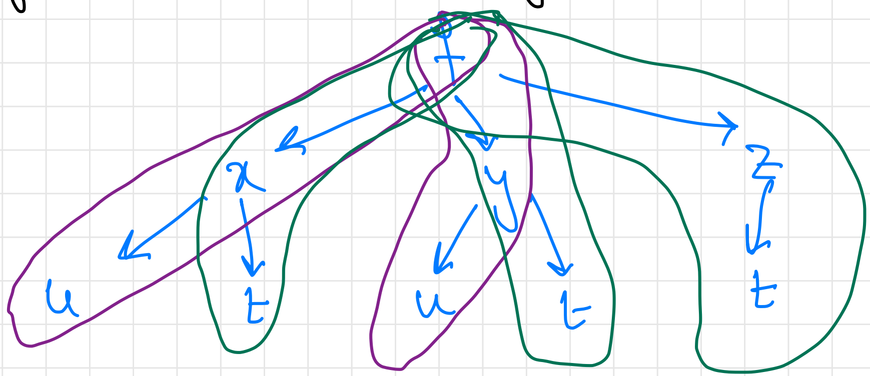
# Lesson 13: Directional Derivative & Gradient (15.5)

**Announcements:**
* Exam 1 on 02/25
* Instructions & study guide
* Feasting with Faculty will Resume on 02/26

**Office Hours:** Monday, Friday: 9:45 AM – 11:00 AM
Thursday: 11:00 AM – 12:00 PM

# Review Example (Chain Rule)

$f(x,y,z)=xy^2+z^3$, $x=\cos u+t^3$, $y=\sin t+u^3$, $z=t^2$. Find $f_u$ and $f_t$



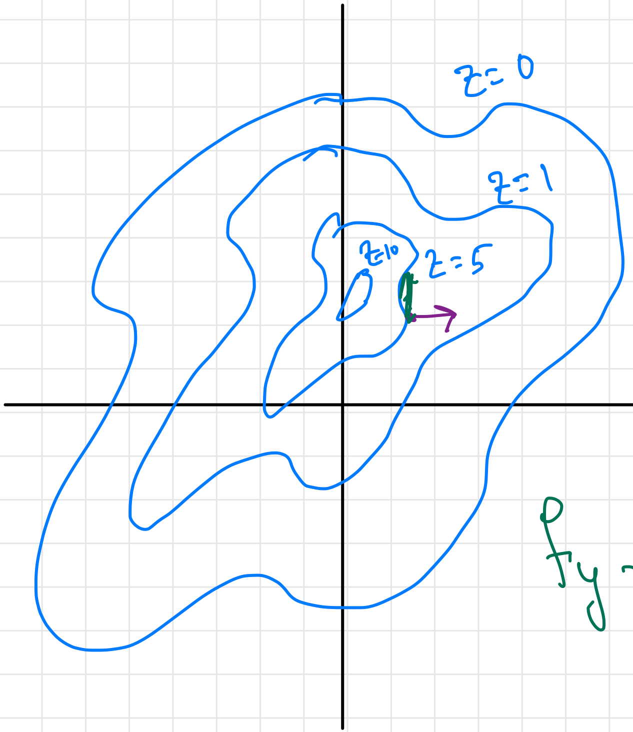

$$f_u = f_x\cdot x_u + f_y\cdot y_u = y^2\cdot(-\sin u) + 2xy\cdot 3u^2$$

$$f_t = f_x\cdot x_t + f_y\cdot y_t + f_z\cdot z_t = y^2\cdot 3t^2 + 2xy\cdot \cos t + 3z^2\cdot 2t\,.$$

# Recall: Partial derivatives

$z = f(x,y)$

Level curves $f(x,y) = k$




$f_x = \frac{\partial f}{\partial x}$ = How fast are you climbing the hill if you walk in x-direction

$= \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$

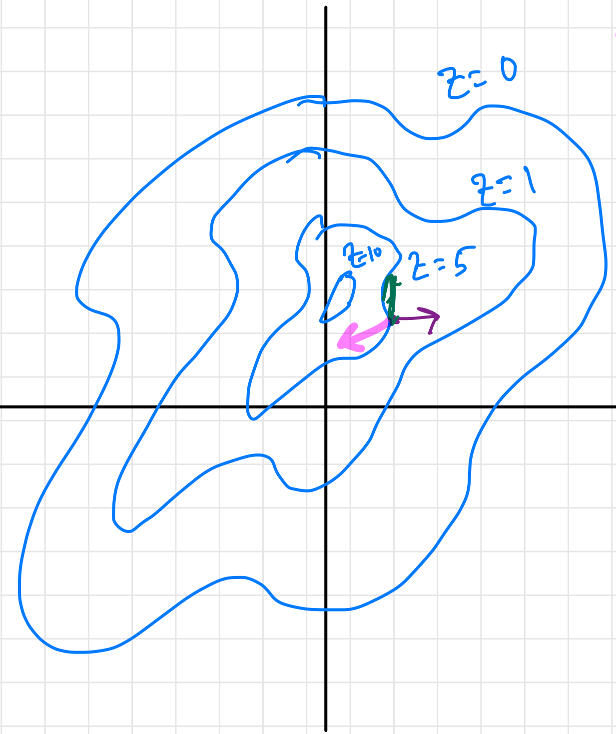
$f_y = \frac{\partial f}{\partial y}$ = How fast are you climbing if you walk in y-direction

$= \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$

# Directional Derivative

$f(x,y) \rightsquigarrow$ two variable function

$\vec{u} = \langle u_1, u_2 \rangle \rightsquigarrow$ unit vector




$D_{\vec{u}} f$ = Rate of change of $f(x,y)$ in the direction of $\vec{u}$

$$= \lim_{h \to 0} \frac{f(x+u_1 h, y+u_2 h) - f(x,y)}{h}$$

Algebra

$$\boxed{D_{\vec{u}} f = f_x u_1 + f_y u_2}$$

# 3 Variable function $f(x, y, z)$

$\vec{u} = \langle u_1, u_2, u_3 \rangle \rightsquigarrow$ unit vector

$D_{\vec{u}} f =$ Derivative in the Direction of $\vec{u}$ $= f_x u_1 + f_y u_2 + f_z u_3$

eg: $f(x,y) = \sin(x^3+7y)$

Directional derivative at $(0, \frac{\pi}{7})$ in direction of $\langle 2,5 \rangle$

unit vector?

① find unit vector

$$\vec{u} = \frac{\langle 2,5\rangle}{|\langle 2,5\rangle|} = \frac{1}{\sqrt{29}}\langle 2,5\rangle = \langle \underbrace{\frac{2}{\sqrt{29}}}_{u_1}, \underbrace{\frac{5}{\sqrt{29}}}_{u_2} \rangle$$

② $f_x = 3x^2\cos(x^3+7y)$

$f_y = 7\cos(x^3+7y)$

③ $D_{\vec{u}}f = f_x u_1 + f_y u_2 = 3x^2\cos(x^3+7y)\cdot\frac{2}{\sqrt{29}} + 7\cos(x^3+7y)\cdot\frac{5}{\sqrt{29}}$

$D_{\vec{u}}f(0, \frac{\pi}{7}) = -\frac{35}{\sqrt{29}}$.

eg: $f(x,y) = x^3y + xy^2$

Directional derivative at $(1,2)$ in direction of $\langle -1,-1 \rangle$

unit vector?

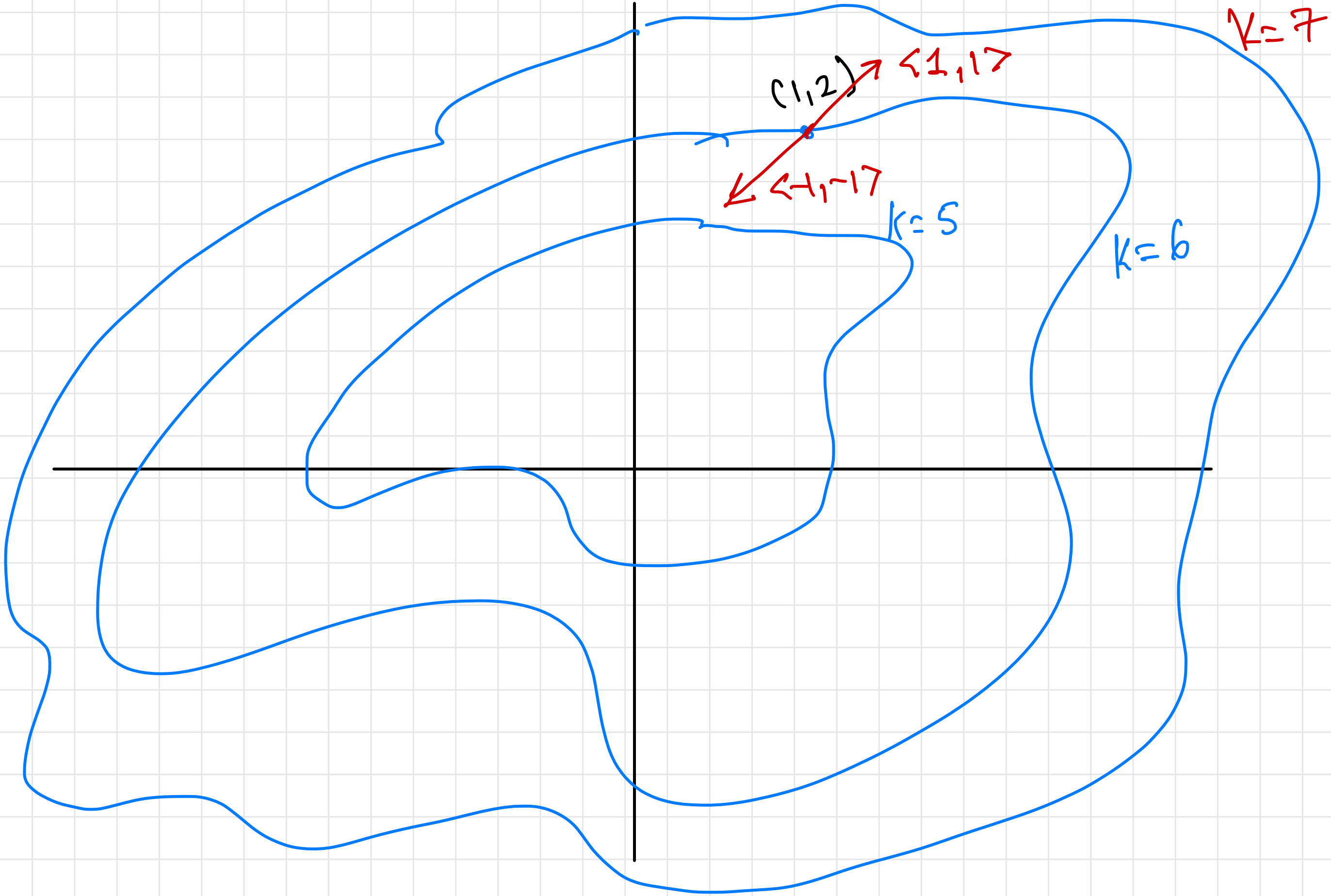
① $\vec{u} = \dfrac{\langle -1,-1\rangle}{|\langle -1,-1\rangle|} = \dfrac{1}{\sqrt{2}}\langle -1,-1\rangle$

$= \left\langle \dfrac{-1}{\sqrt{2}}, \dfrac{-1}{\sqrt{2}} \right\rangle$

② $f_x = 3x^2y + y^2$ , $f_y = x^3 + 2xy$

③ $D_{\vec{u}}f = f_x u_1 + f_y u_2 = (3x^2y+y^2)\left(\dfrac{-1}{\sqrt{2}}\right) + (x^3 + 2xy)\left(\dfrac{-1}{\sqrt{2}}\right)$

$D_{\vec{u}}f(1,2) = 10\left(\dfrac{-1}{\sqrt{2}}\right) + 5\left(\dfrac{-1}{\sqrt{2}}\right) = \dfrac{-15}{\sqrt{2}}$



K=7
(1,2)
<1,1>
<-1,-1>
k=5
k=6

## Gradient:

$f(x,y)$

$$\vec{\nabla} f = \langle f_x , f_y \rangle$$

del
or
Nabla

Similarly for $f(x,y,z)$

$$\vec{\nabla} f = \langle f_x , f_y , f_z \rangle$$

## $D_{\vec{u}} f$ using Gradient:

$u = \langle u_1 , u_2 \rangle$, unit vector

$$D_{\vec{u}} f = f_x u_1 + f_y u_2$$

$$= \vec{\nabla} f \cdot \vec{u}$$

$$\boxed{D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}}$$

**Question:** $f(x,y) = x^3y + xy^2$

Can we find the direction $\vec{u}$ such that $D_{\vec{u}} f(1,2)$ is max.?

$$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}$$

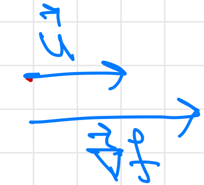
$$= |\vec{\nabla} f| \cdot |\vec{u}| \cos\theta$$

$\theta$ is angle b/w $\vec{u}$ & $\vec{\nabla} f$

$$= |\vec{\nabla} f| \cos\theta$$

max when $\cos\theta = 1$

$\theta = 0$




That is when $\vec{u}$ & $\vec{\nabla} f$ are parallel

min. when $\cos\theta = -1$

$\theta = \pi$


That is when $\vec{u}$ & $\vec{\nabla} f$ are in opp. Direction.

Given $f(x,y)$

Direction Derivative is max. in the Direction of gradient vector $\vec{\nabla} f$ and max. Rate of change is $|\vec{\nabla} f|$

is min in the direction of $-\vec{\nabla} f$ & min. Rate of change is $-|\vec{\nabla} f|$

$$f(x,y) = x^3y + xy^2$$

$$\vec{\nabla} f = \langle 3x^2y + y^2,\ x^3 + 2xy \rangle$$

$$\vec{\nabla} f(1,2) = \langle 10, 5 \rangle$$

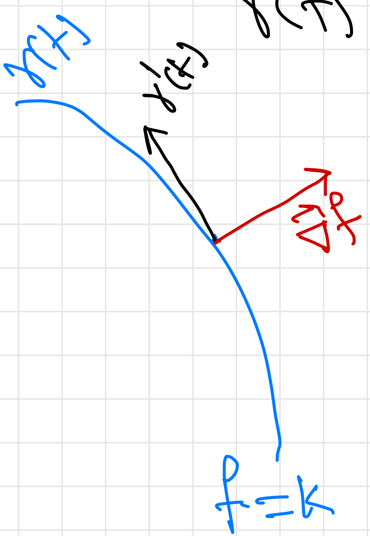
at $(1,2)$ $f(x,y)$ increase the fastest in direction of $\langle 10,5 \rangle$ & decreases the fastest in direction of $-\langle 10,5 \rangle$

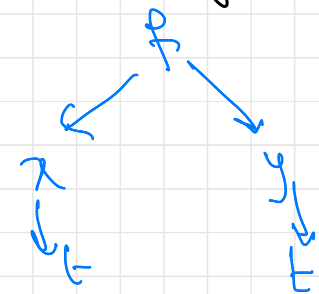
$\vec{\nabla} f$ is perpendicular to the level curves at each point.

Suppose $\gamma(t) = \langle x(t), y(t) \rangle$ represents level curve $f(x,y) = k$

$\gamma'(t) = \langle x'(t), y'(t) \rangle$ is tangent vector

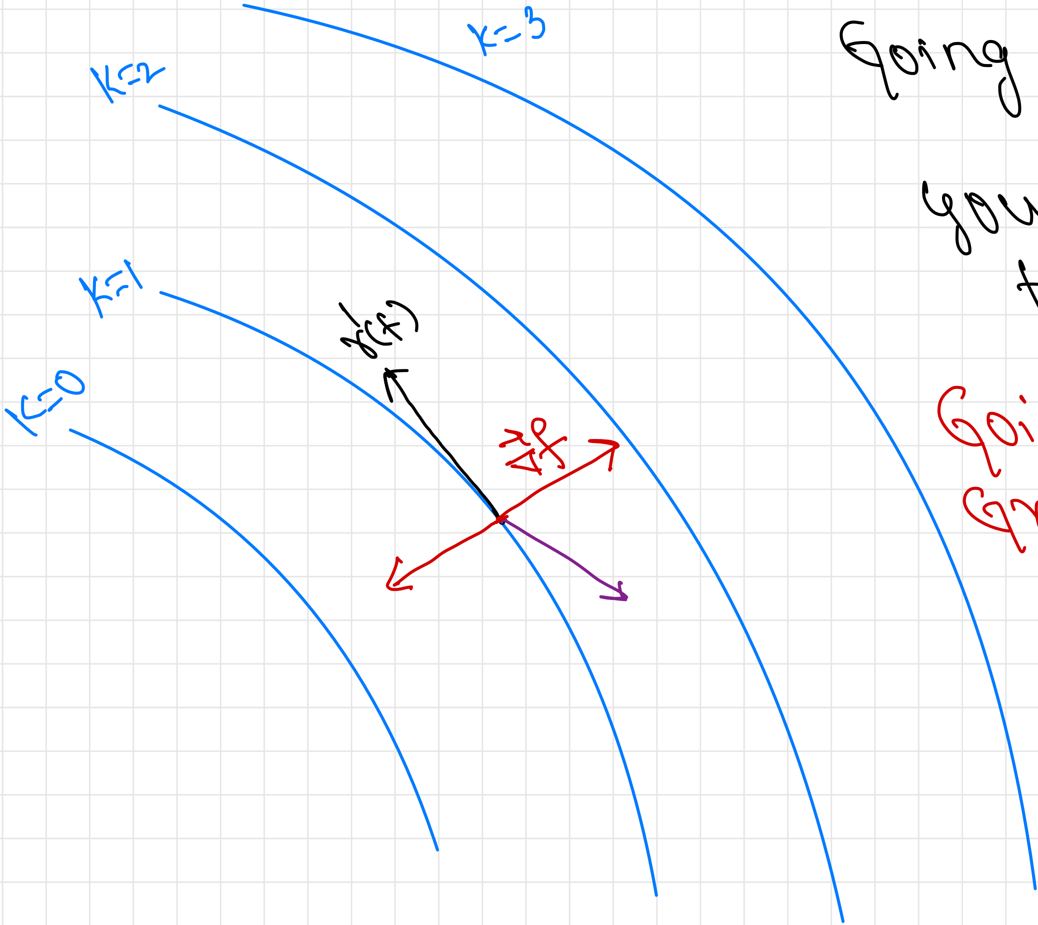
$\gamma(t)$ satisfies $f(x,y) = k \rightsquigarrow f(x(t), y(t)) = k$




$$f_x \cdot \frac{dx}{dt} + f_y \frac{dy}{dt} = 0$$

$$\vec{\nabla} f \cdot \gamma'(t) = 0$$




Going in the tangent Direction you are stuck on the level curve

Going in the Gradient direction you jump level curves the fastest eg: $f(x,y) = x^2y^3$

find the slope of the tangent line to the level curve at $(-1,1)$

is perpendicular to $\vec{\nabla} f$

$\vec{\nabla} f = \langle 2xy^3, 3x^2y^2 \rangle$

$\vec{\nabla} f(-1,1) = \langle -2, 3 \rangle$

tangent vector is perpendicular to $\langle -2, 3 \rangle$

$\langle 3, 2 \rangle$

slope of t. line $= \frac{2}{3}$