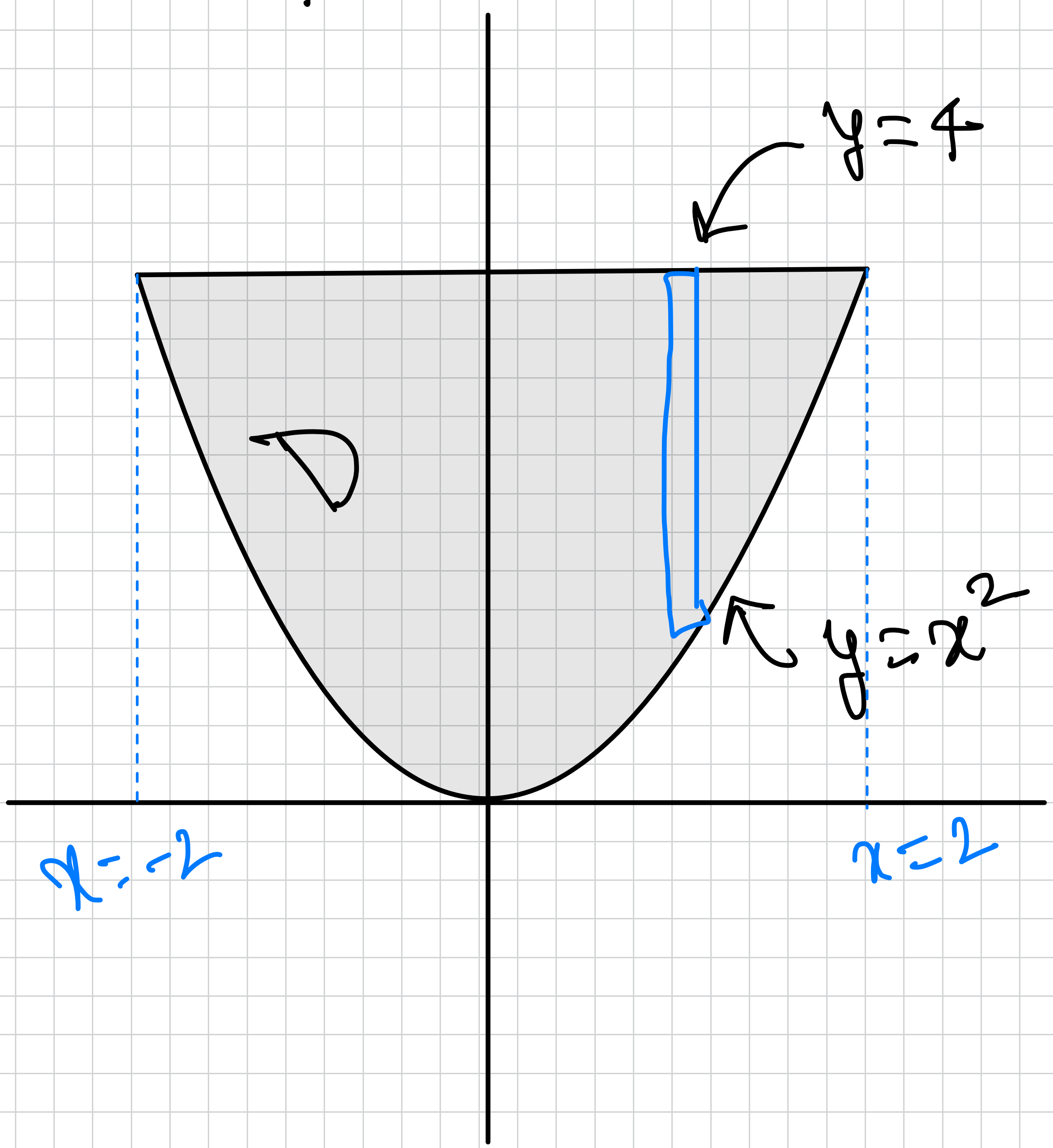
Lesson 21: Triple integrals

Warmup: What is the area between $y = x^2$ and $y = 4$.



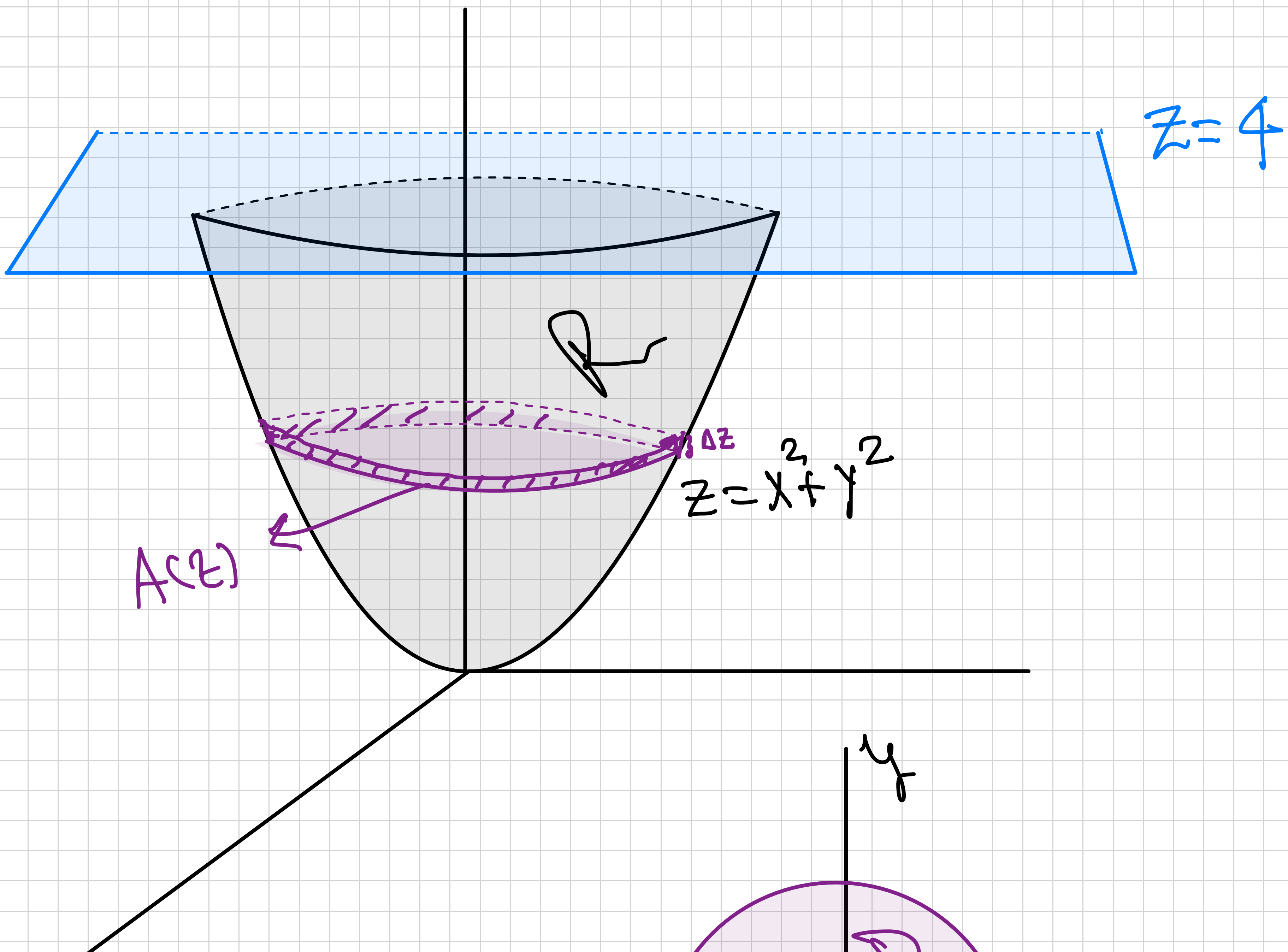
$$= \int_{-2}^{2} (4 - x^2)\, dx$$

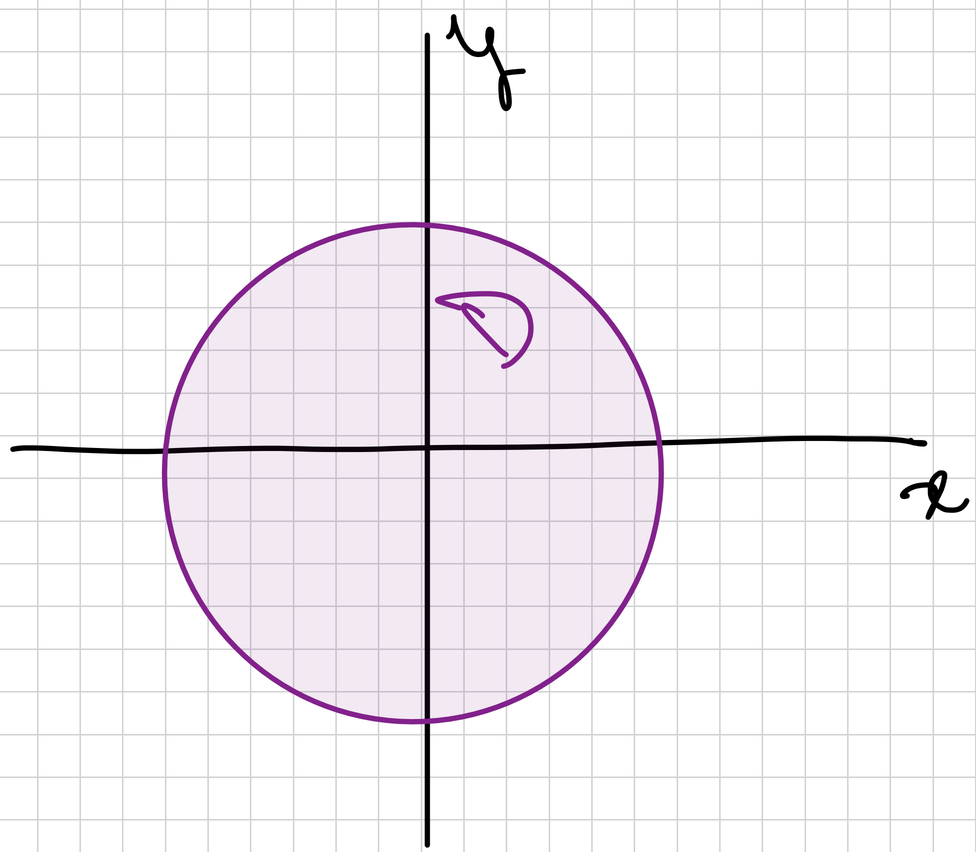
Observe! $4 - x^2 = \int_{x^2}^{4} 1 \; dy$

$$= \int_{-2}^{2} \int_{x^2}^{4} 1 \; dy \; dx$$

$$= \iint_D 1 \; dA$$

What is the Volume of the Region between $z = x^2 + y^2$ & $z = 4$





$$\text{Volume} = \int_0^4 \underbrace{A(z)}_{\substack{\text{Area of cross section} \\ = \text{ Double integral}}} \, dz$$

$$= \int_0^4 \iint_D 1 \; dz$$

$$= \text{Triple integral.}$$

Volume and Mass of a 3d Domain D

$$\text{Volume} = \iiint_D 1 \, dV$$

Suppose $f(x,y,z)$ is the density of the domain

$$\text{Mass} = \iiint_D f(x,y,z) \, dV$$

Triple integral as Riemann Sum:

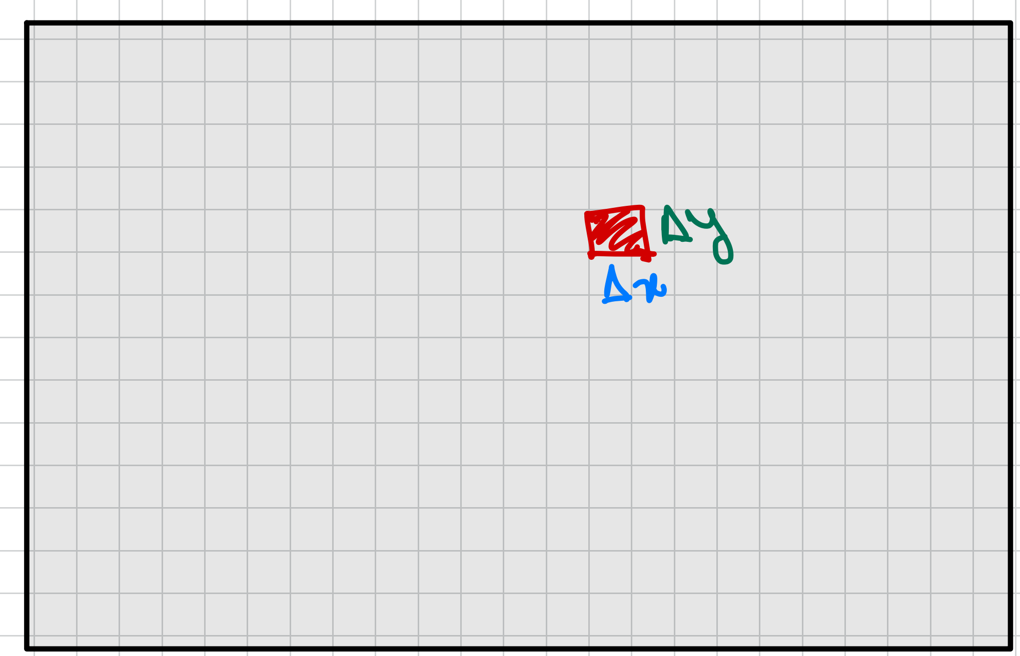
* $f(x)$ on $[a,b]$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

* Divide into small intervals of length Δx

* $f(x_i)$ height at sample point

* $f(x,y)$ on $\underbrace{[a,b] \times [c,d]}_{\text{Divide into small Rectangle of Area } \Delta x \Delta y}$



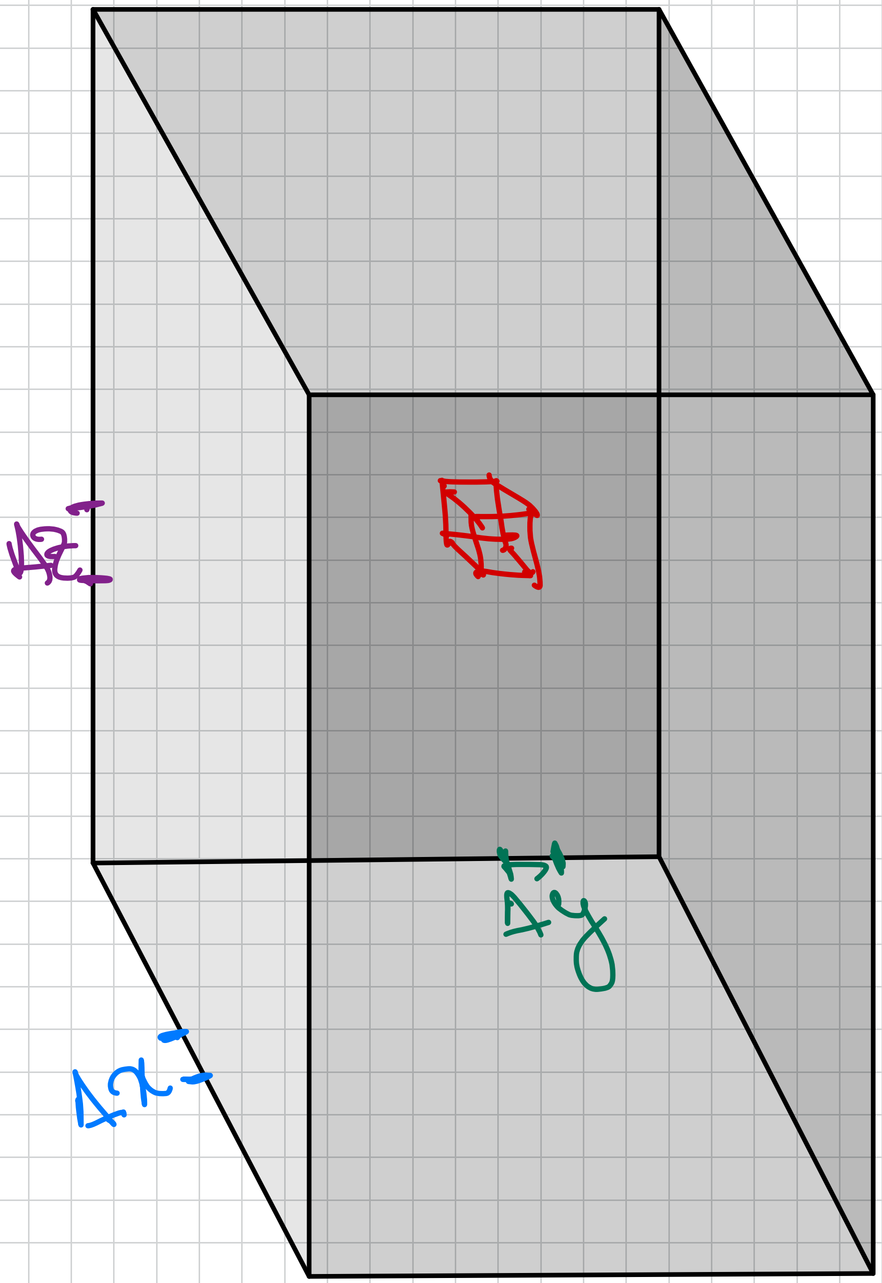
$$\iint_R f(x,y)\,dA = \lim_{m\to\infty} \lim_{n\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_i, y_j)\,\Delta x \Delta y$$

$D = [a,b] \times [c,d] \times [e,f]$

$$\iiint_D f(x,y,z)\, dV$$

Divide the domain to small Boxes of volume

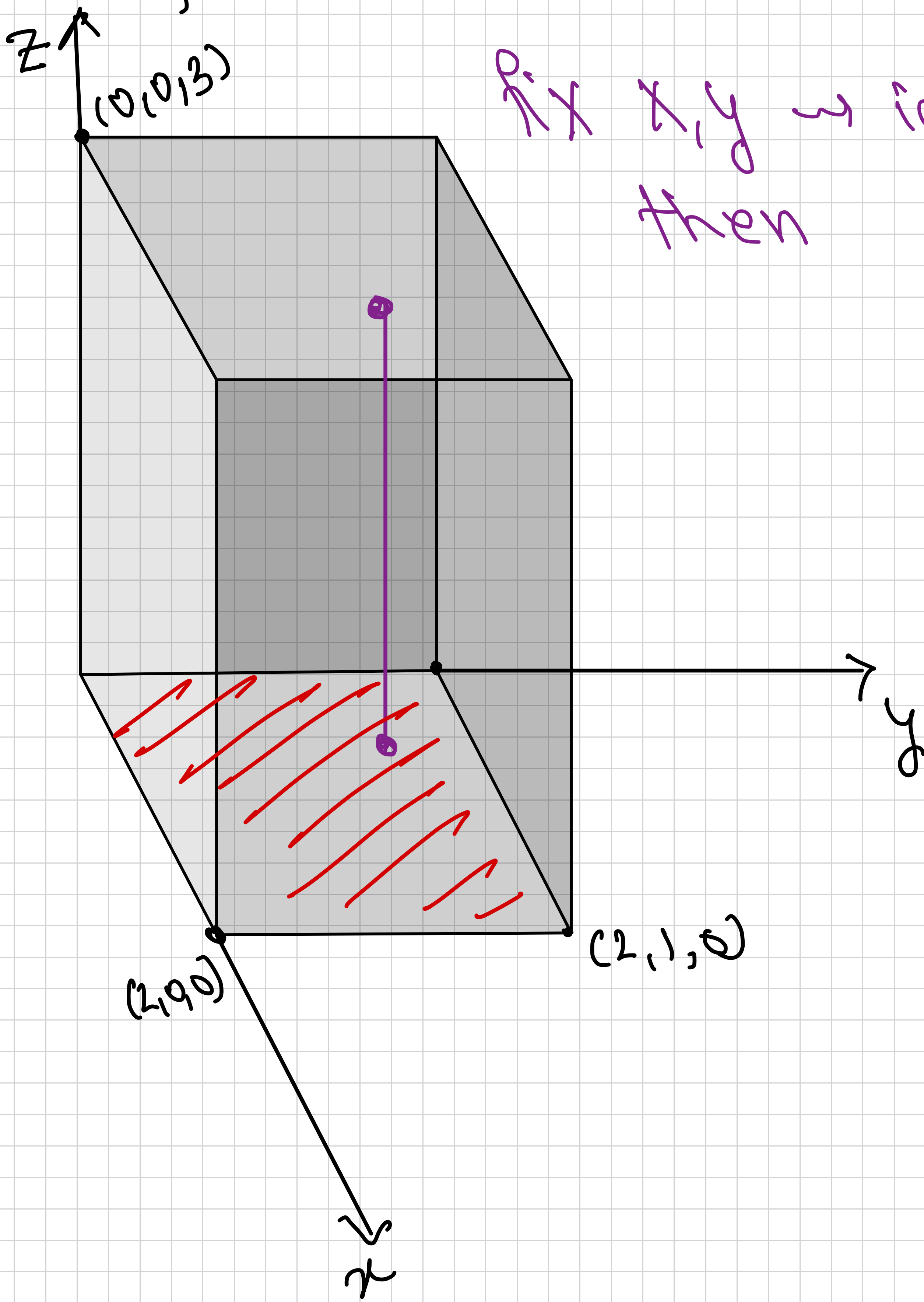
$\Delta V = \Delta x \, \Delta y \, \Delta z$



$$\lim_{l\to\infty} \lim_{m\to\infty} \lim_{n\to\infty} \sum_{k=1}^{l} \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_i, y_j, z_k)\, \Delta x\, \Delta y\, \Delta z$$

Triple integrals as iterated integrals

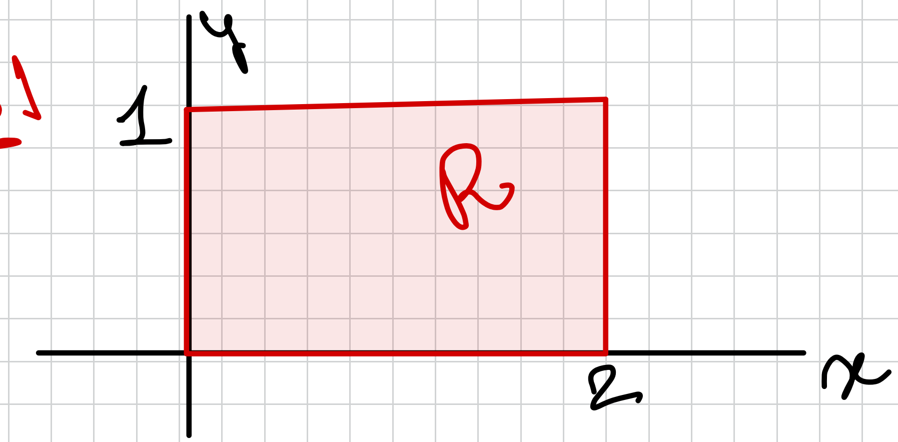
example: $B = [0,2]\times[0,1]\times[0,3]$, evaluate $\displaystyle\iiint_B xyz^2\,dV$



fix x,y ⤳ integrate w.r.t z first
then double integral on the shadow in xy plane

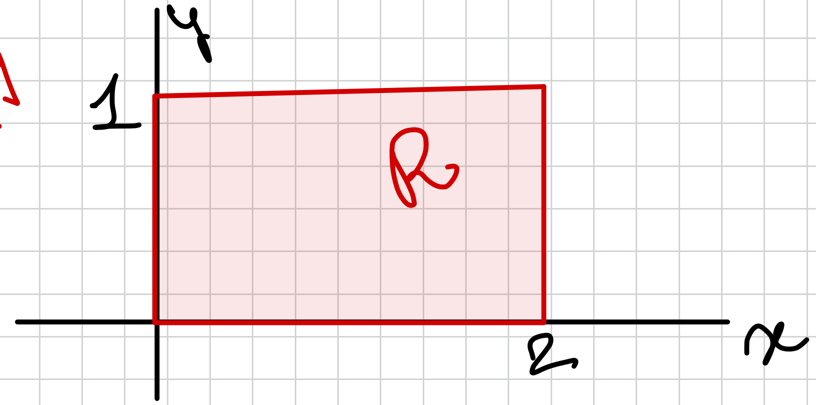
$$0 \le z \le 3$$

shadow!



$$\iiint_B xyz^2\,dV = \iint_R \int_0^3 xyz^2\,dz\,dA = \iint_R 9xy\,dA$$

shadow!



$$\iint_R \int_0^3 xyz^2 \, dz \, dA = \iint_R 9xy \, dA$$

fix y, integrate w.r.t x first
then w.r.t y

$0 \le x \le 2$

$0 \le y \le 1$

$$\int_0^1 \int_0^2 9xy \, dx \, dy$$

$$= \int_0^1 18y \, dy$$

$$= 9$$

fix x, integrate w.r.t y first
on $0 \le y \le 1$

then w.r.t x
on $0 \le x \le 2$

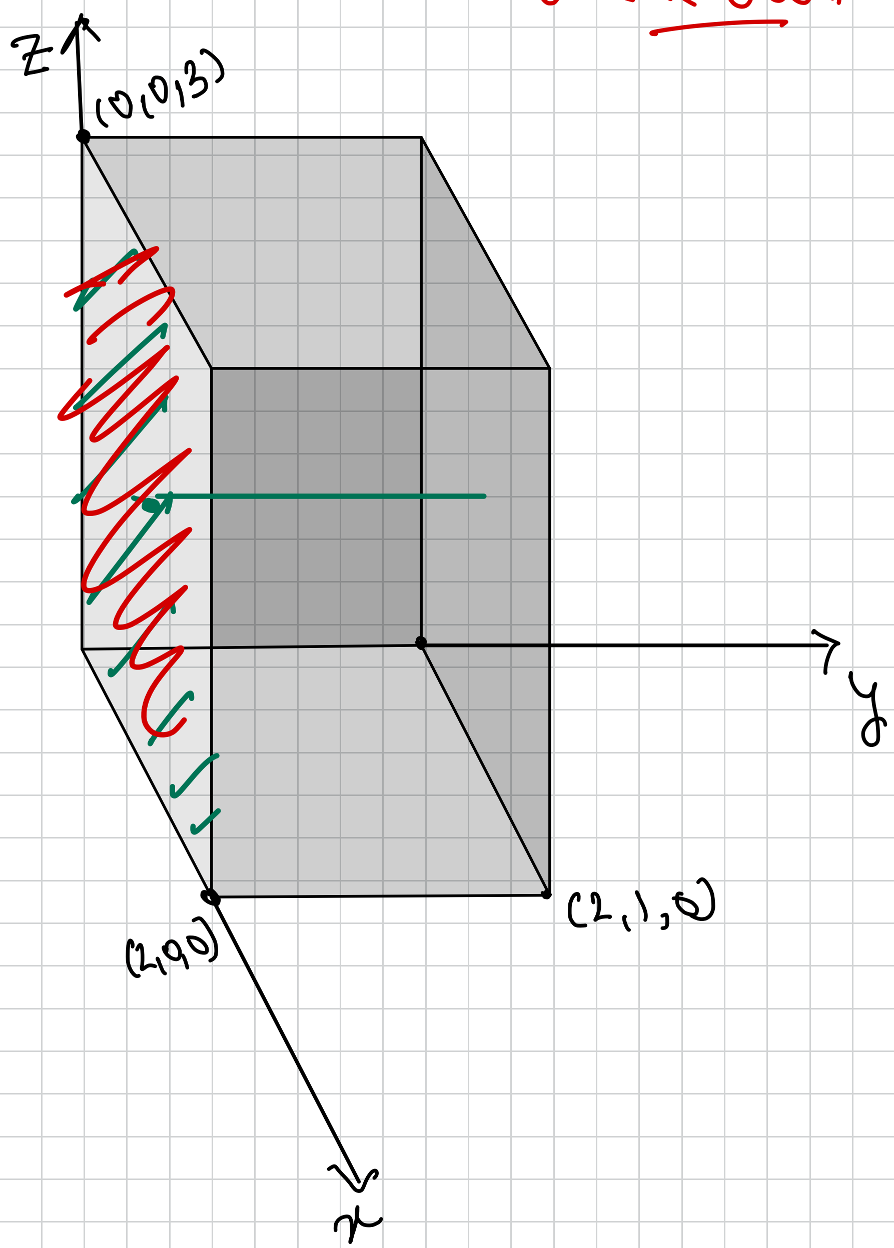
$$= \int_0^2 \int_0^1 9xy \, dy \, dx$$

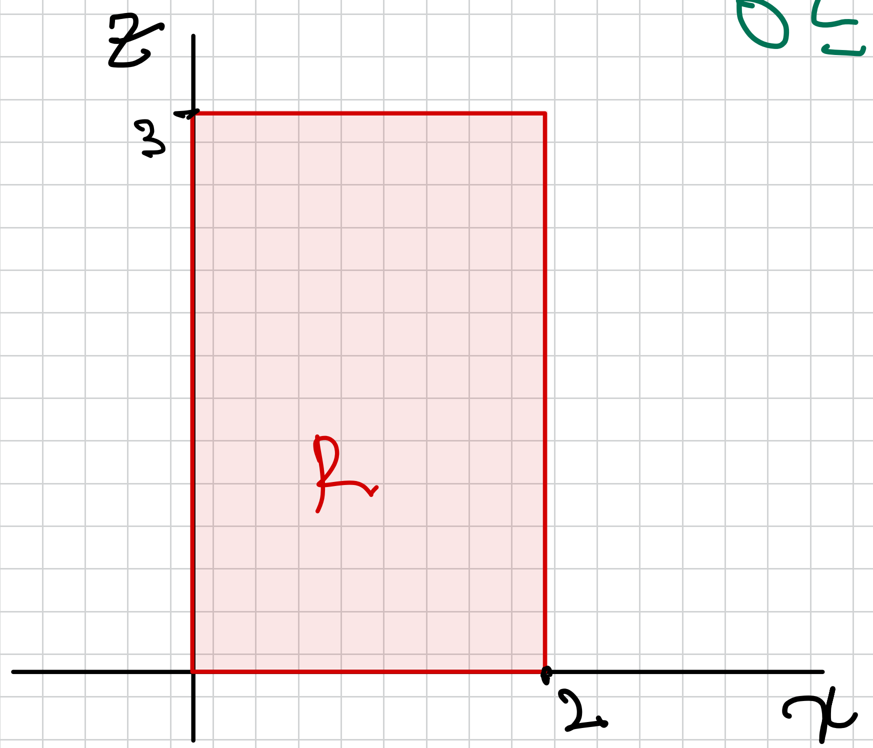
$$= \int_0^2 \frac{9x}{2} \, dx = 9$$

fix x & z first instead & integrate w.r.t y and then on shadow in xz plane

$0 \le y \le 1$

Shadow!



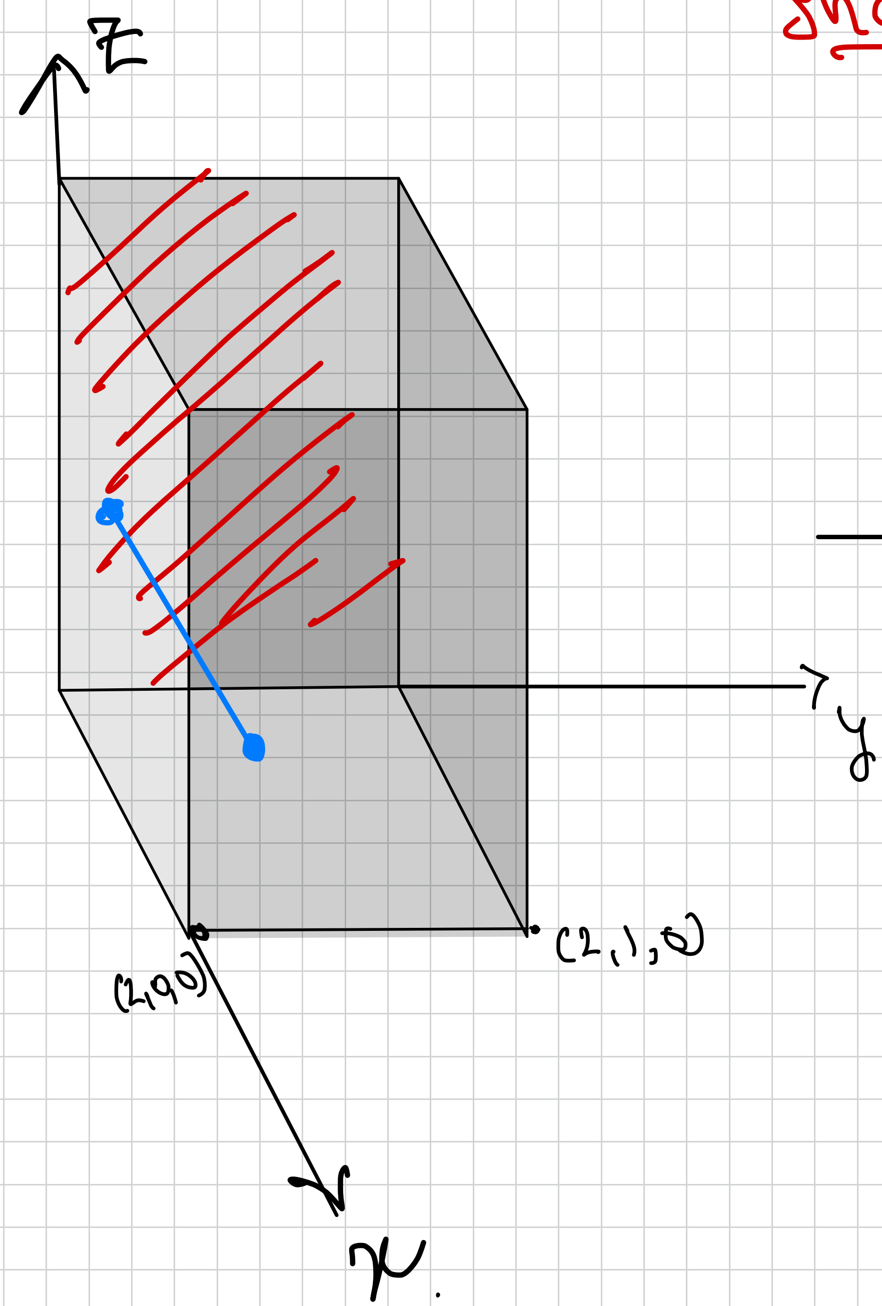


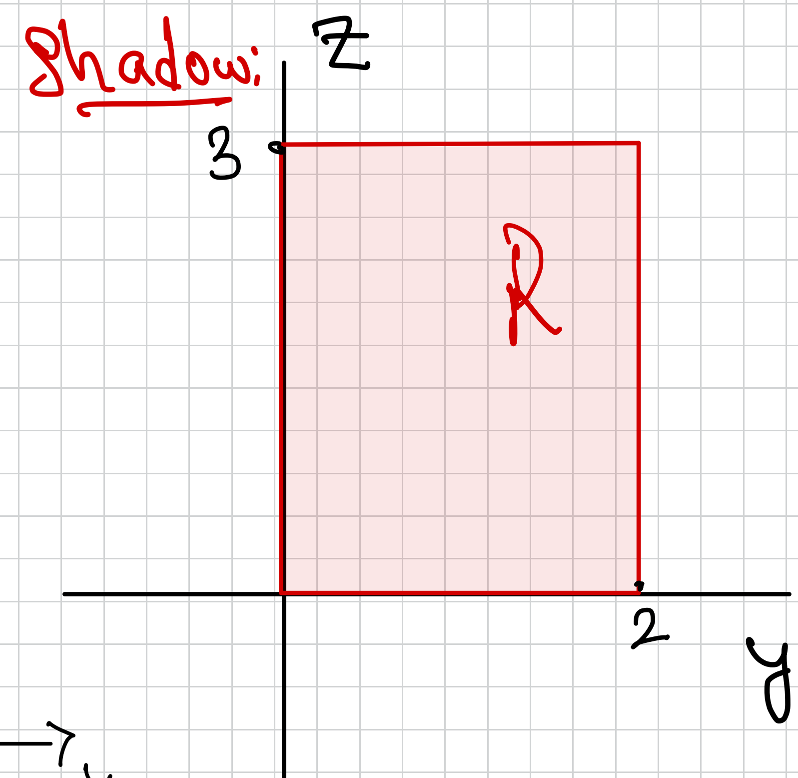
$$\iint_R \int_0^1 xyz^2 \, dy \, dA$$

$$= \int_0^2 \int_0^3 \int_0^1 xyz^2 \, dy \, dz \, dx$$

$$= \int_0^3 \int_0^2 \int_0^1 xyz^2 \, dy \, dx \, dz$$

fix y & z first and integrate w.r.t x and then on shadow in yz plane





$$0 \le x \le 2$$

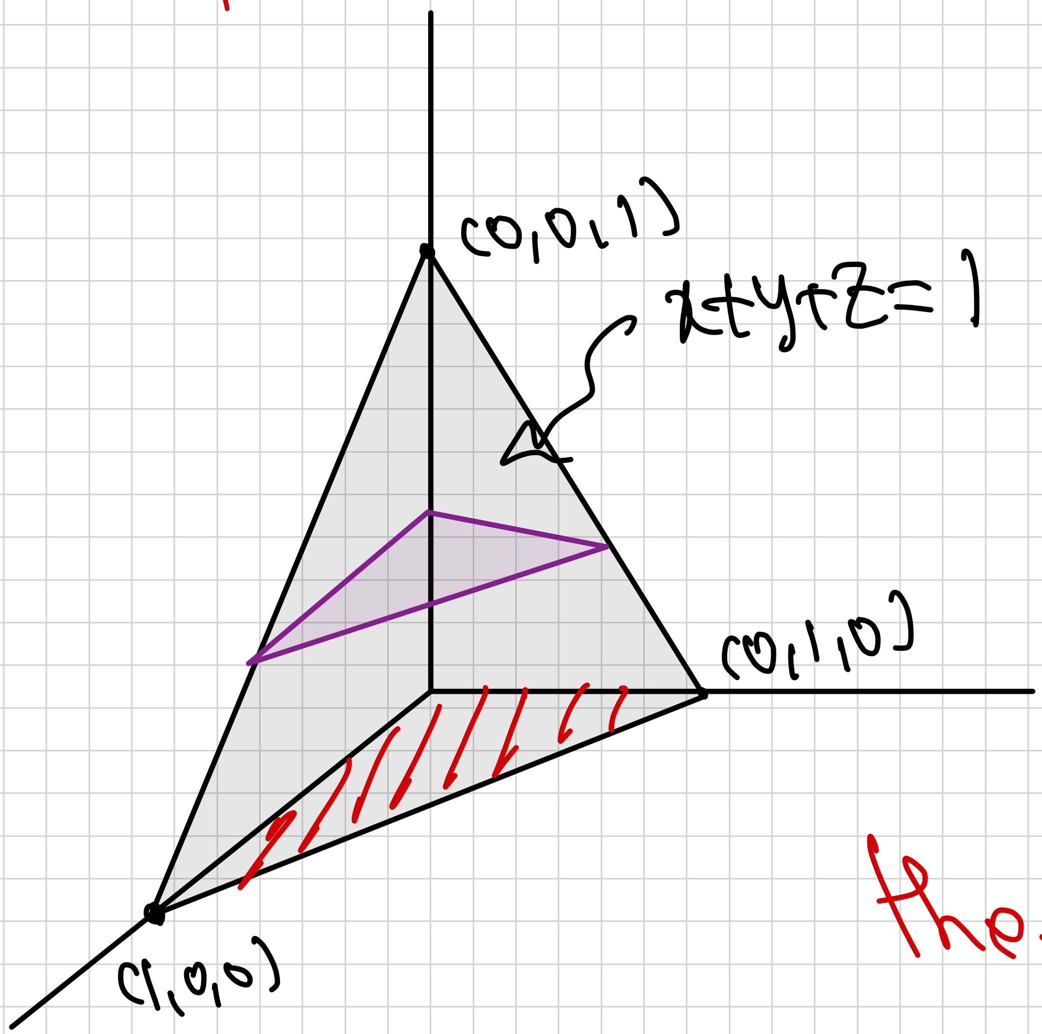
$$\iint_R \int_0^2 xyz^2 \, dx \, dA$$

$$= \int_0^3 \int_0^1 \int_0^2 xyz^2 \, dx \, dy \, dz$$

$$= \int_0^1 \int_0^3 \int_0^1 xyz^2 \, dx \, dz \, dy$$

0

example: find volume of solid under $x+y+z=1$ in 1^{st} octant.



$$\text{Volume} = \iiint_D 1 \, dV$$

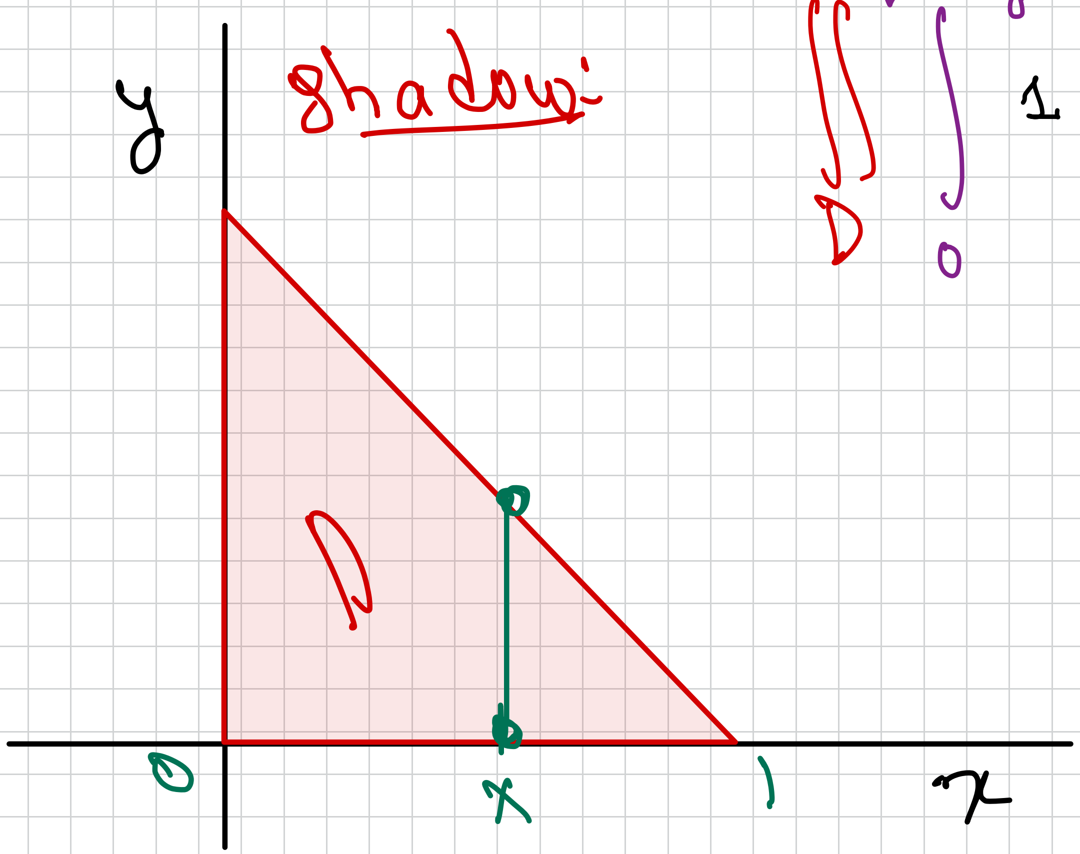
fix x,y & integrate w.r.t z first

$0 \leq z \leq$ hitting plane $x+y+z=1$

$0 \leq z \leq 1-x-y$

then Double integral on shadow in xy

shadow:

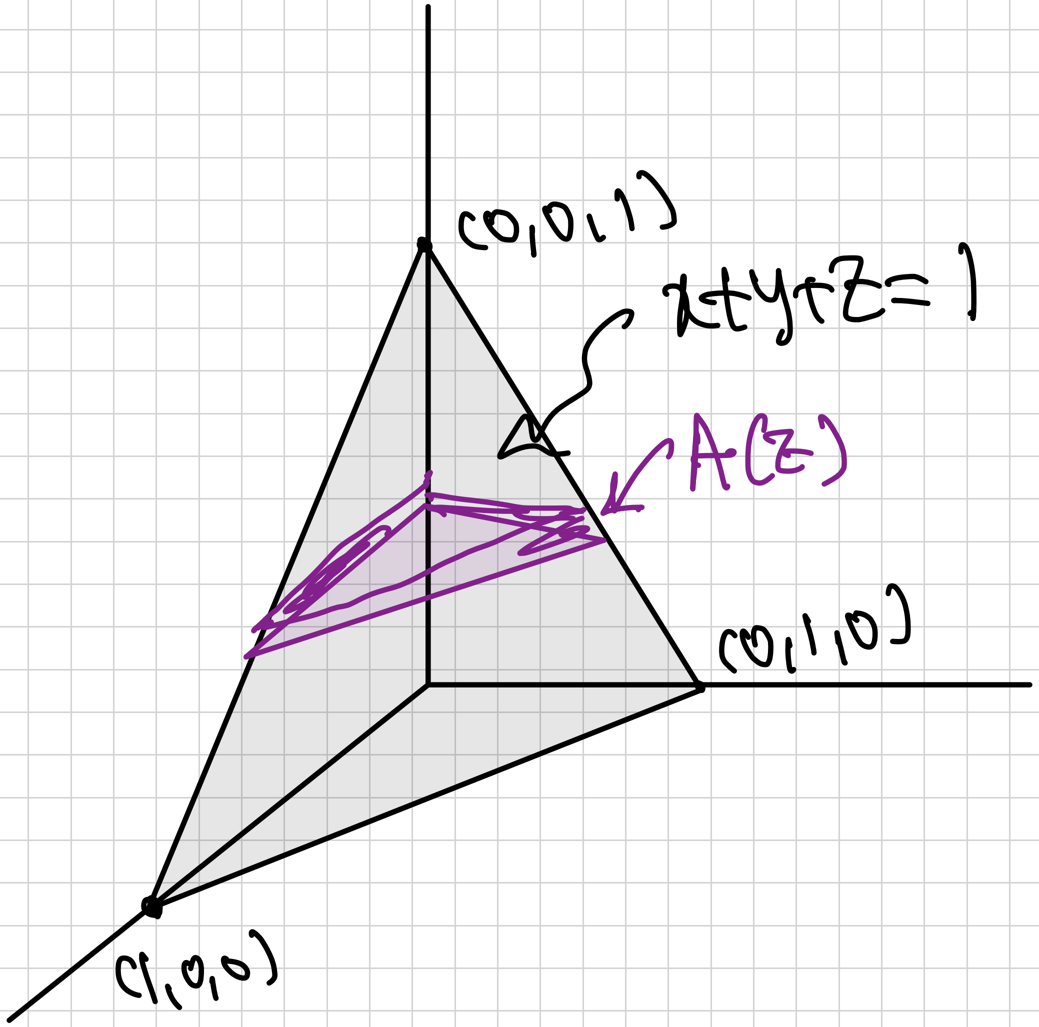


$$\iint_D \int_0^{1-x-y} 1 \, dz \, dA = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1 \, dz \, dy \, dx = \int_0^1 \int_0^{1-x} 1-x-y \, dy \, dx$$

$$= \int_0^1 (1-x)(1-x) - \frac{(1-x)^2}{2} \, dx$$

$$= \int_0^1 \frac{(1-x)^2}{2} \, dx = \frac{1}{6}$$

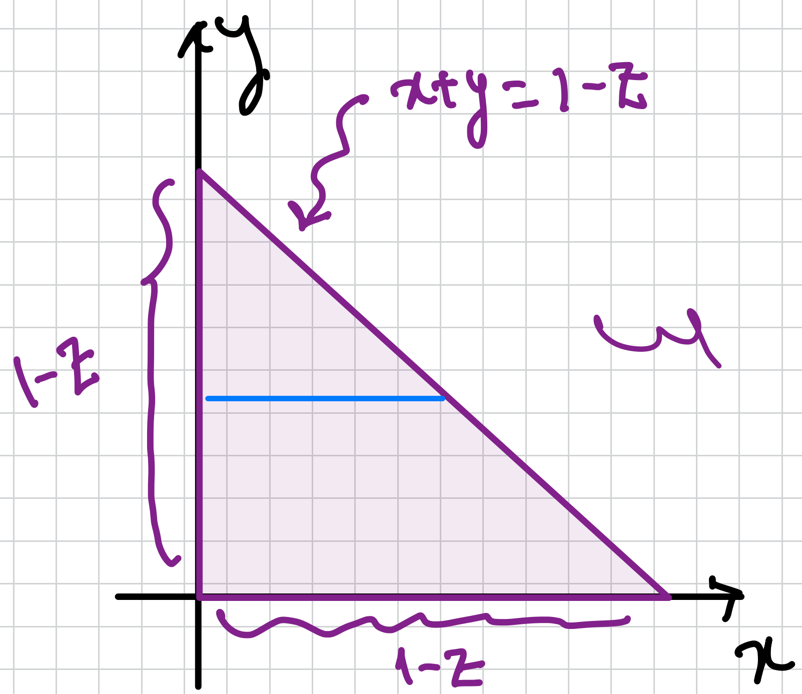
Another Way!



Volume $= \iiint_D 1 \, dV$

$= \int_0^1 A(z) \, dz$ → compute this using double integrals.

Cross-section: $0 \le z \le 1$



$\rightsquigarrow \; A(z) = \frac{(1-z)^2}{2} = \int_0^{1-z} \int_0^{1-z-y} 1 \, dx \, dy$