Lesson 15 : Max and Min Problems - I (15.7)

Announcements:

* Exam 1 on Feb 25th (Wednesday)
  in ELLT 116
  at 6:30 pm

* Instructions, Seating chart, Study guide } posted on Bright space

* Office Hours for Exam 1: 02/16, 02/18, 02/20 9:45 - 11:15
  02/23 9:45 - 11:45
  02/24, 02/25 2:00 pm - 3:00 pm

Warmup Example: $f'(x) = x^2(x-1)(x-2)$

① Find Critical points

$f'(a) = 0$ or $f'(a)$ DNE $\rightsquigarrow$ $x = a$ is critical point

$f'(x) = x^2(x-1)(x-2) = 0 \rightsquigarrow \underbrace{x=0,\ x=1,\ x=2}_{\text{3 Critical points}}$

② Find points where $f(x)$ has local max and local min

Second Derivative test:

$f''(a) > 0 \rightsquigarrow$ local min

$f''(a) < 0 \rightsquigarrow$ local max

$f''(a) = 0 \rightsquigarrow$ inconclusive

$f''(x) = 2x(x-1)(x-2) + x^2(x-2) + x^2(x-1) \rightsquigarrow$

$f''(0) = 0 \rightarrow$ inconclusive

$f''(1) = -1 < 0$

$f''(2) = 4 > 0$

$x=1$ is local max

$x=2$ is local min

two Variable functions: $z=f(x,y)$

Critical points: (a,b) is a critical point if

both $f_x(a,b)=0$ & $f_y(a,b)=0 \rightsquigarrow \vec{\nabla} f(a,b)=\langle 0,0\rangle$.

OR one of the partials DNE

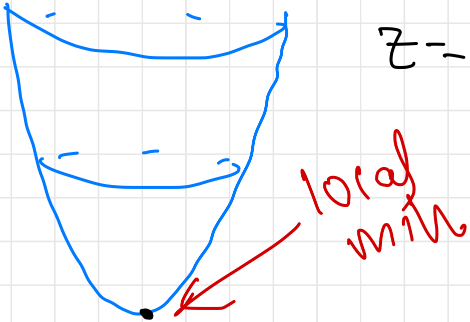
Second Derivative Test: $\underbrace{D = f_{xx}f_{yy} - (f_{xy})^2}$ } Discriminant

$D>0$, $f_{xx}>0, f_{yy}>0 \rightsquigarrow$ Local min

$f_{xx}<0, f_{yy}<0 \rightsquigarrow$ local max

$D<0 \rightsquigarrow$ Saddle

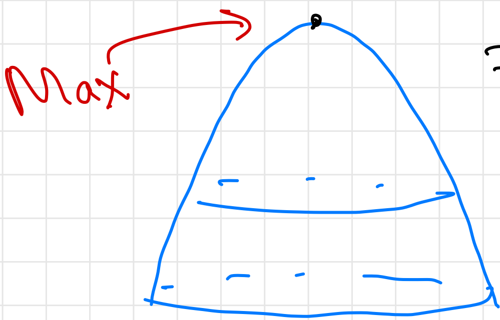
$D=0 \rightsquigarrow$ inconclusive



$z = x^2 + y^2 \rightsquigarrow \left.\begin{array}{l} f_x = 2x = 0 \\ f_y = 2y = 0 \end{array}\right\}$ (0,0) is critical

$f_{xx} = 2 > 0$, $f_{yy} = 2 > 0$, $f_{xy} = 0$

$D = 4 > 0$



$z = -(x^2 + y^2) \rightsquigarrow$ (0,0) is critical

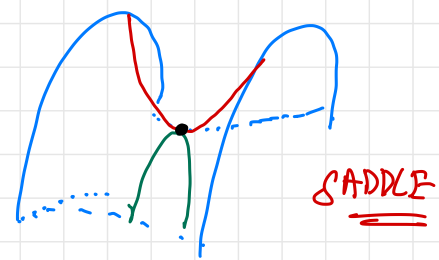
$f_{xx} = f_{yy} = -2 < 0$, $f_{xy} = 0$

$D = 4 > 0$

$z = x^2 - y^2 \rightsquigarrow$ (0,0) is critical

$f_{xx} = 2$, $f_{yy} = -2$, $f_{xy} = 0$

$D = -4 < 0$



eg: $f(x,y) = x^4 + y^4 - 4xy + 1$
#

Critical points:

$f_x = 4x^3 - 4y = 0 \rightsquigarrow \boxed{y = x^3}$ (1)

$f_y = 4y^3 - 4x = 0 \rightsquigarrow \boxed{y^3 = x}$ (2)

Solve for (x,y) satisfying both (1) & (2)

Plug in (1) in (2)

$(x^3)^3 = x \rightsquigarrow x^9 = x \rightsquigarrow x(x^8 - 1) = 0$

$x = 0, \quad x = 1, \quad x = -1$

Plug back in (1)

$$\left.\begin{array}{l} x = 0 \rightsquigarrow y = 0 \\ x = 1 \rightsquigarrow y = 1 \\ x = -1 \rightsquigarrow y = -1 \end{array}\right\}$$

$(0,0)$, $(1,1)$, $(-1,-1)$ are 3 critical points.

Second Derivative Test:

$f_x = 4x^3 - 4y$
$f_y = 4y^3 - 4x$

$f_{xx} = 12x^2$
$f_{yy} = 12y^2$
$f_{xy} = -4$

Want to know the signs of f_{xx} & $D = f_{xx}f_{yy} - (f_{xy})^2$

	f_{xx}	f_{yy}	f_{xy}	D	Classify.
$(0,0)$	0	0	-4	$-16 < 0$	SADDLE
$(1,1)$	$12 > 0$	12	-4	$12^2 - 16 > 0$	LOCAL MIN
$(-1,-1)$	$12 > 0$	12	-4	$12^2 - 16 > 0$	LOCAL MIN.

eg: $f(x,y) = xy\,e^{-(2x^2+8y^2)}$

Critical Points:

$$f_x = y\cdot e^{-(2x^2+8y^2)} + x\cdot y\cdot e^{-(2x^2+8y^2)}\cdot(-4x) = \overbrace{e^{-(2x^2+8y^2)}}^{>0}\cdot y\cdot(1-4x^2) = 0$$

$$\boxed{y(1-4x^2)=0} \quad (1)$$

$$f_y = x\cdot e^{-(2x^2+8y^2)} + xy\cdot e^{-(2x^2+8y^2)}(-16y) = \underbrace{e^{-(2x^2+8y^2)}}_{>0}\cdot x\cdot(1-16y^2) = 0$$

$$\boxed{x\cdot(1-16y^2)=0} \quad (2)$$

(1) $\rightsquigarrow y(1-4x^2)=0 \Rightarrow y=0$ OR $1-4x^2=0$

$y=0$ in (2): $x(1-0)=0 \rightsquigarrow (0,0)$

$x=\frac{1}{2}$ or $x=-\frac{1}{2}$

$x=\frac{1}{2}$ in (2): $\frac{1}{2}(1-16y^2)=0 \rightsquigarrow y=\frac{1}{4}$ or $\frac{-1}{4}$

$\left(\frac{1}{2},\frac{1}{4}\right), \left(\frac{1}{2},\frac{-1}{4}\right)$

$x=-\frac{1}{2}$ in (2): $\frac{1}{2}(1-16y^2)=0 \rightsquigarrow y=\frac{1}{4}$ or $\frac{-1}{4}$

$\left(\frac{-1}{2},\frac{1}{4}\right), \left(\frac{-1}{2},\frac{1}{4}\right)$

Apply Second derivative test:

$$f_{xx} = -4xy(3-4x^2)\, e^{-(2x^2+8y^2)} \quad \geq 0$$

$$f_{yy} = -16xy\,(3-16y^2)\, e^{-(2x^2+8y^2)} \quad \geq 0$$

$$f_{xy} = (1-4x^2)(1-16y^2)\, e^{-(2x^2+8y^2)} \quad \geq 0$$

	f_{xx}	f_{yy}	f_{xy}	$D = f_{xx}f_{yy} - (f_{xy})^2$
$(0,0)$	0	0	1	$-1 < 0$ SADDLE
$(\frac{1}{2}, \frac{1}{4})$	$-\frac{1}{2} * 2 * e^{-1} < 0$	$-2*2*e^{-1}$	0	> 0 LOCAL MAX
$(\frac{1}{2}, -\frac{1}{4})$	$\frac{1}{2} * 2 * e^{-1} > 0$	$2*2*e^{-1}$	0	> 0 LOCAL MIN
$(-\frac{1}{2}, +\frac{1}{4})$	$\frac{1}{2} * 2 * e^{-1} > 0$	$2*2*e^{-1}$	0	> 0 LOCAL MIN
$(-\frac{1}{2}, -\frac{1}{4})$	$-\frac{1}{2} * 2 * e^{-1} < 0$	$-2*2*e^{-1}$	0	> 0 LOCAL MAX

eg: $f(x,y) = \frac{1}{3}y^3 + x^2 + 4xy - 2x - 13y + 7$

Critical:

$f_x = 2x + 4y - 2 = 0 \quad (1)$

$f_y = y^2 + 4x - 13 = 0 \quad (2)$

$(1) \rightsquigarrow 2x + 4y - 2 = 0 \Rightarrow 2x = 2 - 4y \Rightarrow x = 1 - 2y$

in (2)

$y^2 + 4(1-2y) - 13 = 0$

$y^2 - 8y - 9 = 0$

$(y-9)(y+1) = 0$

$y = 9$ OR $y = -1$

in (1): $x = 1 - 2(9) = -17$, $(-17, 9)$

in (1): $x = 1 - 2(-1) = 3$, $(3, -1)$

Second Derivative test:

$f_{xx} = 2$

$f_{yy} = 2y$

$f_{xy} = 4$

	f_{xx}	f_{yy}	f_{xy}
$(-17, 9)$	2	18	4
$(3, -1)$	2	-2	4

$D = f_{xx} f_{yy} - (f_{xy})^2$

$36 - 4^2 > 0$

LOCAL MIN

$-4 - 4^2 < 0$

SADDLE