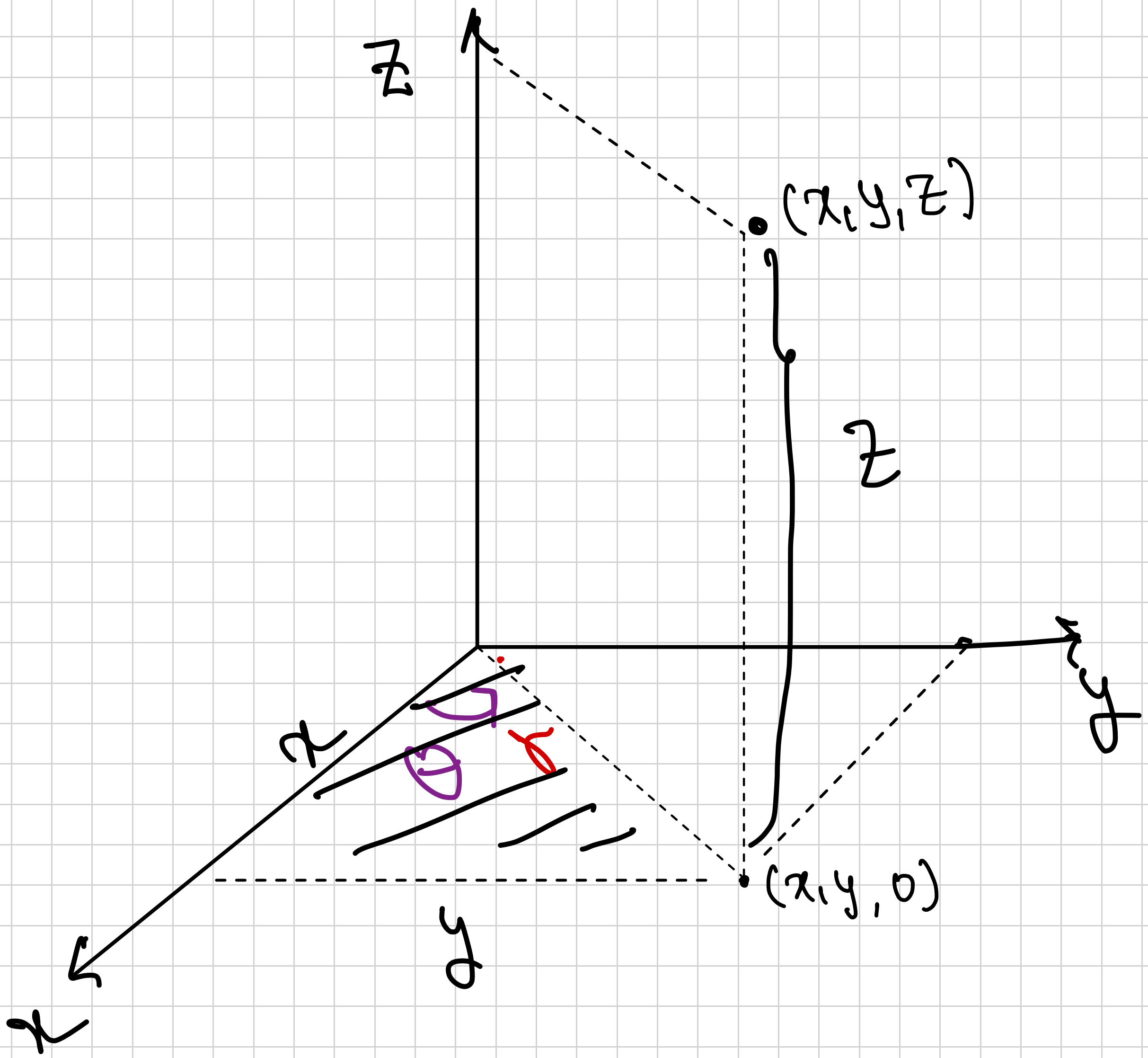
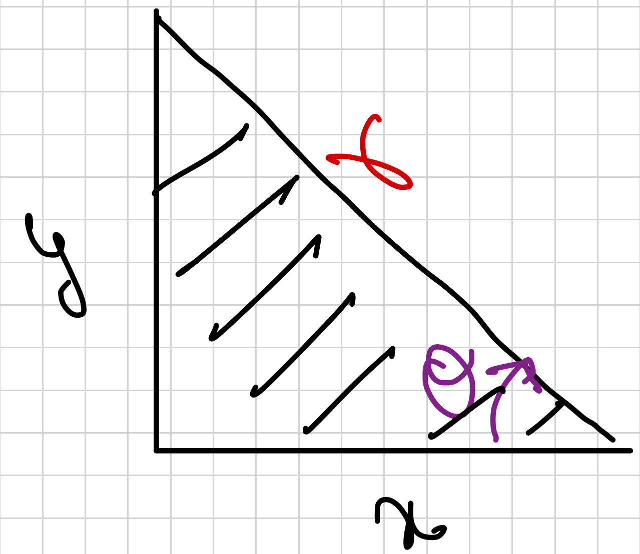
Lesson 23 : Triple Integrals in Spherical Coordinates (16.5)

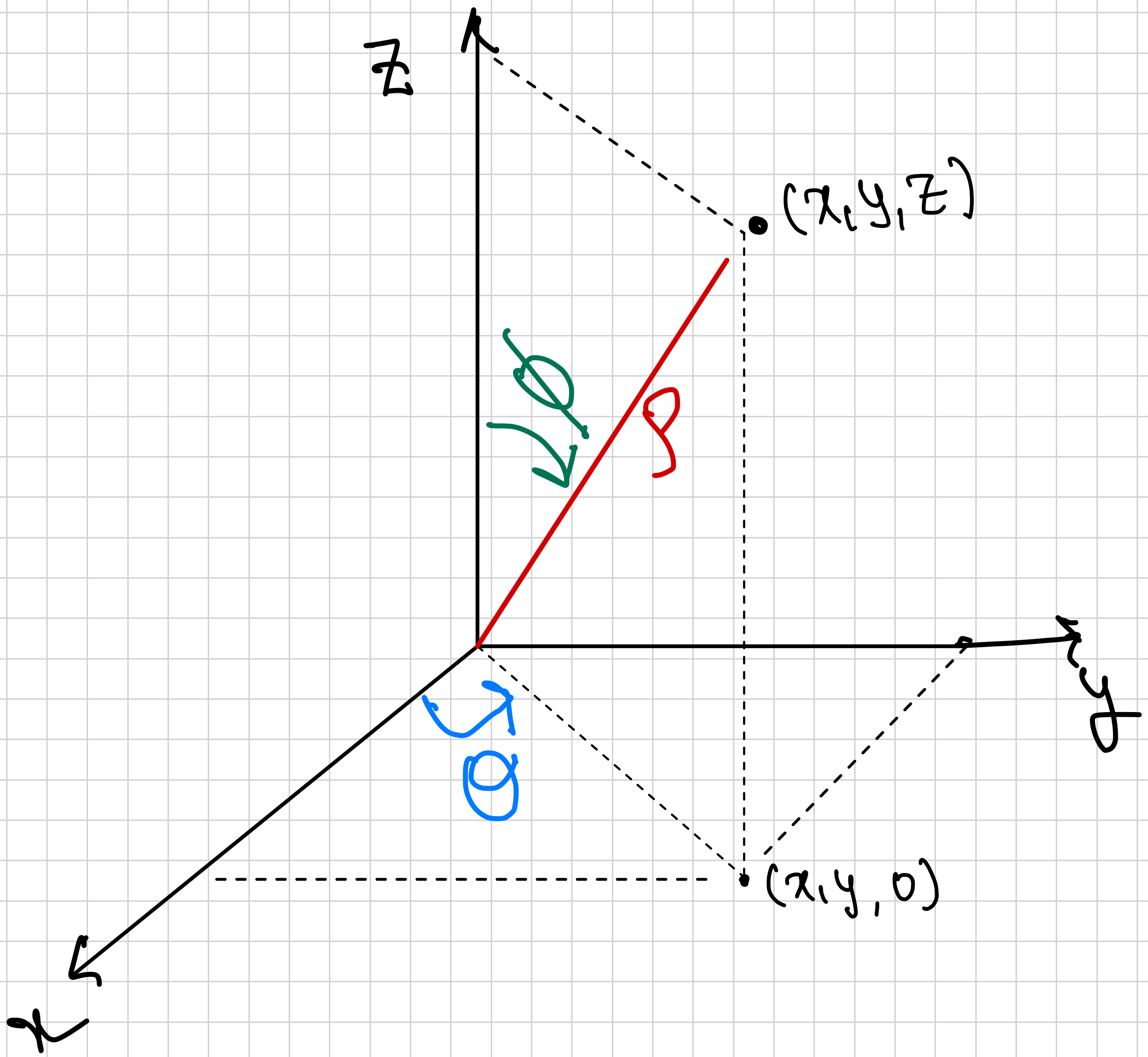
Review: Cylindrical Coordinates





$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned} \quad \longleftrightarrow \quad \begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}(y/x) \\ z &= z \end{aligned}$$

Spherical Coordinates



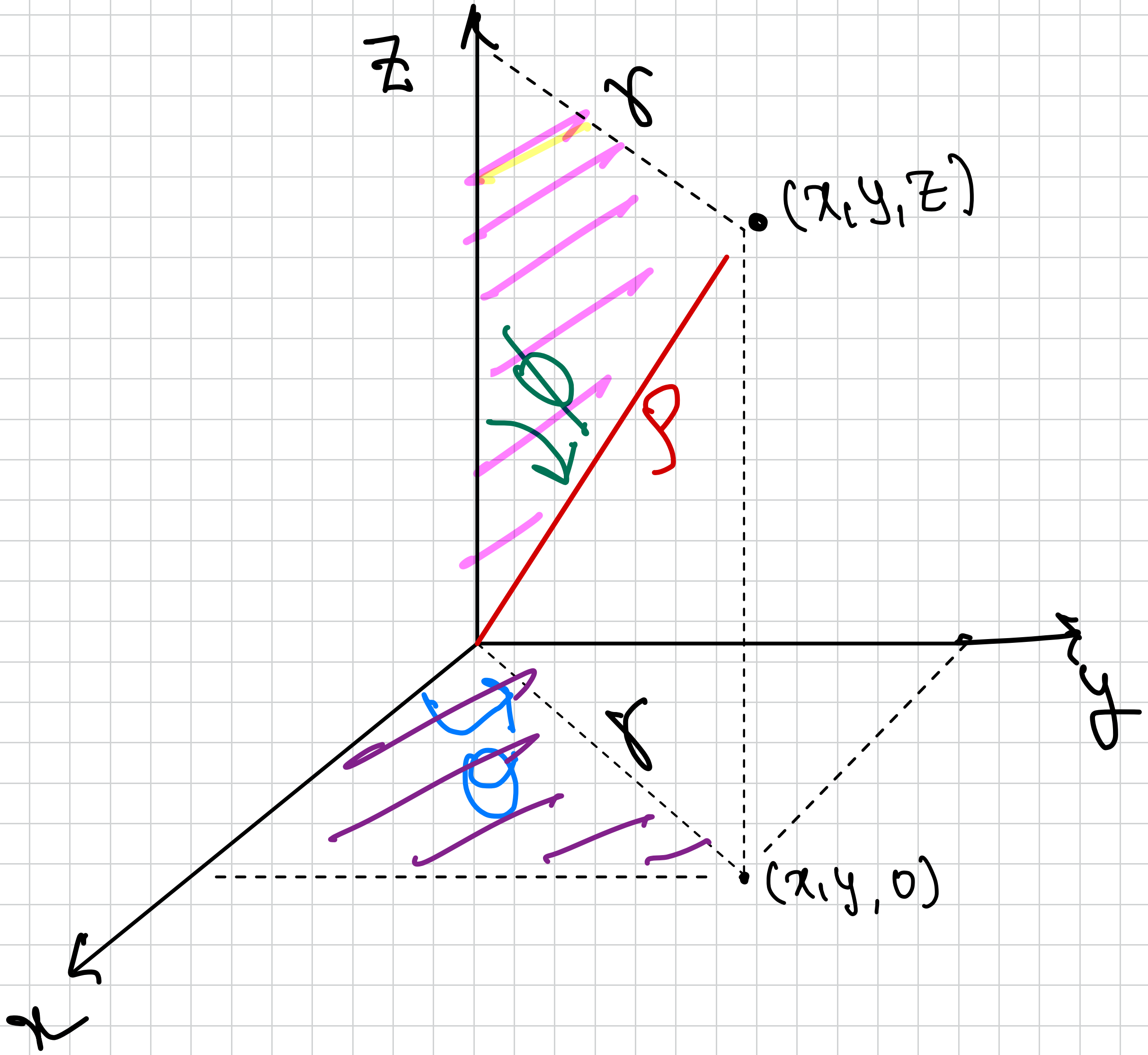
ρ = distance of point > 0
from origin

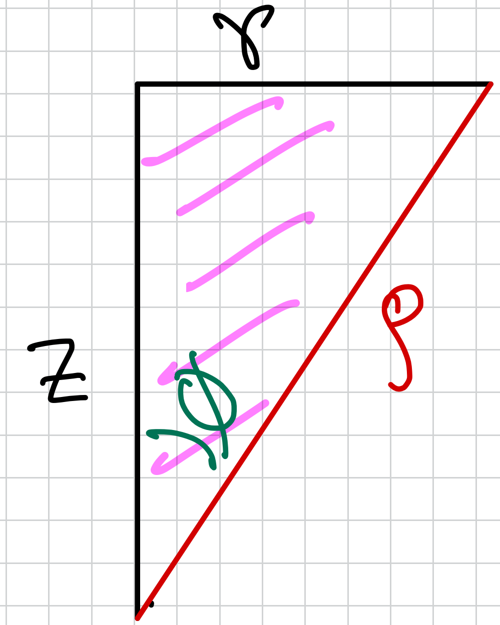
ϕ = Smallest Angle from
the +ve z-axis

$0 \le \phi \le \pi$

θ = Angle from +ve x-axis
to the shadow
line from (0,0,0) to (x,y,0)

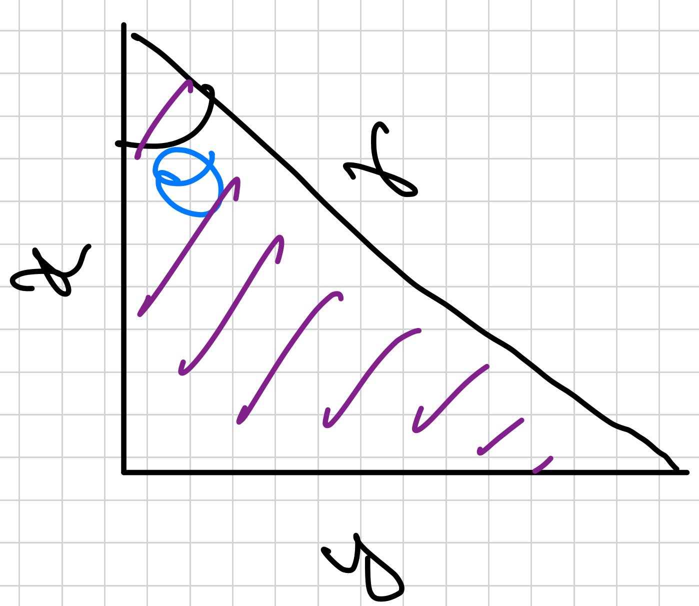
$0 \le \theta \le 2\pi$





$$\frac{z}{\rho} = \cos\phi \rightsquigarrow z = \rho\cos\phi$$

$$\frac{z}{r} = \sin\phi \rightsquigarrow r = \rho\sin\phi$$



$$x = r\cos\theta = \rho\sin\phi\cos\theta$$

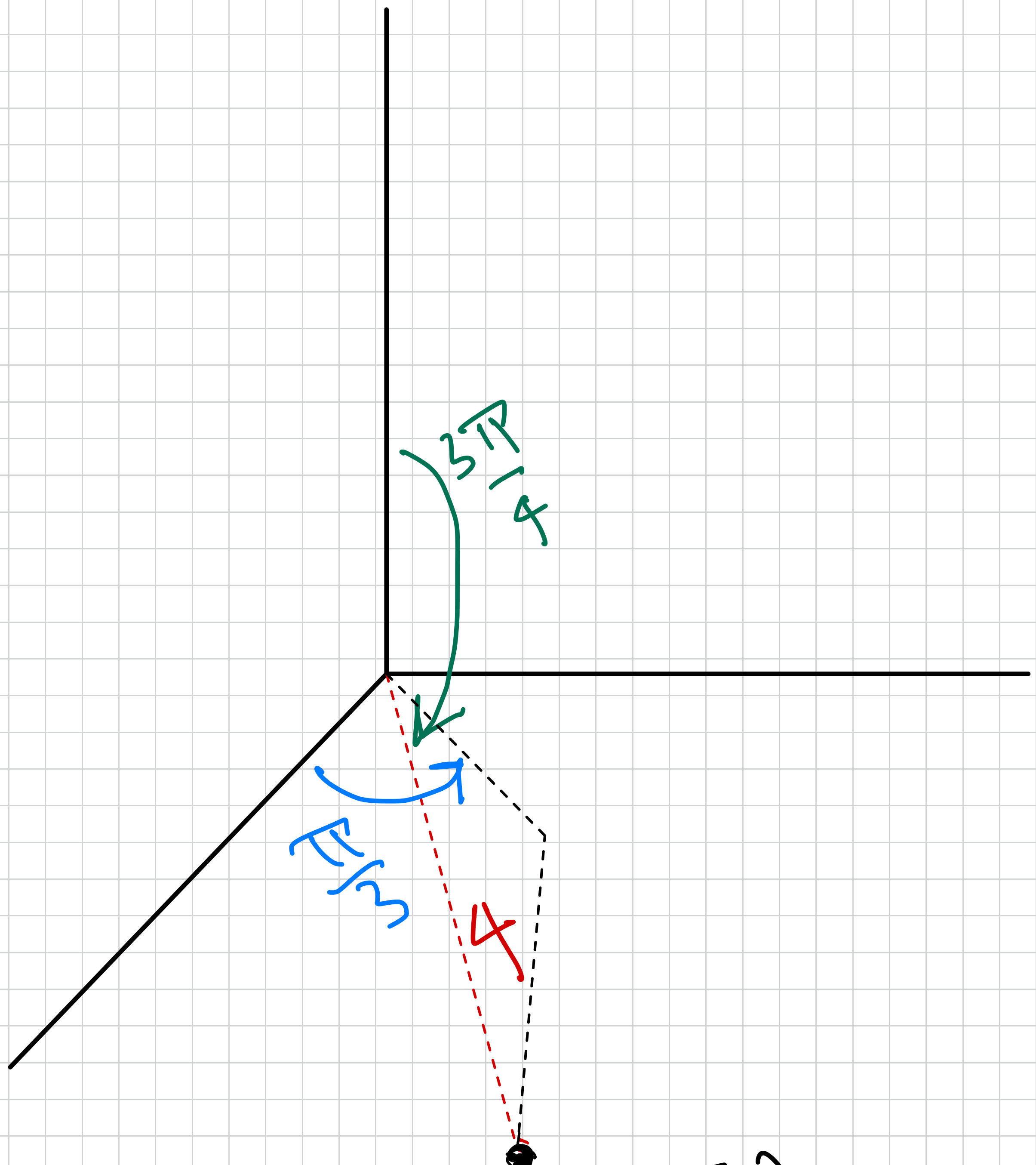
$$y = r\sin\theta = \rho\sin\phi\sin\theta$$

eg: Convert $(\rho, \phi, \theta) = (4, \frac{3\pi}{4}, \frac{\pi}{3})$ to Cartesian Coordinates

$\theta = \frac{\pi}{3} \rightsquigarrow x > 0$

$y > 0$

$\phi = \frac{3\pi}{4} \rightsquigarrow z < 0$



$(\sqrt{2}, \sqrt{6}, -2\sqrt{2})$

$$x = \rho \sin\phi \cos\theta = 4 \sin\left(\frac{3\pi}{4}\right) \cos\left(\frac{\pi}{3}\right)$$

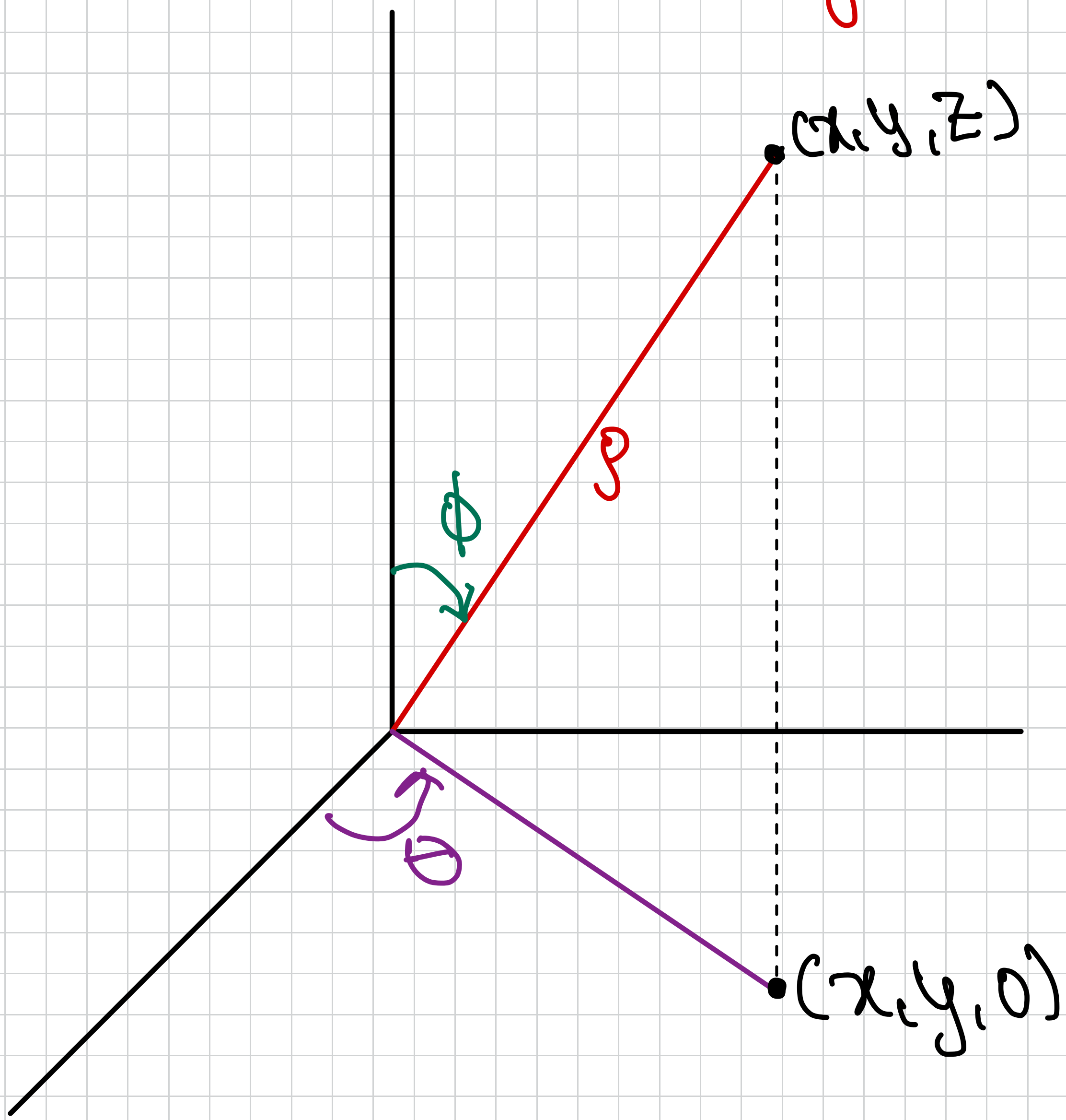
$$= 4 \cdot \frac{\sqrt{2}}{2} * \frac{1}{2}$$

$$= \sqrt{2}$$

$$y = \rho \sin\phi \sin\theta = 4 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = \sqrt{6}$$

$$z = \rho \cos\phi = 4 \cdot \cos\left(\frac{3\pi}{4}\right) = -2\sqrt{2}$$

Summary:



$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

$$\updownarrow$$

$$\rho = \sqrt{x^2+y^2+z^2}$$

$$\phi = \cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

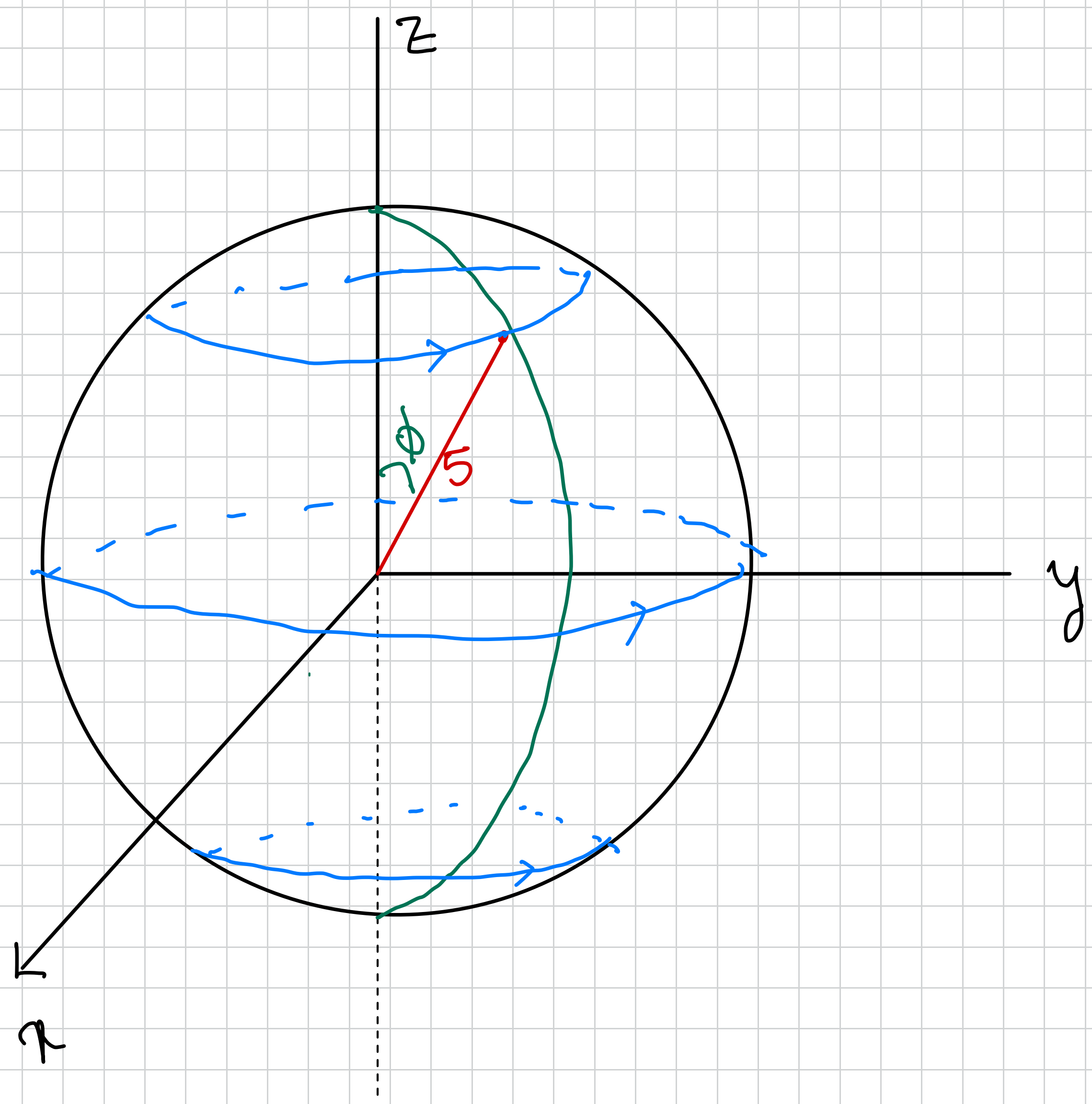
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Graphs in Spherical Coordinates

eg ①: $\rho = 5 \rightsquigarrow \sqrt{x^2+y^2+z^2} = 5$

$$x^2+y^2+z^2 = 25$$

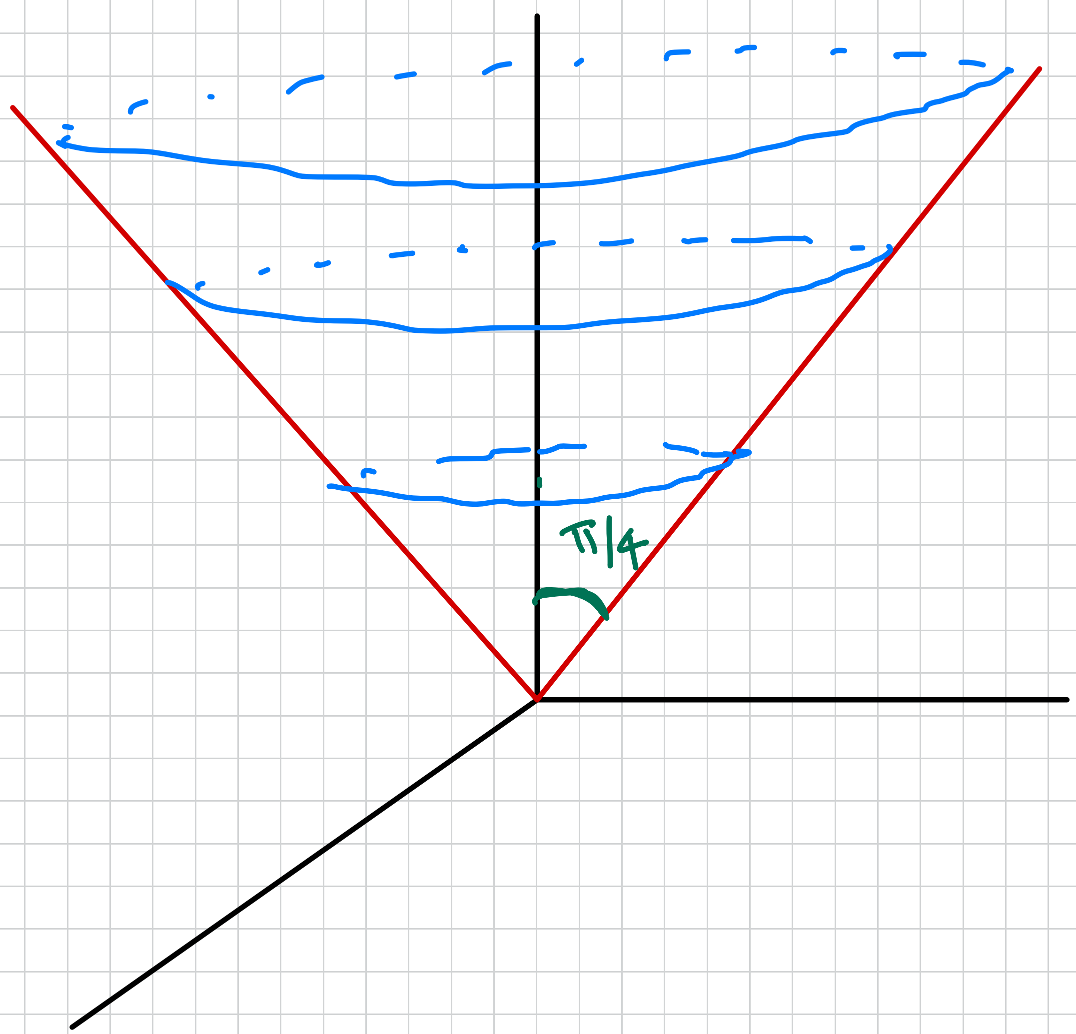
a Sphere of radius 5



in general

$\rho = R$ is a Sphere of Radius R

eg (2): $\phi = \pi/4$



$$\cos\phi = \frac{z}{\rho}$$

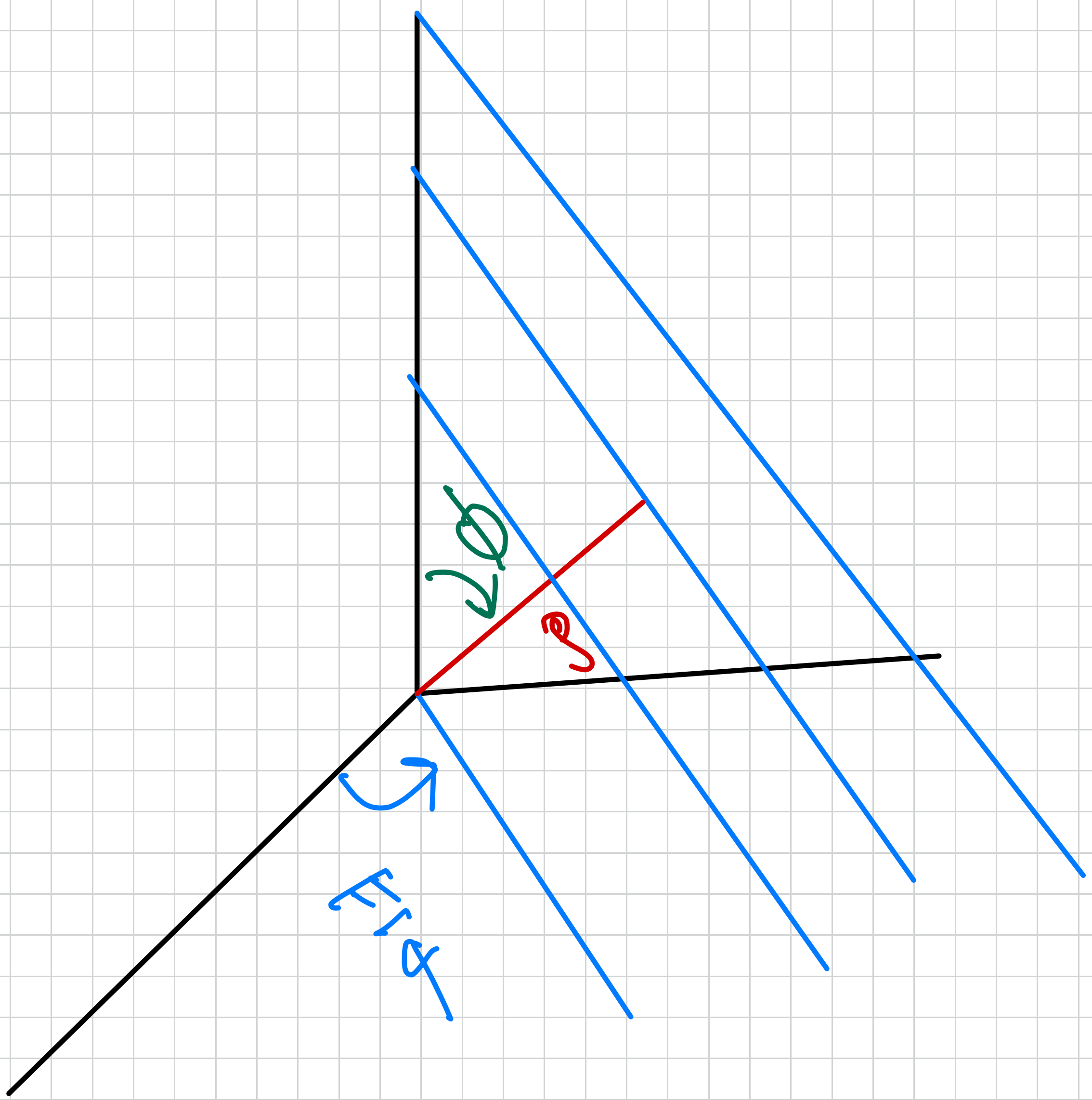
$$\cos\left(\frac{\pi}{4}\right) = \frac{z}{\sqrt{x^2+y^2+z^2}} > 0 \rightsquigarrow z > 0$$

$$\frac{1}{\sqrt{2}} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\left.\begin{array}{l} x^2+y^2+z^2 = 2z^2 \\ x^2+y^2 = z^2, \quad z > 0 \end{array}\right\} \text{Single Cone}$$

$\phi = \frac{\pi}{2} \rightsquigarrow z = 0 \rightsquigarrow xy$ plane

$\phi = \phi_0 \neq \frac{\pi}{2} \rightsquigarrow$ Single Cone eg ③: $\theta = \pi/4$



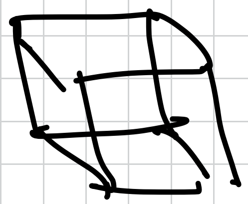
$$\tan\theta = \frac{y}{x}$$

$$\tan\frac{\pi}{4} = \frac{y}{x} \quad , \; y > 0 \; , \; x > 0$$

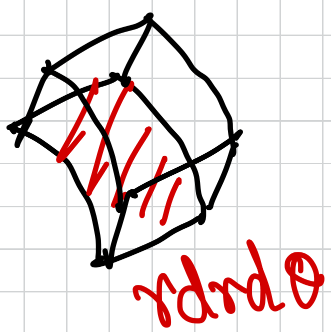
$$\underbrace{y = x}_{\text{half plane}}$$

Triple integrals:

$$\iiint_D f(x,y,z)\, dV$$

Cartesian: 

$$dV = dx\,dy\,dz$$
$$= dy\,dz\,dx$$

Cylindrical: 

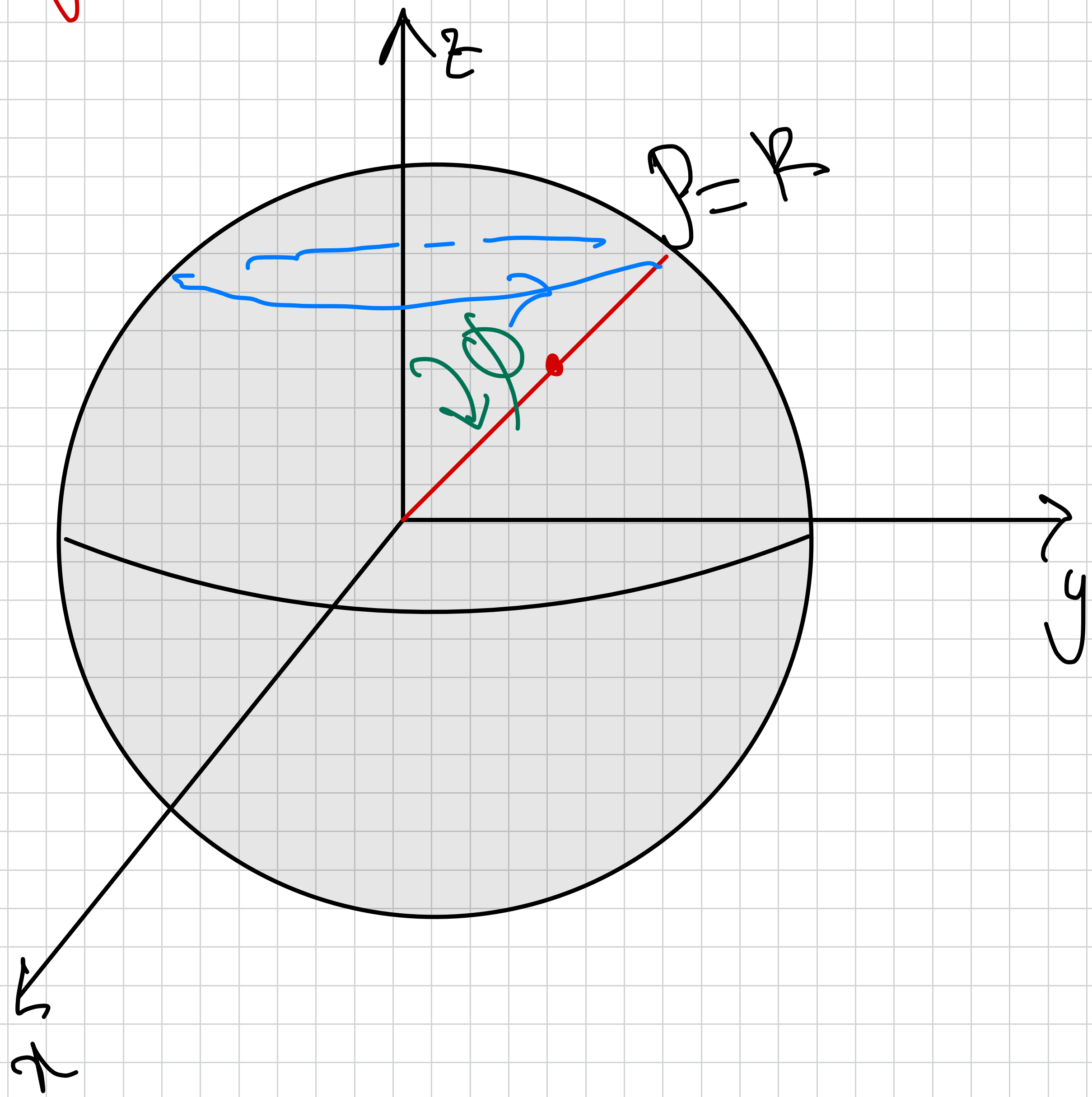
$$dV = r\,dz\,dr\,d\theta$$

Spherical: Spherical sector 

$$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$\iiint_D f(x,y,z)\, dV = \int_\theta \int_\phi \int_\rho f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta .$$

Eg ①: Find Volume of a Sphere of Radius R $\left(\frac{4}{3}\pi R^3\right)$



$$V = \iiint_D 1\, dV$$

$0 \le \rho \le R$

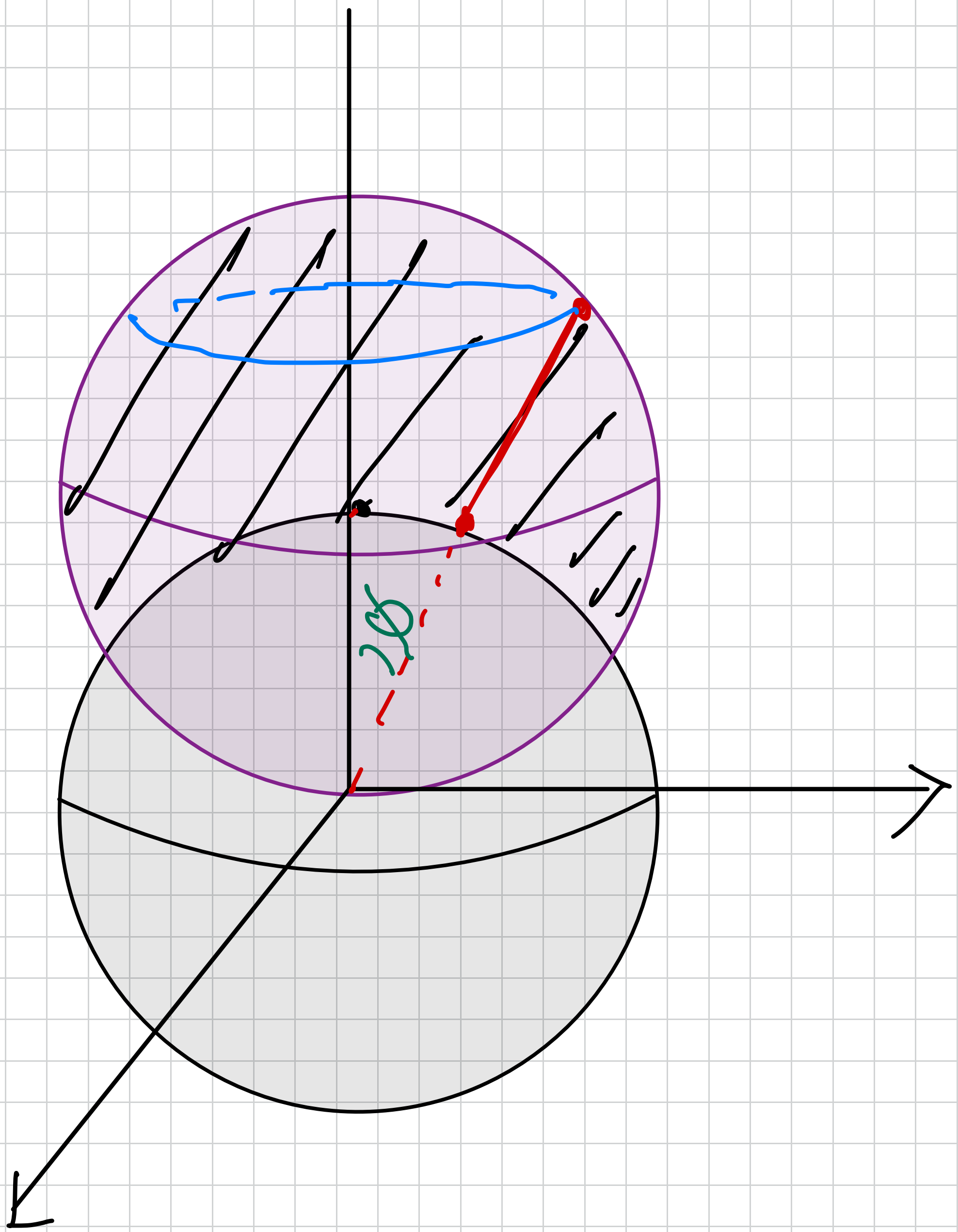
$0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{R^3}{3} \sin\phi \, d\phi \, d\theta = \int_0^{2\pi} \frac{2R^3}{3} \, d\theta = \frac{4\pi R^3}{3}.$$

eg②: Find volume of Region inside $\underbrace{\rho = 4\cos\phi}$ & Outside $\underbrace{\rho = 2}_{x^2+y^2+z^2=4}$.



$$\rho^2 = 4\rho\cos\phi$$

$$x^2+y^2+z^2 = 4z$$

$$x^2+y^2+z^2-4z+4 = 4$$

$$x^2+y^2+(z-2)^2 = 4$$

$$2 \le \rho \le 4\cos\phi$$

$$0 \le \theta \le 2\pi$$

$0 \le \phi \le$ interse ction point

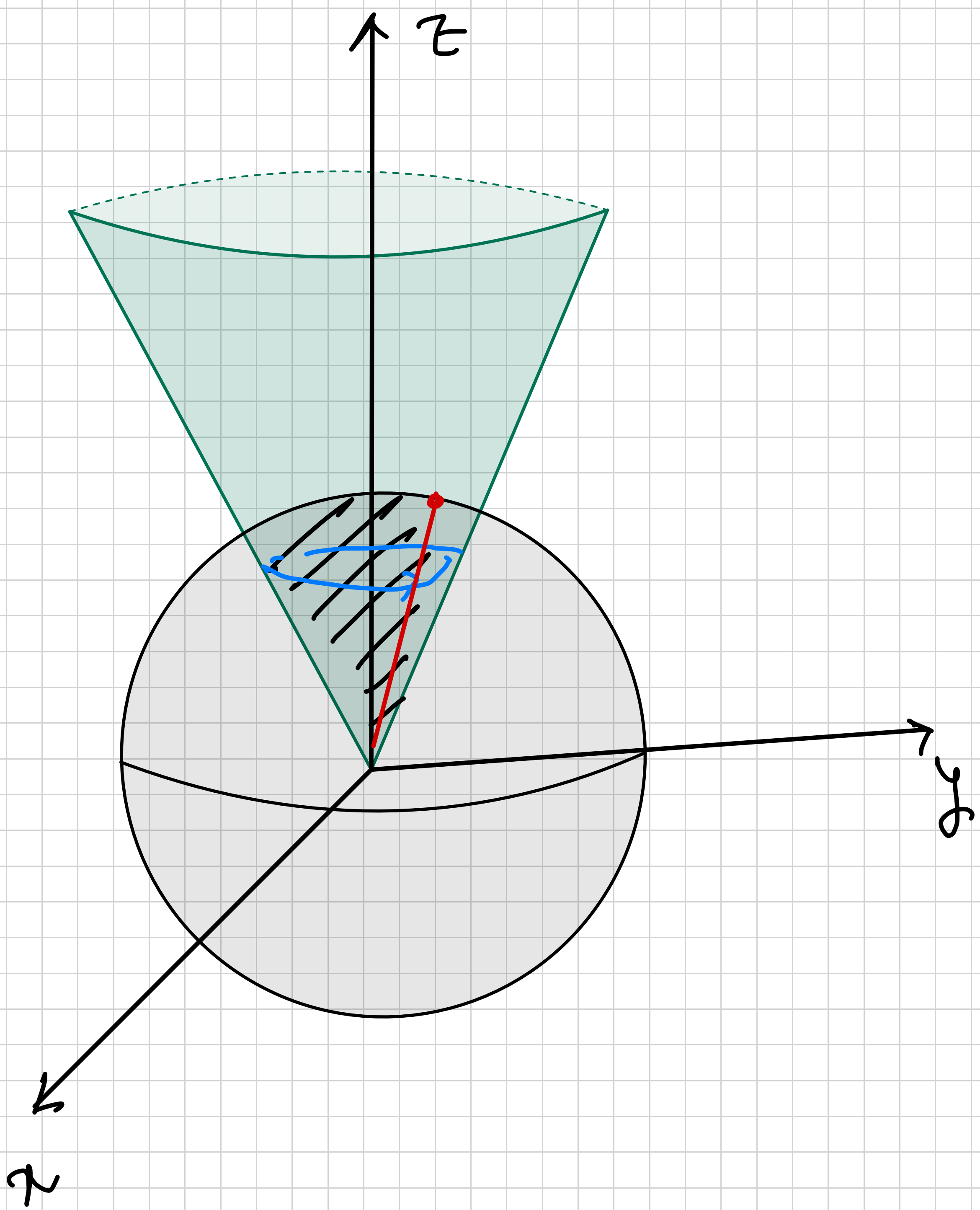
$$4\cos\phi = 2$$

$$\cos\phi = \frac{1}{2}$$

$$\Rightarrow \phi = \frac{\pi}{3}$$

$$\text{Volume} = \int_0^{2\pi}\int_0^{\pi/3}\int_2^{4\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

eg ③: Find Volume of the Region between $\underbrace{\rho=6}_{\text{Sphere}}$ & $\underbrace{\phi=\pi/6}_{\text{Cone}}$.



$0 \le \rho \le 6$

$0 \le \phi \le \pi/6$

$0 \le \theta \le 2\pi$

$$\text{Volume} = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{6} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/6} \frac{6^3}{3} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \frac{6^3}{3}\left[-\cos\left(\frac{\pi}{6}\right)+1\right] d\theta$$

$$= 2\pi * \frac{6^3}{3} * \left[1-\frac{\sqrt{3}}{2}\right].$$