Drunken Birds, Brownian Motion, and Other Random Fun

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Outline

- “Review” of Basic Probability
- Random Walks
- Brownian Motion
- Martingales
- Theorems from MA 530
Random Variables

Informal

A **Probability space** is a the set of possible outcomes of a random event. A **Random Variable** is some number that depends on the outcome.

Formal

A **Probability Space** is a measure space \((\Omega, F, \mathbb{P})\) such that \(\mathbb{P}(\Omega) = 1\). A **Random Variable** is a measurable function defined on \(\Omega\). Measurable sets are called **events**.
### Discrete Random Variables

- Take countably many values, \( \{x_n\}_{n=0}^{\infty} \)
- \( \mu = E(X) = \sum_{n=0}^{\infty} nP(X = n) \)
- \( Var(X) = E((X - \mu)^2) = E(X^2) - E(X)^2 \)
- Ex: Binomial\((n, p)\): \( P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k} \)

### Continuous Random Variables

- Has density \( f(x) \) so \( P(X \in A) = \int_A f(x) \, dx \).
- Cumulative distribution function \( F(x) = \int_{-\infty}^{x} f(t) \, dt = P(X \leq x) \)
- \( \mu = E(X) = \int_{-\infty}^{\infty} xf(x) \, dx = \int_{-\infty}^{\infty} xdF \)
- Ex: \( N(\mu, \sigma^2) : f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \)

Facts: \( E(aX + bY) = aE(X) + bE(Y) \).
Linear Combinations of normal R.V.’s are normal.
Conditional Probability and Filtrations

**Conditional Probability**

- \( P(A|B) = \frac{P(A \cap B)}{P(B)} \)
- \( A \) and \( B \) are independent if \( P(A \cap B) = P(A)P(B) \).
- \( E(X|A) = \frac{E(X1_A)}{P(A)} \)

Let \( X_n \) a sequence of R.V.’s.

**Filtrations**

- \( F_n = \sigma(X_0, \ldots, X_n) = \{ \text{“event's known from observing } X_0, \ldots, X_n \text{.”} \} \)
- \( E(X_{n+k}|F_n) = E(X_{n+k}|X_0, \ldots, X_n) = \text{“Best guess of } X_{n+k} \text{ after watching first } n \text{ steps.”} \)
- \( X \) is independent of \( F_n \) if it is independent of all random variables whose values can be know at time \( n \).
Central Limit Theorem

Let \( X_1, X_2, \ldots \), be i.i.d. R.V.’s with finite mean, \( \mu \), and variance, \( \sigma^2 \); let \( S_n = \frac{X_0 + \ldots + X_n}{n+1} \).

**Theorem:** For large \( n \), \( S_n \approx N(\mu, \sigma^2/n) \). Formally,

\[
\frac{S_n - n\mu}{\sigma \sqrt{n}} \Rightarrow N(0,1). \quad (1)
\]

This is why real world data is often normally distributed, e.g., heights, blood pressures, lengths of manufactured objects.
Simple Symmetric Random Walk on the Integers

**Definition**
- \( \mathbb{P}(\Delta_k = \pm 1) = 1/2 \) (Coin flip)
- \( X_n = \sum_{k=0}^{n} \Delta_k \) is called a Random Walk.

**Properties**
1. \( E(|X_n|) < \infty \) for all \( n \)
2. \( E(X_{n+1} | \mathcal{F}_n) = X_n \)
3. \( \mathbb{P}(X_{n+k} = x | \mathcal{F}_n) = \mathbb{P}(X_{n+k} = x | X_n) \)
4. \( X_0 = 0 \) almost surely
5. \( X_{n+k} - X_n \) is independent of \( \mathcal{F}_n \)
6. \( X_{n+k} - X_n \sim_d X_k \) for all \( n \)

Any Process with Properties 1 and 2 is called a Martingale. Any Process with property 3 is called a Markov Chain.
## History and Applications of BM

### History
- Robert Brown 1827
- Bachelier 1900
- Einstein 1905
- Wiener 1923

### Applications
- Black Scholes Equation
- Gas diffusion
- Other Stochastic Processes
Brownian Motion

**Definition**

A (Standard) Brownian Motion is any continuous time stochastic process, \((B_t)_{t \geq 0}\), which satisfies

1. \(B_0 = 0\)
2. \(t \to B_t\) is a.s. continuous
3. \(B_t - B_s\) is independent of \(\mathcal{F}_s\)
4. \(B_t - B_s \sim_d N(0, t - s)\).

Fact: Brownian Motions exist.

Note: \(B_1 \sim_d N(0, 1)\) but
\(B_1 = (B_1 - B_{1/2}) + B_{1/2} \sim_d N(0, 1/2) + N(0, 1/2)\).

Note: Brownian Motion is the continuous time analog of a random walk. It is both a Markov Process and a martingale.
Properties of Brownian Motion

- Let $0 < \alpha < 1/2$, then, a.s. $\exists C(\omega)$ so
  \[ |B_t(\omega) - B_s(\omega)| < C(\omega)|t - s|^{\alpha}. \]
- $\limsup_{t \to \infty} \frac{B_t}{\sqrt{2t \log \log t}} = 1$ a.s.
- Brownian path's are a.s. nowhere differentiable. Intuitively, differentiability would imply $B_{t+h} - B_t \approx B_t - B_{t-h}$ for small $h$.
- $B_t^2 - t$ is a martingale. So is $\exp(B_t - t/2)$.
- If $f(x, t)$ satisfies $(\frac{1}{2} \triangle - \partial_t)f(x, t) = 0$, $f(B_t, t)$ is a martingale.
- If $\triangle u = 0$, and $B_t$ is complex BM then $u(B_t)$ is a martingale. Thus, if $f$ is complex analytic, $f(B_t)$ is a complex martingale.
Examples of Brownian Paths
Recurrence and Transience

Random Walks
If \( X_n \) is a random walk one or two dimensions, then a.s. \( X_n \) is \textit{recurrent}, i.e. it takes every value infinitely often.
In three or more dimensions, \( X_n \) is \textit{transient}. It takes every value only finitely many times. Moreover, \( |X_n| \to \infty \).

Brownian Motion
In one dimension, \( B_t \) is recurrent. In two dimensions, it is \textit{open set recurrent}, i.e., \( B_t \) visits each open set i.o..
In three or more dimensions, \( B_t \) is transient and \( |B_t| \to \infty \).
Martingale Transforms

Discrete Case

A sequence of R.V.s, $v_k$, is **predictable** if the value of $v_k$ can be known at time $k - 1$, i.e., $v_k \in \mathbb{F}_{k-1}$.

**Theorem:** Let $X_n$ a martingale, $d_k = X_k - X_{k-1}$ so $X_n = \sum_{k=0}^{n} d_k$. Then,

$$(v \ast X)_n = \sum_{k=0}^{n} v_k d_k$$

is a martingale for all $v_k$ bounded and predictable.

Continuous Case

**Theorem:**

$$(H \cdot X)_t = \int_{0}^{t} H_s dX_s$$

is a martingale, for all martingales $X$ with continuous paths and $H_s$ bounded predictable processes.
Representation Theorems

**Time Change**

If $X_t$ is a continuous martingale, there is a unique predictable increasing process $\langle X \rangle_t$ so that $\langle X \rangle_0 = 0$ and $X_t^2 - \langle X \rangle_t$ is a martingale. **Theorem:** If $X_t$ is a continuous-path martingale with $\langle X \rangle_{\infty} = \infty$, then $X_t$ is a time change of a Brownian Motion, in particular, there exists a Brownian Motion so that $X_t = B_{\langle X \rangle_t}$.

**Ito’s Representation Theorem**

**Theorem:** Let $B_t$ a BM with filtration $\mathbb{F}_t$. Then, if $X_t$ is a martingale adapted to $\mathbb{F}_t$, then there exists a predictable sequence $H_s$ so $X_t = X_0 + \int_0^t H_s dB_s$.

Note: In this case, $\langle X \rangle_t = \int_0^t H_s^2 ds$. 
Ito’s formula

Fake Proof of the Fundamental theorem of calculus

Let \( \{t_k\}_{k=0}^n \) a partition of \((0, t)\).

Taylor: \( f(x(t_{k+1})) - f(x(t_k)) = f'(x(t_k))dt + O(dx(t)^2) \)

Summing: \( f(x(t)) - f(x(0)) \approx \sum_{k=0}^n f'(x(t_k))dx \to \int_a^b f'(x(t))dx \)

because \( dx^2 \) is small.

Ito’s formula

\[
f(B(t_{k+1})) - f(B(t_k)) = f'(B(t_k))dB + \frac{1}{2}f''(B(t_k))dB^2 + O(dB^3).\]

\[
f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2} \int_0^t f''(B_s)ds \tag{2}
\]

\[
f(B_t) = f(B_0) + \sum_{i=1}^n \int_0^t f_{x_i}(B_s)dB_s + \frac{1}{2} \int_0^t \triangle f(B_s)ds \tag{3}
\]

Note: \( dB_t^2 = dt \) because \( B_t^2 - t \) is a martingale.
Stopping Times

Definition and Examples
Let $X_t$ be a stochastic process with filtration $\mathbb{F}_t = \sigma\{(X_s)_{s \leq t}\}$. $T$ is called a **Stopping Time** if for all $t$, the event $\{T \leq t\}$ is in $\mathbb{F}_t$.

Examples: If $S$ is a (measurable) set and $B_t$ is a BM, $T = \inf\{t > 0 : B_t \in T\}$. If $S$ and $T$ are stopping times, so is $S \wedge T$.

Results
**Theorem:** If $X_t$ is a martingale and $T$ a stopping time, the stopped process, $X_{t\wedge T}$ is a martingale.

**Theorem:** If $X_t$ is a martingale and $T$ is a bounded stopping time, $E(X_T) = E(X_0)$. 

M. Perlmutter (Purdue)
The Averaging Property of Harmonic Functions

**Theorem:** Let $0 < r < R$. Let $u$ be harmonic on $B(z, R)$. Then,

$$u(z) = \frac{1}{2\pi} \int_{0}^{2\pi} u(z + re^{i\theta}) d\theta.$$ 

**Proof:** Let $B_t$ be complex BM starting at $z$. Let

$$T = \inf\{t > 0 : |B_t - z| \geq r\}.$$ 

Then,

$$u(z) = u(B_0) = Eu(B_0) = Eu(B_T) = \frac{1}{2\pi} \int_{0}^{2\pi} u(z + re^{i\theta}) d\theta.$$
Louiville’s Theorem

**Theorem:** Let \( u \) be a bounded harmonic function defined on \( \mathbb{C} \). Then \( u \) is constant.

**Proof:** Let \( u \) be bounded, harmonic, and non-constant. Let \( B_t \) complex BM, \( X_t = u(B_t) \).

\( X_t \) is a real-valued martingale, so it suffices to show \( \langle X \rangle_\infty = \infty \). Then \( X_t \) is a time change of a BM and thus open set recurrent.

By Itô’s formula, \( u(B_t) = \int_0^t \nabla u(B_s) \cdot dB_s \). So, \( \langle X \rangle_t = \int_0^t |\nabla u(B_s)|^2 ds \).

\( \langle X \rangle_\infty = \int_0^\infty |\nabla u(B_s)|^2 ds = \infty \). Since if the integral converged, that would imply that \( \lim |\nabla u(B)_s| = 0 \), but \( u(B)_s \) is open set recurrent and there is an open set on which \( \nabla u \) is non-vanishing.
THANK YOU!