Lesson 16: Max and Min Problems - II (15.7)

Announcements:

* Exam 1 on Feb 25th (Wednesday)
  in ELLT 116
  at 6:30pm

* Instructions
  Seating chart
  Study guide
  Office Hours for Exam 1

  posted on Brightspace

Review Example:

$f_{xx}(a,b) \neq 0$, $\vec{\nabla} f(a,b) = \vec{0}$ and $f_{xx}(a,b) + f_{yy}(a,b) = 0$

$\langle f_x(a,b), f_y(a,b) \rangle = \langle 0, 0 \rangle$

what can you say about the point (a,b)?

A) Saddle

B) Local max

C) Local min

D) Cannot be Determined

E) I don't know

$f_{xx} = -f_{yy}$

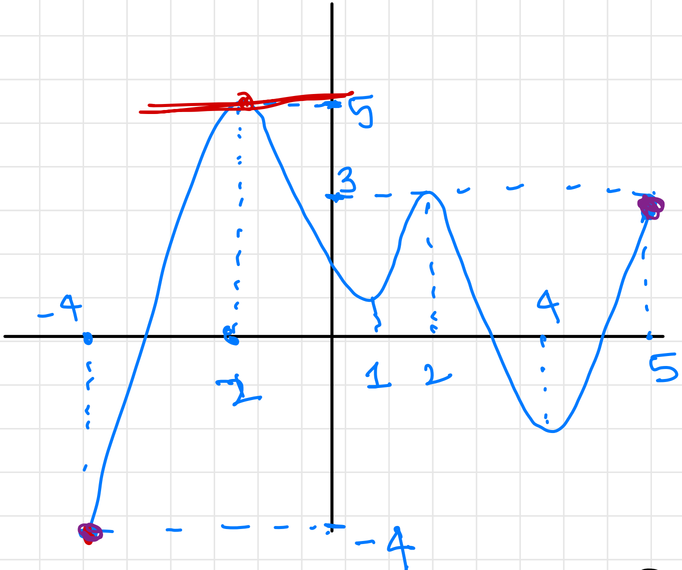
$D = f_{xx} f_{yy} - (f_{xy})^2$

$= -(f_{yy})^2 - (f_{xy})^2 < 0$

$D > 0 \rightsquigarrow$ $f_{xx} > 0$ Min

$f_{xx} < 0$ Max

$D = 0$ inconclusive



$y = f(x)$ on $[a,b]$

Abs Max / Min occur

at critical points inside $[a,b]$

or at End points / Boundary points

Process

① Find critical points

② Evaluate $f(x)$ at critical & End points

③ Compare

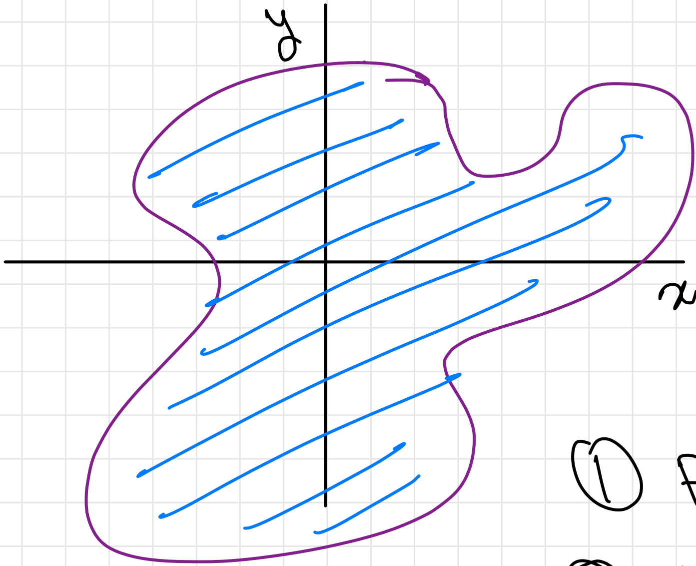
eg #: $g(x) = 2x^2 - 1$ on $[-1, 1]$

① Critical: $g'(x) = 4x = 0 \rightsquigarrow x = 0$

② Evaluate $g(0) = -1 \rightarrow$ Min $= -1$ at $x = 0$

$$\left.\begin{aligned} g(-1) &= 1 \\ g(1) &= 1 \end{aligned}\right\} \text{Max} = 1 \text{ at } x = 1 \,\&\, -1$$

Absolute max/min of $z=f(x,y)$ Over a Region



Abs Max/Min occur either at Critical points inside or at Boundary.

Process:

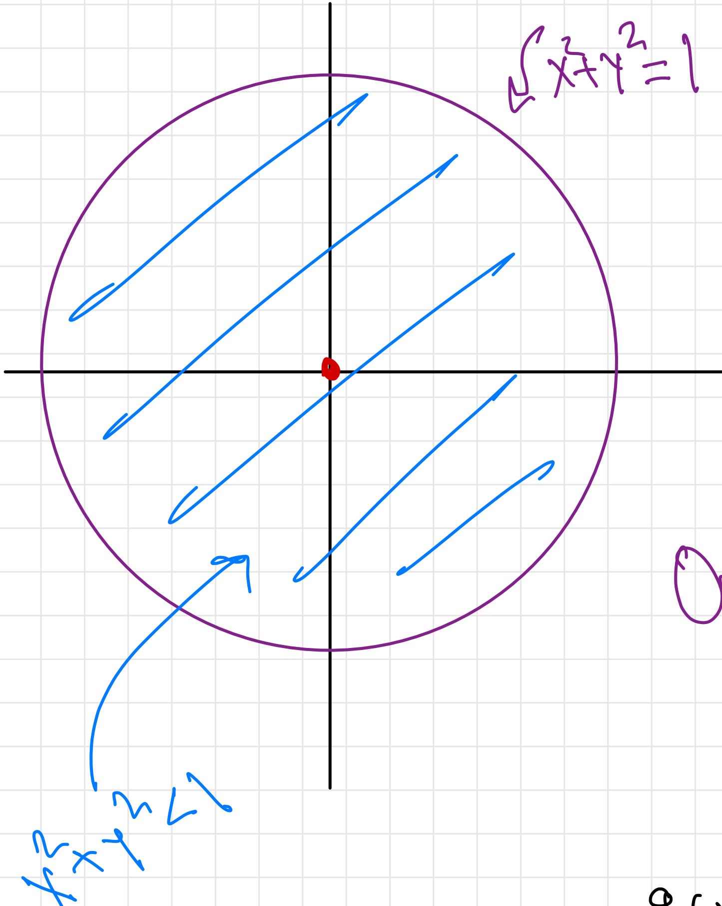
① Find Critical points inside

② Use Boundary to write $f(x,y)$ as function of 1-Variable

+ Use Calc I techniques to find possible Candidates

③ Evaluate & Compare

eg: Find Abs. max/min of $f(x,y) = x^2 - y^2$ on $x^2+y^2 \leq 1$



Critical points:

$f_x = 2x = 0$
$f_y = -2y = 0$

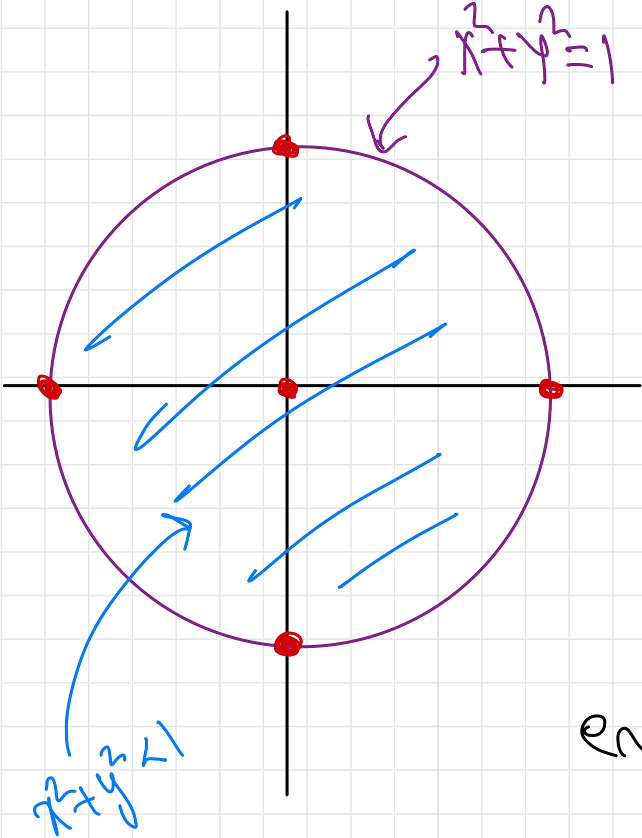
$(0,0)$ is only critical point

On Boundary: $x^2+y^2=1$

$y^2 = 1-x^2$

Substitute in $f(x,y)$ to eliminate y.

$g(x) = x^2 - (1-x^2) = 2x^2 - 1, \quad -1 \leq x \leq 1$



On Boundary: $x^2+y^2=1$

$f(x,y) = x^2 - y^3$

$\downarrow$

$g(x) = 2x^2 - 1, \quad -1 \leq x \leq 1$

Critical: $x = 0$

$\downarrow$ in Boundary

$0^2 + y^2 = 1 \rightsquigarrow y = 1$ or -1 $\Big\}$ $(0,1)$ $(0,-1)$

end points: $x = -1 \rightsquigarrow (-1)^2 + y^2 = 1$
$\Rightarrow y = 0$ $\Big\}$ $(-1,0)$

$x = 1 \rightsquigarrow 1^2 + y^2 = 1 \Rightarrow y = 0$ $\Big\}$ $(1,0)$

$f(x,y) = x^2 - y^2$

possible candidates for max/min

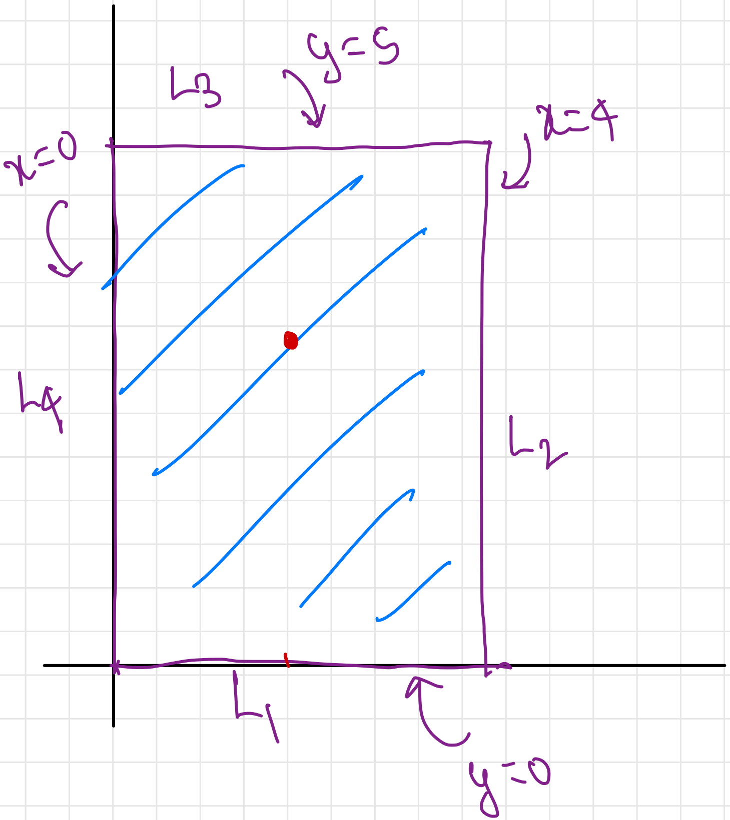
$(0,0)$, $(1,0)$, $(-1,0)$, $(0,1)$, $(0,-1)$

$f(0,0) = 0$

$$\left.\begin{array}{l} f(1,0) = 1 \\ f(-1,0) = 1 \end{array}\right\} \quad \text{Max} = 1 \quad \text{at} \quad (1,0) \ \& \ (-1,0)$$

$$\left.\begin{array}{l} f(0,1) = -1 \\ f(0,-1) = -1 \end{array}\right\} \quad \text{Min} = -1 \quad \text{at} \quad (0,1) \ \& \ (0,-1).$$

eg: find abs. max/min of $f(x,y) = 4x + 6y - x^2 - y^2 + 9$ on the Region $\{(x,y): 0 \le x \le 4,\ 0 \le y \le 5\}$



Critical points:

$f_x = 4 + 0 - 2x + 0 = 0 \Rightarrow x = 2$

$f_y = 6 - 2y = 0 \Rightarrow y = 3$

$(2,3) \Rightarrow$ critical

On Boundary)

L_1: $y = 0,\ 0 \le x \le 4$

L_2: $x = 4,\ 0 \le y \le 5$

L_3: $y = 5,\ 0 \le x \le 4$

L_4: $x = 0,\ 0 \le y \le 5$.

$$f(x,y) = 4x + 6y - x^2 - y^2 + 9$$

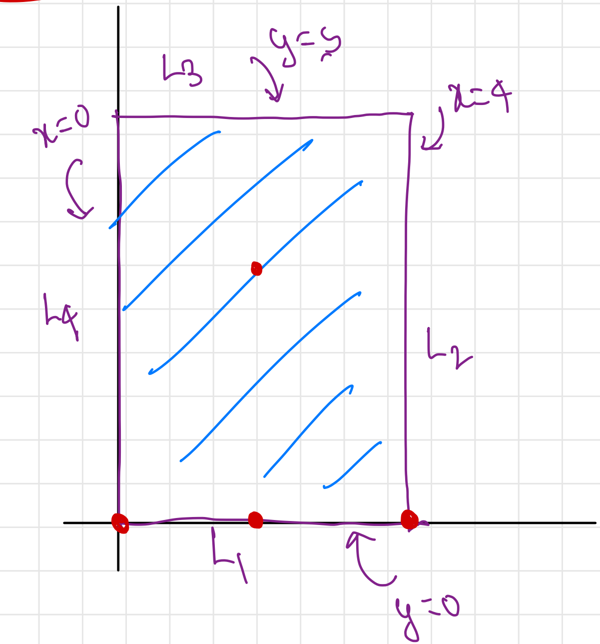
On L_1: $y = 0$, $0 \le x \le 4$

$$g_1(x) = 4x - x^2 + 9 \quad \text{on} \quad [0, 4]$$

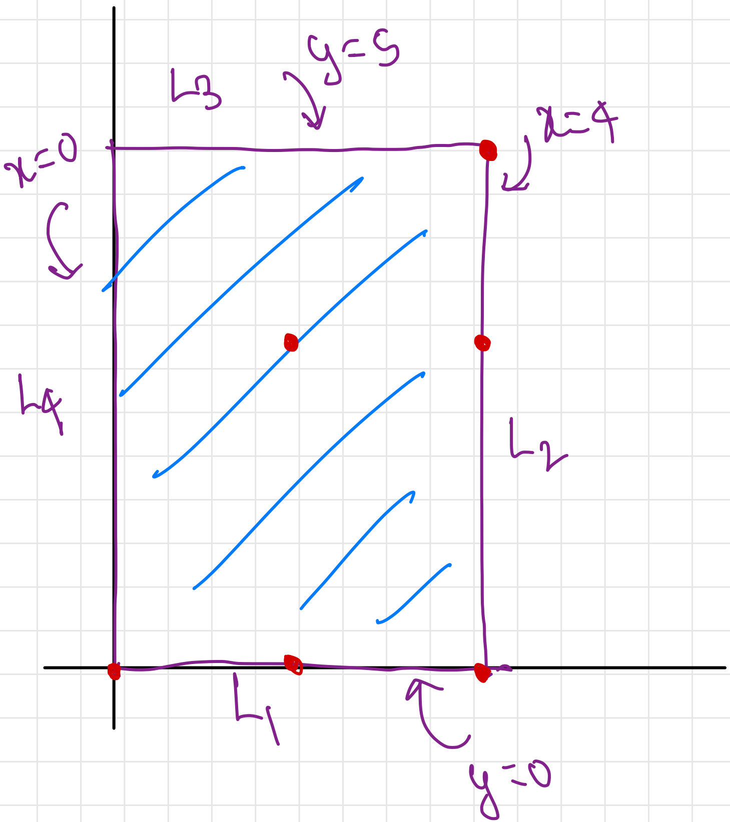
Critical! $g_1'(x) = 4 - 2x = 0 \rightsquigarrow x = 2 \rightsquigarrow (2, 0)$

end points: $x = 0 \rightsquigarrow (0, 0)$

$x = 4 \rightsquigarrow (4, 0)$



$$f(x,y) = 4x + 6y - x^2 - y^2 + 9$$



On L_2!

$$g_2(y) = 16 + 6y - 16 - y^2 + 9$$
$$= 6y - y^2 + 9 \quad \text{on } [0,5]$$

$$g_2'(y) = 6 - 2y = 0 \rightsquigarrow y = 3 \rightarrow (4,3)$$

$$y = 0 \rightsquigarrow (4,0)$$

$$y = 5 \rightsquigarrow (4,5)$$

$$f(x,y) = 4x + 6y - x^2 - y^2 + 9$$

$\underline{L_3}$: $y = 5,\ 0 \le x \le 4$

$$g_3(x) = 4x + 30 - x^2 - 25 + 9,$$

$$= 4x - x^2 + 14 \quad \text{on } [0,4]$$

$g_3'(x) = 0 \rightsquigarrow x = 2 \rightarrow (2,5)$

$x = 0 \rightsquigarrow (0,5)$

$x = 4 \rightsquigarrow (4,5)$

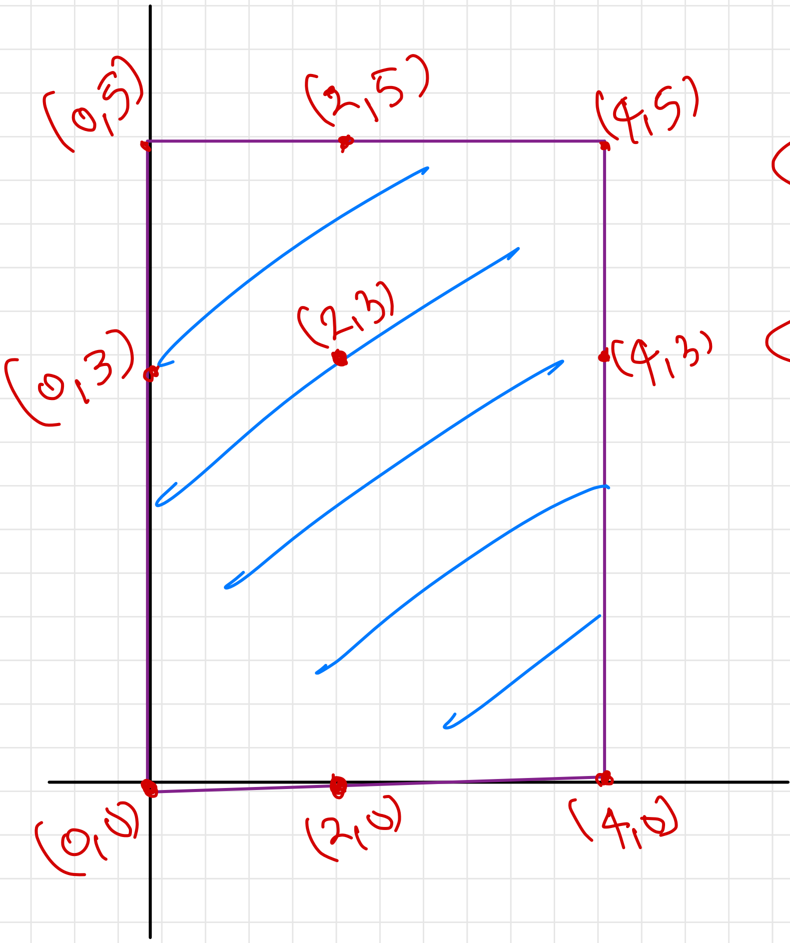
$\underline{L_4}$: $x = 0,\ 0 \le y \le 5$

$$g_4(y) = 6y - y^2 + 9, \quad [0,5]$$

$g_4'(y) = 6 - 2y \rightsquigarrow y = 3 \rightarrow (0,3)$

$y = 0 \rightsquigarrow (0,0)$

$y = 5 \rightsquigarrow (5,0)$



$f(x,y) = 4x + 6y - x^2 - y^2 + 9$

$f(0,0) = 9$

$f(2,0) = 13$

$f(4,0) = 9$

$f(0,3) = 18$

$f(2,3) = 22$

$f(4,3) = 18$

$f(0,5) = 14$

$f(2,5) = 18$

$f(4,5) = 14$

Min $= 9$ at $(0,0)$ & $(4,0)$

Max $= 22$ at $(2,3)$

eg: Find point on $x+2y+z=10$ that is closest to the Origin

distance is Minimum

distance b/w $(0,0,0)$ & (x,y,z)

$$d = \sqrt{x^2+y^2+z^2}$$

but (x,y,z) has to lie on $x+2y+z=10$

$$\Leftrightarrow z = 10-x-2y$$

$$d = \sqrt{x^2+y^2+(10-x-2y)^2}$$

Trick: if d is minimum, d^2 is min.

Rephrase: What is minimum of $f(x,y) = x^2+y^2+(10-x-2y)^2$ on $\mathbb{R}^2$

What is minimum of $f(x,y) = x^2+y^2+(10-x-2y)^2$ on $\mathbb{R}^2$

$$f_x = 2x+0+2(10-x-2y)(-1) = \cancel{4x}+4y-20 = 0 \quad (1)$$
$$f_y = 0+2y+2(10-x-2y)(-2) = \cancel{4x}+10y-40 = 0 \quad (2)$$
$$(-) \quad (-) \quad (+)$$
$$-6y+20=0$$
$$y = \frac{20}{6} = \frac{10}{3}$$

plugin $y=\frac{10}{3}$ in (1) $\rightsquigarrow 4x+\frac{40}{3}-20=0 \rightsquigarrow 4x = \frac{20}{3} \Rightarrow x = \frac{5}{3}$

One critical point $\left(\frac{5}{3}, \frac{10}{3}\right)$ } is minimum because

$f_{xx} = 4 > 0$
$f_{yy} = 10 > 0$
$f_{xy} = 4$
$D = 24 > 0$

point on the plane is $\left(\frac{5}{3}, \frac{10}{3}, \underbrace{10-\left(\frac{5}{3}\right)-2\left(\frac{10}{3}\right)}_{z=10-x-2y}\right) = \left(\frac{5}{3}, \frac{10}{3}, \frac{5}{3}\right)$