Lesson 14: Tangent Plane and Linear Approximation (15.6)

Announcements:

* Exam 1 on Feb 25^{th}, ELLT 116, 6:30pm - 7:30pm

* Instructions
  Seating chart
  Study guide
  } posted on Bright Space

* Different office hours schedule for 02/16 - 02/25

Regular Office Hours: Monday, Friday: 9:45am - 11:00am
Thursday: 11:00 am - 12:00pm

Review example (Directional Derivative)

$f(x,y,z) = z\ln(xy)$, $\vec{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{\sqrt{3}}{\sqrt{5}} \right\rangle$ unit vector? ✓

- $D_{\vec{u}} f(\frac{1}{4}, 4, 1) = \vec{\nabla} f(\frac{1}{4}, 4, 1) \cdot \vec{u}$

$$= \left\langle f_x(\tfrac{1}{4},4,1), f_y(\tfrac{1}{4},4,1), f_z(\tfrac{1}{4},4,1) \right\rangle \cdot \vec{u}$$

$$= \left\langle 4, \tfrac{1}{4}, 0 \right\rangle \cdot \left\langle \tfrac{1}{\sqrt{5}}, \tfrac{1}{\sqrt{5}}, \tfrac{\sqrt{3}}{\sqrt{5}} \right\rangle$$

$$= \frac{1}{\sqrt{5}}\left[4 + \frac{1}{4}\right]$$

$f_x = \frac{z}{xy} \cdot y = \frac{z}{x}$

$f_y = \frac{z}{y}$

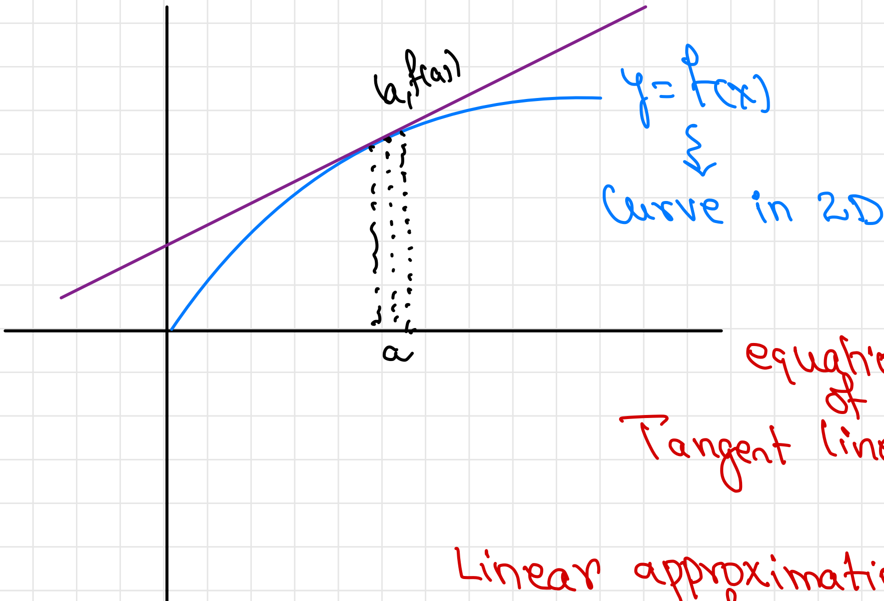
$f_z = \ln(xy)$

- What is the maximum rate of increase at $(\frac{1}{4}, 4, 1)$?

→ in direction of $\vec{\nabla} f$

→ $|\vec{\nabla} f| = |\langle 4, \tfrac{1}{4}, 0 \rangle| = \sqrt{16 + \frac{1}{16}}$

Review: Tangent lines and Linear Approximation



Approximate $y = f(x)$

using the tangent line

line through $(a, f(a))$ with slope $= f'(a)$

equation of Tangent line $\left\{ y = f(a) + f'(a)(x-a) \right.$

Linear approximation of $f(x)$ at $x = a$ $\left\{ L(x) = f(a) + f'(a)(x-a) \right.$

Differential $\left\{ \Delta y \approx f'(x) \Delta x \right.$

eg: Approximate the value of $\sqrt{4.01} \approx 2.002498\ldots$

exact value

Let $f(x) = \sqrt{x}$ ⇝ Linear approximation at $x = 4$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$L(x) = f(a) + f'(a)(x-a)$$

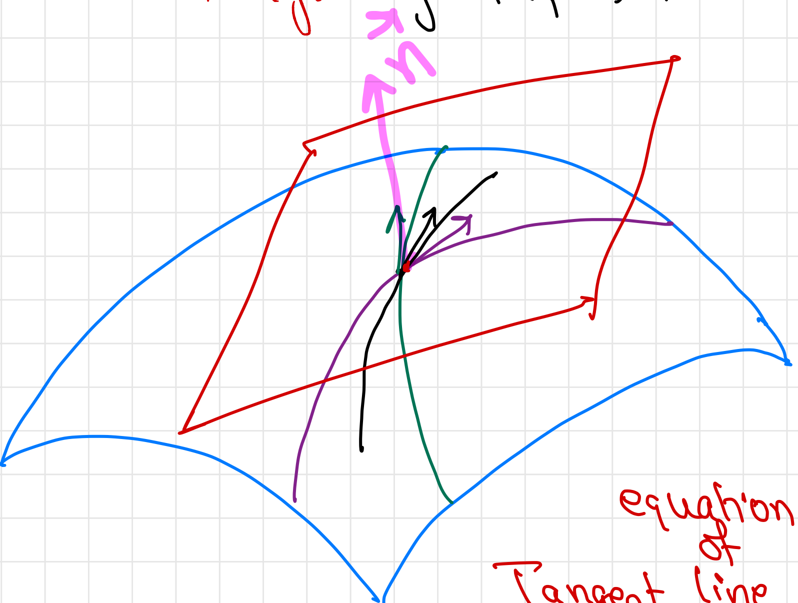
$$= 2 + \frac{1}{4}(x-4)$$

$$\sqrt{4.01} \approx L(4.01) = 2 + \frac{1}{4}(0.01)$$

$$= 2.0025$$

Approximate value

$z = f(x,y)$ } Represents Surface in 3D.



Tangent plane: plane containg all the tangent lines.

Normal vector: Orthogonal to each of the tangent lines

equation of Tangent line $\left\{ z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \right.$

Linear approximation of $f(x)$ at $x=a$ $\left\{ L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \right.$

Differential $\left\{ \Delta z \approx f_x \Delta x + f_y \Delta y \right.$

$z = f(x,y)$ is Surface in 3D

Tangent plane at $(a, b, f(a,b))$

is a plane containing all the tangent vectors to the curves passing through $(a, b, f(a,b))$

Normal vector!

Vector $\perp$ to all tangent vectors

Eq. of Tangent Plane: $z = f(a,b) + f_x(a,b)(x-a) + \boxed{f_y(a,b)(y-b)}$

Linear Approx: $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

Differential: $\Delta z = f_x \Delta x + f_y \Delta y$

e.g.: $f(x,y) = 25 - x^2 - y^2$

- Find the tangent plane at $(2, 3, 12)$ — $a, b, f(a,b)$
- Approximate $f(2.01, 2.98)$

$f_x = -2x, \quad f_y = -2y \Rightarrow f_x(2,3) = -4, \quad f_y(2,3) = -6$

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$z = f(2,3) + f_x(2,3)(x-2) + f_y(2,3)(y-3)$$

$$\boxed{z = 12 - 4(x-2) - 6(y-3)}$$

$$L(x,y) = 12 - 4(x-2) - 6(y-3)$$

$$f(2.01, 2.98) \approx L(2.01, 2.98) = 12 - 4(2.01-2) - 6(2.98-3)$$

$$= 12 - 0.04 + 0.12 = \underline{12.08}$$

$$\Delta z = 0.08$$

eg: let $z = x^3 y$, find approximate change in z when (x,y) changes from $(1,3)$ to $(0.98, 3.04)$

$\Delta x = 0.98 - 1$
$= -0.02$

$\Delta y = 3.04 - 3$
$= 0.04$

$\Delta z = f_x \Delta x + f_y \Delta y$

$f_x = 3x^2 y \Rightarrow f_x(1,3) = 9$

$f_y = x^3 \Rightarrow f_y(1,3) = 1$

$\Delta z = 9 * (0.02) + 1 * 0.04 = -0.18 + 0.04$
$= -0.14$

$f(1,3) = 3$

$\therefore$ Approx of $x^3 y$ at $(0.98, 3.04) = f(1,3) + \underbrace{f_x \Delta x + f_y \Delta y}_{-0.14}$
$= 2.86$.

Finding Normal vector:

$5x+6y-3z=5 \rightsquigarrow$ Normal vector is $\langle 5, 6, -3\rangle$

$ax+by+cz=d \rightsquigarrow$ Nomal vector is $\langle a, b, c\rangle$

$z = f(x,y) \rightsquigarrow$ equation of T plane at (a,b) is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$\Downarrow$$

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) - z + f(a,b) = 0$$

$$\Downarrow$$

$$\vec{n} = \langle f_x(a,b), f_y(a,b), -1\rangle$$

$z=f(x,y) \rightsquigarrow$ Normal vector to T. plane at $(a,b,f(a,b))$ is $\langle f_x(a,b), f_y(a,b), -1\rangle$

$f(x,y) = 25 - x^2 - y^2$

T. plane at $(2,3,12)$

$$\vec{n} = \langle f_x(2,3),\ f_y(2,3),\ -1 \rangle$$

$$= \langle -4,\ -6,\ -1 \rangle$$

Q: Find equation of plane through $(2,3,12)$ with $\vec{n} = \langle -4, -6, -1 \rangle$

$$\boxed{-4(x-2) - 6(y-3) - 1(z-12) = 0}$$

$z = f(x,y)$

Rewrite

$$\underbrace{f(x,y) - z}_{F(x,y,z)} = 0$$

$$F(x,y,z) = f(x,y) - z = 0$$

$$F_x = f_x$$

$$F_y = f_y$$

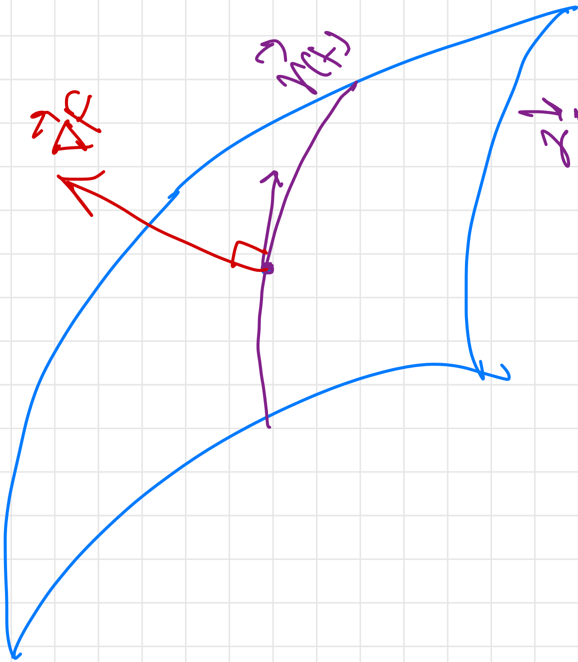
$$F_z = -1$$

$$\underbrace{\vec{\nabla} F = \langle F_x, F_y, F_z \rangle}_{} = \langle f_x, f_y, -1 \rangle$$

Gradient vector is the Normal vector.

Suppose z is given implicitly w/ $F(x,y,z)=0$

then $\vec{\nabla}F(a,b,c)$ Represents the Normal Vector to the Tangent plane at (a,b,c)



<u>proof:</u> Show $\vec{\nabla}F \perp$ tangent vector

Let $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ be a curve through (a,b,c)

$\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$ is tangent vector

$\vec{r}(t)$ lies on $F(x,y,z)=0$

$$\Rightarrow F(x(t), y(t), z(t)) = 0$$

$\downarrow \frac{d}{dt}$, Chain Rule

$$F_x \cdot x'(t) + F_y\, y'(t) + F_z\, z'(t) = 0$$

$$\Rightarrow \underbrace{\langle F_x, F_y, F_z \rangle}_{\vec{\nabla}F} \cdot \underbrace{\langle x'(t), y'(t), z'(t) \rangle}_{\vec{r}\,'(t)} = 0$$

eg: find tangent plane to Surface $2xy^2 + yz^2 + 3x^2 z = 6$ at $(1,1,1)$

$$F(x,y,z) = 2xy^2 + yz^2 + 3x^2 z - 6$$

$$\vec{\nabla} F = \langle F_x, F_y, F_z \rangle$$

$$= \langle 2y^2 + 6xz,\ 4xy + z^2,\ 2yz + 3x^2 \rangle$$

$$\vec{\nabla} F(1,1,1) = \langle 8, 5, 5 \rangle$$

Q: Find Equation of plane through $(1,1,1)$ with normal vector $\langle 8, 5, 5 \rangle$.

$$8(x-1) + 5(y-1) + 5(z-1) = 0$$

$$8x + 5y + 5z = 18.$$

e.g.: find tangent plane to the graph $z = x^3 + y^4$ at $(2,1,9)$

$$z = x^3 + y^4 = f(x,y)$$

$$\vec{n} = \langle f_x, f_y, -1 \rangle$$

$$= \langle 3x^2, 4y^3, -1 \rangle$$

$$F(x,y,z) = x^3 + y^4 - z = 0$$

$$\vec{n} = \vec{\nabla} F = \langle F_x, F_y, F_z \rangle$$

$$= \langle 3x^2, 4y^3, -1 \rangle$$

$$\vec{n} = \langle 12, 4, -1 \rangle$$

* find equation of plane through $(2,1,9)$ with normal vector $\langle 12, 4, -1 \rangle$

$$12(x-2) + 4(y-1) - z(x-9] = 0$$

$$12x + 4y - z - 19 = 0.$$