mathscr{L}) \to \mathbf{G}_a$, where $\mathbf{V}(\mathscr{L})$ is the functor sending $g : \mathrm{Spec}(A) \to \mathrm{Spec}(R)$ to $\Gamma(\mathrm{Spec}(A), g^*\mathscr{L})$. Taking the divided power envelope of the origin we then also get maps

$$u^\sharp : \mathbf{G}_a^\sharp \to \mathbf{V}(\mathscr{L})^\sharp, \quad t^\sharp : \mathbf{V}(\mathscr{L})^\sharp \to \mathbf{G}_a^\sharp.$$

The ring stack $\mathbf{G}_a^{\mathscr{N}}$ is defined by considering diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{G}_a^\sharp & \longrightarrow & W & \xrightarrow{F} & F_*W & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u^\sharp} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathbf{V}(\mathscr{L})^\sharp & \longrightarrow & M & \longrightarrow & F_*W & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle t^\sharp} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle p} & & \\ 0 & \longrightarrow & \mathbf{G}_a^\sharp & \longrightarrow & W & \xrightarrow{F} & F_*W & \longrightarrow & 0, \end{array} \tag{2.1.1}$$

where $(\mathscr{L}, u, t)$ denotes the universal triple over $k^{\mathscr{N}}$, and setting $\mathbf{G}_a^{\mathscr{N}} := \mathrm{cone}(M \to W)$. Here all functors are viewed as sheaves for the fpqc topology and the pushforwards indicate the W-module structure; that is, F_*W associates to a p-nilpotent ring R the ring $W(R)$, viewed as a $W(R)$-algebra by the Frobenius morphism $F : W(R) \to W(R)$.

The stack $X^{\mathcal{N}}$ is defined by transmutation from $\mathbf{G}_a^{\mathcal{N}}$. So for a p-nilpotent algebra R with a map $\mathrm{Spec}(R) \to k^{\mathcal{N}}$ we have $X^{\mathcal{N}}(R) = \mathrm{Map}(\mathrm{Spec}(\mathbf{G}_a^{\mathcal{N}}(R)), X)$. Over the locus $k^{\mathcal{N}}_{u\neq 0} = \mathrm{Spec}(W)$ we have $\mathbf{G}_a^{\mathcal{N}} \simeq W/p$ and therefore we have $X^{\mathcal{N}}_{u\neq 0} \simeq X^{\Delta}$. The resulting open immersion is denoted $j_{HT} : X^{\Delta} \hookrightarrow X^{\mathcal{N}}$. Over the locus $k^{\mathcal{N}}_{t\neq 0}$ we have $\mathbf{G}_a^{\mathcal{N}} \simeq F_*W/p$ and therefore $X^{\mathcal{N}}_{t\neq 0} \simeq \sigma^*X^{\Delta}$. Composing with the σ-linear isomorphism $\sigma^*X^{\Delta} \to X^{\Delta}$, we get another inclusion $j_{dR} : X^{\Delta} \hookrightarrow X^{\mathcal{N}}$. The stack X^{syn} is obtained as the pushout (gluing) of the diagram

$$\begin{array}{ccc} X^{\Delta} \coprod X^{\Delta} & \xrightarrow{j_{HT} \coprod j_{dR}} & X^{\mathcal{N}} \\ \downarrow{\scriptstyle \mathrm{id} \coprod \mathrm{id}} & & \\ X^{\Delta}. & & \end{array} \tag{2.1.2}$$

2.2 (The map π_t)**.** The map $F : W \to F_*W$ induces upon passage to the quotient a map $\mathbf{G}_a^{\mathcal{N}} \to F_*W/p$. Composing this with the map $W/p \to \mathbf{G}_a$, defined when $p = 0$, we get a map $\mathbf{G}_a^{\mathcal{N}} \to \mathbf{G}_a$ over $k^{\mathcal{N}}_{p=0}$. The map $\pi_t : X^{\mathcal{N}}_t \to \phi^*X \simeq X$ is defined to be the map induced by transmutation from the restriction of this map to $k^{\mathcal{N}}_t$.

2.3 (The map π_u)**.** Over the locus $k^{\mathcal{N}}_u = k^{\mathcal{N}}_{t=0} \subset k^{\mathcal{N}}$ we have a map $\alpha_u : \mathbf{G}_a^{\mathcal{N}}|_{k^{\mathcal{N}}_u} \to \mathbf{G}_a$ induced by the commutative diagram

$$\begin{array}{ccccc} \mathbf{G}_a^{\sharp} & \longrightarrow & W & & \\ \downarrow{\scriptstyle u^{\sharp}} & & \downarrow{\scriptstyle d} & \searrow{\scriptstyle 0} & \\ \mathbf{V}(\mathscr{L})^{\sharp} & \longrightarrow & M & & \\ & \searrow{\scriptstyle 0} & \downarrow{\scriptstyle d} & & \\ & & W & \xrightarrow{R} & \mathbf{G}_a. \end{array}$$

By transmutation this map defines a morphism $\pi_u : X^{\mathcal{N}}_u \to X$ for any k-scheme X.

Lemma 2.4. *The equalities* (1.1.1) *hold.*

Proof. The restriction of α_t to $k^{\mathcal{N}}_0 := k^{\mathcal{N}}_{t=u=0}$ is equal to $\alpha_u|_{k^{\mathcal{N}}_0} \circ F$. Applying transmutation to this we obtain the equality 1.1.1. □

Remark 2.5. Let us explain a cohomological perspective on the above constructions. By [4], one can identify $X^{\mathcal{N}}_t \simeq X^{\mathrm{dR},+}$. Suppose that $X := \mathrm{Spec}\, R$ for some quasiregular semiperfect ring R. Then $X^{\mathrm{dR},+} \simeq \mathrm{Spec}\left(\bigoplus_i \mathrm{Fil}^i_{\mathrm{Hodge}} R\Gamma_{\mathrm{dR}}(R)\right)/\mathbf{G}_m$. The map $X^{\mathrm{dR},+} \to X$ induced by π_t constructed above is induced by the map of graded rings $R \to \bigoplus_i \mathrm{Fil}^i_{\mathrm{Hodge}} R\Gamma_{\mathrm{dR}}(R)$, induced by the canonical map $R \simeq \mathrm{Fil}^0_{\mathrm{conj}} R\Gamma_{\mathrm{dR}}(R) \to R\Gamma_{\mathrm{dR}}(R)$. Here we view R as a graded ring concentrated in weight 0. By quasisyntomic descent, for a quasisyntomic scheme X, the map π_t is determined by this concrete description in the case of quasiregular semiperfect algebras. One can give a similar perspective on the map $\pi_u : X^{\mathcal{N}}_u \to X$ for any quasisyntomic k-scheme X by using the isomorphism of k-algebra stacks $\phi^*X^{\mathcal{N}}_u \simeq X^{\mathrm{dR},c}$ (see [4, 2.8.3]).

3. Observations about Dieudonné modules

3.1. Following [14, 3.5.26] let $\mathbf{Vect}^{\mathrm{iso}}_{\{0,1\}}(X)$ denote the category of objects $K \in \mathscr{D}_{\mathrm{qcoh}}(X^{\mathrm{syn}})$ for which the Hodge-Tate weights are in $[0, 1]$ and such that K can quasi-syntomic locally be represented by a two-term complex $\alpha : \mathscr{V}^{-1} \to \mathscr{V}^0$ of vectors bundles placed in degrees -1 and 0, and with the map α an isomorphism after inverting p. One of the fundamental results of [14, 3.5.26] is that the Dieudonné functor[1]

$$\mathrm{FFG}(X) \to \mathbf{Vect}^{\mathrm{iso}}_{\{0,1\}}(X), \quad G \mapsto \mathscr{M}(G^*)$$

is an equivalence of categories, where $\mathrm{FFG}(X)$ denotes the category of finite locally free abelian group schemes G/X of p-power order.

Remark 3.2. The conventions about Hodge-Tate weights are not consistent in the literature. We are following the convention that the Breuil-Kisin twist $\mathcal{O}(1)$ has Hodge-Tate weight -1, whereas in the article [11] the convention is that $\mathcal{O}(1)$ has Hodge-Tate weight 1. To account for this difference the functor in [11, Theorem A, Remark 1.1.4] incorporates a Breuil-Kisin twist.

Lemma 3.3. *Let X/k be a quasi-syntomic p-adic formal scheme. Then X^{Δ}, $X^{\mathcal{N}}$ and X^{syn} are flat over $\mathbf{Z}_p$.*

Proof. This follows, for example, from the more general result [15, Corollary 2.21] and the fact that $k^{\mathcal{N}}$, and therefore also k^{syn}, is flat over $\mathbf{Z}_p$. □

Remark 3.4. This applies, in particular, to a smooth k-scheme X. As discussed in [4, 3.1.1 (6)] a flat cover of X^{Δ} can be obtained as follows. Let $X = \cup_i U_i$ be an open cover such that for each i there exists a lifting $\widetilde{U}_i/W(k)$ of U_i to a smooth p-adic formal scheme over $W(k)$ equipped with a lifting of Frobenius. We then get a map $\coprod U_i \to X^{\Delta}$ which is a flat cover.

3.5. In the case when X/k is quasi-syntomic the complex $\mathscr{M}(G)$ associated to $G \in \mathrm{FFG}(X)$ is a sheaf in the following sense. By 3.3 the stack X^{syn} is flat over $\mathbf{Z}_p$. In particular, if $U \to X^{\mathrm{syn}}$ is a flat cover over which $\mathscr{M}(G)$ can be represented by a two-term complex $\alpha : \mathscr{V}^{-1} \to \mathscr{V}^0$, where U is a p-adic formal scheme, then the restriction $\mathscr{M}(G)_U$ of $\mathscr{M}(G)$ to $U_{\text{ét}}$ is locally given by a map $\alpha_U : \mathscr{V}_U^{-1} \to \mathscr{V}_U^0$. This map α_U must be a monomorphism, since α_U is an isomorphism after inverting p and U is flat over $\mathbf{Z}_p$. It follows that $\mathscr{M}(G)_U$ is isomorphic to $M(G)_U := \mathscr{H}^0(\mathscr{M}(G)_U)$. By descent theory we then get a quasi-coherent sheaf $M(G)$ on X^{syn} such that for any $g : \mathrm{Spec}(R) \to X^{\mathrm{syn}}$ we have $\mathscr{M}(G)_R = \mathbf{L}g^* M(G)$. By definition, we therefore have $M(G)_U = (R^2\pi_*^{\mathrm{syn}}\mathcal{O}_{BG^{\mathrm{syn}}})_U$.

3.6. If we further assume that G is killed by p (in addition to X/k being smooth), then we can also describe $M(G)$ as follows. The stack BG^{syn} is also flat over $\mathbf{Z}_p$, since the map $X^{\mathrm{syn}} \to BG^{\mathrm{syn}}$ induced by the tautological map $X \to BG$ is a flat cover. Let $z_X : X^{\mathrm{syn}}_{p=0} \hookrightarrow X^{\mathrm{syn}}$ and $z_{BG} : BG^{\mathrm{syn}}_{p=0} \hookrightarrow BG^{\mathrm{syn}}$ be the reductions modulo p (so the stacks obtained by restricting to $\mathbf{F}_p$-algebras). For any flat morphism $U \to BG^{\mathrm{syn}}$ we then have a short exact sequence

$$0 \longrightarrow \mathcal{O}_U \xrightarrow{p} \mathcal{O}_U \longrightarrow \mathcal{O}_{U_{p=0}} \longrightarrow 0.$$

Let $\pi^{\mathrm{syn}} : BG^{\mathrm{syn}} \to X^{\mathrm{syn}}$ be the projection. Taking cohomology we get for any flat morphism $V \to X^{\mathrm{syn}}$ an exact sequence

(3.6.1)

$$(R^1\pi_*^{\mathrm{syn}}\mathcal{O}_{BG^{\mathrm{syn}}})_V \longrightarrow z_{X*}(R^1\pi_{p=0,*}^{\mathrm{syn}}\mathcal{O}_{BG^{\mathrm{syn}}_{p=0}})|_V \longrightarrow (R^2\pi_*^{\mathrm{syn}}\mathcal{O}_{BG^{\mathrm{syn}}})_V \xrightarrow{p} (R^2\pi_*^{\mathrm{syn}}\mathcal{O}_{BG^{\mathrm{syn}}})_V.$$

Lemma 3.7. *(i) $(R^1\pi_*^{\mathrm{syn}}\mathcal{O}_{BG^{\mathrm{syn}}})_V = 0$.*

(ii) $R^j\pi_*^{\mathrm{syn}}\mathcal{O}_{BG^{\mathrm{syn}}}$ is killed by p for $j > 0$.

(ii) $M(G)_V \simeq (R^1\pi_{p=0,*}^{\mathrm{syn}}\mathcal{O}_{BG^{\mathrm{syn}}_{p=0}})_{V_{p=0}}$.

[1] Note that we compose the functor in loc. cit. with duality to make it covariant.

Proof. From the simplicial presentation

$$\cdots G^{\mathrm{syn},2} \rightrightarrows G^{\mathrm{syn}} \rightrightarrows X^{\mathrm{syn}} \longrightarrow BG^{\mathrm{syn}}$$

and the spectral sequence of a hypercover, we find that for any sheaf of abelian groups $\mathscr{F}$ on X^{syn}

$$R^1\pi_*^{\mathrm{syn}}\mathscr{F}|_{BG^{\mathrm{syn}}} = \mathrm{Ker}(\mathrm{pr}_1^* + \mathrm{pr}_2^* - \mathrm{pr}_{13}^* : \pi_*^{G^{\mathrm{syn}}}\mathscr{F}|_{G^{\mathrm{syn}}} \to \pi_*^{G^{\mathrm{syn}2}}\mathscr{F}|_{G^{\mathrm{syn}2}}),$$

which is $\mathscr{H}om(G^{\mathrm{syn}}, \mathscr{F})$.

Applying this to $\mathscr{F} = \mathbf{G}_a$, this implies (i) since V is flat over $\mathbf{Z}_p$, which implies that multiplication by p on $\mathscr{H}om(G^{\mathrm{syn}}, \mathbf{G}_a)_V$ is injective, and G^{syn} is killed by p, which implies that multiplication by p on $\mathscr{H}om(G^{\mathrm{syn}}, \mathbf{G}_a)_V$ is 0.

For statement (ii), we show that more generally for any sheaf of abelian groups $\mathscr{F}$ on X^{syn} the sheaf $R^j\pi_*^{\mathrm{syn}}\mathscr{F}|_{BG^{\mathrm{syn}}}$ is killed by p for $j > 0$. For $j = 1$ this follows from the preceding observations, since G is killed by p. For bigger j we then proceed by induction. Indeed if $\mathscr{F} \hookrightarrow \mathscr{I}$ is an inclusion into an injective sheaf of abelian groups with quotient $\mathbb{Q} := \mathscr{I}/\mathscr{F}$ then for $j > 1$ we have

$$R^j\pi_*^{\mathrm{syn}}\mathscr{F} \simeq R^{j-1}\pi_*^{\mathrm{syn}}\mathbb{Q}.$$

Finally (iii) follows from (i) and (ii) and consideration of the sequence (3.6.1). □

Corollary 3.8. *For $G \in \mathrm{FFG}(X)$ killed by p we have $M(G) \simeq z_{X*}R^1\pi_{p=0*}^{\mathrm{syn}}\mathscr{O}_{BG_{p=0}^{\mathrm{syn}}}$.*

□

4. The crystalline result

In this section, we compare the isomorphism in 1.4 with the one constructed in [2].

4.1. Let X/k be a k-scheme which is p-completely a local complete intersection.

Recall that the stack X^{crys} is defined by transmutation from the ring stack $\mathbf{G}_a/\mathbf{G}_a^\sharp \simeq F_*W/p$ over $\mathrm{Spf}(W)$. For any object $U = \mathrm{Spec}(A_0) \hookrightarrow T = \mathrm{Spec}(A)$ of the crystalline site of X/W there is an induced map $g_T : T \to X^{\mathrm{crys}}$ obtained as follows. First, we note that the divided power structure on the ideal $\mathrm{Ker}(A \to A_0)$ defines a factorization of the natural inclusion morphism $\mathrm{Ker}(A \to A_0) \to A$ as $\mathrm{Ker}(A \to A_0) \to \mathbf{G}_a^\sharp(A) \to A$. Using the octahedral axiom, we obtain a map

$$A_0 \to \mathbf{G}_a/\mathbf{G}_a^\sharp(A).$$

Therefore, we obtain a morphism $\mathrm{Spec}(\mathbf{G}_a/\mathbf{G}_a^\sharp(A)) \to X$, which, by definition of transmutation, defines a natural map

(4.1.1) $$g_T : T \to X^{\mathrm{crys}}.$$

In particular, there is a natural map

$$R\Gamma(X^{\mathrm{crys}}, \mathscr{O}) \to R\Gamma((X/W)_{\mathrm{crys}}, \mathscr{O}_{X/W}),$$

which is an isomorphism by [6, 6.4].

4.2. For $G \in \mathrm{FFG}(X)$ let $\underline{G}$ be the sheaf of groups on the big crystalline site $\mathrm{CRIS}(X/W)$ obtained by associating to an object

(4.2.1) $$\begin{array}{ccc} U & \hookrightarrow & T \\ \downarrow & & \\ X & & \end{array}$$

the group $G(U)$. The crystalline site of BG/W consists of diagram (4.2.1) together with a G_U-torsor $P_U \to U$. Therefore, there is a tautological class $\xi \in H^1((BG/W)_{\mathrm{crys}}, \underline{G})$; equivalently, we think of ξ as a morphism in the derived category $\mathbf{Z}_{(BG/W)_{\mathrm{crys}}} \to \pi^*\underline{G}[1]$, where $\pi : (BG/W)_{\mathrm{crys}} \to (X/W)_{\mathrm{crys}}$ is the projection. We therefore get for any complex $\mathscr{F}$ on $CRIS(X/W)$ a map of complexes on $(X/W)_{\mathrm{crys}}$

$$\mathscr{R}Hom_X(\underline{G}, \mathscr{F}) \longrightarrow R\pi_*\mathscr{R}Hom_{BG}(\pi^*\underline{G}, \pi^*\mathscr{F}) \xrightarrow{\xi} R\pi_*\mathscr{R}Hom_{BG}(\mathbf{Z}_{BG}[-1], \pi^*\mathscr{F}) \xrightarrow{\simeq} R\pi_*\pi^*\mathscr{F}[1].$$

In particular, we obtain a map

$$\mathscr{E}xt^j_{(X/W)_{\mathrm{crys}}}(\underline{G}, \mathscr{O}_{X/W}) \to R^{j+1}\pi_*\mathscr{O}_{BG/W}. \tag{4.2.2}$$

Proposition 4.3. *Let $U \to X$ be an étale morphism, let $U \hookrightarrow Y$ be an immersion into a smooth W-scheme Y, and let $U \hookrightarrow D$ be the associated p-adically completed divided power envelope. Then the map*

$$\xi_D : \mathscr{E}xt^1(\underline{G}, \mathscr{O}_{X/W})_D \to (R^2\pi_*\mathscr{O}_{BG/W})_D,$$

obtained by evaluating (4.2.2) *for $j = 1$ on D, is an isomorphism.*

Proof. The assertion is étale local on U so it suffices to consider the case when U and Y, and hence also D, are both affine, and we may further assume that $U = X$.

Let us first prove the surjectivity in this case. Since $(R^2\pi_*\mathscr{O}_{BG/W})_D$ is quasi-coherent on D it suffices to show that a class $\alpha \in H^2((BG_U/D)_{\mathrm{CRIS}}, \mathscr{O}_{BG_U/W})$ is locally on D induced by an extension

$$0 \to \mathscr{O}_{BG_U/D} \to \mathscr{E} \to \underline{G} \to 0.$$

The class α corresponds to a $\mathscr{O}_{BG/D}$-gerbe $\mathfrak{X}_\alpha$ on $\mathrm{CRIS}(BG/D) = \mathrm{CRIS}(X/D)|_{BG}$. By the general formalism of over-categories, such a gerbe corresponds to a gerbe $\widetilde{\mathfrak{X}}_\alpha$ on $\mathrm{CRIS}(X/D)$ with a morphism to BG. Concretely, let $\widetilde{\mathfrak{X}}_\alpha$ denote the stack over $\mathrm{CRIS}(X/D)$ which to any object $U \hookrightarrow T$ associates pairs (P, x), where P is a $\underline{G}$-torsor over U and $x \in \mathfrak{X}_\alpha(U \hookrightarrow T)$ is an object (here $U \hookrightarrow T$ is viewed as an object of $\mathrm{CRIS}(BG/D)$ via P). A morphism $(P, x) \simeq (P', x')$ in $\widetilde{\mathfrak{X}}_\alpha(U \hookrightarrow T)$ is a pair (u, u^b), where $u : P \to P'$ is an isomorphism of G-torsors on U and $u^b : x \to x'$ is an isomorphism in $\mathfrak{X}_\alpha(U \hookrightarrow T)$. It is immediate from the definition that $\widetilde{\mathfrak{X}}_\alpha$ is a gerbe with map to BG sending (P, x) to P.

Lemma 4.4. *The band $\mathscr{E}$ of $\widetilde{\mathfrak{X}}_\alpha$ is abelian and the projection to BG defines a short exact sequence of sheaves of abelian groups on* $\mathrm{CRIS}(X/D)$

$$0 \to \mathscr{O}_{X/D} \to \mathscr{E} \to \underline{G} \to 0. \tag{4.4.1}$$

Proof. It suffices to prove the first statement, for then the short exact sequence (4.4.1) is obtained by taking inertia stacks.

To prove that the band $\mathscr{E}$ is abelian proceed as follows. Note first that for $(U \hookrightarrow T) \in \mathrm{CRIS}(X/D)$ and $(P, x) \in \widetilde{\mathfrak{X}}_\alpha(U \hookrightarrow T)$ we obtain a central extension of sheaves of groups

$$1 \to \mathscr{O}_{X/D} \to \underline{\mathrm{Aut}}_{\widetilde{\mathfrak{X}}_\alpha}(P, x) \to \underline{G} \to 1$$

on $\mathrm{CRIS}(X/D)|_{(U \hookrightarrow T)}$. The possible failure of commutativity of the band is therefore measured by a homomorphism

$$\rho : \underline{G} \times \underline{G} \to \mathscr{O}_{X/D}.$$

This map does not depend on the particular local choices of objects (P, x) and therefore is defined globally in terms of the gerbe $\widetilde{\mathfrak{X}}_\alpha$.

We claim that the map ρ must be the zero map. For this choose an immersion $G \times G \hookrightarrow M$ over Y with $M \to Y$ smooth, and let $G \times G \hookrightarrow E$ be the divided power envelope of $G \times G$ in M. Let $\beta \in (\underline{G} \times \underline{G})(G \times G \hookrightarrow M)$ be the universal object. Then for any object $U \hookrightarrow T$, with U and T affine, and section $\gamma \in (\underline{G} \times \underline{G})(U \hookrightarrow T)$ there exists a commutative diagram

$$\begin{array}{ccc} U & \hookrightarrow & T \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\ G \times G & \hookrightarrow & E \end{array}$$

such that $\gamma = f_0^*(\beta)$. Therefore the map ρ is determined by $\rho(\beta) \in \mathscr{O}_{X/D}(E)$. Since β has finite order and $\mathscr{O}_{X/D}(E)$ is torsion free, by [8, 4.7] and the fact that $G \times G$ is lci over the lci scheme X, it follows that $\rho(\beta) = 0$ and $\rho = 0$. □

From the band of $\widetilde{\mathfrak{X}}_\alpha$ we there obtain an extension (4.4.1) whose associated gerbe $B\mathscr{E}$ defines the class $\xi_D([\mathscr{E}])$ of the extension. To complete the proof of surjectivity of ξ_D it then suffices to note that $\mathfrak{X}_\alpha$ is locally on D isomorphic to $B\mathscr{E}$.

This argument also proves the injectivity: We can recover the class of an extension of sheaves by passing to the inertia stacks of the gerbe defined by the image of the extension under ξ_D.

This completes the proof of 4.3. □

4.5. There is a morphism of ringed topoi

$$\Phi : (X/k)_{\text{CRIS}} \to X_{\text{ÉT}}$$

defined as follows, where the target is the big étale topos of X. On the level of underlying topoi this is simply the usual projection to the étale topos. So for a sheaf of sets $\mathscr{F}$ on $X_{\text{ÉT}}$ we have $\Phi^{-1}\mathscr{F}(U \hookrightarrow T) = \mathscr{F}(U)$, and for $\mathscr{G}$ on $(X/W)_{\text{CRIS}}$ the value of $\Phi_*\mathscr{G}$ on U is the global sections $\Gamma((U/k)_{\text{CRIS}}, \mathscr{G})$.

The map on sheaves of rings incorporates a Frobenius twist. For any object $U \hookrightarrow T$ the Frobenius morphism $\mathscr{O}_T \to \mathscr{O}_T$ factors through a map $\mathscr{O}_U \to \mathscr{O}_T$, since the kernel of $\mathscr{O}_T \to \mathscr{O}_U$ has divided powers, which defines the map of rings $\Phi^{-1}\mathscr{O}_X \to \mathscr{O}_{X/k}$.

Lemma 4.6. *For a sheaf of $\mathscr{O}_X$-modules $\mathscr{G}$ on $X_{\text{ÉT}}$ the sheaf $\Phi^*\mathscr{G}$ on $(X/k)_{\text{CRIS}}$ is canonically isomorphic to the sheaf*

$$(U \hookrightarrow T) \mapsto \Gamma(T, g_T^* \pi_t^{\text{crys}*}\mathscr{G}),$$

where $\pi_t^{\text{crys}} : X_{p=0}^{\text{crys}} \to X$ is the restriction of the previously defined map π_t (which is the map induced by transmutation from the map $F_*W/p \to F_*\mathbf{G}_a$) and g_T is as defined in 4.1.1.

Proof. This is immediate from the definitions. □

4.7. Let $M_{\mathbb{\Delta}}(G)$ denote the pullback of $M(G) \in \mathscr{D}_{\text{qcoh}}(X^{\text{syn}})$ to $X^{\mathbb{\Delta}}$, so $M_{\mathbb{\Delta}}(G) := R^2 q_*^{\mathbb{\Delta}} \mathbf{G}_a$, where $q^{\mathbb{\Delta}} : BG^{\mathbb{\Delta}} \to X^{\mathbb{\Delta}}$ is the natural map of prismatizations. Under the isomorphism $\phi^* X^{\mathbb{\Delta}} \simeq X^{\text{crys}}$ we obtain a quasi-coherent sheaf $M_{\text{crys}}(G)$ on X^{crys}, and it follows from the preceding discussion and Proposition 4.3 that the crystal $M'_{\text{crys}}(G) := \mathscr{E}xt^1_{(X/W)_{\text{crys}}}(\underline{G}, \mathscr{O}_{X/W})$, which is the crystalline Dieudonné module as defined in [2], is given by associating to a thickening (4.2.1) the complex $g_T^* M_{\text{crys}}(G)$.

We will also consider the restriction $M'_{\text{crys}}(G)_{p=0}$ of $M'_{\text{crys}}(G)$ to $\text{CRIS}(X/k)$. This sheaf can also be described as $\mathscr{E}xt^1_{(X/k)_{\text{crys}}}(\underline{G}, \mathscr{O}_{X/k})$.

4.8. The universal class $\xi \in H^1(BG, G)$ also defines a map

$$\mathscr{H}om(G, \mathbf{G}_a) \to R^1\pi_*\mathcal{O}_{BG}, \quad \alpha \mapsto \alpha_*\xi.$$

By [12, Equation 14.4] we have a canonical isomorphism $\mathscr{L}ie_{G^*} \simeq \mathscr{H}om(G, \mathbf{G}_a)$.

Lemma 4.9. *The induced morphism $\mathscr{L}ie_{G^*} \to R^1\pi_*\mathcal{O}_{BG}$ is an isomorphism.*

Proof. This is standard. An inverse to the map can be defined as follows. Let $s : X \to BG$ be the section corresponding to the trivial torsor. Then $R^1\pi_*\mathcal{O}_{BG}$ can be viewed as the sheaf which to any $T \to X$ associates the set of isomorphism classes of pairs (P, σ), where $P \to BG_T$ is a $\mathcal{O}_{BG_T}$-torsor and $\sigma : s_T^*P \simeq \mathcal{O}_T$ is a trivialization (the point of adding the "rigidification" at the identity section is so that these objects have no nontrivial automorphisms). Given a pair (P, σ) over $T \to X$ the natural action of $G_T = \mathrm{Aut}(s_T)$ on s_T^*P induces via σ an action of G_T on $\mathcal{O}_T$ which defines a homomorphism $G_T \to \mathcal{O}_T$. This defines an inverse to the morphism defined above. $\square$

4.10. The map $\pi_t^{\mathrm{crys}} : X^{\mathrm{crys}}_{p=0} \to X$ can also be defined on the classifying stacks, so we get a commutative diagram

$$\begin{array}{ccc} BG^{\mathrm{crys}}_{p=0} & \xrightarrow{\pi_t^{\mathrm{crys}BG}} & BG \\ \downarrow{\scriptstyle \pi^{\mathrm{dR}}} & & \downarrow{\scriptstyle \pi} \\ X^{\mathrm{crys}}_{p=0} & \xrightarrow{\pi_t^{\mathrm{crys}}} & X. \end{array}$$

This defines a pullback map (see Lemma 3.7, (ii))

$$\pi_t^{\mathrm{crys}*}\mathscr{L}ie_{G^*} \simeq \pi_t^* R^1\pi_*\mathcal{O}_{BG} \to R^1\pi^{\mathrm{crys}}_{p=0*}\mathcal{O}_{BG^{\mathrm{crys}}_{p=0}} \simeq M_{\mathrm{crys}}(G)_{p=0} \tag{4.10.1}$$

over $X^{\mathrm{crys}}_{p=0}$. Pulling back further to the crystalline site $\mathrm{CRIS}(X/k)$ and applying the pushforward along $z_{\mathrm{crys}} : (X/k)_{\mathrm{CRIS}} \to (X/W(k))_{\mathrm{CRIS}}$ we get a map

$$z_{\mathrm{crys}*}\Phi^*\mathscr{L}ie_{G^*} \to M'_{\mathrm{crys}}(G). \tag{4.10.2}$$

By construction this map is induced by applying $z_{\mathrm{crys}*}$ to a map $\rho : \Phi^*\mathscr{L}ie_{G^*} \to M'_{\mathrm{crys}}(G)_{p=0}$.

Such a map $z_{\mathrm{crys}*}\Phi^*\mathscr{L}ie_{G^*} \to M'_{\mathrm{crys}}(G)$ is also defined [2, 4.3.6] using a map $\rho' : \Phi^*\mathscr{L}ie_{G^*} \to M'_{\mathrm{crys}}(G)_{p=0}$.

Proposition 4.11. *We have $\rho = \rho'$. In particular, the map* (4.10.2) *agrees with the one defined in* [2, 4.3.6].

Proof. By adjunction, to verify that $\rho = \rho'$ it suffices to show that the adjoint maps

$$\mathscr{H}om(G, \mathcal{O}_X) = \mathscr{L}ie_{G^*} \to \Phi_* M'_{\mathrm{crys}}(G)_{p=0}$$

agree.

The map ρ' can be described as follows. There is an isomorphism [2, 4.3.8]

$$\text{"p"} : M'_{\mathrm{crys}}(G)_{p=0} = \mathscr{E}xt^1(\underline{G}, \mathcal{O}_{X/k}) \to \mathscr{H}om(\underline{G}, \mathcal{O}_{X/k}) \tag{4.11.1}$$

defined by sending the class of an extension

$$0 \to \mathcal{O}_{X/k} \to \mathscr{E} \to \underline{G} \to 0$$

to the map $\delta : \underline{G} \to \mathcal{O}_{X/k}$ obtained from the snake lemma applied to multiplication by p on this sequence, noting that multiplication by p on $\underline{G}$ and $\mathcal{O}_{X/k}$ is 0.

As discussed in [2, Paragraph following 4.3.8] the composition

$$\mathscr{H}om(G,\mathcal{O}_X)\xrightarrow{\rho'}\Phi_*\mathscr{E}xt^1(\underline{G},\mathcal{O}_{X/k})\xrightarrow{\texttt{"p"}}\Phi_*\mathscr{H}om(\underline{G},\mathcal{O}_{X/k})\simeq\mathscr{H}om(G,\Phi_*\mathcal{O}_{X/k}),$$

where the last isomorphism is by adjunction noting that $\underline{G}=\Phi^{-1}G$, is the map induced by $\Phi^b:\mathcal{O}_X\to\Phi_*\mathcal{O}_{X/k}$.

There is also a map

$$\texttt{"p"}:\mathscr{E}xt^1(G,R\Phi_*\mathcal{O}_{X/k})\to\mathscr{H}om(G,\Phi_*\mathcal{O}_{X/k}) \tag{4.11.2}$$

defined similarly: For a morphism $\lambda:G\to R\Phi_*\mathcal{O}_{X/k}[1]$ multiplication by p on cone(λ) defines a morphism

$$G=\mathrm{cone}(R\Phi_*\mathcal{O}_{X/k}\to\mathrm{cone}(\lambda))\to\mathrm{cocone}(\mathrm{cone}(\lambda)\to G)\simeq R\Phi_*\mathcal{O}_{X/k}.$$

By the construction the map (4.11.2) equals the composition of the natural map $\mathscr{E}xt^1(G,R\Phi_*\mathcal{O}_{X/k})\to\Phi_*\mathscr{E}xt^1(\underline{G},\mathcal{O}_{X/k})$ with the map (4.11.1).

Now recall that $\Phi_*=u_{X/S*}$ so we also have a distinguished triangle (using that X/k is smooth; see [3, 7.24])

$$Ru_{X/W(k)*}\mathcal{O}_{X/W}\xrightarrow{p}Ru_{X/W(k)*}\mathcal{O}_{X/W}\longrightarrow R\Phi_*\mathcal{O}_{X/k}\xrightarrow{\partial}Ru_{X/W(k)*}\mathcal{O}_{X/W}[1].$$

The map ∂ induces by composing with the reduction map a morphism $\bar\partial:R\Phi_*\mathcal{O}_{X/k}\to R\Phi_*\mathcal{O}_{X/k}[1]$, which then induces a morphism (which we denote by the same symbol)

$$\bar\partial:\mathscr{H}om(G,\Phi_*\mathcal{O}_{X/k})\simeq\mathscr{H}om(G,R\Phi_*\mathcal{O}_{X/k})\to\mathscr{E}xt^1(G,R\Phi_*\mathcal{O}_{X/k}).$$

Lemma 4.12. *The composition*

$$\mathscr{L}ie_{G^*}\simeq\mathscr{H}om(G,\mathcal{O}_X)\xrightarrow{\Phi^b}\mathscr{H}om(G,\Phi_*\mathcal{O}_{X/k})\xrightarrow{\bar\partial}\mathscr{E}xt^1(G,R\Phi_*\mathcal{O}_{X/k})\longrightarrow\Phi_*\mathscr{E}xt^1(\underline{G},\mathcal{O}_{X/k})$$

is the map adjoint to ρ.

Proof. Let $\xi_{\text{ÉT}}\in H^1(BG_{\text{ÉT}},G)$ be the tautological class, which defines as in 4.2 a map

$$\mathscr{R}Hom(G,R\Phi_*\mathcal{O}_{X/k})\to R\pi_*R\Phi^{BG}_*\mathcal{O}_{BG/k}.$$

The result then follows from the definition of ρ and the fact that the diagram

$$\begin{array}{ccc}
u_*\mathscr{E}xt^1(\underline{G},\mathcal{O}_{X/W}) & \xrightarrow{\xi} & u_*R^2\pi^{\mathrm{crys}}_*\mathcal{O}_{BG/W}\\
\uparrow{\scriptstyle\partial} & & \uparrow{\scriptstyle\partial}\\
u_*\mathscr{H}om(\underline{G},\mathcal{O}_{X/k}) & \xrightarrow{\xi} & u_*R^1\pi^{\mathrm{crys}}_*\mathcal{O}_{BG/k}\\
\uparrow & & \uparrow\\
\mathscr{H}om(G,R\Phi_*\mathcal{O}_{X/k}) & \xrightarrow{\xi_{\text{ÉT}}} & R^1\pi_*R\Phi^{BG}_*\mathcal{O}_{BG/k}
\end{array}$$

commutes, where the bottom vertical arrows are induced by adjunction. □

Lemma 4.13. *The composition*

$$\mathscr{H}om(G,\Phi_*\mathcal{O}_{X/k})\xrightarrow{\bar\partial}\mathscr{E}xt^1(G,R\Phi_*\mathcal{O}_{X/k})\longrightarrow\Phi_*\mathscr{E}xt^1(\underline{G},\mathcal{O}_{X/k})\xrightarrow{\text{"}p\text{"}}\mathscr{H}om(G,\Phi_*\mathcal{O}_{X/k})$$

is the identity map.

Proof. Indeed given a map $\lambda : G \to \Phi_* \mathcal{O}_{X/k}$ (defined locally on X) the image under the composition is obtained by first pulling back the sequence

$$Ru_*\mathcal{O}_{X/W} \xrightarrow{p} Ru_*\mathcal{O}_{X/W} \longrightarrow R\Phi_*\mathcal{O}_{X/k}$$

along λ, and then considering multiplication by p on the resulting extension as in the construction of the map (4.11.2). It follows immediately from this construction that this recovers λ. □

This lemma completes the proof of 4.11 since it implies that both ρ and ρ' compose with the map (4.11.2) to the map induced by Φ^b. □

Corollary 4.14. *If G has height 1 then the map (4.10.1) is an isomorphism.*

Proof. By [2, 4.3.6] and 4.11 the map (4.10.1) becomes an isomorphism after pulling back to the crystalline site of $X/W(k)$. Now recall (as in 3.4) that if $X = \cup_i U_i$ is an open cover for which there exist smooth lifts $\widetilde{U}_i$ with lifts of Frobenius then we get a flat cover $\coprod_i \widetilde{U}_i \to X^{\mathrm{crys}}$. Since each $U_i \hookrightarrow \widetilde{U}_i$ defines an object of $\mathrm{CRIS}(X/W(k))$ it follows that the map (4.10.1) is an isomorphism on a flat cover of X^{crys}. □

5. The case of a point

In the case when $X = \mathrm{Spec}(k)$ the theory reduces to the classical theory [9, 2.3] as we now explain.

5.1. Consider $\mathrm{Spf}(W(k)[u,t]/(ut-p))$ with action of $\mathbf{G}_m$ given by assigning u to have (resp. t) weight 1 (resp. -1), so that $k^{\mathcal{N}} \simeq [\mathrm{Spf}(W(k)[u,t]/(ut-p))/\mathbf{G}_m]$. Then a quasi-coherent sheaf $\mathcal{M}$ on $k^{\mathcal{N}}$ is equivalent to the data of a sequence of quasi-coherent W-modules with maps

$$\cdots \rightleftarrows M^{n+1} \underset{u_n}{\overset{t_n}{\rightleftarrows}} M^n \underset{u_{n-1}}{\overset{t_{n-1}}{\rightleftarrows}} M^{n-1} \rightleftarrows \cdots$$

for which $u_n \circ t_n = p$ and $t_n \circ u_n = p$ for all n. We will write $(M^\bullet, u_\bullet, t_\bullet)$ for such a system.

Given a quasi-coherent sheaf $\mathcal{M}$ on $k^{\mathcal{N}}$ with associated system $(M^\bullet, u_\bullet, t_\bullet)$ we can consider the $W(k)$-modules

$$M^{-\infty} := \mathrm{colim}_t M^n, \quad M^{\infty} := \mathrm{colim}_u M^n.$$

Then $M^{-\infty}$ is the restriction of $\mathcal{M}$ to the open substack $\mathrm{Spf}(W) \simeq k^{\mathcal{N}}_{t\neq 0}$ and M^{∞} is the restriction to $\mathrm{Spf}(W) \simeq k^{\mathcal{N}}_{u\neq 0}$. Therefore to descend $\mathcal{M}$ to a quasi-coherent sheaf on k^{syn} one has to specify an isomorphism

$$\varphi : F_X^* M^{\infty} \simeq M^{-\infty}.$$

Lemma 5.2. *Let $\mathcal{M} \in \mathbf{Vect}^{\mathrm{iso}}_{\{0,1\}}(\mathrm{Spec}(k))$ be an object with associated system $(M^\bullet, u_\bullet, t_\bullet)$. Then the maps $t_n : M^{n+1} \to M^n$ (resp. $u_n : M^n \to M^{n+1}$) are isomorphisms for $n \leq -1$ (resp. $n \geq 1$).*

Proof. Note first that $\mathcal{M}$ is the cone of a map of vector bundles $\alpha : \mathcal{V}^{-1} \to \mathcal{V}^0$ with the $\mathcal{V}^i$ of Hodge-Tate weights in $[0,1]$. This can be verified directly using commutative algebra, or by noting that by [11, 7.1.1] the object $\mathcal{M}$ is the prismatic Dieudonné module of a finite flat abelian group scheme G, and then embedding G into an abelian variety realizing G as the kernel of a map of abelian varieties. Taking Dieudonné modules and using [14, 3.5.5] we then get the presentation of $\mathcal{M}$.

For a given n the functor on quasi-coherent sheaves on $k^{\mathcal{N}}$ sending $\mathcal{M}$ to M^n is exact. It therefore suffices to prove the analogous result for a vector bundle $\mathcal{V}$ on k^{syn} with Hodge-Tate weights in

$[0,1]$. The restriction to $k^{\mathcal{N}}$ of such a vector bundle is of the form $V^0 \otimes_k \mathcal{O}_{k^{\mathcal{N}}} \oplus V^1 \otimes_k \mathcal{O}_{k^{\mathcal{N}}}\{-1\}$ for some W-modules V^0 and V^1 (cf. [14, Prop. 3.3.23, (3.3.8)]), and here the result is immediate. □

5.3. In particular, for $\mathcal{M} \in \mathbf{Vect}^{\mathrm{iso}}_{\{0,1\}}(\mathrm{Spec}(k))$ the natural maps $M^1 \to M^{\infty}$ and $M^0 \to M^{-\infty}$ are isomorphisms. The isomorphism φ therefore defines an identification $M^0 \simeq F^*M^1$, and the maps $t_0 : M^1 \to M^0$ and $u_0 : M^0 \to M^1$ can be viewed as maps $V : M^1 \to F^*M^1$ and $F : F^*M^1 \to M^1$, giving M^1 the structure of a module over the Dieudonné ring $W\langle F, V\rangle$.

Corollary 5.4. *The category* $\mathbf{Vect}^{\mathrm{iso}}_{\{0,1\}}(\mathrm{Spec}(k))$ *is equivalent to the category of finite length (as $W(k)$-modules) Dieudonné modules.*

□

Consider the inclusions $j_t : [\mathrm{Spec}(k[t])/\mathbf{G}_m] \hookrightarrow k^{\mathcal{N}}_{p=0}$, $j_u : [\mathrm{Spec}(k[u])/\mathbf{G}_m] \hookrightarrow k^{\mathcal{N}}_{p=0}$, and $j_0 : B\mathbf{G}_m \to k^{\mathcal{N}}_{p=0}$. Let $\mathcal{M} \in \mathbf{Vect}^{\mathrm{iso}}_{\{0,1\}}(\mathrm{Spec}(k))$ be an F-gauge.

Corollary 5.5. *For an F-gauge $\mathcal{M} \in \mathbf{Vect}^{\mathrm{iso}}_{\{0,1\}}(\mathrm{Spec}(k))$ the sequence*

$$\mathcal{M}_{p=0}|_{k^{\mathcal{N}}} \to j_{t*}j_t^*\mathcal{M}_{p=0} \oplus j_{u*}j_u^*\mathcal{M}_{p=0} \to j_{0*}j_0^*\mathcal{M}_{p=0}$$

is a fiber sequence, where the maps are induced by restriction and all functors are derived.

Proof. The sequence in question is obtained by tensoring the corresponding sequence for $\mathcal{O}_{k^{\mathrm{syn}}}$ so it suffices to show the result for $\mathcal{O}_{k^{\mathrm{syn}}}$. Working locally on $k^{\mathcal{N}}$ the result in this case follows from noting that the sequence

$$0 \to k[u,t]/(ut) \to k[u] \oplus k[t] \to k \to 0$$

is exact. Here, the map $k[u,t]/(ut) \to k[u] \oplus k[t]$ is induced by sending a polynomial $g(u,t) \mapsto (g(u,0), g(0,t))$. Further, the map $k[u] \oplus k[t] \to k$ is defined by $(x(u), y(t)) \mapsto x(0) - y(0)$. □

Corollary 5.6. *A morphism $f : \mathcal{M} \to \mathcal{M}'$ in* $\mathbf{Vect}^{\mathrm{iso}}_{\{0,1\}}(\mathrm{Spec}(k))$ *is an isomorphism if and only if the induced morphism of W-modules $j_{\mathrm{dR}}\mathcal{M} \to j^*_{\mathrm{dR}}\mathcal{M}'$ is an isomorphism.*

Proof. This is equivalent to the statement that the corresponding map of systems $(M^\bullet, u_\bullet, t_\bullet) \to (M'^\bullet, u'_\bullet, t'_\bullet)$ is an isomorphism if and only if the induced map $M^1 \to M'^1$ is an isomorphism, which follows from 5.2. □

5.7. For studying height 1 group schemes we are particularly interested in the Dieudonné module associated to a pair (E, ρ), consisting of a k-vector space E with a map $\rho : F^*E \to E$. In fact, it is convenient to consider a slightly more general situation. Let (E, E', ρ) be a triple consisting of two k-vector spaces E and E' and a map $\rho : E' \to E$. Such a triple (E, E', ρ) defines a system (called a Gauge) $(M^\bullet, u_\bullet, t_\bullet)$ by setting $M^n := E$ for $n > 0$, $M^n = E'$ for $n \leq 0$, $u_n = \mathrm{id}_E$ for $n > 0$, $u_0 = \rho$, $u_n = 0$ for $n < 0$, $t_n = 0$ for $n \geq 0$, and $t_n = \mathrm{id}_{E'}$ for $n < 0$.

Lemma 5.8. *The natural map*

$$M^\bullet \to \mathrm{cocone}(E \otimes_k k[u] \oplus E' \otimes_k k[t] \xrightarrow{\mathrm{id}\otimes\nu - \rho\otimes\nu} E)$$

is an isomorphism, where $\nu : k[u] \to k$ is the canonical quotient map.

Proof. This is immediate from construction. □

6. The case of a smooth scheme X/k

6.1. **The complexes $\mathcal{M}^{\mathcal{N}}_{(\mathcal{E},\mathcal{E}',\rho)}$.** Let X/k be a smooth scheme, and as in 1.2 consider a triple $(\mathcal{E},\mathcal{E}',\rho)$ consisting of a pair of vector bundles $\mathcal{E}$ and $\mathcal{E}'$ on X and a map $\rho:\mathcal{E}\to\mathcal{E}'$. Define $\mathcal{M}^{\mathcal{N}}_{(\mathcal{E},\mathcal{E}',\rho)}\in\mathcal{D}(X^{\mathcal{N}})$ as in (1.2.1).

Lemma 6.2. *Locally on $X^{\mathcal{N}}$ the complex $\mathcal{M}^{\mathcal{N}}_{(\mathcal{E},\mathcal{E}',\rho)}$ is isomorphic to a 2-term complex of vector bundles concentrated in degrees -1 and 0, which is acyclic after inverting p .*

Proof. There is a distinguished triangle

$$(j_{u*}\pi_u^*\mathcal{E}'\xrightarrow{i_u^*}j_{0*}i_u^*\pi_u^*\mathcal{E}')\longrightarrow\mathcal{M}^{\mathcal{N}}_{(\mathcal{E},\mathcal{E}',\rho)}\longrightarrow j_{t*}\pi_t^*\mathcal{E}\xrightarrow{+1}$$

so it suffices to show that the complexes $j_{t*}\pi_t^*\mathcal{E}$ and $j_{u*}\pi_u^*\mathcal{E}'\xrightarrow{i_u^*}j_{0*}i_u^*\pi_u^*\mathcal{E}'$ have projective dimension 1. This can be verified locally when $\mathcal{E}$ and $\mathcal{E}'$ are free, which reduces to the case of trivial vector bundles of rank 1 and $k^{\mathcal{N}}$. In this case, $j_{t*}\mathcal{O}_{k_t^{\mathcal{N}}}$ is represented by the complex

$$W[u,t]/(ut-p)\xrightarrow{u}W[u,t]/(ut-p)$$

and the restriction map $j_{u*}\mathcal{O}_{k_u^{\mathcal{N}}}\to j_{0*}k_0^{\mathcal{N}}$ is represented by the map of complexes

$$\begin{array}{ccccc} & & W[u,t]/(ut-p) & \xrightarrow{t} & W[u,t]/(ut-p)\\ & & \downarrow & & \|\\ W[u,t]/(ut-p) & \xrightarrow{\begin{pmatrix}u\\ t\end{pmatrix}} & W[u,t]/(ut-p)^{\oplus 2} & \xrightarrow{\begin{pmatrix}t & u\end{pmatrix}} & W[u,t]/(ut-p),\end{array}$$

where the first vertical map is the inclusion of the first factor. From this the result follows. □

Corollary 6.3. *For a pair $(\mathcal{E},\rho)$ consisting of a vector bundle $\mathcal{E}$ on X and a map $\rho:F^*\mathcal{E}\to\mathcal{E}$ the associated prismatic F-gauge $\mathcal{M}^{\mathrm{syn}}_{(\mathcal{E},\rho)}\in\mathcal{D}(X^{\mathrm{syn}})$ defined in 1.3 is an object of $\mathbf{Vect}^{\mathrm{iso}}_{\{0,1\}}(X)$.*

Proof. The only part that does not follow immediately from 6.2 is the range of the Hodge-Tate weights. This part can be verified after derived pullback along an arbitrary map $\operatorname{Spec}\Omega\to X$, where Ω is a perfect field of characteristic p, where it follows from 5.8. □

6.4. **Dieudonné module of a group scheme.** Let X/k be a smooth group scheme and let $G\in\mathrm{FFG}(X)$ be a finite flat abelian group scheme killed by p. Let $\mathcal{M}(G)_{p=0}\in\mathcal{D}(X^{\mathrm{syn}}_{p=0})$ denote $R^1\pi^{\mathrm{syn}}_{p=0,*}\mathcal{O}_{BG^{\mathrm{syn}}_{p=0}}$ so that by 3.7 we have $\mathcal{M}(G)=z_*\mathcal{M}(G)_{p=0}$. Let $\mathcal{M}(G)_t$ (resp. $\mathcal{M}(G)_u$, $\mathcal{M}(G)_0$) denote the restriction of $\mathcal{M}(G)_{p=0}$ to $X_t^{\mathcal{N}}$ (resp. $X_u^{\mathcal{N}}$, $X_0^{\mathcal{N}}$).

Lemma 6.5. *The natural map in $\mathcal{D}(X^{\mathcal{N}})$*

$$\mathcal{M}(G)_{p=0}|_{X^{\mathcal{N}}_{p=0}}\to\mathrm{cocone}(j_{t*}\mathcal{M}(G)_t\oplus j_{u*}\mathcal{M}(G)_u\to j_{0*}\mathcal{M}(G)_0)$$

is an isomorphism.

Proof. By the derived version of Nakayama's lemma [16, Tag 0G1U] it suffices to show that the map is an isomorphism over points $\operatorname{Spec}(\Omega)\to X^{\mathcal{N}}$ with Ω an algebraically closed field over k, and furthermore it suffices to consider such points with image in the Hodge-Tate stack $X^{HT}:=X^{\mathcal{N}}_{u=t=0}$. The stack X^{HT} is a gerbe over X (see [6, 5.12]), so any map $\operatorname{Spec}(\Omega)\to X^{HT}$ factors through $\operatorname{Spec}(\Omega)^{HT}$. This therefore reduces the proof to the case of a field which is 5.5. □

Suppose furthermore that G^* has height 1, and let $(\mathscr{E}, \rho)$ denote the Lie algebra $\mathscr{L}ie(G^*)$ with semilinear map $\rho : F_X^* \mathscr{L}ie_{G^*} \to \mathscr{L}ie_{G^*}$.

Pulling back along the maps defined by the commutative squares

$$\begin{array}{ccc} BG_t^{\mathcal{N}} & \xrightarrow{\pi_t^{BG}} & BG \\ \downarrow & & \downarrow \\ X_t^{\mathcal{N}} & \xrightarrow{\pi_t} & X, \end{array}$$

$$\begin{array}{ccc} BG_u^{\mathcal{N}} & \xrightarrow{\pi_u^{BG}} & BG \\ \downarrow & & \downarrow \\ X_u^{\mathcal{N}} & \xrightarrow{\pi_u} & X, \end{array}$$

and

$$\begin{array}{ccc} BG_0^{\mathcal{N}} & \xrightarrow{\pi_0^{BG}} & BG \\ \downarrow & & \downarrow \\ X_0^{\mathcal{N}} & \xrightarrow{\pi_0} & X, \end{array}$$

we get a commutative diagram

$$\begin{array}{ccc} j_{t*}\pi_t^*\mathscr{E} \oplus j_{u*}\pi_u^*\mathscr{E} & \xrightarrow{\rho|_{X_0^{\mathcal{N}}} - i_u^*} & j_{0*}i_u^*\pi_u^*\mathscr{E} \\ \downarrow & & \downarrow \\ j_{t*}\mathscr{M}(G)_t \oplus j_{u*}\mathscr{M}(G)_u & \longrightarrow & j_{0*}\mathscr{M}(G)_0. \end{array}$$

By taking cocones of the horizontal maps, the above diagram defines a map

$$\mathscr{M}^{\mathcal{N}}_{(\mathscr{E},\rho)} \to \mathscr{M}(G).$$

By construction this is compatible with the gluing data and therefore also defines a morphism

$$\mathscr{M}^{\mathrm{syn}}_{(\mathscr{E},\rho)} \to \mathscr{M}(G). \tag{6.5.1}$$

Theorem 6.6. *The map* (6.5.1) *is an isomorphism.*

Proof. As in the proof of 6.5 it suffices to consider the case when X is a point, where the result follows from 4.14, 5.6, and 5.8. □

7. Proof of Theorem 1.8

This is a variation of the proof of [4, 4.4.7].

First let us describe the cohomology

$$R\Gamma(X^{\mathcal{N}}, \mathscr{M}^{\mathcal{N}}_{(\mathscr{E},\mathscr{E}',\rho)}\{m\})$$

for a triple $(\mathscr{E}, \mathscr{E}', \rho)$ and an integer $m \geq 0$. For this it suffices to calculate $R\Gamma(X_t^{\mathcal{N}}, \pi_t^*\mathscr{E}\{m\})$, $R\Gamma(X_u^{\mathcal{N}}, \pi_u^*\mathscr{E}'\{m\})$, $R\Gamma(X_0^{\mathcal{N}}, i_u^*\pi_u^*\mathscr{E}'\{m\})$, and the restriction maps relating these. Let $\mathrm{Spec}(R) \to X$

be an étale morphism and let $\mathscr{E}_R$ (resp. $\mathscr{E}'_R$) be the R-module defined by $\mathscr{E}$ (resp. $\mathscr{E}'$). Then by the projection formula and [4, 2.8.6] we have

$$R\Gamma((\mathrm{Spec}(R))^{\mathcal{N}}_t, \pi_t^*\mathscr{E}\{m\}) = \mathscr{E}_R \otimes_R R\Gamma(\mathrm{Spec}(R)^{\mathcal{N}}_t, \mathcal{O}_{X_t^{\mathcal{N}}}\{m\}) \simeq \mathscr{E}_R \otimes \mathrm{Fil}^m_{Hodge}\Omega^\bullet_{R/k},$$

where $\mathrm{Fil}^m_{Hodge}\Omega^\bullet_{R/k}$ denotes the m-th step of the Hodge filtration viewed as a complex of R-modules via the Frobenius morphism on R. Similarly we have

$$R\Gamma(X^{\mathcal{N}}_u, \pi_u^*\mathscr{E}'\{m\}) \simeq \mathscr{E}'_R \otimes_R \mathrm{Fil}^m_{conj}\Omega^\bullet_{R/k},$$

where $\mathrm{Fil}^m_{conj}\Omega^\bullet_{R/k}$ denotes the m-th step of the conjugate filtration, and

$$R\Gamma(X^{\mathcal{N}}_0, i_u^*\pi_u^*\mathscr{E}'\{m\}) \simeq \mathscr{E}'_R \otimes \Omega^m_{R/k}[-m].$$

Furthermore, by loc. cit. the restriction map

$$\rho : \mathscr{E}_R \otimes \mathrm{Fil}^m_{Hodge}\Omega^\bullet_{R/k} \to \mathscr{E}' \otimes_R \Omega^m_{R/k}[-m]$$

is the tensor product of ρ with the natural map $\mathrm{Fil}^m_{Hodge}\Omega^\bullet_{R/k} \to \Omega^m_{R/k}[-m]$ (which we suppress from the notation), and the restriction map

$$C : \mathscr{E}'_R \otimes_R \mathrm{Fil}^m_{conj}\Omega^\bullet_{R/k} \to \mathscr{E}'_R \otimes_R \Omega^m_{R/k}[-m]$$

is the tensor product of the identity map on $\mathscr{E}'_R$ (which we also suppress from the notation) and the map $\mathrm{Fil}^m_{conj}\Omega^\bullet_{R/k} \to \Omega^m_{R/k}[-m]$ given by the Cartier operator.

Taking limits over the category of étale morphisms $\mathrm{Spec}(R) \to X$ we find that $R\Gamma(X^{\mathcal{N}}, \mathscr{M}^{\mathcal{N}}_{(\mathscr{E},\mathscr{E}',\rho)}\{m\})$ is isomorphic to

$$\mathrm{cocone}\left(R\Gamma(X, \mathscr{E} \otimes \mathrm{Fil}^m_{Hodge}\Omega^\bullet_{X/k}) \oplus R\Gamma(X, \mathscr{E}' \otimes \mathrm{Fil}^m_{conj}\Omega^\bullet_{X/k}) \xrightarrow{\rho - C} R\Gamma(X, \mathscr{E}' \otimes \Omega^m_{X/k}[-m]) \right).$$

Now in the case of $\mathscr{M}^{\mathrm{syn}}_{(\mathscr{E},\rho)}$ this implies that $\mathrm{R}\Gamma(X^{\mathrm{syn}}, \mathscr{M}^{\mathrm{syn}}_{(\mathscr{E},\rho)}\{m\})$ is given by first forming the cocone (incorporating the gluing of the two copies of X^{Δ})

$$\mathscr{K} := \mathrm{cocone}\left(R\Gamma(X, \mathscr{E} \otimes \mathrm{Fil}^m_{Hodge}\Omega^\bullet_{X/k}) \oplus R\Gamma(X, \mathscr{E} \otimes \mathrm{Fil}^m_{conj}\Omega^\bullet_{X/k}) \to R\Gamma(X, \Omega^\bullet_{X/k})\right)$$

and then taking the cocone of the induced map $\rho - C : \mathscr{K} \to R\Gamma(X, \Omega^m_{X/k}[-m])$. Now observe that the restriction map

$$\mathscr{E} \otimes \mathrm{Fil}^m_{Hodge}\Omega^\bullet_{X/k} \oplus \mathscr{E} \otimes \mathrm{Fil}^m_{conj} \to \mathscr{E} \otimes \Omega^\bullet_{X/k}$$

is an isomorphism in all degrees except degree m, where the map is given by

$$\mathscr{E} \otimes \Omega^m_{X/k} \oplus \mathscr{E} \otimes F_{X*}Z^m_{X/k} \to \mathscr{E} \otimes \Omega^m_{X/k}.$$

It follows that $\mathscr{K} \simeq R\Gamma(X, \mathscr{E} \otimes F_{X*}Z^m_{X/k}[-m])$ with the restriction map to $R\Gamma(X, \Omega^m_{X/k}[-m])$ given by $\rho - C$.

From this 1.8 follows. □

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