READING AND LEARNING SEMINAR ON PERFECTOID SPACES

Meeting time: Friday, 1:30 PM **Location:** ESB 4133 at PIMS

This is a reading and learning seminar on perfectoid spaces at UBC organised by Sujatha Ramdorai and Shubhodip Mondal, during Winter term 1, 2023. The goal for our seminar would be to go through certain parts of Scholze's paper on perfectoid spaces [Sch12]. In particular, we will go over the equivalent material of section 3-5 in detail. This will mostly cover the "algebraic" aspects of the theory and will span over the first 8 talks. The main goal of these talks would be to prove the "tilting equivalence" which builds a bridge between characteristic 0 and characteristic p. Talk 9 and Talk 10 will discuss adic spaces (following e.g., [Wed]) and introduce the notion of perfectoid spaces. More details will be decided as the talks progress. All references below are from [Sch12] unless otherwise mentioned. Each talks are supposed to last around an hour with some time for questions and discussions afterwards.

Talk 1 (*Sujatha*/Sept. 29). Recall the definition of complete non-archimedean fields, residue field, pseudo-uniformizer etc (Chapter 1 of [Bha]). Introduce the perfection functors, Fontaine's "tilting" operation and prove Lemma 2.0.6 following Chapter 2 [Bha]. End with the definition of a perfectoid field (Definition 3.1 and Lemma 3.2 from [Sch12]) and Example 3.1.2 from [Bha].

Reading for the thanksgiving break (no meeting on Oct. 6): The proof of Proposition 3.8 in [Sch12] uses the theory of Newton Polygons to justify why certain irreducible polynomials have coefficients in the ring of integers, without having to do any algebraic manipulations. For this week, instead of having a meeting, we will read about Newton polygons. There are several references for this (e.g., [Cas86, Chapter 6, §3]) and you are encouraged to use your own. The only thing that we need is the definition of Newton polygons and the result that if a polynomial is irreducible then its Newton polygon is "pure", i.e., a line, see [Cas86, Chapter 6, §3, Corollary 1].

Talk 2 (*Amin*/Oct. 13). Prove Lemma 3.4 (recall that parts of the proof of Lemma 3.4 is already covered in Talk 1) and state Theorem 3.7. Prove Proposition 3.8. Include Example 3.2.12 from [Bha] that shows that tilt of $\widehat{\mathbb{Q}_p(\mu_{p^\infty})}$ is $\mathbb{F}_p(\widehat{(t^{\frac{1}{p^\infty}})})$.

Talk 3 (*Atharva*/Oct. 20). This talk will be on the cotangent complex. The classical reference for this material is [Ill71] and [Ill72]. We will follow [Bha]. Briefly cover Section 6.1 from [Bha]: cover Construction 6.1.1, Definition 6.1.2, state the properties 1-8 (without proof), Theorem 6.1.3 (without proof), Proposition 6.1.4 and Corollary 6.1.5 (with proofs). Time permitting, cover Example 6.1.6 from [Bha] giving a deformation theory perspective on p-typical Witt vectors.

Talk 4 (*Xiaohan*/Oct. 27). This talk will be on almost ring theory ([GR03]) following Ch. 4 from [Sch12]. Cover untill Def. 4.13 in [Sch12]. For background on quotients of abelian categories see [SP23, Tag 02MS]. Also see [Bha, Construction 4.1.5] for a concrete description of the category of almost modules.

Talk 5 (*Amin*/Nov. 3). Prove [Sch12, Prop. 3.8] (this was only stated in Talk 2). Introduce the three apriori different definition of perfectoid algebras ([Sch12, Def. 5.1]). Discuss [Bha, Example 6.2.4] as examples of perfectoid algebras (without proving it). State [Sch12, Thm. 5.2], which says that all these definitions are equivalent, which leads to the "tilting equivalence". Rest of the talk will be towards explaining the equivalence

$$K - \mathrm{Perf} \simeq K^{\circ a} - \mathrm{Perf}.$$

State [Sch12, Lemma 5.3] (without proof). Prove [Sch12, Lemma 5.5], explaining that it gives a functor $K - \mathrm{Perf} \to K^{\circ a} - \mathrm{Perf}$.

Talk 6 (*Emanuele*/Nov. 10). Prove [Sch12, Lemma 5.6]. This constructs the inverse functor $K^{\circ a} - \mathrm{Perf} \to K - \mathrm{Perf}$ and gives the equivalence $K - \mathrm{Perf} \simeq K^{\circ a} - \mathrm{Perf}$. Prove [Sch12, Prop. 5.8, Prop. 5.9]. Rest of the talk will state and go towards the proof of [Sch12, Thm. 5.10], i.e., the equivalence

$$K^{\circ a} - \mathrm{Perf} \simeq (K^{\circ a}/\varpi) - \mathrm{Perf}.$$

This is done via a deformation theory argument and requires an almost version of the cotangent complex. Recall [Sch12, Prop. 5.13], which was proved in Talk 3. State [Sch12, Cor. 5.16], and time permitting, discuss the proof.

Talk 7 (*Ben*/Nov. 17). Finish the proof of [Sch12, Thm. 5.10]. Note that we have obtained a proof of the tilting equivalence [Sch12, Thm. 5.2] (the middle isomorphism in [Sch12, Thm. 5.2] follows from [Sch12, lemma 3.4 (iii)] and was covered in Talk 2). Cover [Sch12, Prop. 5.17], which explicitly describes the tilting operation from char. 0 to char. p.

Talk 8 (*Tam*/Nov. 24). Define affinoid algebra and its adic spectrum. Introduce adic spaces.

Talk 9) (*Tam*/Dec. 1). Disucss examples of adic spaces. State the definition of perfectoid spaces as an adic spaces [Sch12, Def. 6.15]. Introduce tilting for perfectoid spaces by glueing the notion from previous talks and deduce the tilting equivalence ([Sch12, Def. 6.16, Prop. 6.17]).

Talk 10 (*Shubhodip*/Dec. 8). Discuss application of perfectoid techniques in the classification of p-divisible groups via Dieudonné theory.

References

[Bha] B. Bhatt. Lecture notes for a class on perfectoid spaces. available at http://www-personal.umich.edu/ bhattb/teaching/mat679w17/lectures.pdf. 1, 2

[Cas86] J. W. S. Cassels. *Local fields*, volume 3 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1986. 1

[GR03] O. Gabber and L. Ramero. Almost ring theory. Lecture Notes in Math. 1800, Springer, 2003. 1

[Ill71] L. Illusie. *Complexe cotangent et déformations. I*. Lecture Notes in Mathematics, Vol. 239. Springer-Verlag, Berlin-New York, 1971. 1

[Ill72] L. Illusie. *Complexe cotangent et déformations. II*. Lecture Notes in Mathematics, Vol. 283. Springer-Verlag, Berlin-New York, 1972. 1

[Sch12] P. Scholze. Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.*, 116:245–313, 2012. 1, 2

[SP23] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`, 2023. 1

[Wed] T. Wedhorn. Adic spaces. https://arxiv.org/abs/1910.05934. 1