Lesson 17: Lagrange Multipliers (15.8)

Not in exam 1

Announcements:

* Exam 1 on Feb 25th (Wednesday)
  in ELLT 116
  at 6:30 pm

* Instructions
  Seating chart
  Study guide
  } posted on Bright space

* Office Before Exam 1: Today, Monday } 9:45AM - 11:15AM
  Tuesday, Wednesday } 2:45PM - 4:15PM

Warmup example: Constrained Optimization

Find max/min of x^2+y^2 on the curve $xy=1$

Objective

Constraint

Substitute

$y=\frac{1}{x}, x\neq 0$

Find ~~max~~/min of $g(x)=x^2+\frac{1}{x^2}, \quad x\neq 0$

Do not have max as we can make $g(x)$ as large as we want

$g'(x)=2x-\frac{2}{x^3}=0 \Rightarrow x^4=1$

$x=1 \xrightarrow{y=1/x} y=1$: $(1,1)$

$x=-1 \xrightarrow{y=1/x} y=-1$: $(-1,-1)$

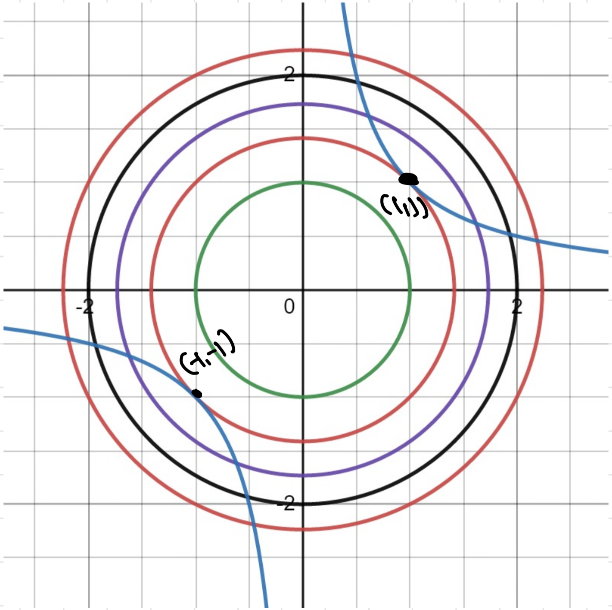
Minimum occur at $(1,1)$ & $(-1,-1)$

Lagrange Multipliers:

Find max/min of $f(x,y)=x^2+y^2$ on the curve $g(x,y)=xy=1$

$\underbrace{f(x,y)=x^2+y^2}$: Level curves are circles of increasing radius

$\underbrace{g(x,y)=xy=1}$: Hyperbola



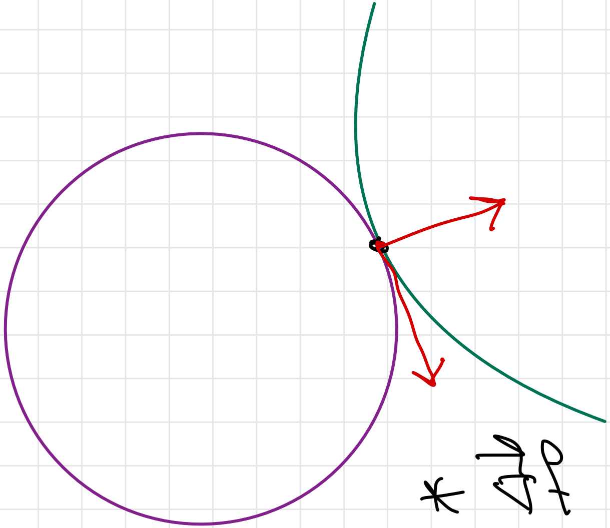
* if you move along $xy=1$ from $(1,1)$ & $(-1,-1)$ you reach higher level curves

Observe! Level curve & $xy=1$ "touch"

tangent is parallel

Recall! tangent $\perp$ Gradient

$\Rightarrow$ Gradients of f & g are parallel at $(1,1)$, $(-1,-1)$.



$f(x,y) = x^2+y^2$

$\vec{\nabla} f = \langle 2x, 2y \rangle$

$\vec{\nabla} f(1,1) = \langle 2,2 \rangle$

$\vec{\nabla} f(-1,-1) = \langle -2,-2 \rangle$

$g(x,y) = xy$

$\vec{\nabla} g = \langle y, x \rangle$

$\vec{\nabla} g(1,1) = \langle 1,1 \rangle$

$\vec{\nabla} g(-1,-1) = \langle -1,-1 \rangle$

* $\vec{\nabla} f$ & $\vec{\nabla} g$ are parallel to each other

* $\vec{\nabla} f$ & $\vec{\nabla} g$ are scalar multiples of each other

$$\boxed{\vec{\nabla} f = \lambda \vec{\nabla} g}$$

at points of max/min

Method of Lagrange Multipliers

Find max/min of $\underbrace{f(x,y,z)}_{\text{objective}}$ Subject to $\underbrace{g(x,y,z)=k}_{\text{constraint}}$.

① Solve the System of equations

$$\vec{\nabla} f = \lambda \vec{\nabla} g \quad \text{or} \quad \langle f_x, f_y, f_z \rangle = \lambda \langle g_x, g_y, g_z \rangle$$

$$\left.\begin{aligned}
&\text{I} \quad f_x = \lambda g_x \\
&\text{II} \quad f_y = \lambda g_y \\
&\text{III} \quad f_z = \lambda g_z \\
&\text{IV} \quad g(x,y,z) = k
\end{aligned}\right\}$$

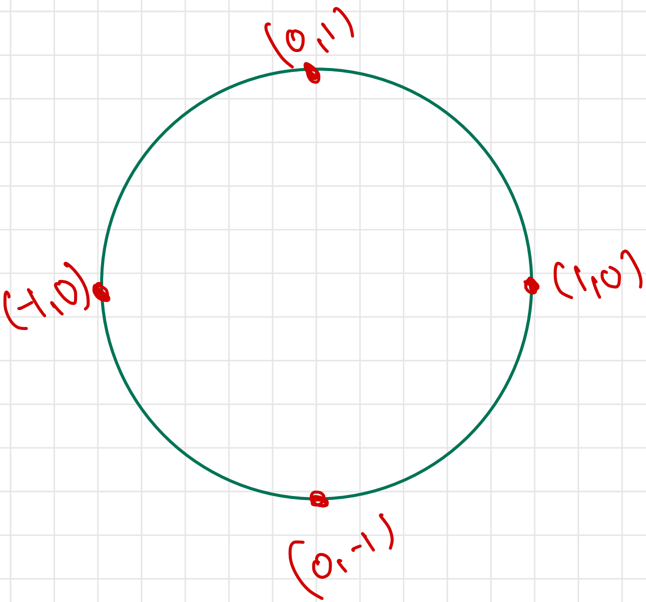
find all possible (x,y,z) & λ satisfying the four equations

② Evaluate $f(x,y,z)$ on all the points

Highest = Max.

Lowest = Min.

find max/min of $f(x,y) = x^2 - y^2$ subject to $x^2+y^2=1$



Last time: $x^2+y^2=1 \rightsquigarrow y^2 = 1-x^2$

in $f(x,y)$

$$g(x,y) = x^2 - (1-x^2) = 2x^2 - 1 \text{ on } [-1,1]$$

solve for critical points

$(-1,0)$, $(1,0)$, $(0,1)$, $(0,-1)$

Lets verify this using Lagrange Multipliers find max/min of $\underbrace{f(x,y) = x^2 - y^2}$ Subject to $\underbrace{x^2+y^2=1}$

$$\vec{\nabla} f = \langle 2x, -2y \rangle$$

$$g(x,y) = x^2+y^2$$

$$\vec{\nabla} g = \langle 2x, 2y \rangle$$

Solve:

$$2x = \lambda 2x \quad \text{I}$$

$$-2y = \lambda 2y \quad \text{II}$$

$$x^2+y^2=1 \quad \text{III}$$

$\rightsquigarrow (\lambda-1)\cdot(2x)=0 \Rightarrow \lambda = 1$ OR $x=0$

$\lambda = 1$: plug in II

$$-2y = 2y$$

$$\Rightarrow y = 0$$

plug in III

$$x^2 + 0^2 = 1$$

$$\Rightarrow x=1 \text{ or } x=-1$$

$$(1,0), (-1,0)$$

$x=0$: plug in III

$$0^2+y^2=1$$

$$\Rightarrow y=1 \text{ OR } y=-1$$

$$(0,1), (0,-1)$$

Evaluate: $f(x,y) = x^2-y^2$ at $(1,0), (-1,0), (0,1), (0,-1)$

$$\left.\begin{aligned} f(1,0) &= 1 \\ f(-1,0) &= 1 \end{aligned}\right\} \text{Max}$$

$$\left.\begin{aligned} f(0,1) &= -1 \\ f(0,-1) &= -1 \end{aligned}\right\} \text{Min}$$

eg: Find Max/Min of $f(x,y) = xy$ subject to $x^2+xy+y^2=1$

Objective Constraint

$g(x,y) = x^2+xy+y^2$

$$\vec{\nabla} f = \langle y, x \rangle = \lambda \vec{\nabla} g = \lambda \langle 2x+y, x+2y \rangle$$

I $\quad y = \lambda(2x+y) \longrightarrow \lambda = \dfrac{y}{2x+y}$

$\downarrow$ in II

II $\quad x = \lambda(x+2y)$

III $\quad x^2+xy+y^2=1$

$$x = \left(\frac{y}{2x+y}\right)(x+2y)$$

$$x(2x+y) = y(x+2y)$$

$$2x^2+\cancel{xy} = \cancel{xy}+2y^2$$

$$x^2+y^2$$

Make sure $2x+y \neq 0$.

If $2x+y=0$

I $\Rightarrow y=0$

back in $2x+y=0$

$x=0$

$x=0, y=0$ in III

$0=1$ X.

Solving the System of equations:

I $\quad y = \lambda(2x+y) \longrightarrow \lambda = \frac{y}{2x+y}$

II $\quad x = \lambda(x+2y)$

III $\quad x^2+xy+y^2=1$

$\downarrow$ in II

$$x = \left(\frac{y}{2x+y}\right)(x+2y)$$

$$x(2x+y) = y(x+2y)$$

$$2x^2+\cancel{xy} = \cancel{xy}+2y^2$$

$$x^2 = y^2$$

$y = x$

$\downarrow$ in III

$$3x^2 = 1$$

$$x = \frac{1}{\sqrt{3}} \quad \text{or} \quad \frac{-1}{\sqrt{3}}$$

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

$$\underbrace{f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = f\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) = \frac{1}{3}}_{\text{MAX}}$$

$y = x$

$\downarrow$ in III

$$x^2 = 1$$

$$x = 1 \quad \text{or} \quad -1$$

$$(1,-1), \quad (-1,1)$$

$$\underbrace{f(1,-1) = -1 = f(-1,1)}_{\text{MIN}}$$

eg: Find max/min of $f(x,y,z) = x + 5y + 3z$ subject to $x^2+y^2+z^2=1$

$\vec{\nabla} f = \langle 1, 5, 3 \rangle$

$g(x,y,z) = x^2+y^2+z^2$

$\vec{\nabla} g = \langle 2x, 2y, 2z \rangle$

(I) $1 = \lambda(2x) \Rightarrow x = \frac{1}{2\lambda}$

(II) $5 = \lambda(2y) \Rightarrow y = \frac{5}{2\lambda}$

(III) $3 = \lambda(2z) \Rightarrow z = \frac{3}{2\lambda}$

(IV) $x^2+y^2+z^2=1$

($\lambda \neq 0$, if $\lambda = 0$ (I) $\Rightarrow 1 = 0$)

plug in (IV) $\Rightarrow \frac{1}{4\lambda^2} + \frac{25}{4\lambda^2} + \frac{9}{4\lambda^2} = 1$

$\frac{35}{4\lambda^2} = 1$

$\lambda^2 = \frac{35}{4}$

$\lambda = \frac{\sqrt{35}}{2}$

$\downarrow$ in (I), (II), (III)

$(x,y,z) = \left(\frac{1}{\sqrt{35}}, \frac{5}{\sqrt{35}}, \frac{3}{\sqrt{35}}\right)$

$f\left(\frac{1}{\sqrt{35}}, \frac{5}{\sqrt{35}}, \frac{3}{\sqrt{35}}\right) = \sqrt{35}$ } MAX

$\lambda = -\frac{\sqrt{35}}{2}$

$\downarrow$ in (I), (II), (III)

$(x,y,z) = \left(\frac{-1}{\sqrt{35}}, \frac{-5}{\sqrt{35}}, \frac{-3}{\sqrt{35}}\right)$

$f\left(\frac{-1}{\sqrt{35}}, \frac{-5}{\sqrt{35}}, \frac{-3}{\sqrt{35}}\right) = \underbrace{-\sqrt{35}}_{\text{MIN}}$