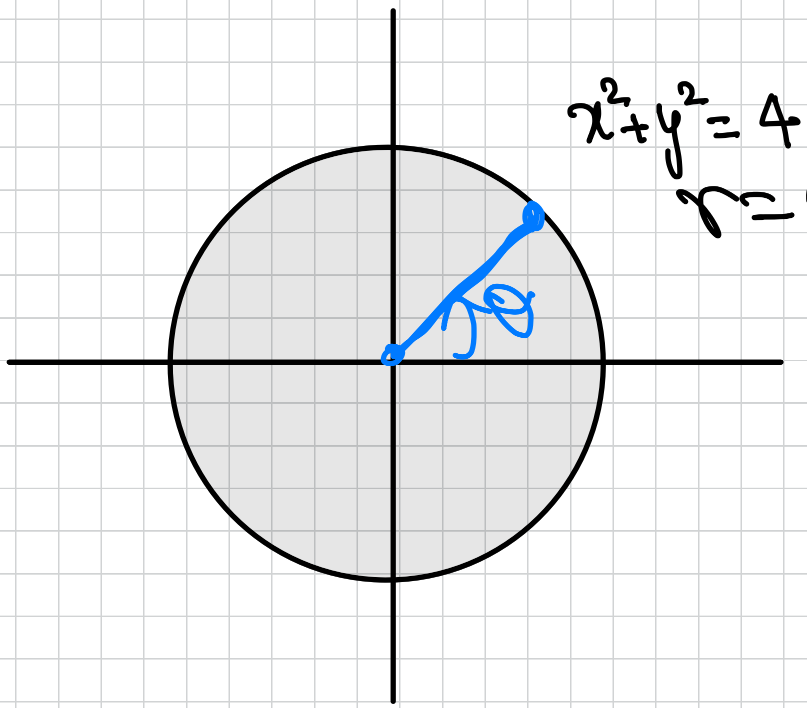
Lesson 22: Triple Integrals in Cylindrical Coordinates (16.5)

Warmup: Evaluate $\iint_D 4 - x^2 - y^2 \, dA$, D is the region below



Using polar coordinates:

$x = r\cos\theta$, $r^2 = x^2+y^2$

$y = r\sin\theta$, $\tan\theta = \frac{y}{x}$

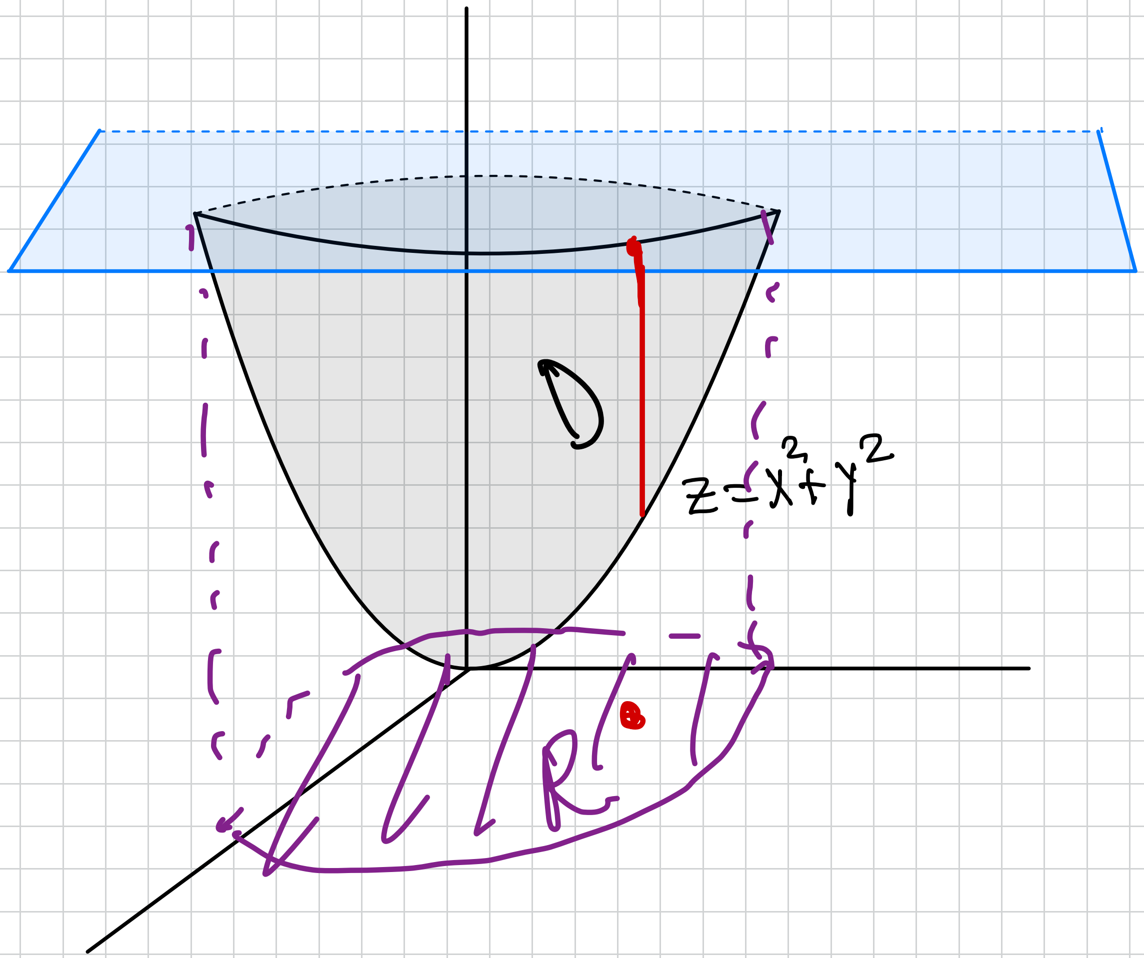
$$\iint_D f(x,y)\,dA = \int_{\theta \text{ Bounds}} \int_{r \text{ Bounds}} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Bounds: $0 \le r \le 2$, $0 \le \theta \le 2\pi$

$$\iint_D 4 - x^2 - y^2 \, dA = \int_0^{2\pi}\int_0^2 4 - r^2 \; r\, dr\, d\theta$$

$$= \int_0^{2\pi} 2r^2 - \frac{r^4}{4}\Big|_0^2 \, d\theta \quad = \underline{\underline{8\pi}}$$

example: Find volume of the Region between $z = x^2+y^2$ & $z=4$



$$V = \iiint_D 1 \, dV$$

fix x, y in the shadow

$$x^2+y^2 \le z \le 4$$

Boundary of shadow = intersection

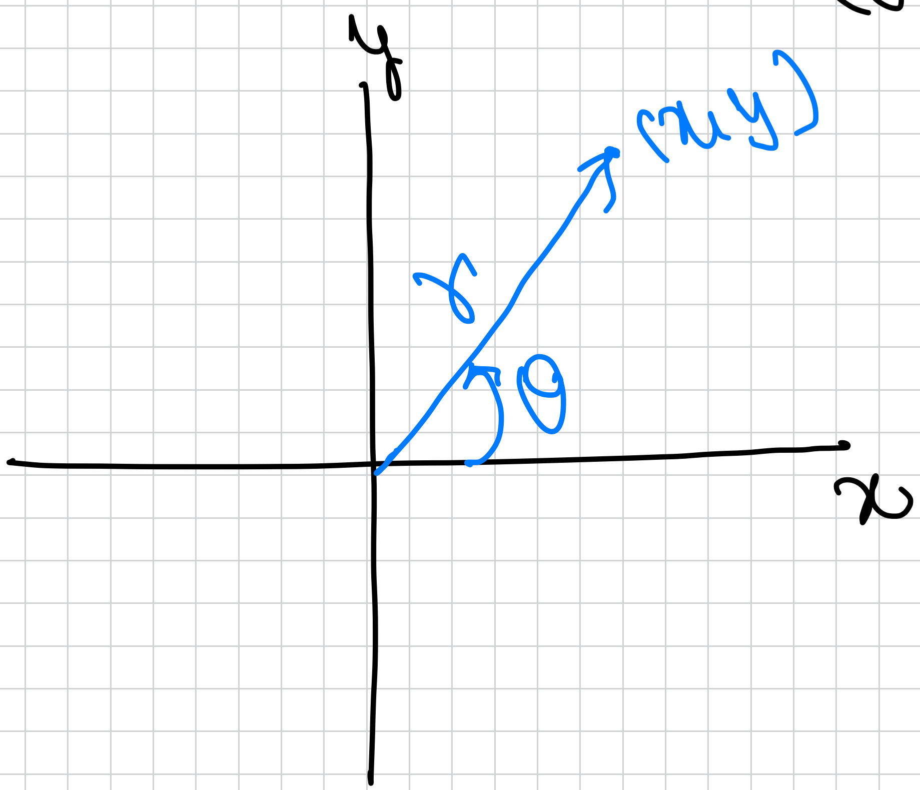
$$x^2+y^2=4.$$

$$\iiint_D 1 \, dV = \iint_R \left[\int_{x^2+y^2}^{4} 1 \, dz \right] dA = \underbrace{\iint_R 4 - x^2 - y^2 \, dA}_{\text{Use polar Coordinates to compute}}$$

Cylindrical Coordinates

↳ Combination of Cartesian & polar

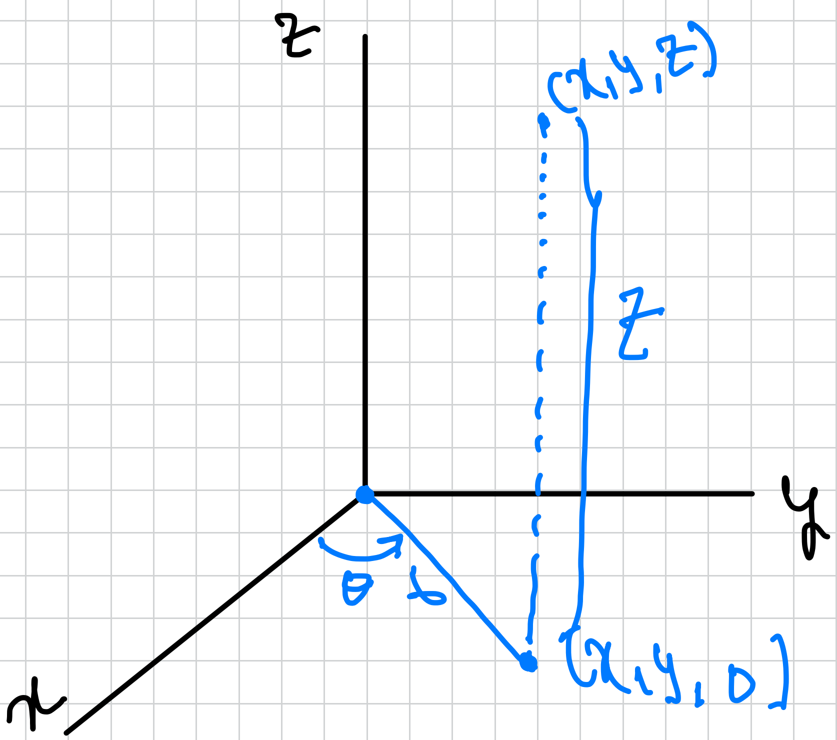
(in 1 variable) (in 2 Variables)



$$x = r\cos\theta \qquad\qquad r = \sqrt{x^2+y^2}$$

$$y = r\sin\theta \quad \longleftrightarrow \quad \theta = \tan^{-1}(y/x)$$

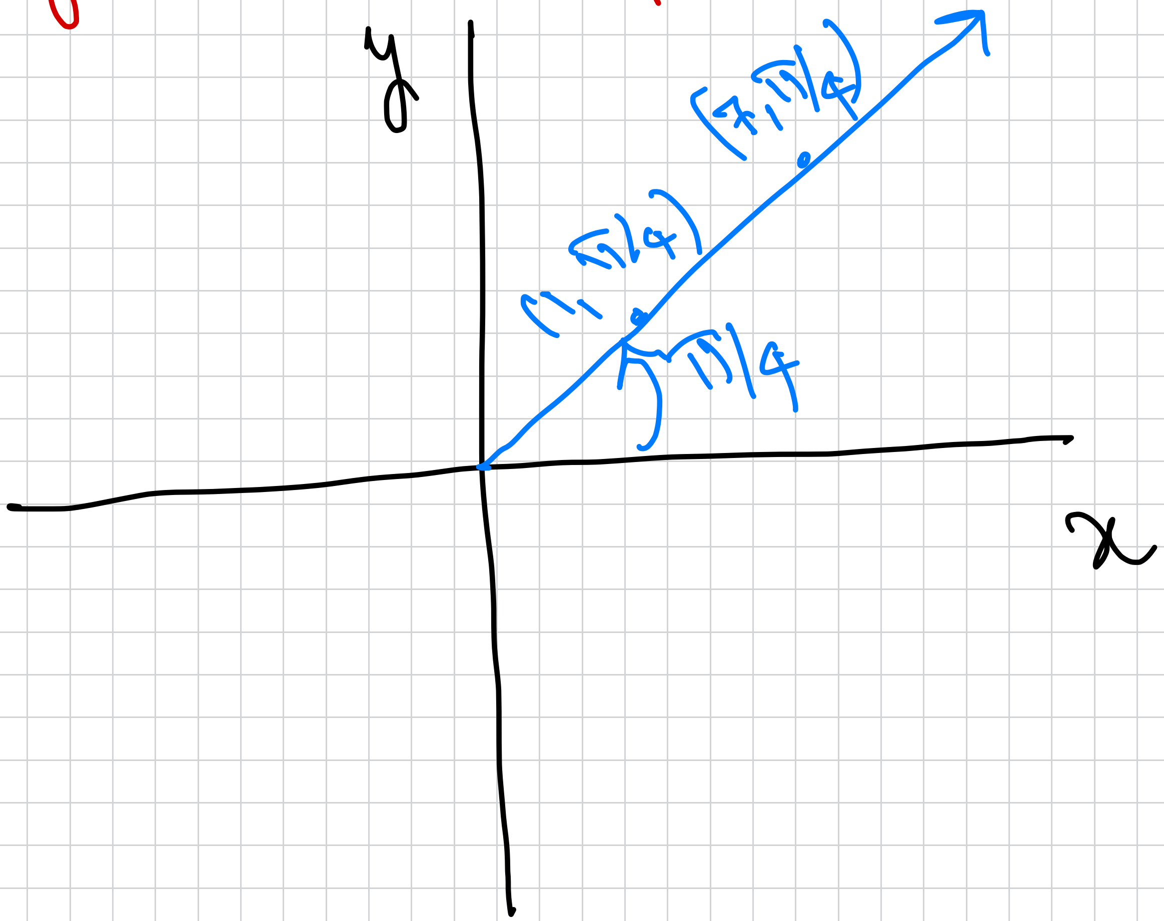
$$z = z \qquad\qquad z = z$$

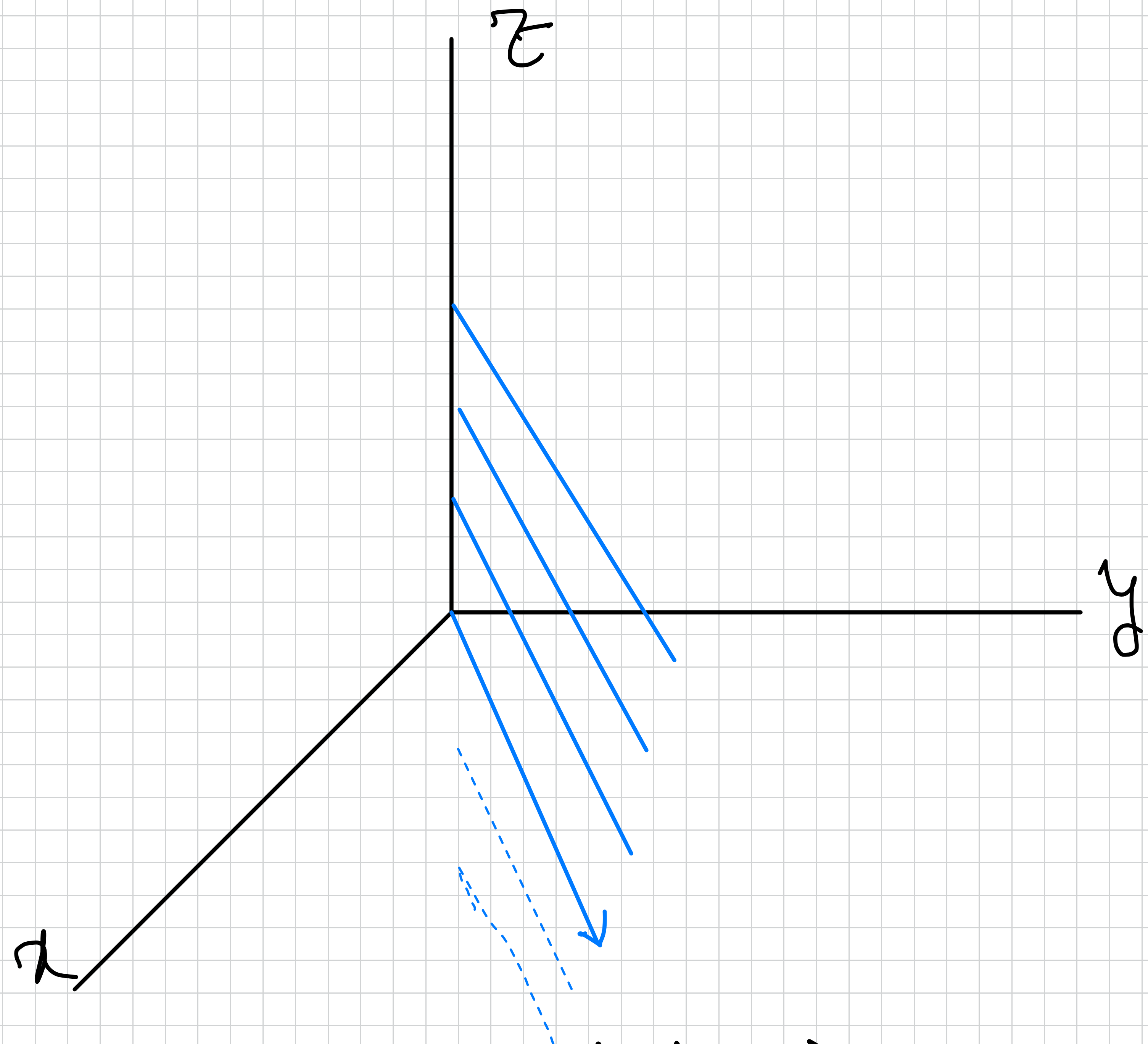


$$\iiint_D f(x,y,z)\,dV = \int_{\theta\text{ Bounds}} \int_{r\text{ Bounds}} \int_{z\text{ Bounds}} f(r\cos\theta, r\sin\theta, z)\cdot r\,dz\,dr\,d\theta$$

Graphs in Cylindrical Coordinates

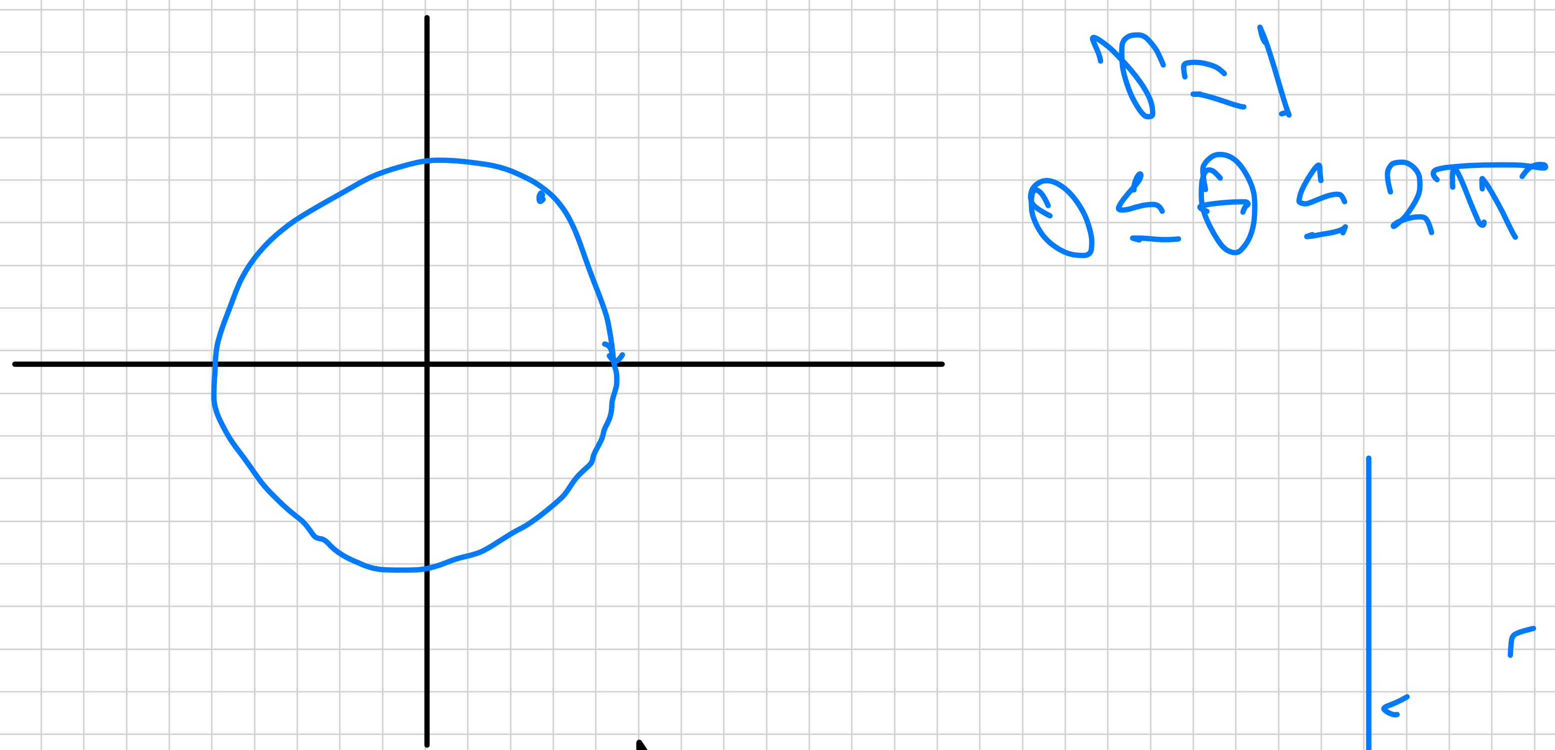
eg①: $\theta = \pi/4$ w/ r & z are free



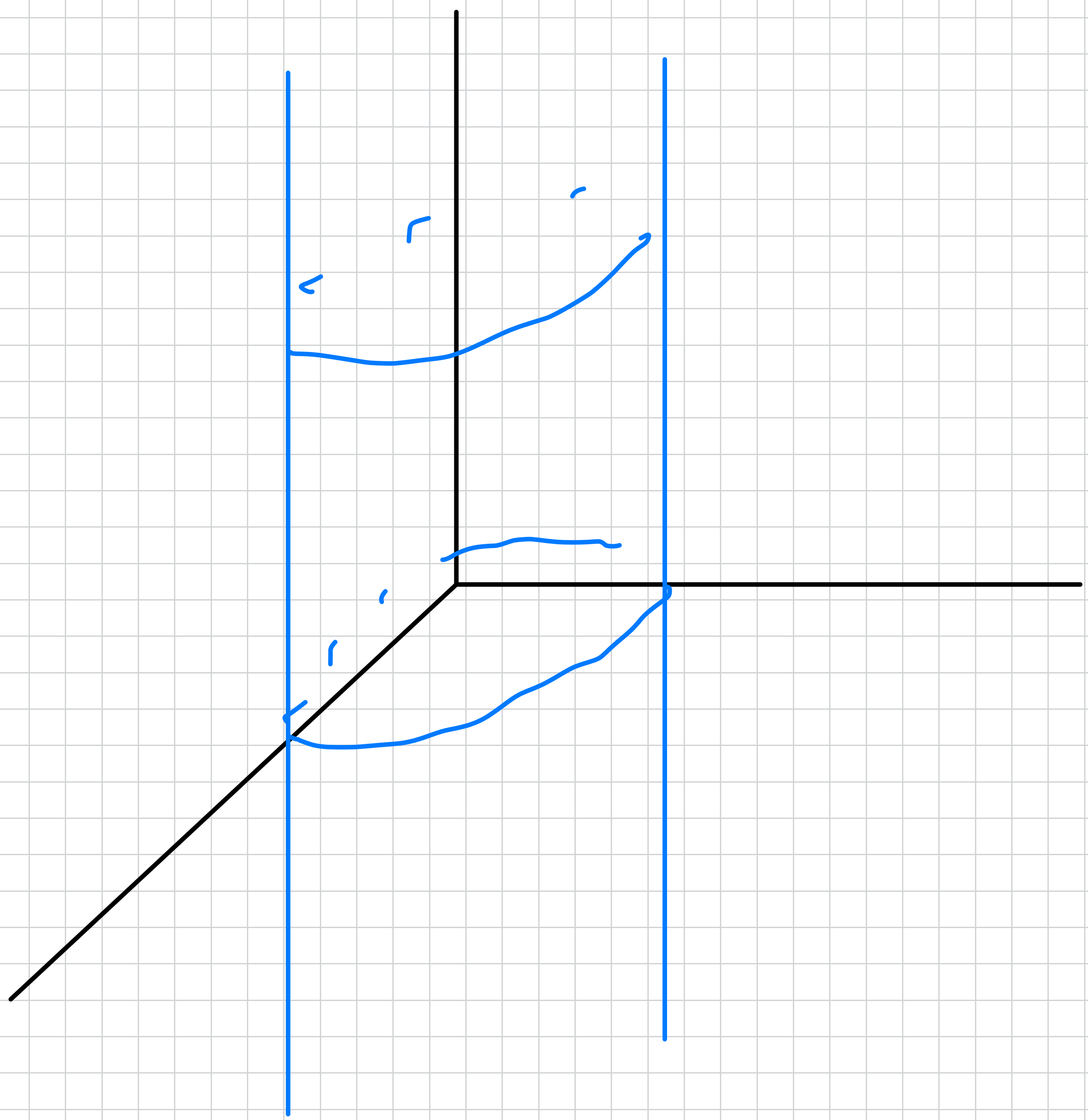


Represents a plane: $\theta = \frac{\pi}{4} \Rightarrow \tan\left(\frac{\pi}{4}\right) = \frac{y}{x} \Rightarrow y = x$.

eg ②: $r = 1$ in 2D Circle of Radius 1

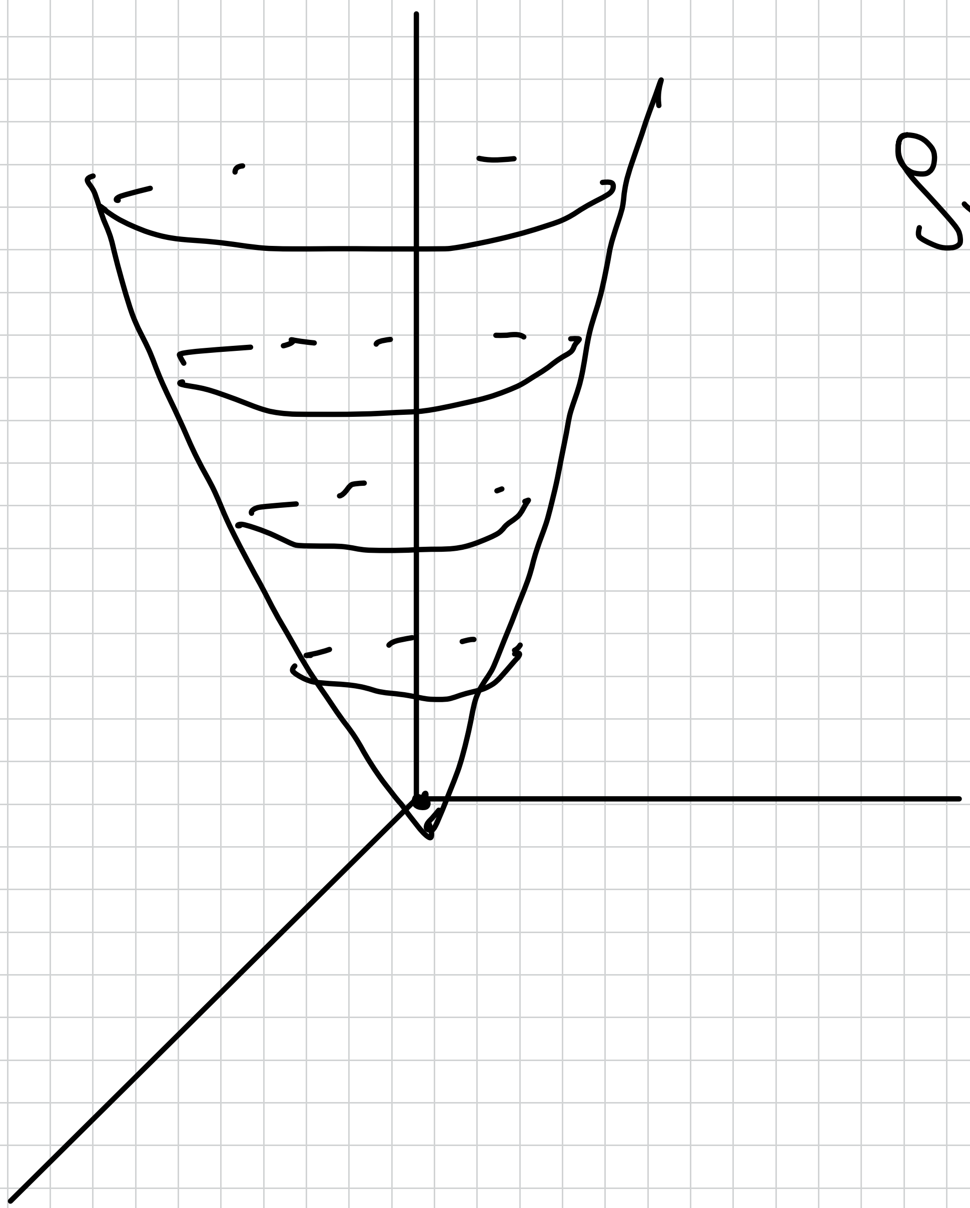


in 3D! z is "free"



Circular Cylinder.

eg ③: $z = r$ or $z = \sqrt{x^2+y^2}$

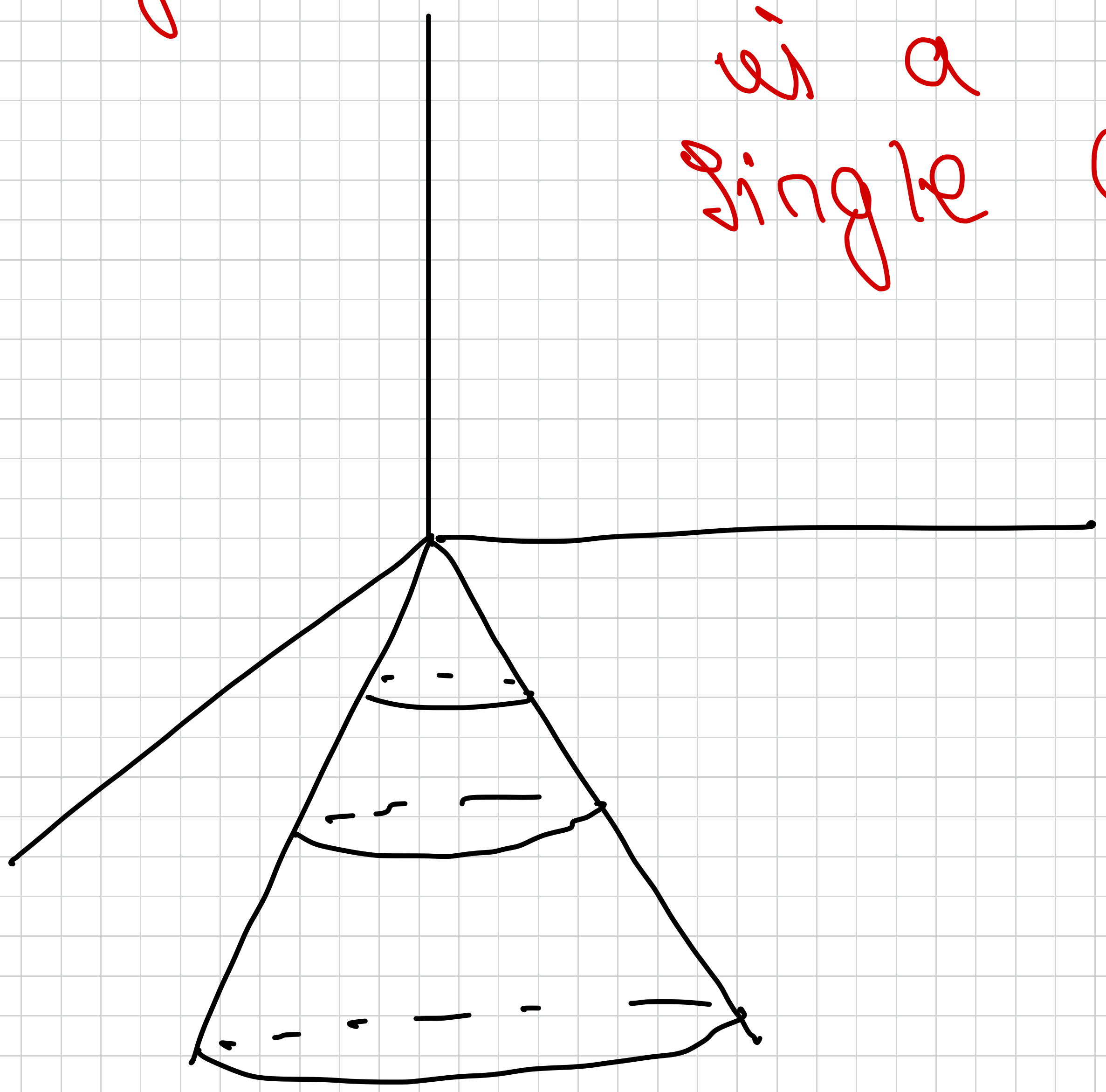


Single Cone.

Similarly: $z = -r = -\sqrt{x^2+y^2}$

is a

Single Cone

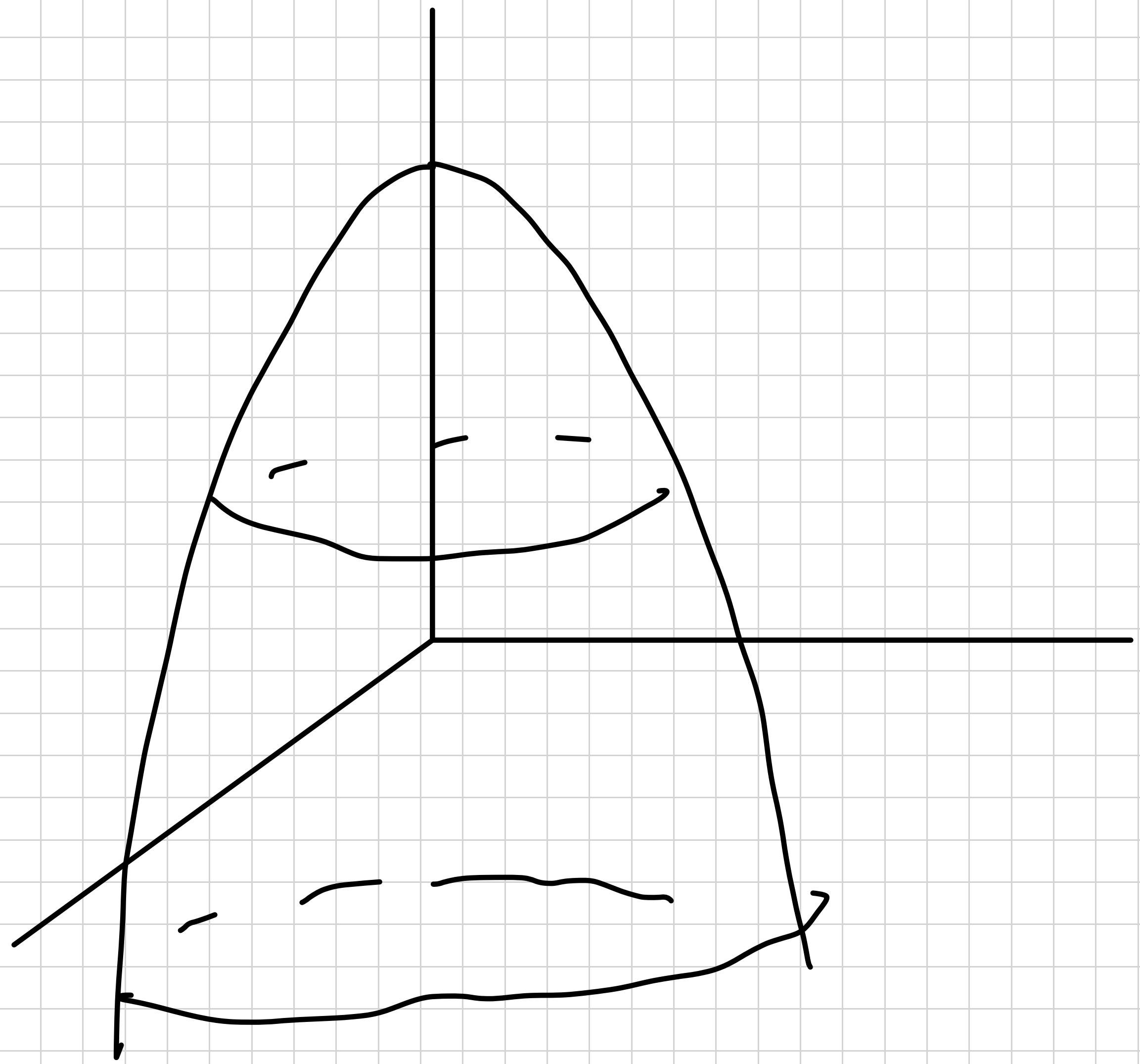


eg ④: $z = 6 - r^2 = 6 - x^2 - y^2$

$z = 6 \rightsquigarrow$ $6 = 6 - x^2 - y^2 \rightsquigarrow$ $x^2 + y^2 = 0$

$z > 6 \rightsquigarrow$ $z = 6 - x^2 - y^2 \rightsquigarrow$ $x^2 + y^2 = 6 - z$ X

$z < 6 \rightsquigarrow$ $x^2 + y^2 = 6 - z \rightsquigarrow$ Circles



Paraboloid facing down.

Evaluating Triple integrals using Cylindrical Coordinates

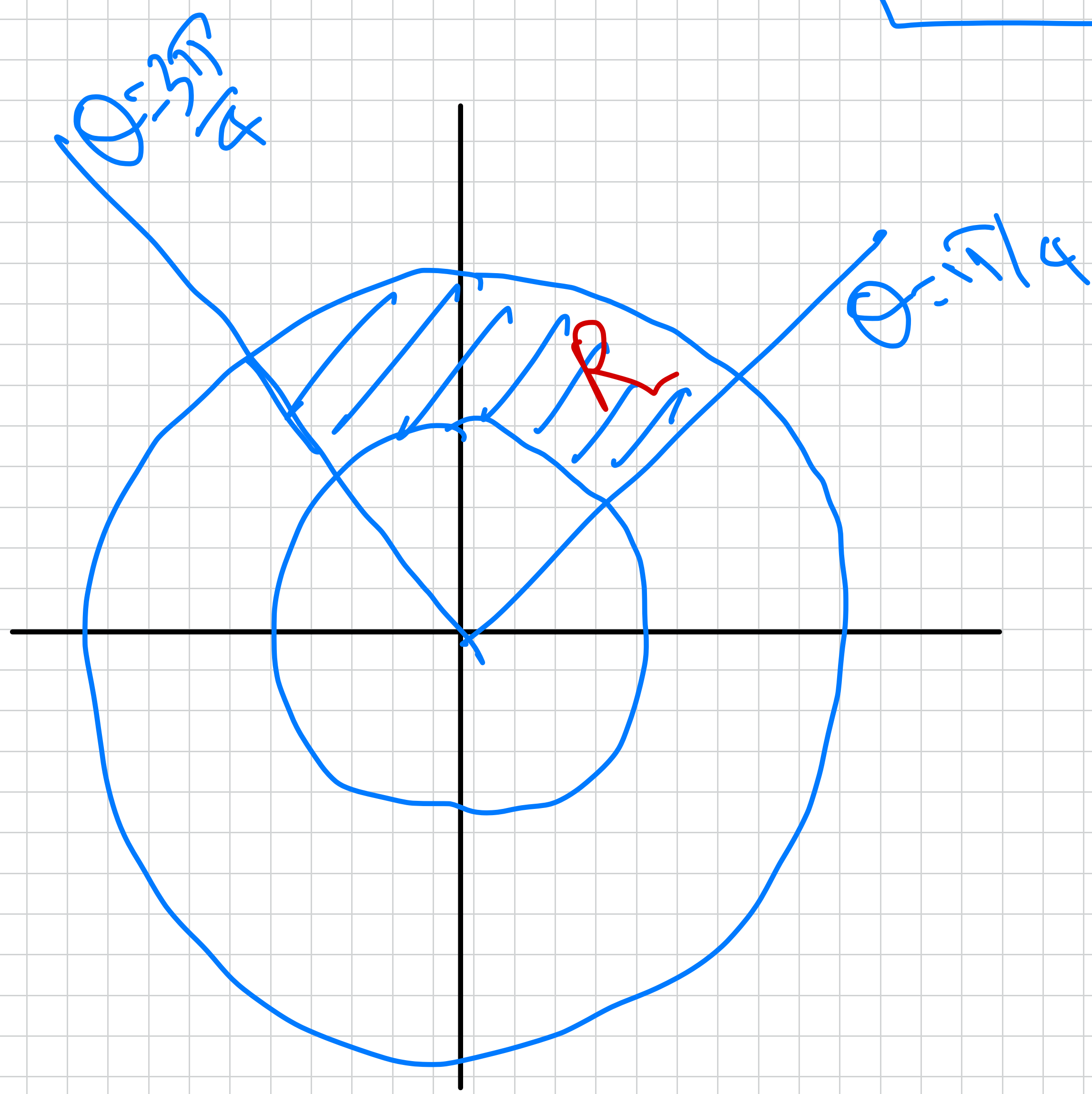
eg①: find volume of the region $D = \{(r,\theta,z) : 1 \le r \le 5;\ \frac{\pi}{4} \le \theta \le \frac{3\pi}{4};\ 2 \le z \le 7\}$

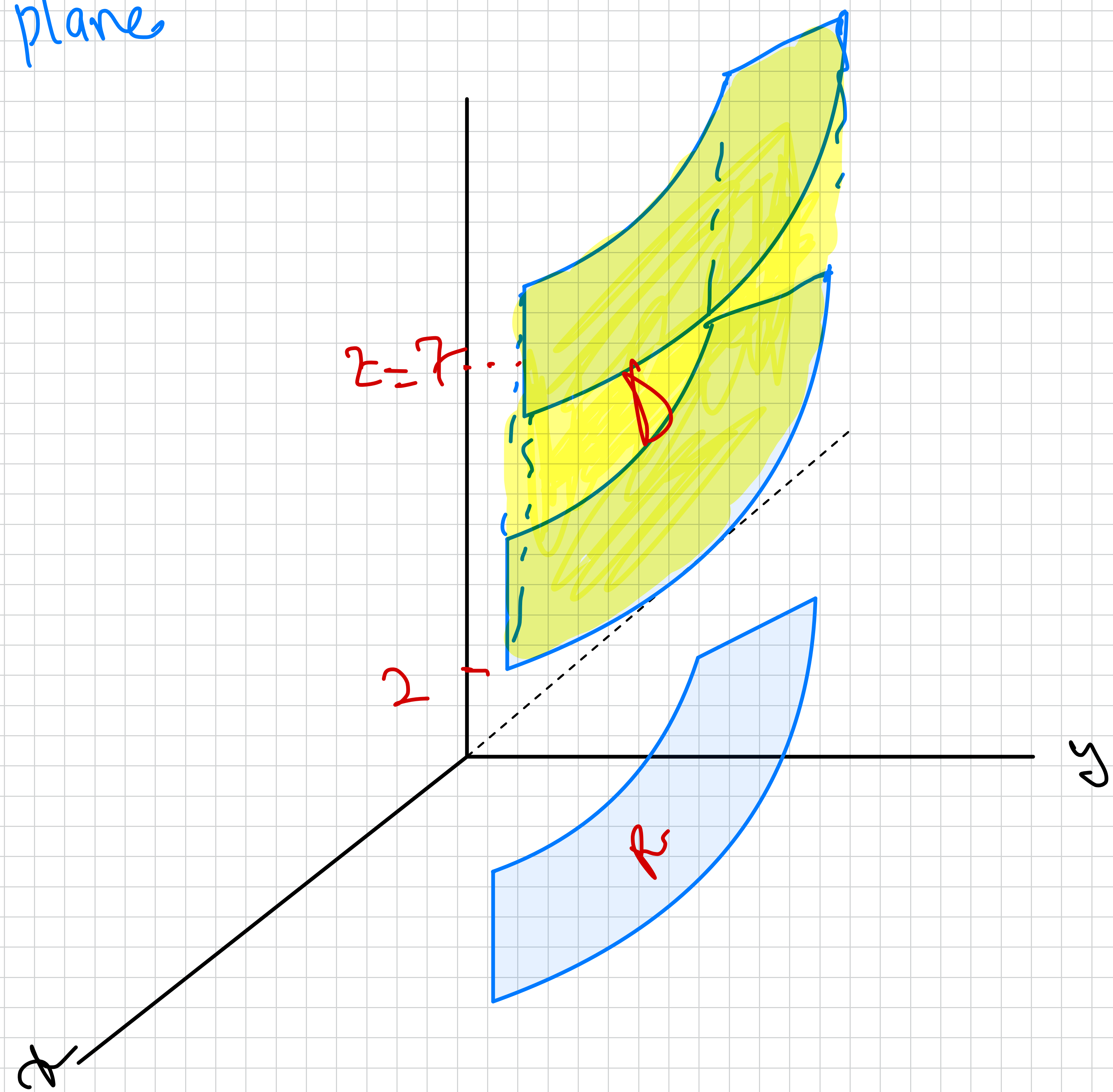
$$\iiint_D 1 \, dV = \int_{\pi/4}^{\frac{3\pi}{4}} \int_1^5 \int_2^7 1 \cdot r \, dz \, dr \, d\theta$$

$$= \int_{\pi/4}^{\frac{3\pi}{4}} \int_1^5 5r \, dr \, d\theta = \int_{\pi/4}^{3\pi/4} \left. \frac{5r^3}{2} \right|_1^5 d\theta$$

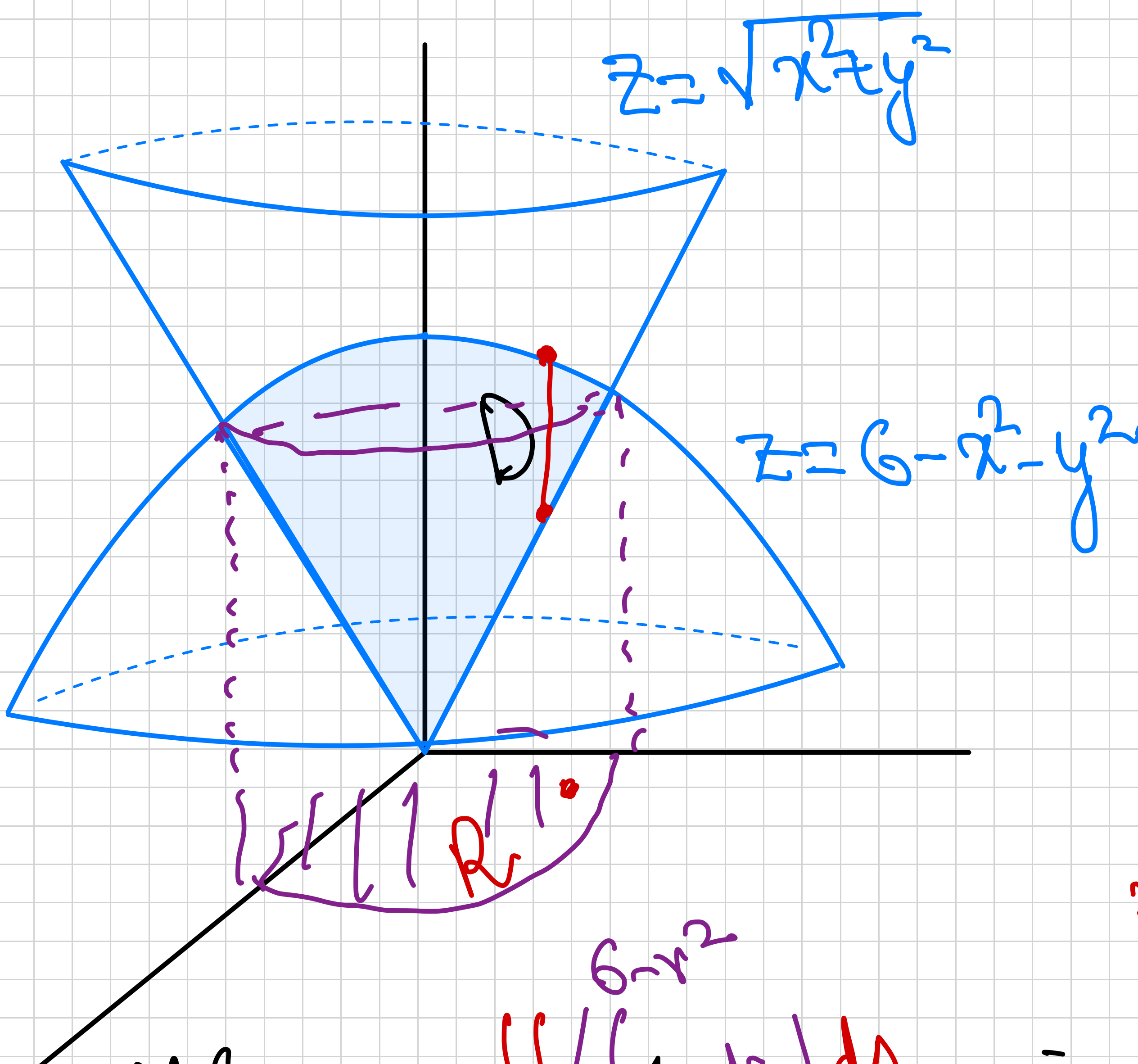
$$= \int_{\pi/4}^{\frac{3\pi}{4}} 60 \, d\theta = 60 \cdot \frac{\pi}{2}.$$

$$D = \{(r,\theta,z) : \underbrace{1 \le r \le 5 ;\ \frac{\pi}{4} \le \theta \le \frac{3\pi}{4}}_{\text{xy plane}} ;\ 2 \le z \le 7\}$$





eg②: Find volume of region between $\underbrace{z=\sqrt{x^2+y^2}}_{z=r}$ and $\underbrace{z=6-x^2-y^2}_{z=6-r^2}$



$$V = \iiint_D 1 \, dV$$

Boundary of shadow
= intersection of two surfaces

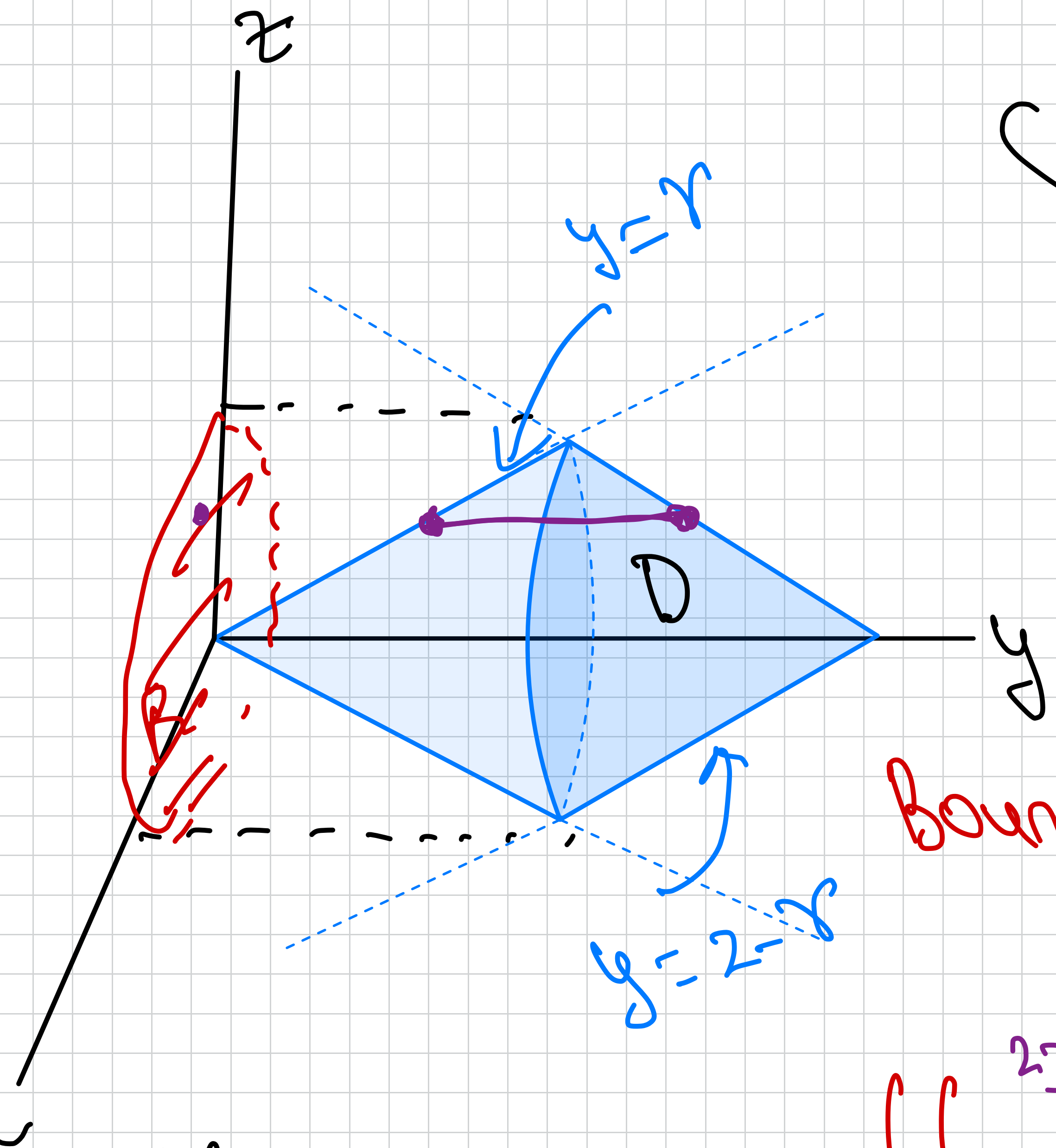
$$6-r^2 = r \implies r^2+r-6 = 0$$
$$(r+3)(r-2) = 0$$
$$\underline{\underline{r=2}}.$$

$$\iiint_D 1 \, dV = \iint_R \left(\int_r^{6-r^2} 1 \, dz \right) dA = \int_0^{2\pi} \int_0^2 \int_r^{6-r^3} 1 \cdot r \, dz \, dr \, d\theta$$

fix a point in R

$r \le z \le 6-r^2$

eg (3): Find volume of the Region between $y = \sqrt{x^2+z^2}$
& $y = 2 - \sqrt{x^2+z^2}$



Cartesian in y, polar in xz

$x^2+z^2 = r^2$

$y = \sqrt{x^2+z^2} = r$

$y = 2 - \sqrt{x^2+z^2} = 2 - r$

Boundary of Shadow = intersection

$r = 2 - r \rightsquigarrow r = 1$

$$V = \iiint_D 1 \, dV = \iint_R \int_r^{2-r} 1 \, dy \, dA = \int_0^{2\pi} \int_0^1 \int_r^{2-r} r \, dy \, dr \, d\theta.$$

R = Disk & Radius 1

Ex(4): $\displaystyle\int_0^{\frac{\sqrt{2}}{2}}\int_y^{\sqrt{1-y^2}}\int_0^{1-x^2-y^2} 1\; dz\, \underbrace{dx\, dy}_{r\,dr\,d\theta}$, evaluate by converting to Cylindrical Coordinates

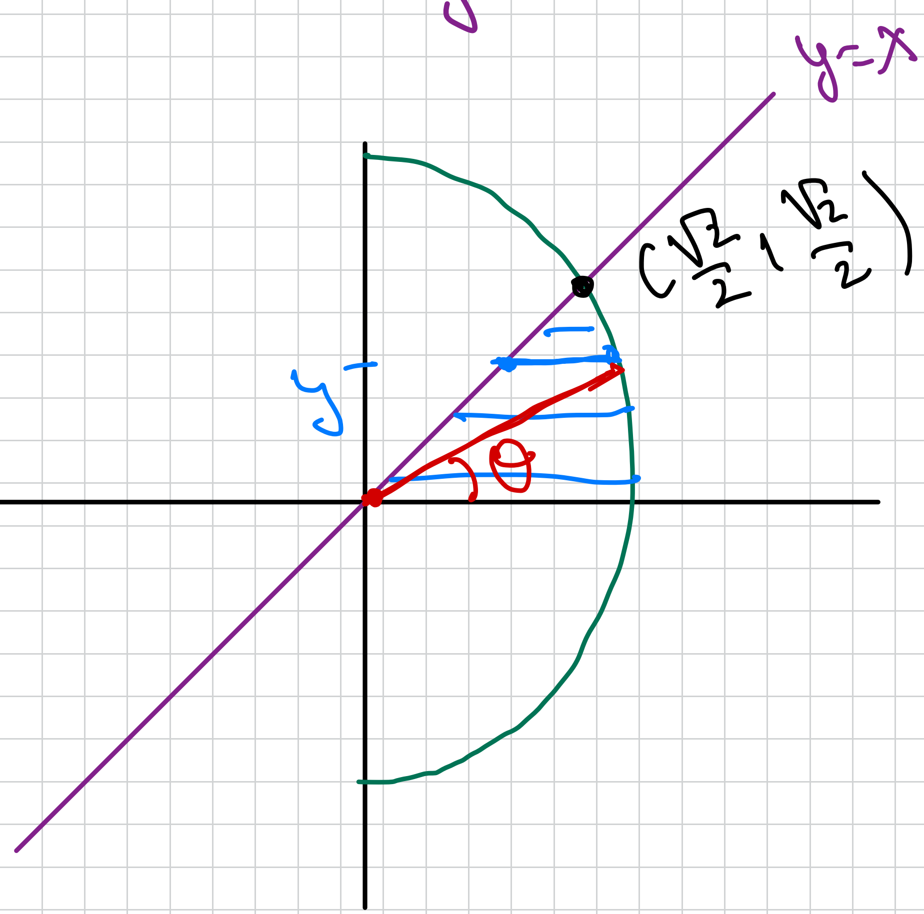
Convert to Bounds in polar

$$\underbrace{y \le x}_{\text{Left Bound}} \le \underbrace{\sqrt{1-y^2}}_{\text{Right}}$$

Left Bound is $y = x$

Right bound is $x = \sqrt{1-y^2}$

$\Rightarrow x^2 + y^2 = 1$



Bounds in polar: $0 \le r \le 1$

$0 \le \theta \le \pi/4$

$$\int_0^{\frac{\sqrt{2}}{2}}\int_y^{\sqrt{1-y^2}}\int_0^{1-x^2-y^2} 1\; dz\,dx\,dy = \int_0^{\frac{\pi}{4}}\int_0^{1}\int_0^{1-r^2} 1\cdot r\; dz\,dr\,d\theta.$$