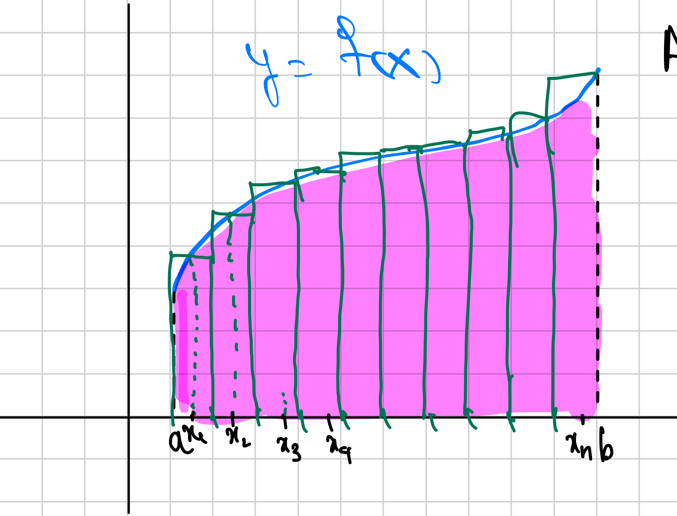
Lesson 18: Double integrals over Rectangular Regions (16.1)

Warmup:

Area under the curve. $\int_a^b f(x)\,dx$



Approximate area: * Divide $[a,b]$ into n small intervals each of size $\Delta x = \frac{b-a}{n}$

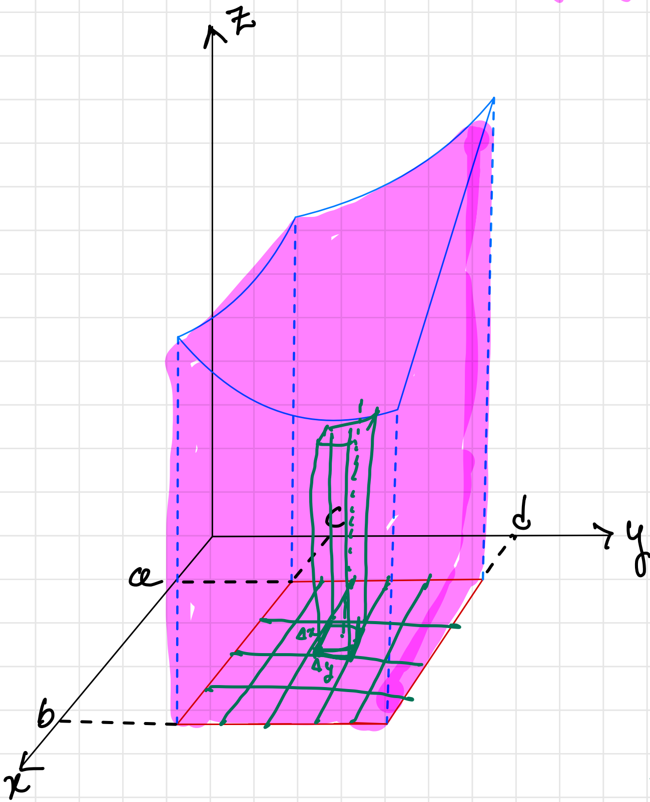
* pick Sample points in each of the intervals

* Compute Area of each Rectangle and Add.

$$\sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$Z = f(x,y)$ on $R = [a,b]\times[c,d] = \{(x,y)\in\mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d\}$



volume under Surface = Double integral of $f(x,y)$ over R

$$= \iint_R f(x,y)\,dA$$

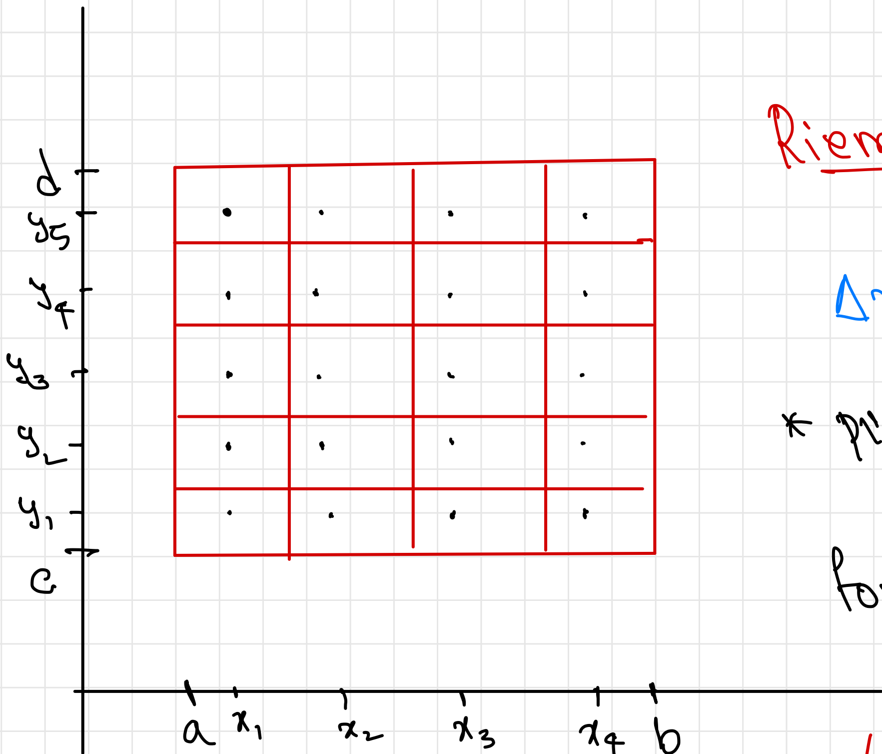
Approximate volume | Riemann Sum:

* Divide $[a,b]$ on x-axis, $[c,d]$ on y-axis into smaller intervals } get a grid of small Rectangles
* take a Sample point in each of the Rectangles, form a Box
* Compute volume of each box and add

Approximating by Dividing [a,b] into 4 intervals
[c,d] into 5 intervals

$\Downarrow$

20 Rectangles



Riemann Sum using 20 Rectangles

$\Delta x = \frac{b-a}{4}$, $\Delta y = \frac{d-c}{5}$

* pick Sample point (x_i, y_j) in each of the Rectangles

form a Box & Volume.

$f(x_i, y_j)\,\Delta x\,\Delta y$

Approx. Volume/Riemann Sum = $\sum_{j=1}^{5} \sum_{i=1}^{4} f(x_i, y_j)\,\Delta x\,\Delta y$

Approx. Volume / Riemann Sum using m intervals on X-axis

n - intervals on Y-axis

↳ total mn Rectangles

$\Delta x = \frac{b-a}{m}$, $\Delta y = \frac{d-c}{n}$

$f(x_i, y_j)$ = height of the Box in ij^{th} Rectangle

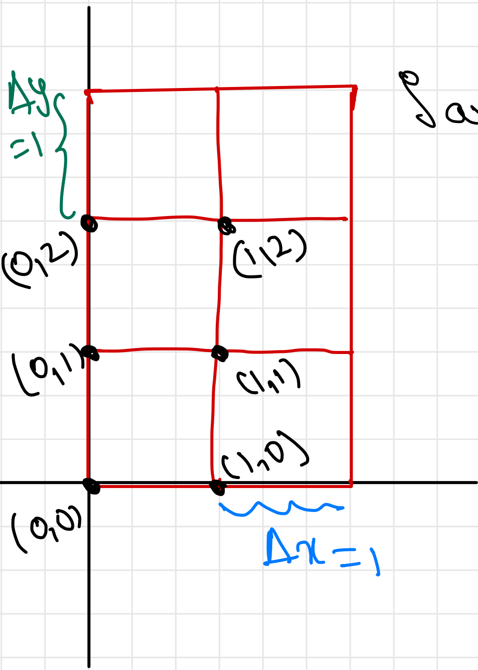
$$\boxed{\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta x\, \Delta y}$$

Riemann Sum using mn Rectangles

$$\iint_R f(x,y)\, dA = \lim_{m\to\infty} \lim_{n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta y\, \Delta x$$

eg: $Z = \underbrace{16 - x^2 - y^2}_{f(x,y)}$, $R = [0,2] \times [0,3]$ estimate volume using 6 Rectangles & bottom left corner point

$\Delta x = \frac{2-0}{2} = 1$, $\Delta y = 1$



Sample points: (0,0), (1,0)
(0,1), (1,1)
(0,2), (1,2)

$x_1 = 0$, $x_2 = 1$

$y_1 = 0$, $y_2 = 1$, $y_3 = 2$

$$\sum_{j=1}^{3} \sum_{i=1}^{2} f(x_i, y_j) \underbrace{\Delta x}_{1} \underbrace{\Delta y}_{1}$$

$$\sum_{j=1}^{3} f(x_1, y_j) + f(x_2, y_j)$$

$$= f(x_1, y_1) + f(x_2, y_1) + f(x_1, y_2) + f(x_2, y_2) + f(x_1, y_3) + f(x_2, y_2)$$

$$= f(0,0) + f(1,0) + f(0,1) + f(1,1) + f(0,2) + f(1,2)$$

$$= 16 + 15 + 15 + 14 + 12 + 11 = \underline{\underline{83}}$$

Evaluating double integrals

$f(x,y) = 16 - x^2 - y^2$ on $\underbrace{[0,2] \times [0,3]}_{R}$

$$\iint_R f(x,y)\, dA$$

fix x, integrate w.r.t y

$$\int_0^3 16 - x^2 - y^2 \, dy$$

$$= 16y - x^2 y - \frac{y^3}{3} \Big|_0^3$$

$$= 48 - 3x^2 - 9$$

$$= 39 - 3x^2$$

integrate w.r.t x $\rightsquigarrow$

$$\int_0^2 39 - 3x^2 \, dx = 39x - x^3 \Big|_0^2$$

$$= \underline{\underline{70}}.$$

$$f(x,y) = 16 - x^2 - y^2 \quad \text{on} \quad [0,2] \times [0,3]$$

Fix y, integrate w.r.t x

$$\int_0^2 16 - x^2 - y^2 \, dx = 16x - \frac{x^3}{3} - xy^2 \Big|_0^2$$

$$= 32 - \frac{8}{3} - 2y^2$$

integrate w.r.t y

$$\int_0^3 32 - \frac{8}{3} - 2y^2 \, dy = 32y - \frac{8}{3}y - \frac{2y^3}{3} \Big|_0^3$$

$$= 96 - 8 - 18$$

$$= \underline{\underline{70}}$$

Fubini's Theorem

if $f(x,y)$ is continuous on $R = [a,b] \times [c,d]$

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Average Value of a Function

$f(x)$ on $[a,b]$ $\rightsquigarrow$ $f_{ave} = \dfrac{1}{b-a} \displaystyle\int_a^b f(x)\,dx$

Length of interval

$f(x,y)$ on R inside $\mathbb{R}^2$

$$f_{ave} = \frac{1}{A} \iint_R f(x,y)\,dA$$

Area of Region

If $R = [a,b] \times [c,d]$ $\rightsquigarrow$ Rectangle

Area $= (b-a) \times (d-c)$

eg: $f(x,y) = x - 3y^2$ on $\underbrace{R = [0,2] \times [1,2]}$, find f_{ave}

Area of $R = (2-0)(2-1) = 2$

$$f_{ave} = \frac{1}{\text{Area of } R} \iint_R x - 3y^2 \, dA$$

$$= \frac{1}{2} \iint_R x - 3y^2 \, dA$$

$$= \frac{1}{2} \int_1^2 \left(\int_0^2 x - 3y^2 \, dx \right) dy$$

$$= \frac{1}{2} \int_1^2 \left[\frac{x^2}{2} - 3y^2 \cdot x \right]_0^2 dy = \frac{1}{2} \int_1^2 2 - 6y^2 \, dy$$

$$= \frac{1}{2} \left[2y - 2y^3 \right]_1^2 = \frac{1}{2} \left[-12 - 0 \right]$$

$$= -6.$$