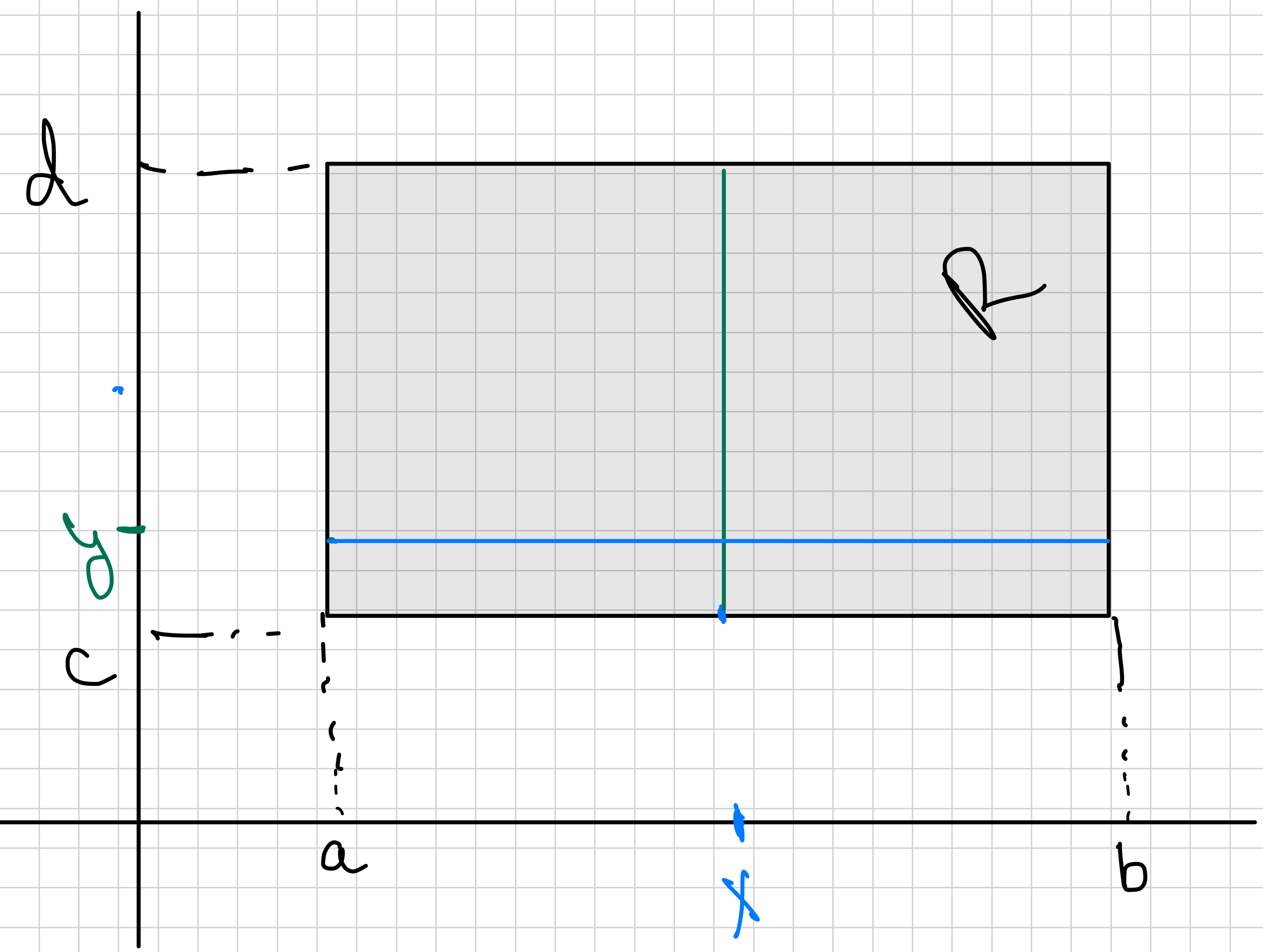
# Lesson 19: Double integrals over General Regions (16.2)

Last time: $\iint_R f(x,y)\,dA$, where $R$ is a rectangular Region
$R=[a,b]\times[c,d]$




① fix $x$, integrate w.r.t $y$ first on $c \le y \le d$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx$$

② fix $y$, integrate w.r.t $x$ first on $a \le x \le b$

$$\int_c^d \int_a^b f(x,y)\,dx\,dy$$

**Review example:** Evaluate $\iint_R y\sin(xy)\,dA$, $R=[1,2]\times[0,\pi]$

$$\int_1^2 \int_0^{\pi} y\sin(xy)\,dy\,dx$$

↓

integration by parts

⋮

$$\int_0^{\pi} \int_1^2 y\sin(xy)\,dx\,dy$$

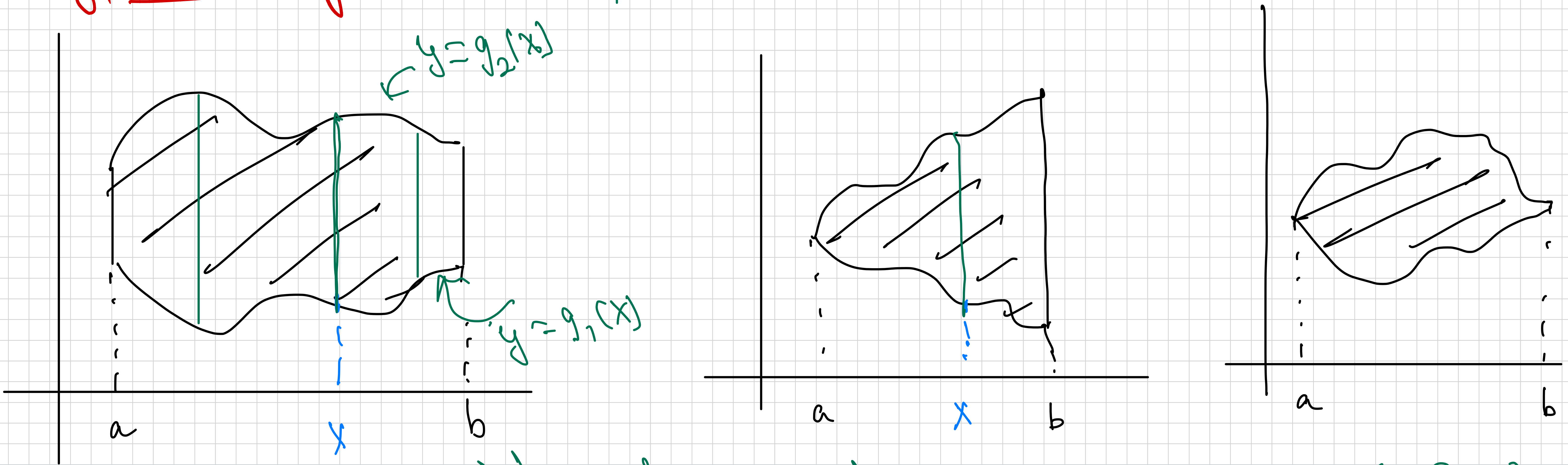
$$= \int_0^{\pi} y\left(\frac{-\cos(xy)}{y}\right)\Bigg|_1^2\,dy$$

$$= \int_0^{\pi} -\cos(2y) + \cos(y)\,dy$$

$$= -\frac{\sin(2y)}{2} + \sin(y)\Bigg]_0^{\pi}$$

$$= 0$$

# Type I Regions (top & bottom curve)

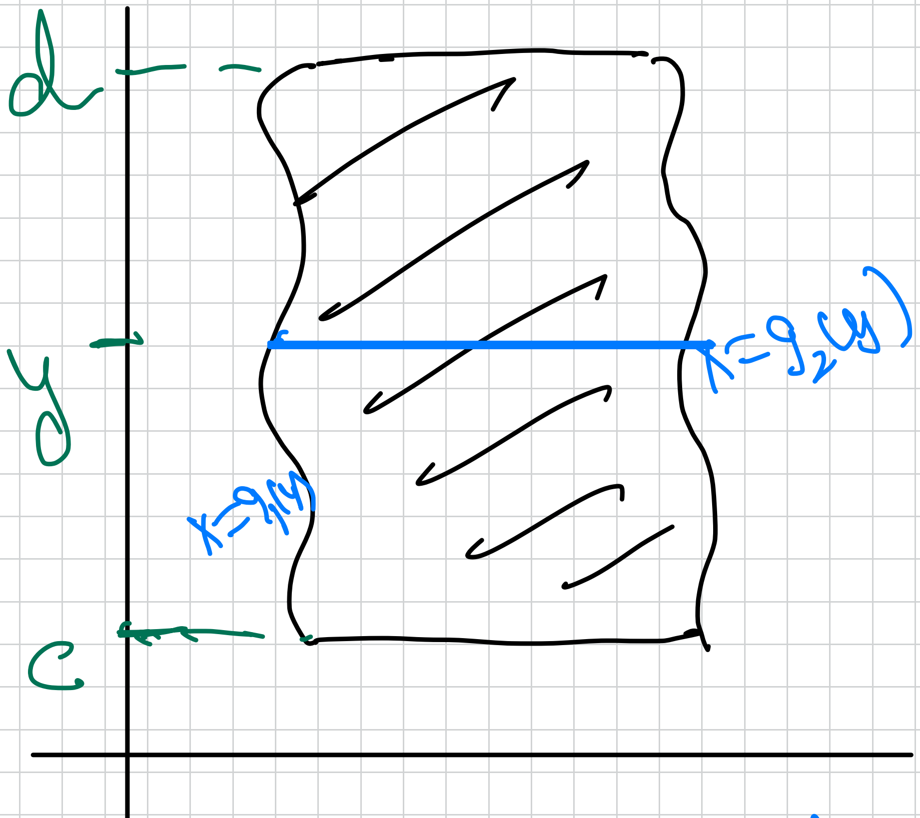


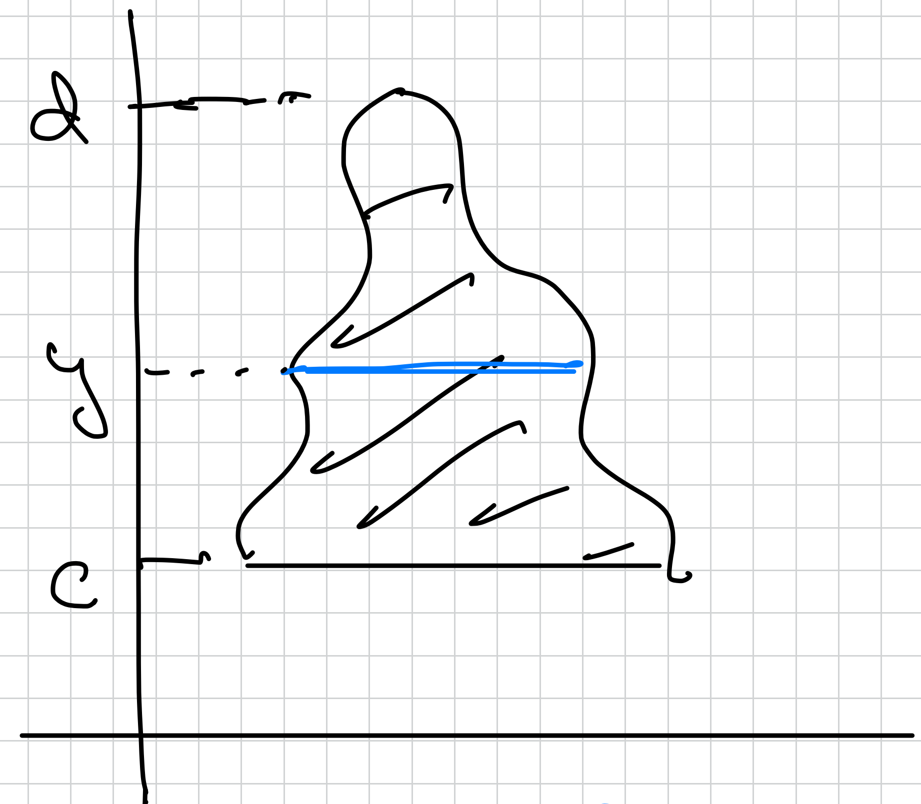

Fix $x$ $\rightsquigarrow$ integrate w.r.t $y$ on $g_1(x) \le y \le g_2(x)$

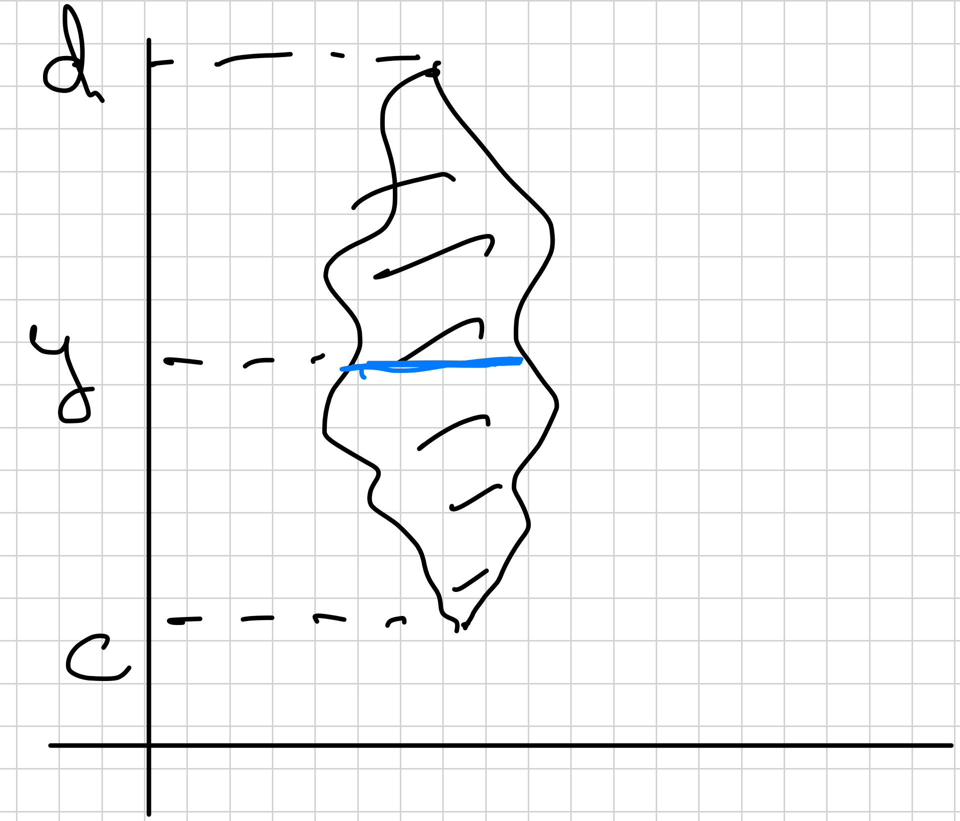
then integrate w.r.t $x$ on $a \le x \le b$

$$\iint_D f(x,y)\, dA = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x,y)\, dy\, dx$$

# Type II Regions Right & Left Curve



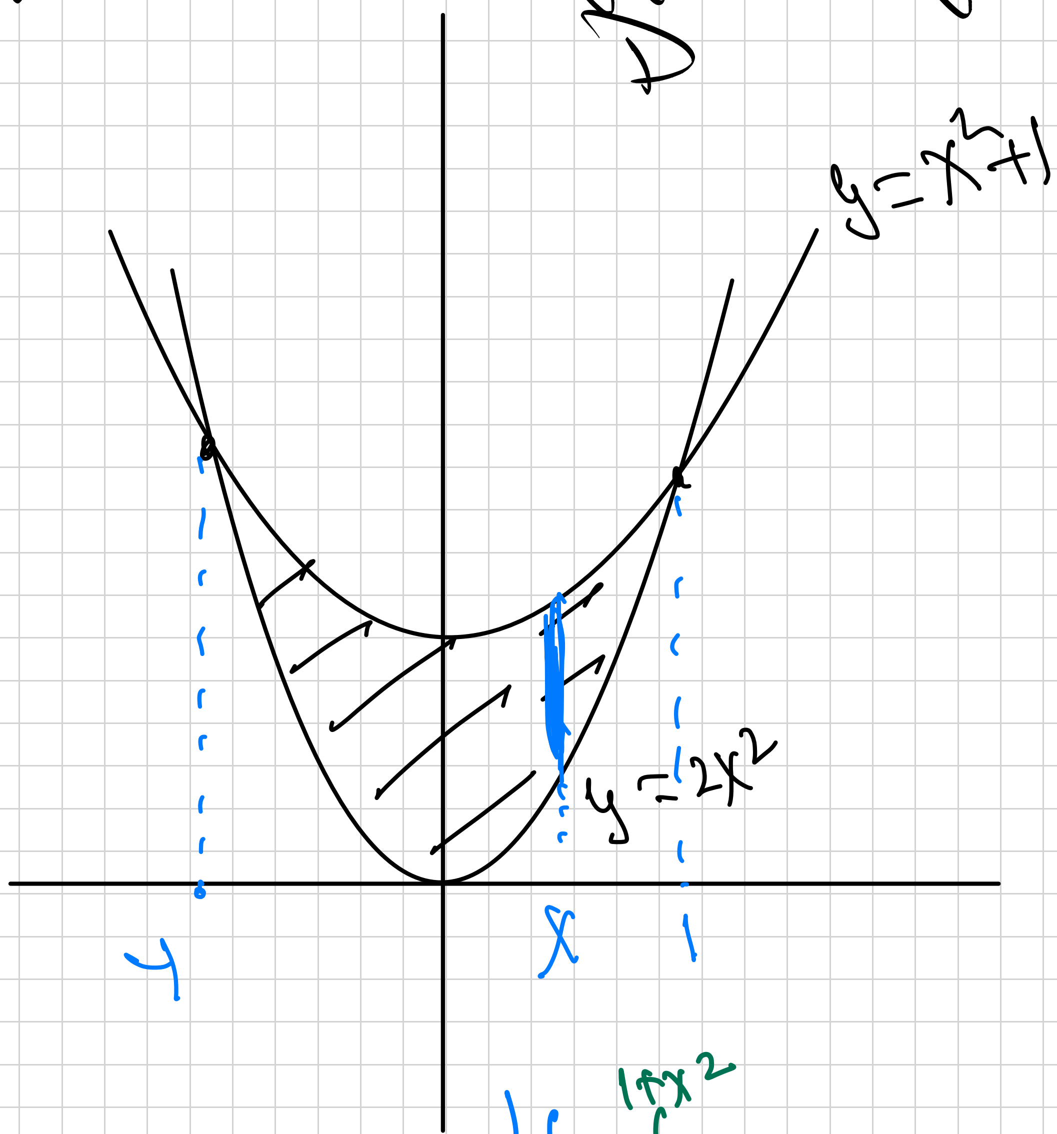




fix y, integrate w.r.t x first on $g_1(y) \le x \le g_2(y)$
then integrate w.r.t y on $c \le y \le d$

$$\iint_D f(x,y)\, dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\, dx\, dy$$

eg: evaluate $\iint_D (x+2y)\,dA$, D is bounded by $y=2x^2$ & $y=1+x^2$




fix $x \rightsquigarrow 2x^2 \le y \le 1+x^2$

$$\int \int_{2x^2}^{1+x^2} x+2y \; dy \; dx$$

x bounds

find x-bounds by finding points of intersection:

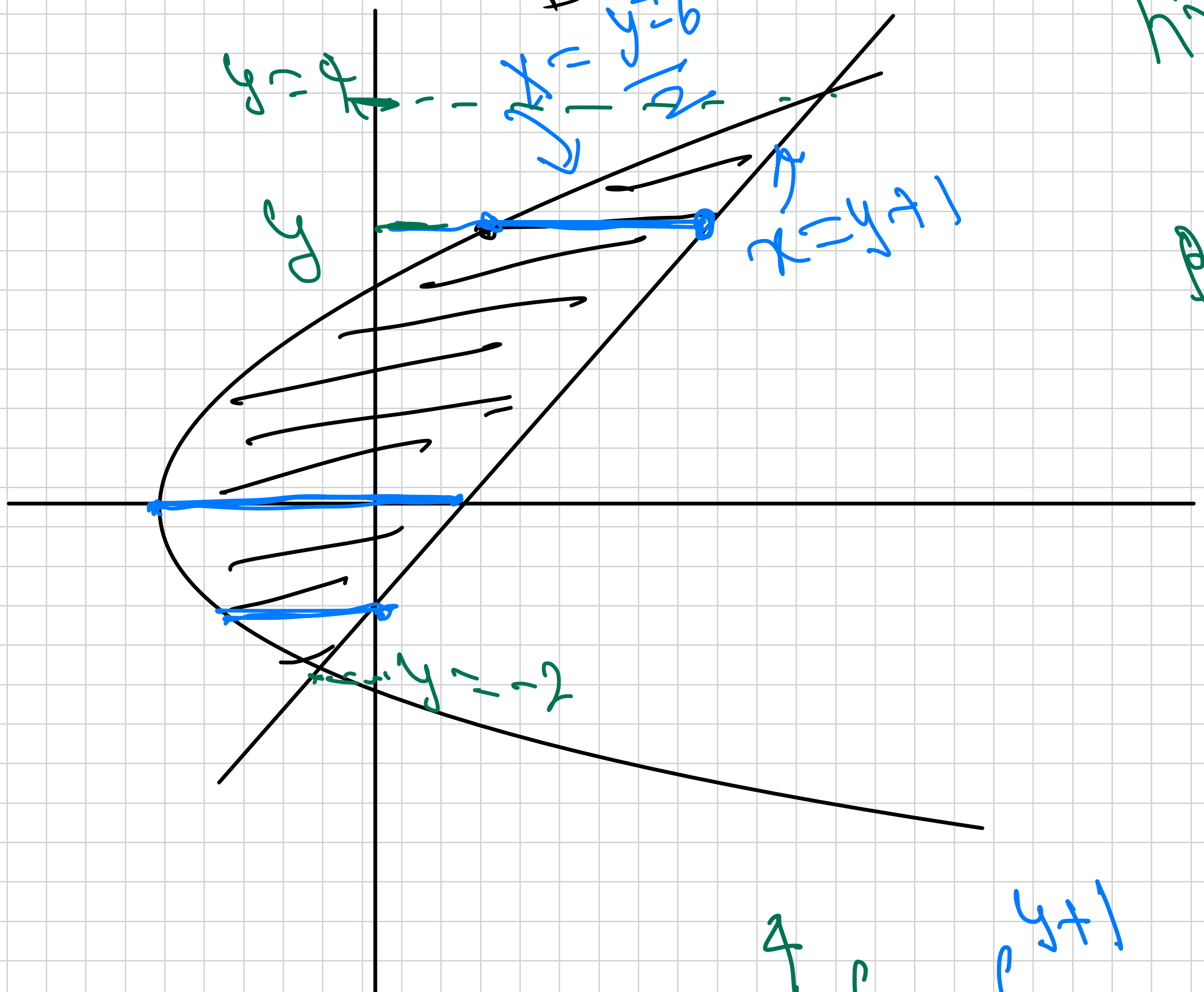
$2x^2 = 1+x^2 \rightsquigarrow x^2 = 1$

$x = 1$ or $-1$

$$\iint_D x+2y\,dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} x+2y\;dy\;dx$$

$$= \int_{-1}^{1} xy + y^2 \Big]_{y=2x^2}^{1+x^2} dx = \int_{-1}^{1} \left[x(1+x^2)+(1+x^2)^2\right] - \left[x(2x^2)+(2x^2)^2\right] dx$$

Eg: Evaluate $\iint_D xy\, dA$, D is bounded by $y = x-1$ & $y^2 = 2x+6$




fix y, $\frac{y^2-6}{2} \le x \le y+1$

Bounds for y
find points of intersection

$y+1 = \frac{y^2-6}{2} \rightsquigarrow y^2 - 2y - 8 = 0$

$(y-4)(y+2) = 0$

$y = -2, \; y = 4$

$$\iint_D xy\, dA = \int_{-2}^{4} \int_{\frac{y^2-6}{2}}^{y+1} xy \, dx \, dy = \int_{-2}^{4} \left. \frac{x^2 y}{2} \right|_{x=\frac{y^2-6}{2}}^{x=y+1} dy$$

$$= \int_{-2}^{4} \frac{(y+1)^2 y}{2} - \frac{(y^2-6)^2 y}{8} \, dy$$

eg: Switch the order of integration and evaluate

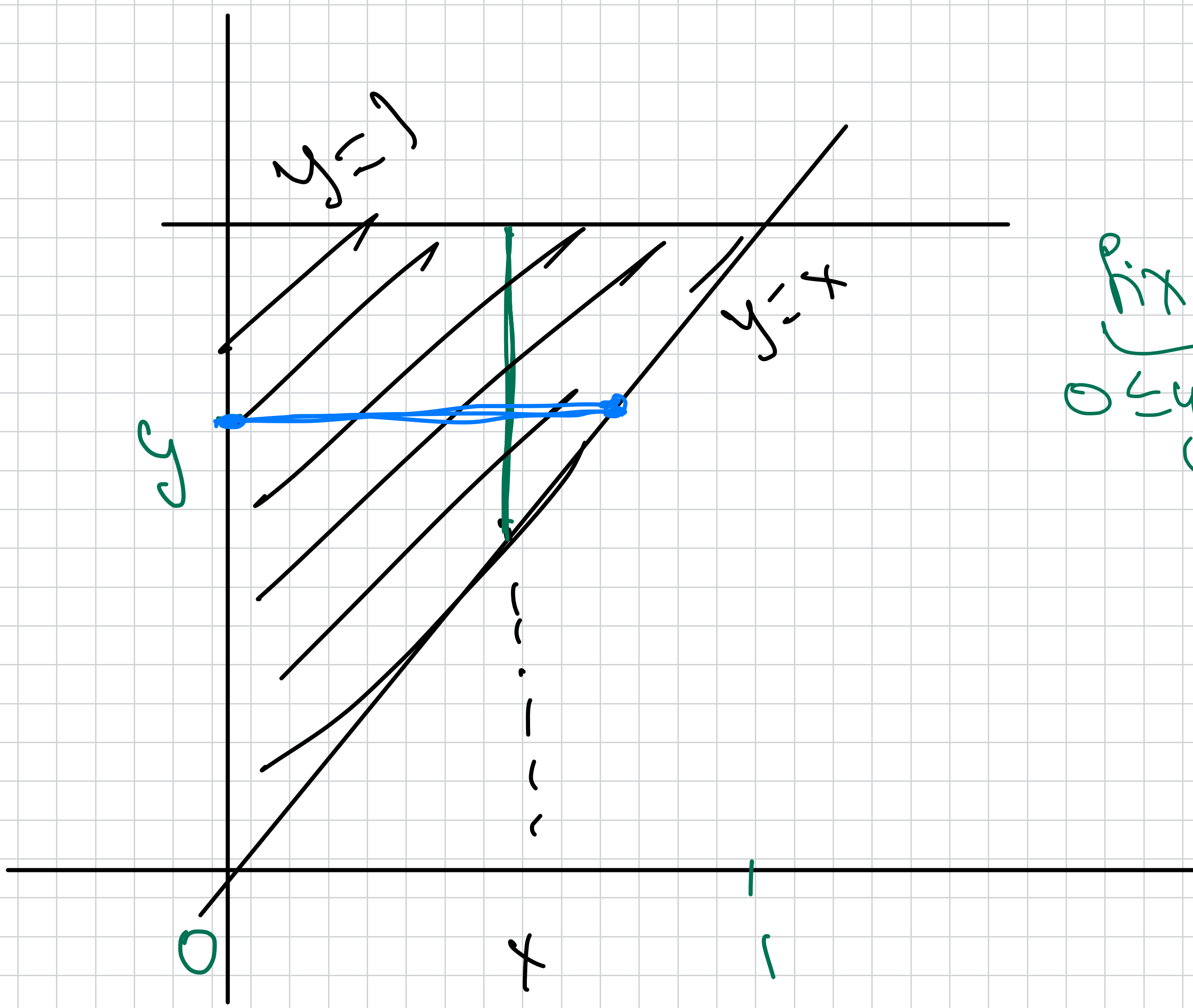
$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$$

Sketch the Region, using the bounds given

fix $x$ w/ $x \le y \le 1$, $0 \le x \le 1$

$$\int \int \sin(y^2)\, dx\, dy$$

y bounds  x bounds




fix $y$ w/ $0 \le x \le y$

$0 \le y \le 1$

$$\int_0^1 \int_0^y \sin(y^2)\, dx\, dy = \int_0^1 x \sin(y^2) \Big|_{x=0}^{x=y} dy$$

$$= \int_0^1 y \sin(y^2)\, dy$$

$$= -\frac{\cos(y^2)}{2} \Big|_0^1$$

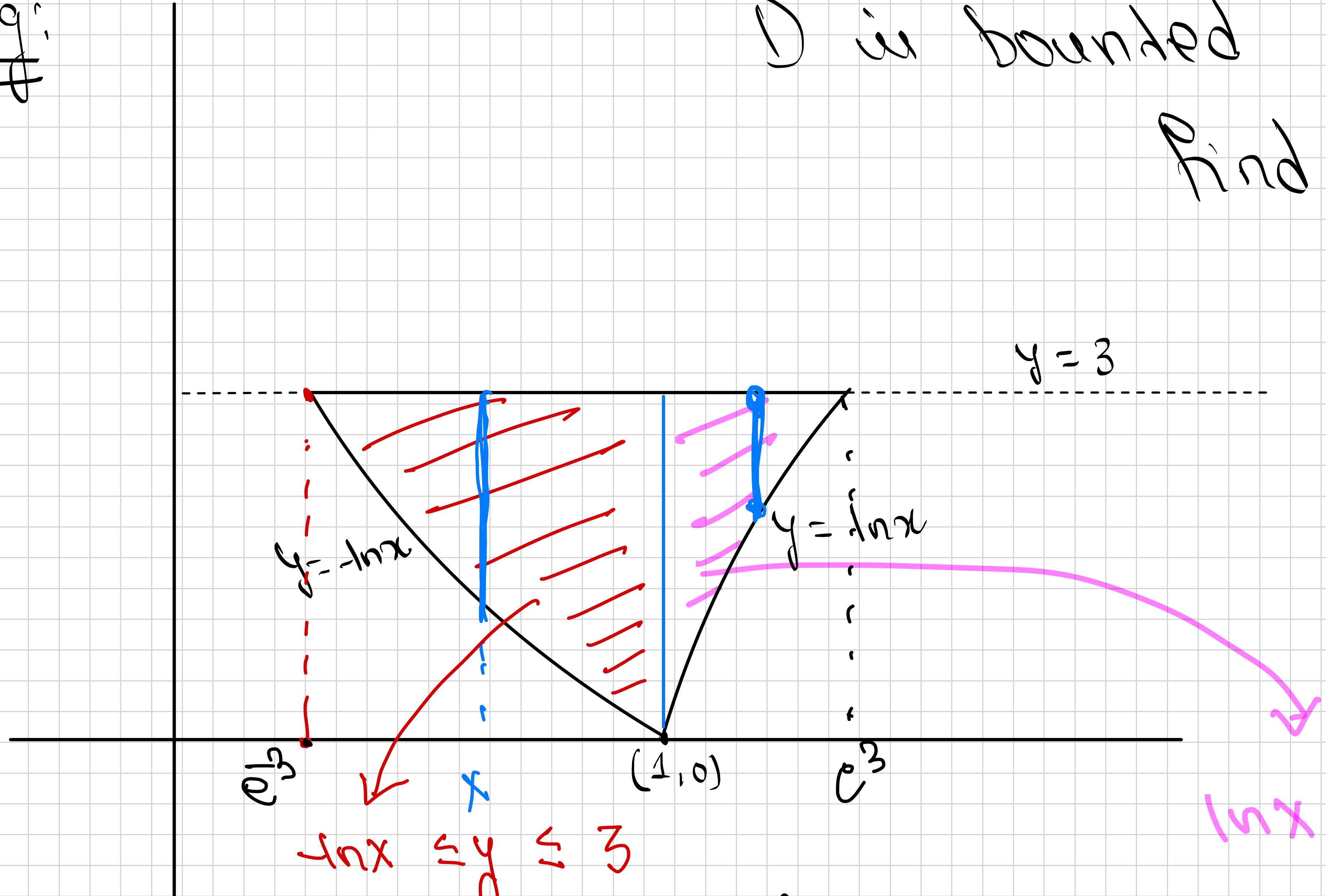
$$= \frac{-\cos(1) + 1}{2}$$

eg:

D is bounded by $y=-\ln x$, $y=\ln x$ & $y=3$

Find bounds to write $\iint_D f(x,y)\,dA$

$$\iint f(x,y)\,dy\,dx$$




$-\ln x \le y \le 3$

$3=-\ln x \Rightarrow x=e^{-3}$

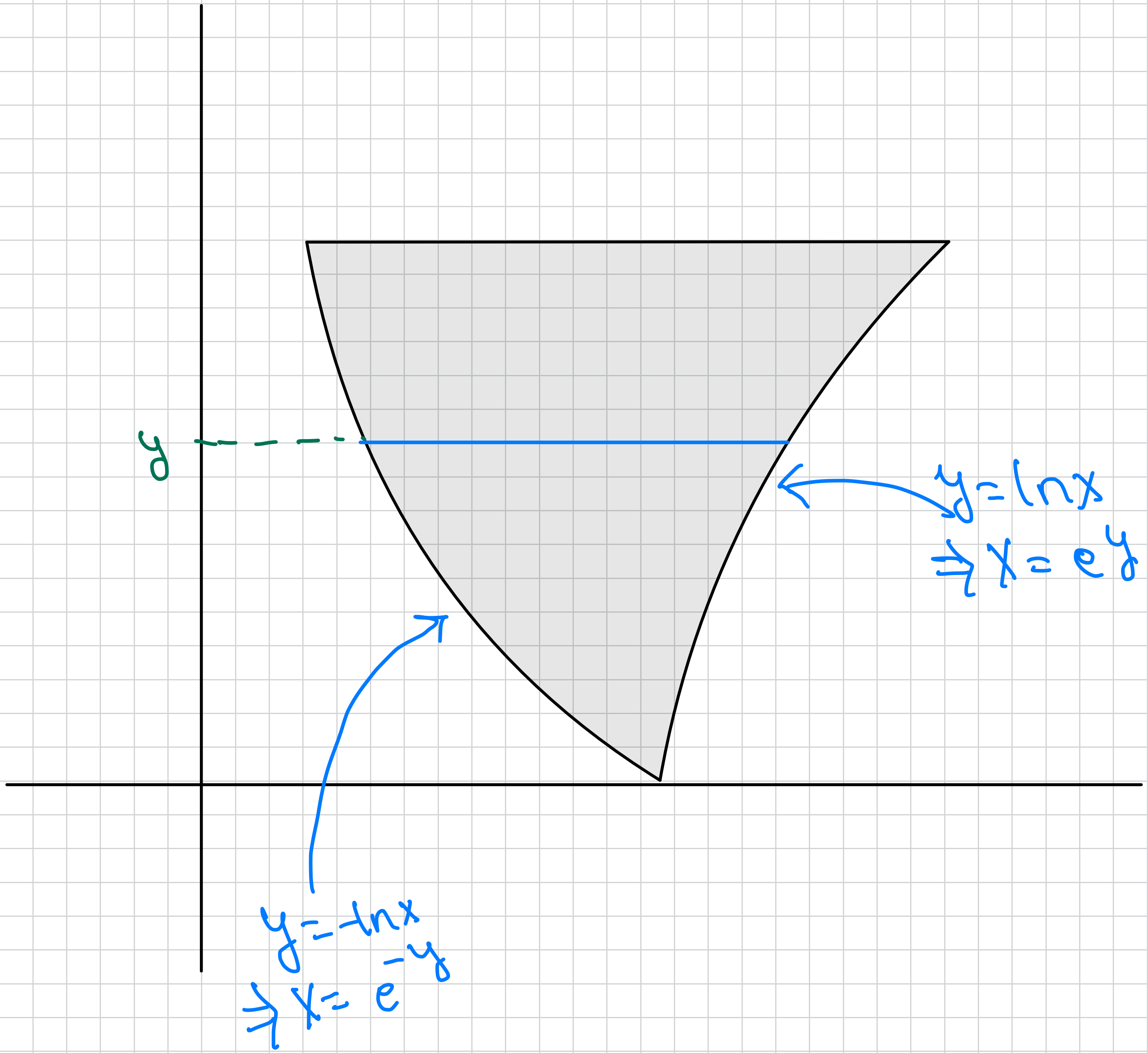
$e^{-3} \le x \le 1$

$\ln x \le y \le 3$

$1 \le x \le e^3$

$$\int_{e^{-3}}^{1}\int_{-\ln x}^{3} f(x,y)\,dy\,dx + \int_{1}^{e^3}\int_{\ln x}^{3} f(x,y)\,dy\,dx$$

How about $\iint f(x,y)\,dx\,dy$ ?




fix y,

$e^{-y} \le x \le e^{y}$

$0 \le y \le 3$

$$\iint_D f(x,y)\,dA = \int_0^3 \int_{e^{-y}}^{e^{y}} f(x,y)\,dx\,dy$$