Lesson 6: Calculus of Vector-Valued Functions (14.2)

Motion in Space (14.3) - Part I

Warmup: $\vec{r}(t) = \langle \underbrace{\cos t}_{x}, \underbrace{\sin t}_{y}, \underbrace{2t}_{z} \rangle \rightsquigarrow \vec{r}(t)$ is a Helix on $x^2 + y^2 = 1$

What do you think $\frac{d}{dt}(\vec{r}(t))$ should be?

$\frac{d}{dt}\cos t = -\sin t$

$\frac{d}{dt}\sin t = \cos t$, $\frac{d}{dt}2t = 2$

What do you think $\int \vec{r}(t)\,dt$ should be?

Differentiate and integrate componentwise

Office Hours: MWF: 10:00 AM - 11:15 AM, MATH 842

Review: Find the points of intersection of

$\underbrace{\vec{r}(t) = \langle t\cos t, t\sin t, t\rangle}_{\text{Curve}}$ $\quad 0 \le t \le 2\pi$ and $\underbrace{x - y = 0}_{\text{Plane}}$

$x = t\cos t$
$y = t\sin t$
$z = t$

plug in

$t\cos t - t\sin t = 0$

$t[\cos t - \sin t] = 0$

$t = 0$ OR $\cos t - \sin t = 0$

$\cos t = \sin t$

$t = \frac{\pi}{4}$ OR $t = \frac{5\pi}{4}$

total 3 points of intersection.

$t = 0 \rightsquigarrow \vec{r}(0) = \langle 0, 0, 0\rangle$

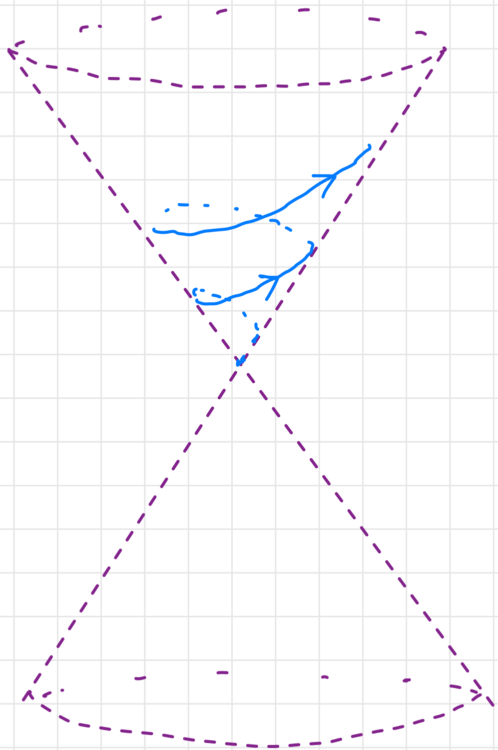
$t = \frac{\pi}{4} \rightsquigarrow \vec{r}\left(\frac{\pi}{4}\right) = \left\langle \frac{\pi\sqrt{2}}{8}, \frac{\pi\sqrt{2}}{8}, \frac{\pi}{4} \right\rangle$

$t = \frac{5\pi}{4} \rightsquigarrow \vec{r}\left(\frac{5\pi}{4}\right) = \left\langle -\frac{5\pi\sqrt{2}}{8}, -\frac{5\pi\sqrt{2}}{8}, \frac{5\pi}{4} \right\rangle$

Sketch $\vec{r}(t) = \langle \underbrace{t\cos t}_{x}, \underbrace{t\sin t}_{y}, \underbrace{t}_{z} \rangle$, $0 \le t \le 2\pi$

Eliminat "t"

$$x^2+y^2 = t^2[\cos^2 t + \sin^2 t] = t^2 = z^2 \rightsquigarrow \underbrace{x^2+y^2=z^2}_{\text{Double Cone}}$$



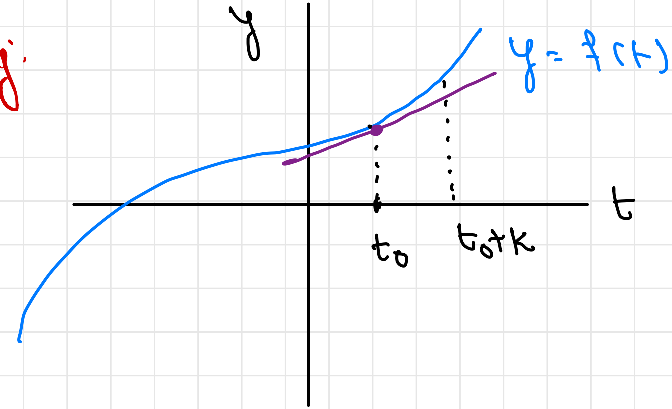
Curve $\vec{r}(t)$ lies on the Double Cone

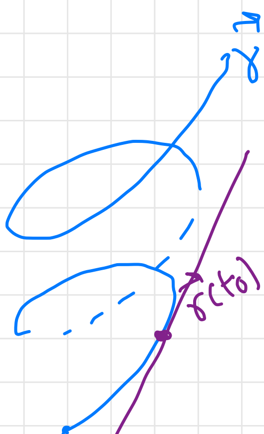
Derivative and its meaning:

$f(t) \rightsquigarrow$ Scalar function

$f'(t_0) \rightsquigarrow$ Slope of tangent line through $(t_0, f(t_0))$

$$f'(t_0) = \lim_{k\to 0} \frac{f(t_0+k) - f(t_0)}{k}$$





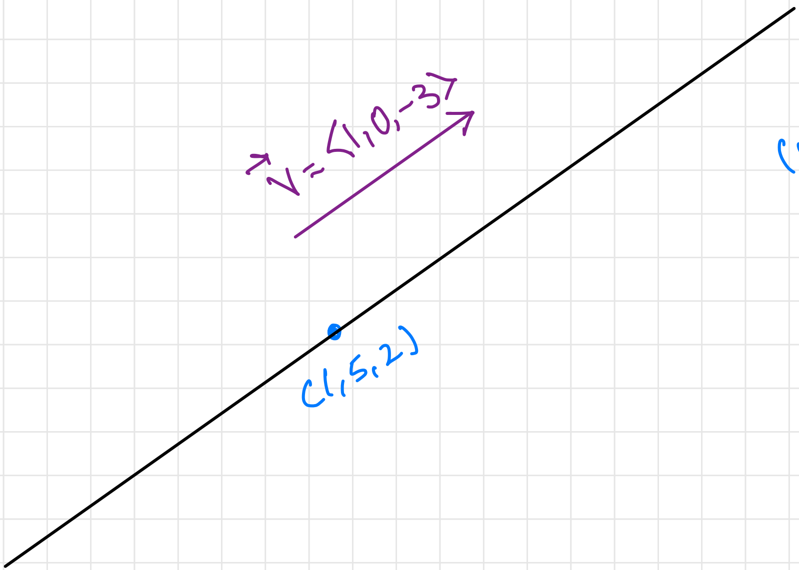
$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$ } VVF Represent curve in 3D

$\vec{r}'(t_0) \rightsquigarrow$ Direction vector of tangent line through $\vec{r}(t_0)$

$$\vec{r}'(t_0) = \langle f'(t_0), g'(t_0), h'(t_0) \rangle$$

$$T(t_0) = \frac{\vec{r}'(t_0)}{|\vec{r}'(t_0)|}$$ } Unit tangent Vector

Recall: Equation of a line.



Equation of Line through (1,5,2), parallel to $\langle 1,0,3\rangle$

$$\langle x,y,z\rangle = t\underbrace{\langle 1,0,-3\rangle}_{\text{Direction vector}} + \underbrace{\langle 1,5,2\rangle}_{\text{point on Line.}}$$

eg: $\vec{r}(t) = \langle \cos t, \sin t, 2t \rangle$

find equation of tangent line at $\underbrace{(0, 1, \pi)}_{\text{point on } \vec{r}(t) \text{ with } t = \pi/2}$

Recall: Equation of line through (x_0, y_0, z_0) with Direction vector $\langle a, b, c \rangle$

is $\langle x, y, z \rangle = t\langle a, b, c \rangle + \langle x_0, y_0, z_0 \rangle$.

Direction vector is the Derivative at $t = \pi/2$

$$\vec{r}'(t) = \left\langle \frac{d}{dt}\cos t, \frac{d}{dt}\sin t, \frac{d}{dt}2t \right\rangle = \langle -\sin t, \cos t, 2 \rangle$$

$$\vec{r}'\left(\frac{\pi}{2}\right) = \langle -1, 0, 2 \rangle$$

Equation of tangent line: $\langle x, y, z \rangle = t\langle -1, 0, 2 \rangle + \langle 0, 1, \pi \rangle$.

Derivative Properties

$$\frac{d}{dt}\left(c\,\vec{r}(t)\right) = c\,\frac{d}{dt}\vec{r}(t)$$

$$\frac{d}{dt}\left(\vec{r}_1(t) \pm \vec{r}_2(t)\right) = \frac{d}{dt}\left(\vec{r}_1(t)\right) \pm \frac{d}{dt}\left(\vec{r}_2(t)\right)$$

$$\frac{d}{dt}\left(\vec{r}_1(t)\cdot\vec{r}_2(t)\right) = \frac{d}{dt}\vec{r}_1(t)\cdot\vec{r}_2(t) + \vec{r}_1(t)\cdot\frac{d}{dt}\vec{r}_2(t)$$

$$\frac{d}{dt}\left(\vec{r}_1(t)\times\vec{r}_2(t)\right) = \frac{d}{dt}\vec{r}_1(t)\times\vec{r}_2(t) + \vec{r}_1(t)\times\frac{d}{dt}\vec{r}_2(t)$$

ORDER is important.

$$\frac{d}{dt}\left(f(t)\,\vec{r}(t)\right) = f'(t)\,\vec{r}(t) + f(t)\,\vec{r}\,'(t)$$

$$\frac{d}{dt}\left[\vec{r}\left(f(t)\right)\right] = \vec{r}\,'\left(f(t)\right)\cdot f'(t)$$

Scalar Multiplication

Integrals:

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ → integrate component wise

eg: $\vec{r}(t) = \langle \cos t, \sin t, 2t \rangle$

$$\int \vec{r}(t)\,dt = \left\langle \int \cos t\,dt, \int \sin t\,dt, \int 2t \right\rangle$$

$$= \langle \sin t + c_1, -\cos t + c_2, t^2 + c_3 \rangle$$

$$= \langle \sin t, -\cos t, t^2 \rangle + \vec{C}$$

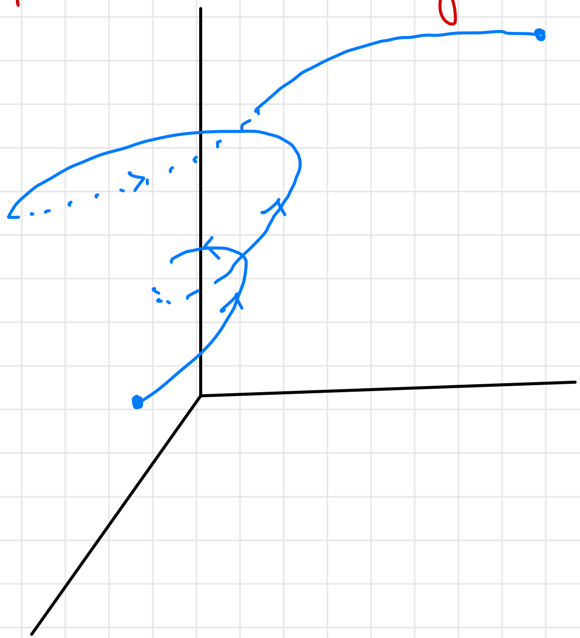
constant vector of integration.

eg: $\vec{r}(t) = \langle e^t, t^2, t^{1/2} \rangle$

Indefinite Integral $\int \vec{r}(t)\,dt = \left\langle e^t, \frac{t^3}{3}, \frac{2}{3}t^{3/2} \right\rangle + \vec{C}$

Definite Integral $\int_0^1 \vec{r}(t)\,dt = \left\langle e^1, \frac{1^3}{3}, \frac{2}{3}1^{3/2} \right\rangle - \left\langle e^0, \frac{0^3}{3}, \frac{2}{3}0 \right\rangle = \left\langle e-1, \frac{1}{3}, \frac{2}{3} \right\rangle.$

Particle moving in Space [3D]



$\vec{r}(t)$ = position vector.

$\frac{d}{dt}\vec{r}(t) = \vec{v}(t)$ = velocity vector

$\frac{d}{dt}\vec{v}(t) = \vec{a}(t)$ = acceleration vector.

Eg! $\vec{a}(t) = \langle e^{-t}, \sin t, t \rangle$, $\underbrace{\vec{V}(0) = \langle 1, 2, 3 \rangle}_{\text{initial Velocity}}$, $\underbrace{\vec{r}(0) = \langle -1, 3, -4 \rangle}_{\text{initial position}}$

Find $\underbrace{\vec{r}(t)}_{\text{position Vector}}$

$$\vec{V}(t) = \int \vec{a}(t)\,dt = \langle -e^{-t}, -\cos t, \frac{t^2}{2} \rangle + \vec{C}$$

$$\langle 1, 2, 3 \rangle = \vec{V}(0) = \langle -e^{-0}, -\cos(0), \frac{0^2}{2} \rangle + \vec{C} = \langle -1, -1, 0 \rangle + \vec{C}$$

$$\Rightarrow \quad \vec{C} = \langle 2, 3, 3 \rangle$$

$$\vec{V}(t) = \langle -e^{-t}, -\cos t, \frac{t^2}{2} \rangle + \langle 2, 2, 3 \rangle,$$

$$\vec{r}(t) = \int \vec{V}(t)\,dt = \langle e^{-t}, -\sin t, \frac{t^3}{6} \rangle + \langle 2t, 2t, 3t \rangle + \vec{D}$$

$$\langle -1, 3, -4 \rangle = \vec{r}(0) = \langle e^{-0}, -\sin(0), \frac{0^3}{6} \rangle + \langle 2\cdot 0, 2\cdot 0, 3\cdot 0 \rangle + \vec{D}$$

$$= \langle 1, 0, 0 \rangle + \langle 0, 0, 0 \rangle + \vec{D}$$

$$\Rightarrow \quad \vec{D} = \langle -2, 3, -4 \rangle$$

$$\vec{r}(t) = \langle e^{-t}, -\sin t, \frac{t^3}{6} \rangle + \langle 2t, 2t, 3t \rangle + \langle -2, 3, -4 \rangle$$