Knot Theory and Problems on Affine Space

Takumi Murayama

September 27, 2016

Abstract

Affine n-space $\mathbf{A}^n$ is one of the most basic objects in algebraic geometry. Unfortunately, there are many basic things we don't know about its geometry and its symmetries: for example,
(1) Is every embedding of $\mathbf{A}^k$ into $\mathbf{A}^n$ linearizable?
(2) What is the automorphism group of $\mathbf{A}^n$?
(3) (The Jacobian Conjecture) Is every endomorphism of $\mathbf{A}^n$ with invertible Jacobian an isomorphism?
The Abhyankar–Moh theorem provides answers to some of these and other related questions in dimension $n = 2$. What's surprising is that even though the statement is purely algebraic, the shortest known proof utilizes knot theory, specifically that of torus knots. We will explain how torus knots naturally appear in algebraic geometry, and present Rudolph's knot-theoretic proof of the Abhyankar–Moh theorem.

We work over the complex numbers $\mathbf{C}$.

1 Introduction

In algebraic geometry, the first, most basic variety you study is affine n-space, $\mathbf{A}^n$. There are some very basic open problems about $\mathbf{A}^n$, some of which are the following (see [Kra96, §1] for more):

Cancellation Problem. *Does* $Y \times \mathbf{A}^k \cong \mathbf{A}^n$ *imply* $Y \cong \mathbf{A}^{n-k}$*?*

Embedding Problem. *Is every closed embedding* $\iota\colon \mathbf{A}^k \hookrightarrow \mathbf{A}^n$ *equivalent to the standard embedding*

$$e\colon (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0),$$

that is, does there exist $\varphi \in \operatorname{Aut} \mathbf{A}^n$ *such that* $e = \varphi \circ \iota$*?*

Automorphism Problem. *What is the automorphism group* $\operatorname{Aut} \mathbf{A}^n$ *of* $\mathbf{A}^n$*?*

Here is perhaps the most famous of them all:

Jacobian Conjecture. *Is every map* $\varphi\colon \mathbf{A}^n \to \mathbf{A}^n$ *with invertible Jacobian an isomorphism?*

For $n = 1$, the Cancellation Problem is trivial, and the rest follow from the following:

Proposition 1. $\operatorname{Aut} \mathbf{A}^1 \cong \mathrm{GA}_1$*, the group of affine transformations.*

Proof. An automorphism of $\mathbf{A}^1$ is an automorphism of $\mathbf{C}[t]$, which must be affine, i.e., of the form $t \mapsto at + b$ for $a \neq 0$ to be surjective. Alternatively, use the fundamental theorem of algebra: the fact that it is bijective implies that there is exactly one zero, but then this zero must be of order 1, for otherwise in a neighborhood of that zero it would not be injective. □

So the first non-trivial case is the case $n = 2$. In this case, the automorphism group is much larger: anything of the form $(x, y) \mapsto (x, y + f(x))$ is an automorphism! And so you need more information about what automorphisms should look like.

The basic result is:

Theorem 2 (Claimed by B. Segre [Seg56], claimed to be fixed by Canals–Lluis [CL70], finally proved by Abhyankar–Moh [AM75] and independently by Suzuki [Suz74])**.** *Consider a closed embedding*

$$\begin{aligned} \mathbf{A}^1 &\hookrightarrow \mathbf{A}^2 \\ t &\longmapsto (p(t), q(t)) \end{aligned}$$

defined by two polynomials $p(t), q(t) \in \mathbf{C}[t]$*. Then, either* $\deg p \mid \deg q$ *or* $\deg q \mid \deg p$*.*

Remark 3. This is true for any field over characteristic zero, but is false (as written here) in characteristic p [vdE04, §4]: take $p(t) = t^{p^2}$ and $q(t) = t^{p(p+1)} + t$.

Remark 4. AM has the distinction of having very many alternative proofs published since it first appeared: we mention [Ric86] in particular.

The power of this theorem is evidenced by the following:

Theorem 5. *If* $n = 2$*, we have the following diagram of implications:*

$$\begin{array}{ccc} & \text{AM} & \\ \swarrow & \Downarrow & \searrow \\ \text{EP} & \text{AP} & \text{JC}^\dagger \\ & \Downarrow & \\ & \text{CP} & \end{array}$$

Remark 6. $\text{JC}^\dagger$ is a weak form of the Jacobian conjecture, that assumes that $\varphi \in \operatorname{End} \mathbf{A}^2$ is also *injective on a line* [Gwo93]. Miyanishi in [Miy73] claimed that a positive answer to the Cancellation Problem (plus the Serre Conjecture, which is now the Quillen–Suslin theorem) would imply the Jacobian Conjecture. Van den Esssen in [vdE04] pointed out that there is a flaw in the argument.

We give some indication to part of the proof:

Proof that AM $\Rightarrow$ EP*.* We want to show $t \mapsto (p(t), q(t))$ is equivalent to $t \mapsto (t, 0)$. We proceed by induction on $d \coloneqq \deg p + \deg q$. If $d = 1$, then composing by an affine transformation gives the standard embedding. Otherwise, by the Abhyankar–Moh theorem, we have that $\deg p \mid \deg q$, say, and so after a dilation we have

$$p(t) = t^n + \cdots \quad \text{and} \quad q(t) = t^{kn} + \cdots .$$

The transformation $(x, y) \mapsto (x, y - x^k)$ then reduces the degree d, so we are done by induction. ☐

How do we think about the Embedding Theorem? It says that an embedding of a complex line can be "unknotted" somehow until it actually becomes a line; in fact, it shows that if Γ is the image of $\mathbf{A}^1$, then the pairs $(\mathbf{A}^2, \Gamma)$ and $(\mathbf{A}^2, \mathbf{A}^1 \times \{0\})$ are diffeomorphic. Rudolph's realization in [Rud82] was that this can be made precise using knot theory, and that this insight can be used to prove the Abhyankar–Moh theorem.

2 Knot Theory and Proof

To set up the proof, we give a short summary of results from knot theory that we will need. First of all, recall

Definition 7. A *knot* K is a smooth embedding $S^1 \to S^3$, or the image thereof.

Example 8. The unknot O.

Example 9. The trefoil $O\{2, 3\}$, and general torus knots $O\{m, n\}$. You can form iterated torus knots $O\{m_1, n_1; m_2, n_2\}$ by taking a torus knot $O\{m_1, n_1\}$, splitting up each strand into m_2 strands, and making those rotate n_2 times each time you go around the torus.

Example 10 (Knots at infinity)**.** Let B_R denote the polydisc of radius R. Then,

$$\partial B_R = \{|x| = R, |y| \leq R\} \cup \{|x| \leq R, |y| = R\} \simeq S^3$$

Now let Γ be the image of a morphism $\mathbf{A}^1 \to \mathbf{A}^2$. We claim

$$K_R \coloneqq \Gamma \cap \partial B_R$$

for $R \gg 0$ is a knot! To see this, let's do an example: consider the curve

$$\begin{cases} x(t) = t^2 \\ y(t) = t^3 \end{cases}$$

Then, the intersection with the first component of ∂B_R is empty, and so we are interested in the region

$$\{|x| \leq R, |y| = R\}.$$

Parametrize $y = Re^{i\theta}$. We obtain $t = \zeta_3^k R^{1/3} e^{1/3\, i\theta}$, and so the knot K_R is given by

$$\begin{cases} x(\theta) = \zeta_3^k R^{2/3} e^{2/3\, i\theta} \\ y(\theta) = Re^{i\theta} \end{cases}$$

We can imagine this living on a torus $S^1 \times S^1$, where the "large" circle is the x coordinate.

1. There are three possible x values for a given θ, corresponding to the three roots of unity in x;
2. When θ goes from 0 to π, the argument for x goes from 0 to $2\pi/3$, that is you go a third of the way around the big circle.

This is a trefoil knot $O\{2, 3\}$!

Example 11 (Iterated torus knots)**.** Consider $\Gamma = \{(t^4 + t^3, t^6\}$. Then, the leading terms are (t^4, t^6), and so the parametrization is "almost" a double cover of the trefoil knot. Like before, we have

$$K_R = \{|t^4 + t^3| \leq R, |t^6| = R\},$$

This implies $t = \zeta_6^k R^{1/6} e^{i\theta}$, and so

$$\begin{cases} x(\theta) = \zeta_3^{2k} R^{2/3} e^{4i\theta} + \zeta_2^k R^{1/2} e^{3i\theta} \\ y(\theta) = Re^{6i\theta} \end{cases}$$

Note that this makes sense: $|t^4 + t^3| \leq R^{2/3} + R^{1/2} \leq 2R^{2/3} \leq R$ for $R \gg 0$. Now, the dominant term in x is the $R^{2/3}$ term. Ignoring the other term, you just have $O\{2, 3\}$ from before. The second term in x is saying that for each θ value, there are two more "satellite" strands that rotate around $O\{2, 3\}$ at 1/2 the speed of the large rotation. So we get an iterated torus knot $O\{2, 3; 2, 2\}$.

Note 12. In the presentation $O\{2, 3; 2, 2\}$, we saw that the first two terms m_1, n_1 are $m/\gcd(m, n)$ and $n/\gcd(m, n)$. This is true in general!

We can actually say that in general, you get a well-defined isomorphism class of a knot for $R \gg 0$.

Proposition 13. *Let Γ have a parametrization*

$$x(t) = t^m + \cdots \quad \textit{and} \quad y(t) = t^n + \cdots,$$

where $m < n$. Then, for $R \gg 0$,

1. *The map $K_R \coloneqq \Gamma \cap \partial B_R \to \mathbf{R}$ is smooth on (R, ∞), hence has constant diffeomorphism type (Sard's theorem, then Ehresmann's theorem);*
2. *$y(t)$ has an nth root s, and so*

$$x = r(s) = s^m + c_{m-1}s^{m-1} + \cdots + c_0 + c_{-1}s^{-1} + \cdots \quad \textit{and} \quad y = s^n,$$

where $r(s)$ is a Laurent series going infinitely in the negative direction (complex analysis); and

3. *K_R is an iterated torus knot of type $O\{m_1, n_1; m_2, n_2; \cdots; m_k, n_k\}$ for some sequence of m_i, n_i, where $m_1 = m/\gcd(m,n)$ and $n_1 = n/\gcd(m,n)$.*

Sketch of 3. The point is that first, you can work as in the iterated torus knot example, and have $m_1 = m/\gcd(m,n)$ and $n_1 = n/\gcd(m,n)$. More terms corresponds to more strands splitting; after a while, though, the roots of unity stop appearing, and so you don't get any more strands splitting up. □

If this looks familiar to you, it is similar to how you can use Puiseux series to study isolated singularities, and use this to resolve curve singularities.

We can now use this to prove the Abhyankar–Moh theorem, following [Rud82].

Facts 14. We blackbox some basic facts about the Alexander polynomial $\Delta_K(t)$, which is a knot invariant:

1. For $O\{m,n\}$,
$$\Delta_{O\{m,n\}}(t) = \frac{(t^{mn}-1)(t-1)}{(t^m-1)(t^n-1)}$$
2. For $K\{m,n\}$,
$$\Delta_{K\{m,n\}}(t) = \Delta_K(t^n)\Delta_{O\{m,n\}}(t)$$
3. If K bounds a disc, then $\Delta_K(t) = F(t)F(t^{-1})$.

Corollary 15. *The only way $O\{m_1, n_1; \cdots\}$ can be the unknot is if m_1 or n_1 equals 1.*

Proof. Use the facts about the Alexander polynomial above. □

Theorem 16. *Let $p = t^m + \cdots, q = t^n + \cdots$ parametrize a curve, such that $m < n$. Then, the degree m of p divides the degree n of q.*

Proof Sketch of AM. Since the map $t \mapsto (p(t), q(t))$ is an embedding, it is in particular a homeomorphism onto its image, so the image K_R of the circle in $\mathbf{A}^1$ bounds a disc in $\mathbf{A}^2$. But the Corollary implies K_R is the unknot, hence one of m_1 or n_1 equals 1. We recall that $m_1 = m/\gcd(m,n)$ and $n_1 = n/\gcd(m,n)$, so this implies that one of m, n divides the other. Since $m < n$, this means $\deg p \mid \deg q$. □

3 Other Applications

Similar considerations about topological types can be used to prove the following:

Theorem 17 (Danielewski)**.** *The two surfaces*
$$X = \{xy - (1-z^2) = 0\} \subset \mathbf{C}^3$$
$$Y = \{x^2y - (1-z^2) = 0\} \subset \mathbf{C}^3$$
are not isomorphic, but $X \times \mathbf{A}^1 \cong Y \times \mathbf{A}^1$.

The proof computes the first homology group $H_1^\infty(\cdot)$ at infinity, defined as
$$H_1^\infty(X) = \varprojlim_{K \subset\subset X} H_1^\infty(X \setminus K) = \mathbf{Z}/2\mathbf{Z}$$
and $\mathbf{Z}/4\mathbf{Z}$ for Y. Showing that $X \times \mathbf{A}^1 \cong Y \times \mathbf{A}^1$ uses a variant of Hilbert 90 for vector bundles, by realizing X and Y as two affine bundles over the line V with two origins and looking at
$$\begin{array}{ccc} X \times_V Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & V \end{array}$$
X and Y are affine, and $X \times_V Y$ is an affine bundle over both, hence must be $X \times \mathbf{A}^1$ and $Y \times \mathbf{A}^1$.

References

[AM75] S. S. Abhyankar and T. T. Moh. "Embeddings of the line in the plane." *J. Reine Angew. Math.* 276 (1975), pp. 148–166. ISSN: 0075-4102. MR: 0379502.

[CL70] I. Canals and E. Lluis. "On a result of Segre." *An. Inst. Mat. Univ. Nac. Autónoma México* 10 (1970), pp. 1–15. MR: 0286794.

[Gwo93] J. Gwoździewicz. "Injectivity on one line." *Bull. Soc. Sci. Lett. Łódź Sér. Rech. Déform.* 15.1-10 (1993), pp. 59–60. ISSN: 0459-6854. MR: 1280980.

[Kra96] H. Kraft. "Challenging problems on affine n-space." *Astérisque* 237 (1996). Séminaire Bourbaki, Vol. 1994/95, Exp. No. 802, 5, 295–317. ISSN: 0303-1179. MR: 1423629.

[Miy73] M. Miyanishi. "Some remarks on polynomial rings." *Osaka J. Math.* 10 (1973), pp. 617–624. ISSN: 0030-6126. MR: 0337957.

[Ric86] D. R. Richman. "On the computation of minimal polynomials." *J. Algebra* 103.1 (1986), pp. 1–17. ISSN: 0021-8693. DOI: 10.1016/0021-8693(86)90166-3. MR: 860686.

[Rud82] L. Rudolph. "Embeddings of the line in the plane." *J. Reine Angew. Math.* 337 (1982), pp. 113–118. ISSN: 0075-4102. DOI: 10.1515/crll.1982.337.113. MR: 676044.

[Seg56] B. Segre. "Corrispondenze di Möbius e trasformazioni cremoniane intere." *Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat.* 91 (1956/1957), pp. 3–19. ISSN: 0001-4419. MR: 0095175.

[Suz74] M. Suzuki. "Propriétés topologiques des polynômes de deux variables complexes, et automorphismes algébriques de l'espace $\mathbf{C}^2$." *J. Math. Soc. Japan* 26 (1974), pp. 241–257. ISSN: 0025-5645. MR: 0338423.

[vdE04] A. van den Essen. "Around the Abhyankar-Moh theorem." In: *Algebra, arithmetic and geometry with applications (West Lafayette, IN, 2000).* Berlin: Springer, 2004, pp. 283–294. ISBN: 3-540-00475-0. MR: 2037095.