Lesson 12: Chain Rule (15.4)

Review example: $f(x,y,z) = e^{xy^2z^3}$, find f_x, f_y, f_z

f_x: y, z constants

f_y: x, z constants

f_z: x, y constants

$$\frac{d}{dx}\left(e^{5x}\right) = e^{5x} \cdot 5$$

$$\frac{\partial}{\partial x} e^{xy^2z^3} = e^{xy^2z^3} \cdot y^2z^3$$

$$\frac{\partial}{\partial y} e^{xy^2z^3} = e^{xy^2z^3} \cdot 2xyz^3$$

$$\frac{\partial}{\partial z} e^{xy^2z^3} = e^{xy^2z^3} \cdot 3xy^2z^2$$

Review Chain Rule

eg: $f(x) = e^x$, $g(t) = 3t^2$

$f'(x) = e^x$ $\qquad$ $g'(t) = 6t$

$h(t) = f(g(t))$

$h'(t) = f'(g(t)) \cdot g'(t) = e^{3t^3} \cdot 6t$

Tree with a Branch?

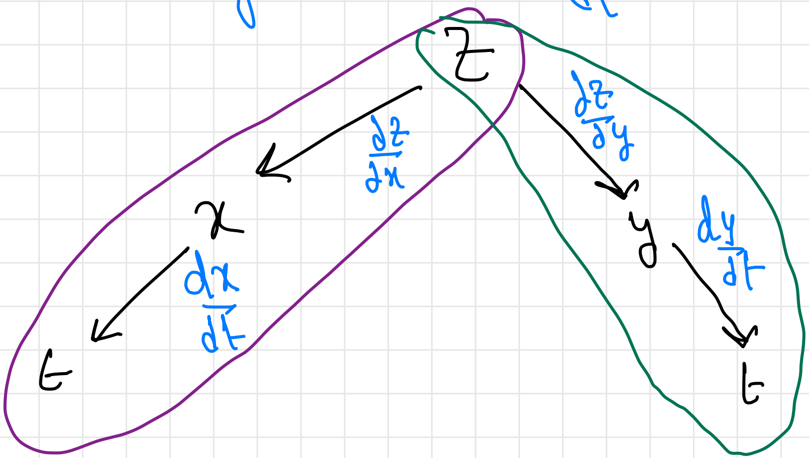
$$\begin{array}{c} f \\ \Big\downarrow \tfrac{df}{dx} \\ x \\ \Big\downarrow \tfrac{dx}{dt} \\ t \end{array}$$

Chain Rule: product of derivatives that appear on the branches.

Chain Rule: two or more Variables

eg: $z = x^2 - y^2$, $x = \tan t$, $y = \sec t$ $\leadsto$ find $\frac{dz}{dt}$

$\frac{\partial z}{\partial x} = 2x$, $\frac{\partial z}{\partial y} = -2y$ $\qquad \frac{dx}{dt} = \sec^2 t$ $\qquad \frac{dy}{dt} = \sec t \tan t$



Chain Rule!

Apply 1-variable Chain Rule along each of the Branches and add them

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} = (2x)(\sec^2 t) + (-2y)(\sec t \tan t)$$

$$= 2\tan t\sec^2 t - 2\sec^2 t \tan t$$

$$= 0.$$

eg: $z = x^2y + 3xy^4$, $x = \sin t$, $y = \cos t$. Find $\dfrac{dz}{dt}$ at $t = 0$

$t = 0 \rightsquigarrow x = 0$

$\qquad\qquad y = 1$

$$\begin{tikzcd}[column sep=small]
 & z \arrow[dl, "\frac{\partial z}{\partial x}"'] \arrow[dr, "\partial z/\partial y"] & \\
x \arrow[d, "\frac{dx}{dt}"] & & y \arrow[d, "\frac{dy}{dt}"] \\
t & & t
\end{tikzcd}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= (2xy + 3y^4)\cos t + (x^2 + 12xy^3)(-\sin t)$$

$$= (2\cdot 0\cdot 1 + 3\cdot 1^4)(\cos(0)) + (0^2 + 12\cdot 0\cdot 1)(-\sin 0) = 3.$$

eg: $z = e^x \sin y$, $x = st^2$, $y = s^2 t$, find $\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}$

$$\begin{tikzcd}[column sep=small]
 & & & z \arrow[dll] \arrow[drr] & & & \\
 & x \arrow[dl] \arrow[dr] & & & & y \arrow[dl] \arrow[dr] & \\
s & & t & & s & & t
\end{tikzcd}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y) \cdot (2st)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2)$$

Chain Rule when you have more than 2 functions

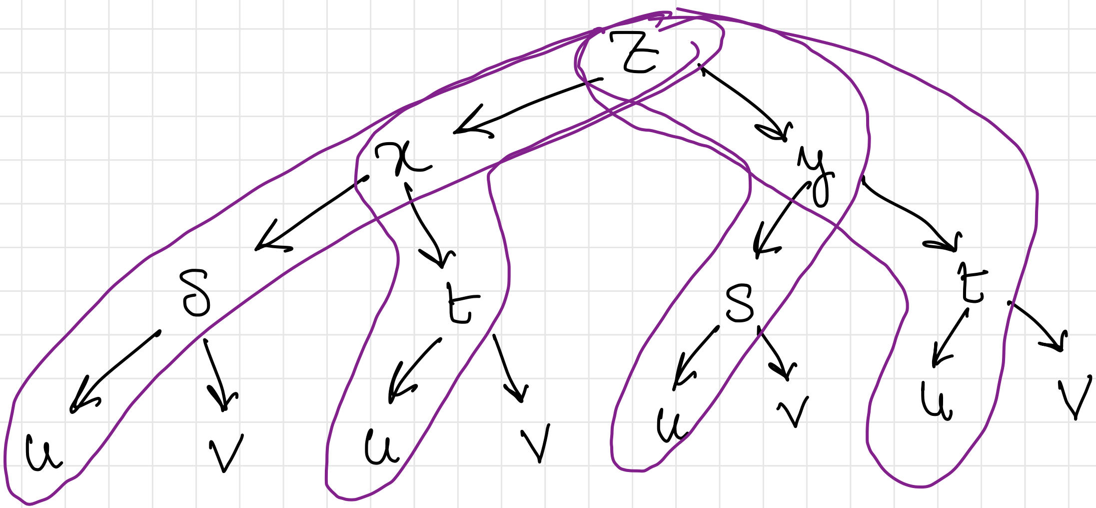
$$\frac{d}{dx}\sin\left(e^{x^3}\right) = \cos\left(e^{x^3}\right) \cdot e^{x^3} \cdot 3x^2$$

$$\begin{tikzcd}
f \arrow[d, "\frac{df}{dx}"] \arrow[ddd, bend right=60, "\frac{df}{du}"'] \\
x \arrow[d, "\frac{dx}{dt}"] \\
t \arrow[d, "\frac{dt}{du}"] \\
u
\end{tikzcd}$$

$$\frac{df}{du} = \frac{df}{dx} \cdot \frac{dx}{dt} \cdot \frac{dt}{du}$$

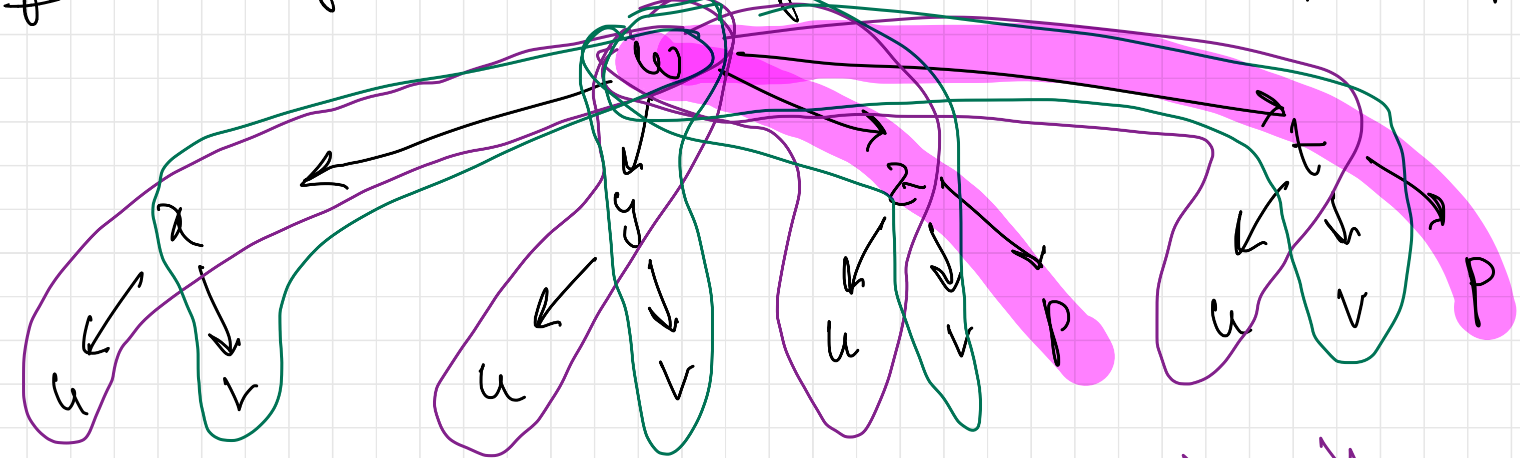
Suppose $z = f(x,y)$, $x = g(s,t)$, $y = h(s,t)$

$s = m(u,v)$, $t = n(u,v)$



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t}\frac{\partial t}{\partial u} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial s}\cdot\frac{\partial s}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\cdot\frac{\partial t}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\frac{\partial s}{\partial u}$$

eg: $w = f(x,y,z,t)$, $x = f_1(u,v)$, $y = f_2(u,v)$, $z = f_3(u,v,p)$, $t = f_4(u,v,p)$



$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\,\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial v}$$

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial p} + \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial p}$$

eg: y is a function of x, given implicitly as

$x^3 + y^3 - 6xy = 0$. Find $\frac{dy}{dx}$

$$x^3 + [y(x)]^3 - 6x \cdot y(x) = 0$$

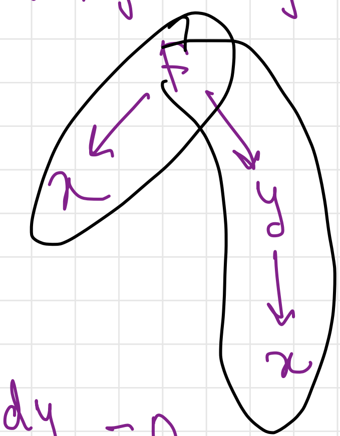
$$3x^2 + 3y^2 \cdot \frac{dy}{dx} - 6y - 6x\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$$

in Calc I

Using Chain Rule

$$F(x,y) = x^3 + y^3 - 6xy = 0$$



$$F_x + F_y \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{-(3x^2 - 6y)}{3y^2 - 6x}$$

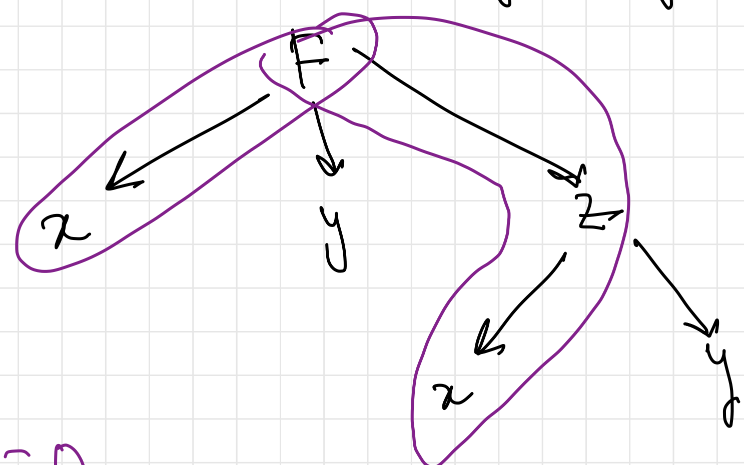
eg: z is a function of x and y given implicitly

$$xyz = x^2y^2 + y^2z^2 \quad \text{find } \frac{\partial z}{\partial x}$$

write as

$$F(x,y,z) = 0$$

$$F(x,y,z) = xyz - x^2y^2 - y^2z^2 = 0$$

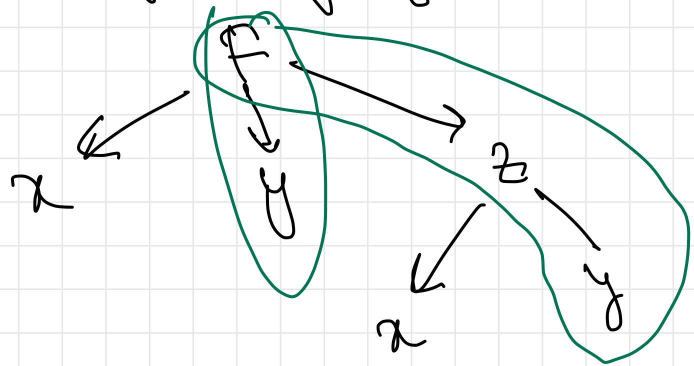


$$F_x + F_z \cdot \frac{\partial z}{\partial x} = 0$$

$$\rightsquigarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{-(yz - 2xy^2)}{(xy - 2y^2z)}$$

Similarly, we can find $\frac{\partial z}{\partial y}$

$$F(x,y,z) = xyz - x^2y^2 - y^2z^2 = 0$$



$$F_y + F_z \cdot \frac{\partial z}{\partial y} = 0 \rightsquigarrow \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

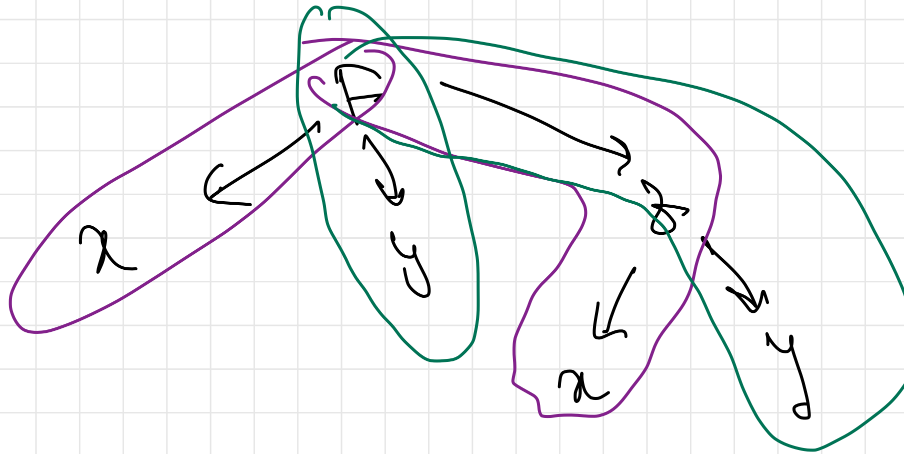
$$= \frac{-(xz - 2x^2y - 2yz^2)}{(yz - 2y^2z)}$$

Summary: z is implicity given as a function of x, y

① Write given equation as a function of 3-variables

$$F(x,y,z)=0$$

② Apply Chain Rule



$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

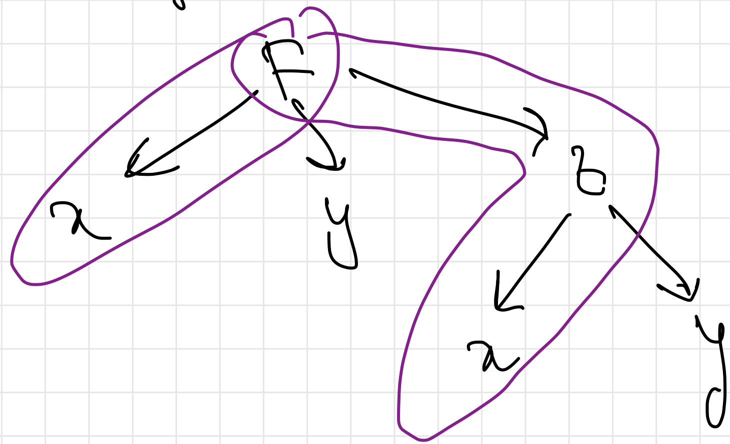
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

eg. z is a function of x and y given implicitly

$$x^3 z + 3(\cos y) z^2 + 2xyz = 11$$

find $\frac{\partial z}{\partial x}$ at $\underbrace{(2,0,1)}_{\substack{x=2\\ y=0\\ z=1}}$

① $F(x,y,z) = x^3 z + 3\cos y \cdot z^2 + 2xyz - 11 = 0$



② $F_x + F_z \frac{\partial z}{\partial x} = 0$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

$$\frac{\partial z}{\partial x} = \frac{-(3x^2 z + 2yz)}{(x^3 + 6\cos y \cdot z + 2xy)}$$

$$\rightsquigarrow \left.\frac{\partial z}{\partial x}\right|_{(2,0,1)} = -\frac{(3\cdot 2^2\cdot 1 + 2\cdot 0\cdot 1)}{(2^3 + 6\cdot \cos 0 \cdot 1 + 2\cdot 2\cdot 0)} = \frac{-12}{14}$$