Lesson 11: Partial Derivatives (15.3)

Warmup: Evaluate the following derivatives

$$\frac{d}{dx}\left(\sin(5)\,x^3\right) = (\sin 5)\cdot 3x^2$$

$$\frac{d}{dx}\left(\sin(8)\,x^3\right) = (\sin 8)\cdot 3x^2$$

$$\frac{d}{dy}\left(8\sin y\right) = 8\cos y$$

$$\frac{d}{dy}\left(27\sin y\right) = 27\cos y$$

Today! finding derivatives when we have two or more variables

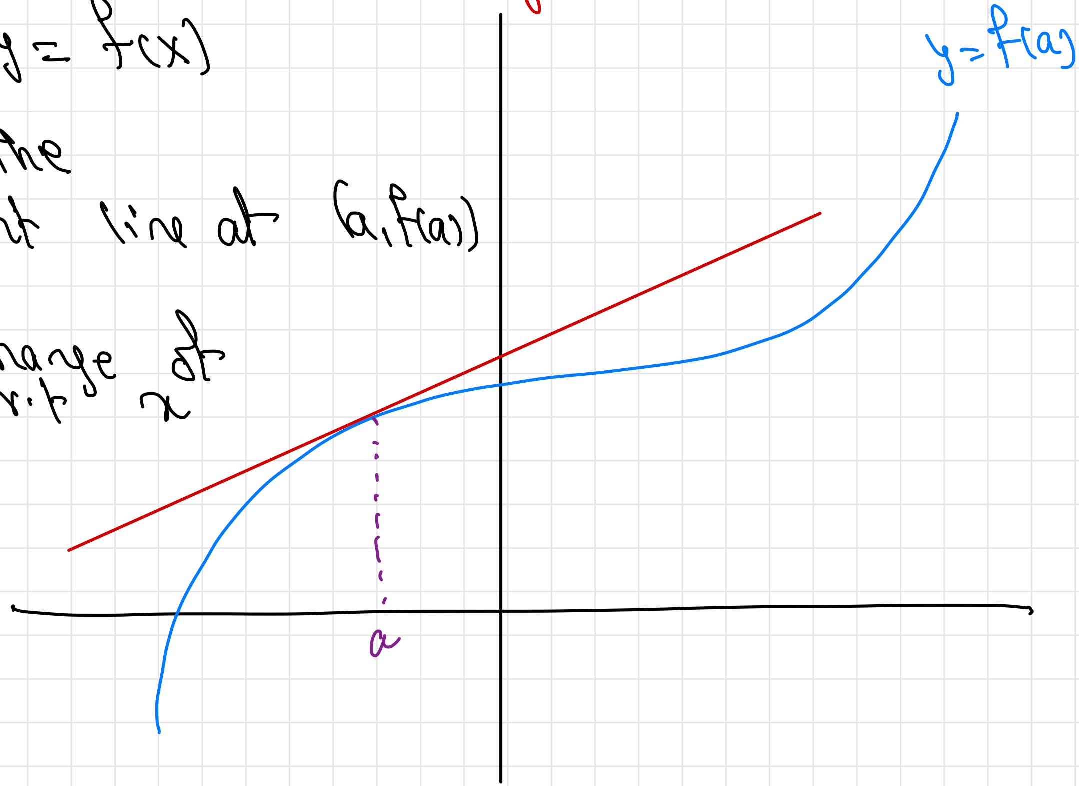
Recall: Derivative and its Meaning

$y = f(x)$

$f'(a)$ = Slope of the tangent line at $(a, f(a))$

= Rate of change of y w.r.t x

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



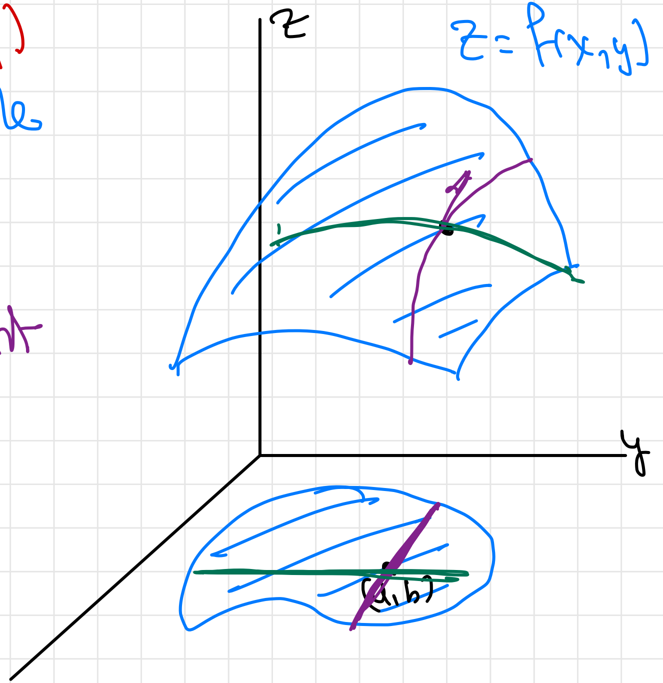
* $f' > 0 \leadsto f \uparrow$
* $f' < 0 \leadsto f \downarrow$

Partial Derivatives of $z = f(x,y)$

take derivative w.r.t one variable keeping the other constant

$\frac{\partial f}{\partial x} = f_x =$ partial w.r.t x
Keep y constant

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$



$\frac{\partial f}{\partial y} = f_y =$ partial w.r.t y
Keeping x constant

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

Finding Partial Derivatives

eg: $f(x,y) = x^3 \sin y$

$$f_x = \frac{\partial}{\partial x}\left(x^3 \sin y\right) = 3x^2 \sin y$$

$$f_y = \frac{\partial}{\partial y}\left(x^3 \sin y\right) = x^3 \cos y$$

eg: $f(x,y) = y^2x^3 + \sin(xy)$

$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$

$$f_x = \frac{\partial}{\partial x}\left(y^2x^3 + \sin(xy)\right)$$

$$= \frac{\partial}{\partial x}(y^2x^3) + \frac{\partial}{\partial x}(\sin(xy))$$

$$= 3x^2y^2 + \cos(xy) \cdot \frac{\partial}{\partial x}(xy) = 3x^2y^2 + y\cos(xy)$$

$$f_y = \frac{\partial}{\partial y}\left(y^2x^3 + \sin(xy)\right)$$

$$= 2yx^3 + \cos(xy) \cdot \frac{\partial}{\partial y}(xy) = 2yx^3 + x\cos(xy)$$

eg: $f(x,y) = \sin\left(\frac{x}{1+y}\right)$, find $f_x\left(\frac{\pi}{2}, 1\right)$, $f_y\left(\frac{\pi}{2}, 1\right)$

$$f_x = \frac{\partial}{\partial x}\left(\sin\left(\frac{1}{1+y}\cdot x\right)\right) = \cos\left(\frac{1}{1+y}\cdot x\right)\cdot \frac{\partial}{\partial x}\left(\frac{1}{1+y}\cdot x\right)$$

$$= \cos\left(\frac{1}{1+y}\cdot x\right)\cdot \frac{1}{1+y}$$

$$f_x\left(\frac{\pi}{2}, 1\right) = \cos\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)\cdot\frac{1}{2} = \frac{\sqrt{2}}{4}$$

$$f_y = \frac{\partial}{\partial y}\left(\sin\left(x\cdot\frac{1}{1+y}\right)\right) = \cos\left(x\cdot\frac{1}{1+y}\right)\cdot\frac{\partial}{\partial y}\left(x\cdot\frac{1}{1+y}\right)$$

$$= \cos\left(x\cdot\frac{1}{1+y}\right)\cdot x\cdot\frac{-1}{(1+y)^2}$$

$$f_y\left(\frac{\pi}{2}, 1\right) = \cos\left(\frac{\pi}{2}\cdot\frac{1}{2}\right)\cdot\frac{\pi}{2}\cdot\frac{-1}{2^2} = \frac{-\pi\sqrt{2}}{16}$$

$$\frac{d}{dy}\left(\frac{1}{1+y}\right) = \frac{0\cdot(1+y) - 1\cdot 1}{(1+y)^2}$$

$$= \frac{-1}{(1+y)^2}$$

Higher Order partial derivatives

$$f(x,y) = y^2x^3 + \sin(xy)$$

$$f_x = 3y^2x^2 + y\cos(xy) \quad , \quad f_y = 2yx^3 + x\cos(xy)$$

functions of x & y variables - we can take partials

$$\frac{\partial}{\partial x} f_x = f_{xx} \qquad \frac{\partial}{\partial x} f_y = f_{yx}$$

$$\frac{\partial}{\partial y} f_x = f_{xy} \qquad \frac{\partial}{\partial y} f_y = f_{yy}$$

Similarly, f_{xxyxxy}

$$f_x = 3y^2x^2 + y\cos(xy)$$

$$f_{xx} = \frac{\partial}{\partial x}\left(3y^2x^2 + y\cdot\cos(xy)\right)$$

$$= 6y^2x + y\cdot(-\sin(xy))\cdot\frac{\partial}{\partial x}(xy)$$

$$= 6y^2x - y^2\sin(xy)$$

$$f_{xy} = \frac{\partial}{\partial y}\left(3y^2x^2 + y\cos(xy)\right)$$

$$= 6yx^2 + \left(\frac{\partial y}{\partial y}\right)\cos(xy) + y\cdot\frac{\partial}{\partial y}\cos(xy)$$

$$= 6yx^2 + \cos(xy) - xy\sin(xy).$$

$$f_y = 2yx^3 + x\cos(xy)$$

$$f_{yx} = \frac{\partial}{\partial x}\left(2yx^3 + x\cos(xy)\right)$$

$$= 6yx^2 + \cos(xy) - xy\sin(xy)$$

$$f_{yy} = \frac{\partial}{\partial y}\left(2yx^3 + x\cos(xy)\right)$$

$$= 2x^3 - x^2\sin(xy)$$

Observe! $f_{xy} = 6yx^2 + \cos(xy) - xy\sin(xy)$.

$$f_{xy} = f_{yx}$$

Clairauts theorem: $\underbrace{f_{xy} = f_{yx}}$ ← provided they are continuous

Meaning: Order in which we Differentiate Does not matter.

Generalization: $f_{xxx} = (f_{xx})_x$

$f_{xyx} = (f_{xy})_x = (f_{yx})_x = f_{yxx}$

$f_{xyy} = (f_{xy})_y = (f_{yx})_y = f_{yxy}$

$f_{yyy} = (f_{yy})_y$

Order does not mater as long as they are continuous

$f_{xyx} = f_{yxx} = f_{xxy}$

$f_{xyy} = f_{yxy} = f_{yyx}$

eg: $f(x,y,z) = xy^2z^3$, find f_{xyz}, f_{yzx}, f_{zyx}

$$f_x = \frac{\partial}{\partial x}(xy^2z^3) = y^2z^3$$

$$f_{xx} = 0$$

$$\boxed{f_{xy} = 2yz^3}$$

$$\boxed{f_{xz} = 3y^2z^2}$$

$$f_{xyz} = 6yz^2$$

$$f_y = 2xyz^3$$

$$\boxed{f_{yx} = 2yz^3}$$

$$f_{yy} = 2xz^3$$

$$\boxed{f_{yz} = 6xyz^2}$$

$$f_{yzx} = 6yz^2$$

$$f_z = \frac{\partial}{\partial z}(xy^2z^3) = 3xy^2z^2$$

$$\boxed{f_{zx} = 3y^2z^2}$$

$$\boxed{f_{zy} = 6xyz^2}$$

$$f_{zz} = 6xy^2z$$

$$f_{zyx} = 6yz^2$$