Lesson 8: Length of Curves (14.4) and Curvature (14.5)

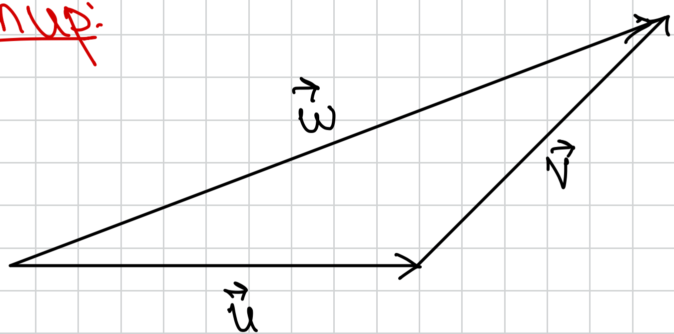
Today: Arc length
Arc length parametrization
Curvature

Next: functions of several variables

Office Hours: Monday, Friday 9:45 PM - 11:00 PM
Thursday 11:00 AM - 12:00 PM

Warmup:

1)



$$\vec{w}-\vec{u}=\vec{v}$$

2)

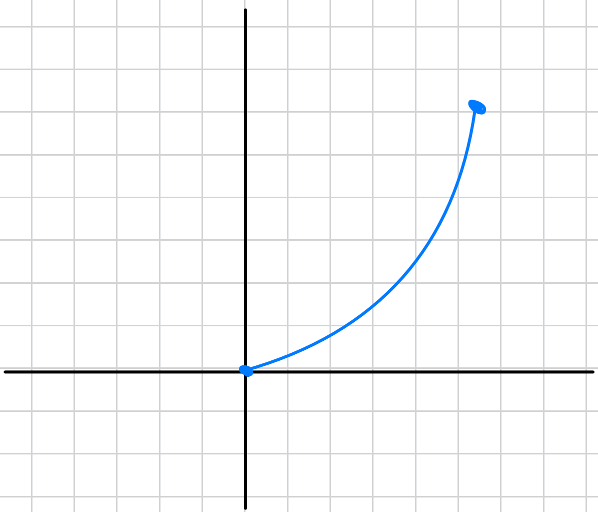
$$F(x)=\int_a^x f(t)\,dt$$

$$F'(x)=f(t) \Big\} \text{FTC}$$

Review: Arc length (from Calc 2)

$y = f(x), \quad a \le x \le b$

$$\text{Arc Length} = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$



eg: $y = x^2; \quad 0 \le x \le 1$

$$\text{Arc length} = \int_0^1 \sqrt{1 + (f'(x))^2}\, dx$$

$$= \int_0^1 \sqrt{1 + 4x^2}\, dx$$

$y = x^2, \quad 0 \le x \le 1 \rightsquigarrow$ Arc length $= \int_0^1 \sqrt{1+4x^2}\, dx$

parametrize

$\vec{r}(t) = \langle t, t^2 \rangle, \quad 0 \le t \le 1$

$\vec{r}'(t) = \langle 1, 2t \rangle$

$|\vec{r}'(t)| = \sqrt{1+4t^2}$

Arc Length $= \int_0^1 \sqrt{1+4x^2}\, dx = \int_0^1 \sqrt{1+4t^2}\, dt = \int_0^1 |\vec{r}'(t)|\, dt$

Arc length formula:

$$\vec{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle \Big\}$$ parametrized curve in 3D

Arc Length of the curve from $t=a$ to $t=b$ is:

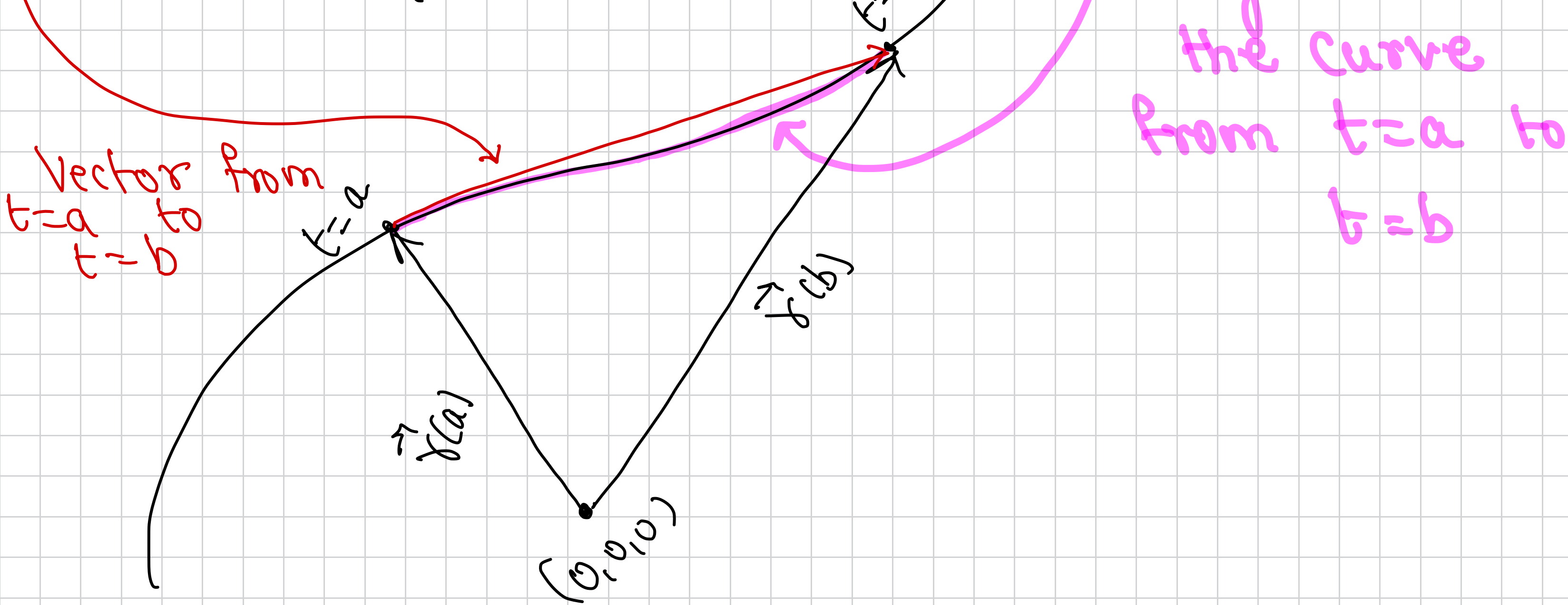
$$\boxed{\int_a^b |\vec{r}\,'(t)|\, dt}$$

$$\boxed{\int_a^b \vec{r}\,'(t)\,dt} \quad \text{Vector}$$

and

$$\int_a^b |\vec{r}\,'(t)|\,dt \quad \text{Scalar}$$

$$= \vec{r}(b) - \vec{r}(a)$$



Summary: $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is a curve in 3D

$$\int_a^b \vec{r}\,'(t)\,dt$$ $\rightsquigarrow$ Represents vector from $\vec{r}(a)$ to $\vec{r}(b)$

$$\int_a^b |\vec{r}\,'(t)|\,dt$$ $\rightsquigarrow$ Represents length of curve from $t=a$ to $t=b$.

eg: $\vec{r}(t) = \langle \cos t, \sin t \rangle$, Compute

$$\int_0^{\frac{\pi}{2}} \vec{r}\,'(t)\, dt$$

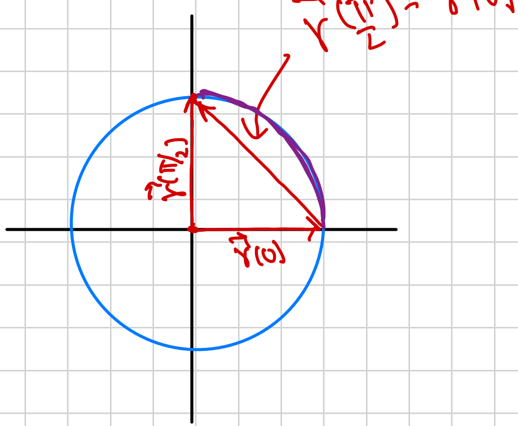
and

$$\int_0^{\frac{\pi}{2}} |\vec{r}\,'(t)|\, dt$$

$$= \vec{r}\left(\tfrac{\pi}{2}\right) - \vec{r}(0)$$

$$= \langle 0, 1 \rangle - \langle 1, 0 \rangle$$

$$= \langle -1, 1 \rangle$$



$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$|\vec{r}\,'(t)| = 1$

$$\int_0^{\frac{\pi}{2}} 1\, dt = \frac{\pi}{2}$$

Length of Curve from $t=0$ to $t=\pi/2$

eg: $\vec{r}(t) = \langle \cos t, \sin t, 2t \rangle$; Compute arc length from $t=0$ to 2π

$$\text{Arclength} = \int_a^b |\vec{r}\,'(t)|\,dt$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 2 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 4} = \sqrt{5}$$

$$L = \int_0^{2\pi} \sqrt{5}\,dt = (2\pi)(\sqrt{5}).$$

Find a formula to compute arclength from 0 to t

$$\vec{r}(t) = \langle \cos t, \sin t, 2t \rangle$$

$$\int_0^t |\vec{r}\,'(u)|\, du = \int_0^t \sqrt{5}\, du$$

$$= \sqrt{5}\, t$$

Arc length function:

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\boxed{s(t) = \int_a^t |\vec{r}\,'(u)|\, du}$$

$\rightsquigarrow$ Arc length function measured from $t = a$

input $= t$
output $=$ Arc length from a to t

Note: $s'(t) = |\vec{r}\,'(u)|$ } using FTC

Arc Length parametrization

$\vec{r}(t) = \langle \cos t, \sin t, 2t \rangle$

$$s(t) = \int_0^t |\vec{r}'(u)|\, du = \int_0^t \sqrt{5}\, du = \sqrt{5}\, t$$

Arc length function measured from $t=0$.

$s = \sqrt{5}\, t \rightsquigarrow \boxed{t = \frac{s}{\sqrt{5}}}$

using this, we can reparametrize $\vec{r}$, with s as new parameter

Substitute in $\vec{r}(t)$

$$\vec{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{5}}\right), \sin\left(\frac{s}{\sqrt{5}}\right), \frac{2s}{\sqrt{5}} \right\rangle$$

Arc length parametrization

Regular Parametrization v/s Arclength Parametrization

Regular Parametrization	Arclength Parametrization
* parameter = "time"	* parameter = length of curve
* given t, $\vec{r}(t)$ Represents position on the curve at time t	* given s, $\vec{r}(s)$ Represents position on curve when you travel s units from start.
* Speed $= \lvert r'(t) \rvert \neq 1$	* Speed $= \lvert r'(s) \rvert = 1$

Speed $= |\gamma'(s)| = 1$ in Arclength parametrization

$$\vec{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{5}}\right), \sin\left(\frac{s}{\sqrt{5}}\right), \frac{2s}{\sqrt{5}} \right\rangle$$

$$\vec{r}'(s) = \left\langle \frac{-1}{\sqrt{5}} \sin\left(\frac{s}{\sqrt{5}}\right), \frac{1}{\sqrt{5}} \cos\left(\frac{s}{\sqrt{5}}\right), \frac{2}{\sqrt{5}} \right\rangle$$

$$|r'(s)| = \sqrt{\frac{1}{5}\sin^2\left(\frac{s}{\sqrt{5}}\right) + \frac{1}{5}\cos^2\left(\frac{s}{\sqrt{5}}\right) + \frac{4}{5}}$$

$$= 1$$

eg: Reparametrize $\vec{r}(t) = \langle \cos t, \sin t, 2t \rangle$ with respect to arclength as measured from $\langle -1, 0, 2\pi \rangle$ in t direction

$\uparrow$ $t = \pi$ is starting point

① Arc length function

$$S(t) = \int_{\pi}^{t} |\vec{r}'(u)|\, du = \int_{\pi}^{t} \sqrt{5}\, du = \sqrt{5}(t - \pi)$$

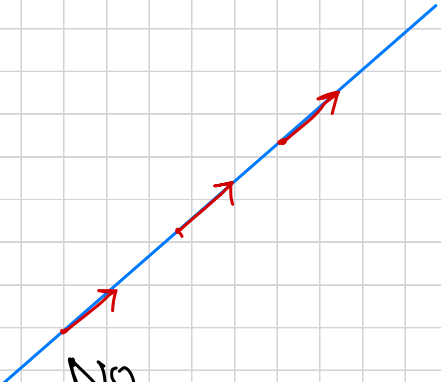
② Write t in terms of s

$$t = \frac{s}{\sqrt{5}} + \pi$$

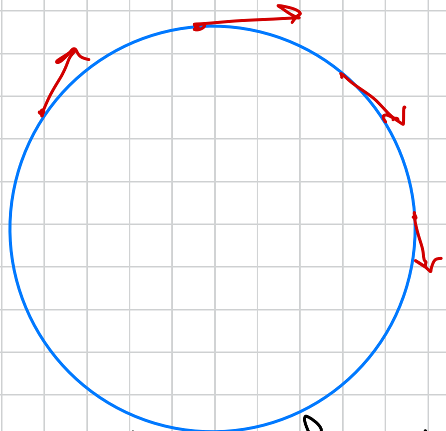
③ plug in to $\vec{r}(t)$

$$\vec{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{5}} + \pi\right), \sin\left(\frac{s}{\sqrt{5}} + \pi\right), 2\left(\frac{s}{\sqrt{5}} + \pi\right) \right\rangle.$$

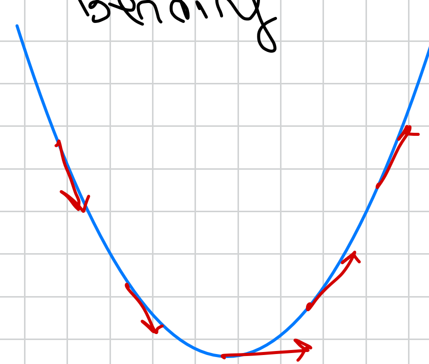
Curvature → How much is curve Bending



No Bending



Bends Uniformly



Bends less at top
Bends more at Bottom.

Curvature $= \kappa =$ Magnitude of the Rate of Change of Unit tangent Vector $= |T'(s)| = |r''(s)|$

Other formulae for Curvature:

$$* \ \kappa = |\gamma''(s)| = |T'(s)| = \left|\frac{d\vec{T}}{ds}\right| = \frac{\left|\frac{d\vec{T}}{dt}\right|}{\left|\frac{ds}{dt}\right|} = \boxed{\frac{|T'(t)|}{|\gamma'(t)|}}$$

s here is arc length parameter

if t is the parameter using chain Rule

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\frac{ds}{dt}$$

$$\frac{ds}{dt} = |\gamma'(t)|$$

$$* \quad \kappa = \boxed{\frac{|\gamma' \times \gamma''|}{|\gamma'|^3}}$$

proof in text book

eg: $\vec{r}(t) = \langle 3, -4t, 5t^2 \rangle$, find curvature at $t=0$

$$r' = \langle 0, -4, 10t \rangle$$
$$r'' = \langle 0, 0, 10 \rangle$$

$$r' \times r'' = \langle -40, 0, 0 \rangle$$
$$|r' \times r''| = 40$$
$$|r'| = \sqrt{16 + 100t^2}$$

$$k = \frac{|r' \times r''|}{|r'|^3} = \frac{40}{\left(\sqrt{16+100t^2}\right)^3}$$

k at $t=0$

$$k = \frac{40}{\left(\sqrt{16}\right)^3} = \frac{40}{64} = \frac{5}{8}.$$