Math 731: Topics in Algebraic Geometry I Berkovich Spaces

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Course Description

Berkovich spaces are analogues of complex manifolds that appear when replacing complex numbers by the elements of a general valued field, e.g. *p*-adic numbers or formal Laurent series. They were introduced in the late 1980s by Vladimir Berkovich as a more honestly geometric alternative to the rigid spaces earlier conceived by Tate. In recent years, Berkovich spaces have seen a large and growing range of applications to complex analysis, tropical geometry, complex and arithmetic dynamics, the local Langlands program, Arakelov geometry, ...

The first part of the course will be devoted to the basic theory of Berkovich spaces (affinoids, gluing, analytifications). In the second part, we will discuss various applications or specialized topics, partly depending on the interest of the audience.

Contents

\mathbf{Sep}	tember 6	5
1.1	Introduction	5
1.2	Valued fields	5
1.3	What is a space?	7
1.4	The Berkovich affine space (as a topological space)	7
	1.4.1 The Berkovich affine line	8
\mathbf{Sep}	tember 8	9
2.1	Banach rings [Ber90, §1.1]	9
	2.1.1 Seminormed groups	9
	2.1.2 Seminormed rings	10
	2.1.3 Spectral radius and uniformization	11
	2.1.4 Seminormed modules	12
	2.1.5 Complete tensor products	12
2.2	The spectrum [Ber90, $\S1.2$]	12
Sep	tember 13	14
3.1^{-1}	The spectrum (continued) $[Ber90, \S1.2]$	14
3.2	Complex Banach algebras [Jon16, App. G]	15
3.3	The Gelfand transform [Ber90, pp. 13–15]	16
3.4	Properties of the spectrum [Ber90, §1.3]	17
	3.4.1 Fibers [Jon16, pp. 13–14]	18
	3.4.2 Ground field extension [Jon16, pp. 14–17]	18
	Sep 1.1 1.2 1.3 1.4 Sep 2.1 2.2 Sep 3.1 3.2 3.3 3.4	September 6 1.1 Introduction 1.2 Valued fields 1.3 What is a space? 1.4 The Berkovich affine space (as a topological space) 1.4.1 The Berkovich affine line September 8 2.1 Banach rings [Ber90, §1.1] 2.1.1 Seminormed groups 2.1.2 Seminormed rings 2.1.3 Spectral radius and uniformization 2.1.4 Seminormed modules 2.1.5 Complete tensor products 2.1.6 Complete tensor products 2.1 The spectrum [Ber90, §1.2] September 13 Salin 3.1 The spectrum (continued) [Ber90, §1.2] 3.2 Complex Banach algebras [Jon16, App. G] 3.3 The Gelfand transform [Ber90, p13–15] 3.4 Properties of the spectrum [Ber90, §1.3] 3.4.1 Fibers [Jon16, pp. 13–14] 3.4.2 Ground field extension [Jon16, pp. 14–17]

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4	September 15 19 $4.1 \mathcal{M}(\mathbf{Z}) \text{ [Ber90, Ex. 1.4.1]} \dots \dots$
5	September 20 23 5.1 The Berkovich closed disc (continued) [Ber90, Ex. 1.4.4] 23 5.1.1 The non-trivially valued, algebraically closed case [BR10, Ch. 1] 24 5.1.2 The tree structure on the Berkovich closed disc [BR10, §1.4] 27 5.1.3 Invariants of non-Archimedean fields [Jon15, §3.1.3] 28 5.1.4 Invariants of points [Ber90, p. 19] 28
6	September 22 29 6.1 The Berkovich closed disc (continued) [Ber90, Ex. 1.4.4] 29 6.1.1 Invariants of points (continued) [Ber90, p. 19] 29 6.1.2 The non-trivially valued, non-algebraically closed case [Ber90, §4.2] 30 6.2 Potential theory [BR10] 32 6.3 Further properties of the Berkovich disc [BR10, §§1.4–1.5] 33 6.3.1 Metrizability/embedding into affine space [BR10, §1.5] 33
7	September 27 33 7.1 Affinoid algebras [Ber90, §2.1; Ber] 34 7.1.1 Properties of the Tate algebra [Ber, §§2.1,2.3] 36
8	September 29388.1Affinoid algebras (continued) [Ber90, $\S2.1$; Ber]388.1.1Properties of Noetherian Banach k-algebras [Ber, $\S§2.3, 2.5$]388.1.2Properties of strictly k-affinoid algebras [Ber, $\S§2.4-2.5$]398.1.3Constructions on strictly k-affinoid algebras [Ber, $\S2.5$]408.1.4The spectral radius [Ber, $\S§2.6-2.7$]41
9	October 4429.1Affinoid algebras (continued) [Ber90, §2.1; Ber]429.1.1The spectral radius [Ber, §§2.6–2.7]429.1.2Properties of k-affinoid algebras [Ber, §2.7]43
10	October 6 45 10.1 Affinoid algebras (continued) [Ber90, §2.1; Ber] 45 10.1.1 Properties of k-affinoid algebras (continued) [Ber, §2.7] 45 10.1.2 Finite modules and algebras over k-affinoid algebras; finite maps [Ber, §2.7] 47 10.2 Weierstrass, Laurent, and rational domains [Ber90, §2.2] 48
11	October 11 49 11.1 The category of k-affinoid spaces [Tem05, §3] 49 11.2 Weierstrass, Laurent, and rational domains (continued) [Ber90, §2.2] 50 11.3 The Tate acyclicity theorem for rational domains [BGR84, Ch. 8] 51 11.4 Affinoid domains [Tem05, §3] 52 11.5 The Gerritzen–Grauert theorem [Tem05, §3] 53
12	October 13 54 12.1 Affinoid domains (continued) [Tem05, §3] 54

13 October 25 13.1 Affinoid domains (continued) [Tem05, §3] 13.2 13.2 Strictly affinoid domains [Tem05, §3] 13.2.1 Ground field extension 13.2.1 Ground field extension 13.3 G-topologies [BGR84, §9.1] 13.3.1 The weak G-topology [Ber90, p. 30] 13.3.2 The "special" G-topology [Ber90, p. 30]	56 57 58 58 59 59
14 October 27 14.1 The structure sheaf on a k-affinoid space [Ber90, §2.3]	60 60 61 63
15 November 1 15.1 Noetherianity of local rings of affinoid spaces [Ber93, §2.1] 15.1.1 Proof of Noetherianity in the strict case [BGR84, 7.3.2] 15.1.2 Proof of Noetherianity in the general case [BGR84, 7.3.2]	64 65 65 67
16 November 3 16.1 The Zariski topology on k-affinoid spaces [Ber93, p. 28]16.2 Further properties of local rings [Ber93, $\S2.2$]16.2.1 Proof of Theorem 16.9 for P = Reg	68 69 69 70
17 November 8 17.1 The reduction map [Ber90, §2.4] 17.1.1 Reduction for strictly k-affinoid algebras [BGR84, §6.3] 17.1.2 The reduction map for strictly k-affinoid spaces	71 71 73 73
18 November 10 18.1 Coherent sheaves [Ber90, §2.3] 10 18.2 The relative interior and boundary [Ber90, §2.5] 10 10	75 76 77
19 November 15 19.1 Global Berkovich spaces [Ber93, §1] 19.1.1 Quasinets and nets [Ber93, §1.1] 19.1.2 k-analytic spaces and morphisms of k-analytic spaces [Ber93, §1.2] 19.1.3 Analytic domains [Ber93, §1.3] 19.1.4 The structure sheaf on a k-analytic space [Ber93, §1.3] 19.1.5 Further constructions/objects [Ber93, §§1.3]	79 79 80 80 82 83 83
20 November 17 20.1 Global Berkovich spaces [Ber93, §1] 20.1 20.1.1 Gluing [Ber93, Prop. 1.3.3] 20.1.2 20.1.2 Futher constructions [Ber93, §1.4] 20.1.2 20.1.2 Futher constructions [Ber90, §3.4] 20.1.2 20.1.2 Futher constructions [Ber90, §3.2] 20.1.2 20.1.2 Futher constructions [Ber90, §3.2] 20.1.2 20.1.2 Futher constructions [Ber90, §3.2] 20.1.2 Futher co	83 83 84 85 85 86
21 November 22 21.1 Coherent modules [Ber93, pp. 25–26] 21.2 Finite morphisms and closed immersions [Ber93, pp. 27–28] 21.2 Finite morphisms [Ber93, pp. 30–31] 21.4 The relative interior and proper morphisms [Ber90, p. 50; Ber93, p. 34] 21.4 The relative interior and proper morphisms [Ber90, p. 50; Ber93, p. 34] 21.4 The relative interior and proper morphisms [Ber90, p. 50; Ber93, p. 34] 21.4 The relative interior and proper morphisms [Ber90, p. 50; Ber93, p. 34] 21.4 The relative interior and proper morphisms [Ber90, p. 50; Ber93, p. 34] 21.4 The relative interior and proper morphisms [Ber90, p. 50; Ber93, p. 34] 21.4 The relative interior and proper morphisms [Ber90, p. 50; Ber93, p. 34] 21.4 The relative interior and proper morphisms [Ber90, p. 50; Ber93, p. 34] 21.4 The relative interior and proper morphisms [Ber90, p. 50; Ber93, p. 34] 21.4 The relative interior and proper morphisms [Ber90, p. 50; Ber93, p. 34] 21.4 The relative interior and proper morphisms [Ber90, p. 50; Ber93, p. 34] 21.4 The relative interior and proper morphisms [Ber90, p. 50; Ber93, p. 34] 21.4 The relative interior and proper morphisms [Ber90, p. 50; Ber93, p. 34] 21.4 The relative interior and proper morphisms [Ber90, p. 50; Ber93, p. 34] 21.4 The relative interior and proper morphisms [Ber90, p. 50; Ber93, p. 34] 21.4 The relative interior and proper morphisms [Ber90, p. 50; Ber93, p. 34] 21.4 The relative interior [Ber90, p. 50; Ber93, p. 34] 21.4 The relative interior [Ber90, p. 50; Ber93, p. 34] 21.4 The relative interior [Ber90, p. 50; Ber93, p. 34] 21.4 The relative interior [Ber90, p. 50; Ber93, p. 34] 21.4 The relative [Ber90, p. 34]	87 88 88 89 90

22 November 29		
22.1 The analytification functor [Ber90, §§3.4–3.5; Ber93, §2.6]	91	
22.1.1 Further properties	92	
22.1.2 GAGA (Géométrie algébrique et géométrie analytique)	92	
22.2 Analysis of Berkovich spaces via models	94	
23 December 1	94	
23.1 Berkovich spaces through models [BFJ16, §2]	94	
23.1.1 Divisorial points	95	
24 December 8	97	
24.1 Berkovich spaces through models	97	
24.1.1 Functions on X^{an}	98	
24.1.2 Dual complexes and the Berkovich space	99	
25 December 13 1	00	
25.1 Graded reduction and the Riemann–Zariski spaces	.00	
25.2 Berkovich spaces through models	.00	
25.2.1 Further results	01	
References 1		

1 September 6

Office Hours: Tuesday/Thursday 2-3:30, 3076 East Hall.

To really learn from a class like this, it is important to do some homework problems. They are usually grouped so there are basic problems, then some harder problems.

The main reference is [Ber90], which is a red book, and [BGR84], a yellow book. The course website is at:

http://math.lsa.umich.edu/~mattiasj/731

There are other useful references, including lecture notes by Brian Conrad [Con08] that introduce different approaches to non-archimedean geometry, even beyond Berkovich spaces. See the webpage for more references.

You can think of non-archimedean analysis (the subject of [BGR84]) as the analogue of commutative algebra for algebraic geometry: it is the main set of tools used in the study of Berkovich spaces (and non-archimedean geometry in general). We will take most of this material for granted; some will be assigned as exercises.

1.1 Introduction

Today we want to give an overview of the course. The main idea is that we want to have an analogue of complex geometry and complex manifolds for non-archimedean field.

Recall 1.1. A complex manifold is a topological space with an open covering by sets U_{α} which are biholomorphic (i.e., homeomorphic, with analytic maps in both directions) with open subsets of \mathbb{C}^n . Here, analytic means $\overline{\partial} = 0$ (the Cauchy–Riemann equations), or that the functions are given by power series. You can then construct a structure sheaf which gives a notion of analytic functions on X.



Figure 1.2: A complex manifold.

We will try to generalize this to other fields. The fields we will consider will come with a valuation.

1.2 Valued fields

Definition 1.3. A valued field is a pair $(k, |\cdot|)$ where k is a field, and $|\cdot|: k \to \mathbb{R}_{\geq 0}$ is a norm or valuation (or absolute value, or...), that is, a function that satisfies:

- |a| = 0 if and only if a = 0;
- $|a b| \le |a| + |b|;$
- $|ab| = |a| \cdot |b|$, that is, $|\cdot|$ is multiplicative (this implies |1| = 1).

Definition 1.4. $(k, |\cdot|)$ is complete if k is a complete metric space under $|\cdot|$; if $(k, |\cdot|)$ is an arbitrary valued field, then you can define the completion $(\hat{k}, |\cdot|)$ by using the standard Cauchy sequence construction.

We give some examples of valued fields:

Example 1.5. Let **C** be the complex numbers, and let $|\cdot|_{\infty}$ be the standard Euclidean norm, that is, $|a + ib|_{\infty} = \sqrt{a^2 + b^2}$. Then, $(\mathbf{C}, |\cdot|_{\infty})$ is a valued field. You can replace **C** with subfields such as **R** or **Q** to get other valued fields. The completion $(\widehat{\mathbf{Q}}, |\cdot|_{\infty})$ of $(\mathbf{Q}, |\cdot|_{\infty})$ is $(\mathbf{R}, |\cdot|_{\infty})$, pretty much by definition.

Example 1.6. Let $k \subset \mathbf{C}$ be any subfield of the complex numbers, and let $|\cdot| = |\cdot|_{\infty}^{\epsilon}$ where $0 < \epsilon \leq 1$. Then, $(k, |\cdot|)$ is a valued field.

Example 1.7. Let k be any field, and consider the trivial valuation $|\cdot|_0$, defined by $|a|_0 = 1$ for all $a \in k^* \coloneqq k \setminus \{0\}$. Then, $(k, |\cdot|_0)$ is a valued field.

The reason why people started looking at valuations is because of the p-adic numbers. We won't say too much about it now, but it's an important example:

Example 1.8 (*p*-adic numbers). Fix p a prime number. Then, any $r \in \mathbf{Q}$ can be written $r = p^n \cdot \frac{a}{b}$, where $p \nmid a, b$, and so we can define a valuation $|\cdot|_p$ by $|r|_p = c^n$ where $c \in (0, 1)$ is a fixed number (e.g., $c = \frac{1}{p}$). This is a norm on \mathbf{Q} , but is not complete. The completion of \mathbf{Q} with respect to the norm $|\cdot|_p$ is \mathbf{Q}_p , the *p*-adic numbers.

Here is another example which is arguably more geometric, as opposed to arithmetic:

Example 1.9 (Laurent series). Let \tilde{k} be any field, and consider

$$k = \tilde{k}((T)) = \left\{ f \in \sum_{-\infty}^{\infty} a_i T^i \mid \begin{array}{l} a_i \in \tilde{k} \\ a_i = 0 \text{ for } i \ll 0 \end{array} \right\}.$$

Fix $c \in (0, 1)$, e.g., c = 1/e. Then, we can define the order of a Laurent series by $\operatorname{ord}_0(f) \coloneqq \min\{i \mid a_i \neq 0\}$, which defines a norm $|f| \coloneqq c^{\operatorname{ord}_0(f)}$. The valued field $(k, |\cdot|)$ is complete. We note \tilde{k} is notated in this way since it is an example of a residue field.

Definition 1.10. $(k, |\cdot|)$ is non-archimedean (NA) if $|a + b| \le \max\{|a|, |b|\}$ for all $a, b \in k$; a valued field or norm that is not non-archimedean is called archimedean.

We will not prove the following:

Theorem 1.11. If $(k, |\cdot|)$ is archimedean and complete, then $k = \mathbf{R}$ or \mathbf{C} , and $|\cdot| = |\cdot|_{\infty}^{\epsilon}$ for some $0 < \epsilon \leq 1$.

The following nice proposition explains why non-archimedean fields are called as such:

Proposition 1.12. $(k, |\cdot|)$ is non-archimedean if and only if $|n| \leq 1$ for all $n \in \mathbb{Z}$.

So non-archimedean fields violate the archimedean principle.

Proof. " \Rightarrow " clear: $|n| = |1 + 1 + \dots + 1| \leq \dots \leq 1$. " \Leftarrow " Pick $a, b \in k$. Then, by the binomial theorem and the triangle inequality,

$$|a+b| = |(a+b)^n|^{1/n} = \left|\sum_{0}^n \binom{n}{i} a^n b^{n-i}\right|^{1/n} \le \left(\sum_{0}^n \underbrace{\left|\binom{n}{i}\right|}_{\le 1} |a|^i |b|^{n-i}\right)^{1/n}$$

where the bound on norms of binomial coefficients is by assumption. Bounding the right-hand side above, we obtain

$$|a+b| \le \left((n+1) \max\{|a|, |b|\}^n \right)^{1/n} = (n+1)^{1/n} \max\{|a|, |b|\}$$

Taking $n \to \infty$ and using a bit of calculus, we see that $|a + b| \le \max\{|a|, |b|\}$, i.e., $|\cdot|$ is non-archimedean. \Box Note that in the proof above, we didn't use all of the axioms for norms.

1.3 What is a space?

We note some more properties of non-archimedean fields that distinguish them from archimedean fields like \mathbf{R} or \mathbf{C} , which we are more used to:

Exercise 1.13. If $(k, |\cdot|)$ is NA, then

- $|a| > |b| \implies |a+b| = |a|;$
- balls in k are both open and closed;
- k is totally disconnected.

These present issues when we try to define a non-archimedean analogue for complex manifolds:

Naïve "Definition" 1.14. Let $(k, |\cdot|)$ be a non-archimedean field. Then, a k-manifold X of dimension n is a topological space where

- analytic functions on $U \subseteq k^n$ are given locally by convergent power series (in analogy to analytic functions); and
- locally, X looks like pieces of k^n with bianalytic homeomorphisms (in analogy to holomorphic charts).

Problems 1.15. There are two problems that we face with the previous naïve definition:

- 1. k^n is totally disconnected, so X is totally disconnected (you would expect a nice notion of a space to have a nice topology);
- 2. there are too many analytic functions (every point can have a different value, without any condition on compatibility), so there is no meaningful notion of analytic continuation.

We therefore need to define things a bit differently.

We list some general properties a nice notion of a space should satisfy. A space X should consist of (i) a set X;

- (ii) a topology on X; and
- (*iii*) a structure sheaf \mathcal{O}_X on X, where sections $\mathcal{O}_X(U)$ on an open set $U \subseteq X$ consist of analytic functions on U.

There are various approaches to non-archimedean geometry to actually do this (we will only focus on one). For an overview of these different approaches, see [Con08].

- 1. J. Tate (1960s): rigid spaces. These do not have a topology; it is replaced by a Grothendieck topology.
- 2. M. Raynaud (1970s): formal models of rigid spaces.
- 3. V. Berkovich (1980s): k-analytic spaces (Berkovich spaces) (these are very good topologically).
- 4. R. Huber (1980s): *adic spaces* (Bhargav will talk about perfectoid spaces next semester, which involves some of this).

We have to start from scratch at some point, but we first consider a very concrete example.

1.4 The Berkovich affine space (as a topological space)

Let $(k, |\cdot|)$ be a complete valued field, and consider $R = k[T_1, \ldots, T_n]$, the polynomial ring in n variables. In algebraic geometry, you can cook up spaces based on R: you can take

 $Max R = \{maximal ideals of R\} \quad (\cong k^n \text{ if } k \text{ is algebraically closed})$

or if you are a more sophisticated algebraic geometer:

Spec $R = \{ \text{prime ideals of } R \}$ (= the algebro-geometric version of affine space $\mathbf{A}_k^n \}$)

Both are endowed with the Zariski topology.

The Berkovich version is as follows:

Definition 1.16. The Berkovish affine space over k, denoted $\mathbf{A}_{k}^{n,\mathrm{an}}$, is defined (as a set) as

$$\mathbf{A}_{k}^{n,\mathrm{an}} \coloneqq \left\{ \begin{array}{ll} \text{multiplicative seminorms on } R\\ \text{extending the norm on } k \end{array} \right\} = \left\{ |\cdot|_{x} \colon R \to \mathbf{R}_{\geq 0} \left| \begin{array}{c} |f + g|_{x} \le |f|_{x} + |g|_{x} & \forall f, g \in R\\ |fg|_{x} = |f|_{x}|g|_{x} & \\ |a|_{x} = |a| & \forall a \in k \end{array} \right\}$$

Note that above, the superscript an stands for "analytification," and that the definition for a seminorm does not require that $|f|_x = 0$ only if f = 0, which is required in the definition for a norm.

The topology on $\mathbf{A}_k^{n,\mathrm{an}}$ is the weakest one such that

$$\mathbf{A}_{k}^{n,\mathrm{an}} \longrightarrow \mathbf{R}_{\geq 0}$$
$$|\cdot|_{x} \longmapsto |f|_{x}$$

is continuous for all $f \in R$.

We will prove the following facts later, but they show that $\mathbf{A}_{k}^{n,\mathrm{an}}$ is in fact a very nice space:

Fact 1.17. $\mathbf{A}_{k}^{n,\mathrm{an}}$ is Hausdorff, locally compact, and path connected.

Lemma 1.18. Max $R \hookrightarrow \mathbf{A}_k^{n, \mathrm{an}}$ (as sets).

Sketch. For simplicity, assume k is algebraically closed. Then, $\operatorname{Max} R \cong k^n$, and given $a \in k^n$, we can associate a multiplicative seminorm $|\cdot|_a \in \mathbf{A}_k^{n,\operatorname{an}}$ by setting $|f|_a \coloneqq |f(a)|$, where we think of f as a function $k^n \to k$, for all $f \in R$.

So we don't yet know all the points of the Berkovich space, but at least we know all maximal ideals of R sit in $\mathbf{A}_k^{n,\mathrm{an}}$. You can still make sense of this even if k is not algebraically closed.

One satisfying feature of Berkovich's approach is the following:

Fact 1.19. If $(k, |\cdot|) = (\mathbf{C}, |\cdot|_{\infty})$, then $\mathbf{A}_{\mathbf{C}}^{n, \text{an}} \cong \mathbf{C}^n$. More precisely, the map in Lemma 1.18 is surjective, and is a homeomorphism which induces isomorphisms on structure sheaves.

Proof. Use the Gelfand–Mazur Theorem 3.12 (from complex Banach algebras).

1.4.1 The Berkovich affine line

We now specialize further to the case when n = 1, that is, the case of the Berkovich affine line. It is possible to classify all points in $\mathbf{A}_{k}^{1,\mathrm{an}}$, which we will later; here, we will content ourselves with a very special case. Assume that

- k is algebraically closed; and
- $|\cdot| = |\cdot|_0$ is the trivial norm.

We claim we can classify points in $\mathbf{A}_{k}^{1,\text{an}}$ into three cases, which can be put together to form the sketch of $\mathbf{A}_{k}^{1,\text{an}}$ in Figure 1.20:



Figure 1.20: A picture of $\mathbf{A}_{k}^{1,\mathrm{an}}$.

Fix $x \in \mathbf{A}_k^{1,\mathrm{an}}$, corresponding to a multiplicative seminorm $|\cdot|_x$ on k[T], trivial on k (which implies it is NA by Proposition 1.12). Now the fact that k is algebraically closed and that $|\cdot|_x$ is multiplicative implies x is determined by $|T - a|_x$ for $a \in k$. There are three possibilities:

Case 1. $|\cdot|_x$ is the trivial norm on k[T], that is, $|f|_x = 1$ for all $f \neq 0$.

Case 2. $|T|_x > 1$, so $r \coloneqq |T|_x \in (1, \infty)$. Then $|T - a|_x = r$ (since $|T|_x > 1 \ge |a|_x$) for all $a \in k$. Thus, x is uniquely determined by r.

Case 3. There exists $a \in k$ such that $|T - a|_x < 1$. Set $r := |T - a|_x \in (0, 1)$, then $|T - b|_x = 1$ for $b \neq a$ since $|T - b|_x = |(T - a) + (a - b)|_x$, and $|a - b|_x = |a - b| = 1$. Thus, x is uniquely determined by (a, r).

Remark 1.21. $k \hookrightarrow \mathbf{A}_k^{1,\mathrm{an}}$ corresponds to Case 3 with r = 0, depicted at the bottom of Figure 1.20.

The topology can be described as follows: we need $x \mapsto |T - a|_x$ to be continuous for all x. Thus, the sets

 $\{s < |T - a|_x < t\}$

should be open, which means on each "leg" isomorphic to $(1, \infty)$ or [0, 1) in Figure 1.20, you get the usual topology. Any open neighborhood of the point corresponding to Case 1 contains all but finitely many "legs." We will later discuss what the analytic functions are.

2 September 8

Last time, we gave an introduction to what Berkovich spaces are, focusing on the example of $\mathbf{A}_{k}^{n,\mathrm{an}}$. Today, we will start defining them from scratch, following [Ber90]. Mattias recommends reading (parts of) the introduction, which he finds enlightening; he also has annotations for the book [Jon16] on his website.

Let \mathscr{A} be a Banach ring. We want to define a space $X = \mathscr{M}(\mathscr{A})$, the Berkovich spectrum of \mathscr{A} , such that \mathscr{A} is the set of functions on X. This is analogy to situations more familiar to us:

Analogies 2.1.

- 1. If A is a (commutative) ring (with unit), then we can consider X = Spec A, the prime spectrum of A.
- 2. If A is a complex Banach algebra (see §3.2), then we can consider X = Max A, the maximal ideal spectrum of A.

Hypothetically we should go through all of [BGR84] to serve as a foundation for Berkovich spaces, but we will instead just give some of that material as homework problems. This is just like how in algebraic geometry, you need to know some commutative algebra to really learn the subject.

2.1 Banach rings [Ber90, §1.1]

2.1.1 Seminormed groups

We first define the notion of a seminormed group.

Definition 2.2. Let M be an abelian group. A function $\|\cdot\|: M \to \mathbf{R}_+$ is a seminorm on M if

• ||0|| = 0; and

• $||f - g|| \le ||f|| + ||g||;$ and is a non-archimedean (NA) seminorm if also • $||f - g|| \le \max\{||f||, ||g||\}.$

The function $\|\cdot\|$ is a norm if $\|f\| = 0 \iff f = 0$.

Fact 2.3. The norm topology on M is Hausdorff if and only if $\|\cdot\|$ is a norm.

You can do various things with seminorms; by using the standard Cauchy sequence construction, you can construct completions with respect to a seminorm, but these act a bit funny when in the non-Hausdorff setting (i.e., when the seminorm is not a norm). Instead, we introduce the following:

Fact 2.4. There exists a group \widehat{M} and a norm $\|\cdot\|$ such that \widehat{M} is Hausdorff and complete with respect to this norm. We call $(\widehat{M}, \|\cdot\|)$ the separated completion.

Note that Berkovich just calls this the completion.

Example 2.5. If $\|\cdot\| = 0$, then $\widehat{M} = 0$.

Two norms can induce the same topology on M, and so we introduce the following notion to get rid of some of this redundancy.

Definition 2.6. Two norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent if there exists C > 0 such that $C^{-1}\|\cdot\| \le \|\cdot\|' \le C\|\cdot\|$.

If $N \subset M$ is a subgroup, then we get a residue seminorm on M/N:

$$||f|| \coloneqq \inf\{||g|| \mid g \in \pi^{-1}(f)\}$$

where $\pi: M \to M/N$ is the quotient map.

To form a category of seminormed groups, you need a notion of a morphism between seminormed groups. There are two important classes of morphisms:

Definition 2.7. Let $\varphi \colon M \to N$ be a homomorphisms of groups. Suppose M, N are seminormed. Then, φ is bounded if there exists a constant C such that for all $f \in M$, we have $\|\varphi(f)\| \leq C \cdot \|f\|$. We say φ is admissible if

 $(M/\ker\varphi, \text{residue norm}) \cong (\operatorname{im}\varphi, \text{subspace norm}),$

i.e., there exists a constant C > 0 such that

$$C^{-1}\inf\left\{\|f+g\|\mid g\in \ker\varphi\right\} \le \|\varphi(f)\| \le C \cdot \|f\|.$$

2.1.2 Seminormed rings

Now we know what seminormed groups are, we can define seminormed rings, for which the seminorm is compatible with the multiplicative structure on the ring. These are what fill the role of commutative rings in algebraic geometry, but now we have the added data of a seminorm.

Let \mathscr{A} be a ring, and let $\|\cdot\|$ be a seminorm on \mathscr{A} as an additive group. We also assume that $\|\cdot\|$ is submultiplicative, that is, $\|fg\| \leq \|f\| \cdot \|g\|$, which in particular implies that $\|1\| \leq 1$.

Remark 2.8. It is in fact true that either ||f|| = 0 for all $f \in \mathscr{A}$, or ||1|| = 1.

Definition 2.9. $\|\cdot\|$ is power-multiplicative if $\|f^n\| = \|f\|^n$ for all $n \ge 1$, and is multiplicative if $\|fg\| = \|f\|\|g\|$ for all $f, g \in \mathscr{A}$. Usually multiplicative norms are assumed to satisfy $\|1\| = 1$.

Here is an arguably non-standard definition:

Definition 2.10. A valuation on \mathscr{A} is a multiplicative norm.

We are therefore ignoring non-rank 1 valuations (which are important for adic spaces).

Definition 2.11. $(\mathscr{A}, \|\cdot\|)$ is a Banach ring if $\|\cdot\|$ is a complete norm.

Examples 2.12. We give some examples of Banach rings, mostly from [Ber90, Ex. 1.1.1]:

- (i) Any ring \mathscr{A} gives a Banach ring $(\mathscr{A}, |\cdot|_0)$, where $|\cdot|_0$ is the trivial norm.
- (*ii*) The integers **Z** form a Banach ring $(\mathbf{Z}, |\cdot|_{\infty})$, where $|\cdot|_{\infty}$ is the usual Euclidean norm. Note this is archimedean, in contrast to what we ordinarily consider.
- (α) The zero ring 0, together with the only possible norm $|\cdot| = 0$, is a Banach ring. So while we won't be studying the field with one element, we can still consider the ring with one element.
- (β) The real numbers **R** from a Banach ring (**R**, $|\cdot|_{\infty}^{\epsilon}$) for any $0 < \epsilon \leq 1$, where $|\cdot|_{\infty}$ is again the usual Euclidean norm.
- (γ) The *p*-adic numbers \mathbf{Q}_p and the Laurent series $\mathbf{C}((T))$ form Banach rings with norms as defined last time in Examples 1.8–1.9.
- (δ) We can define a different norm on the complex numbers to get a Banach algebra ($\mathbf{C}, \max\{|\cdot|_{\infty}, |\cdot|_0\}$). This is "in between" the archimedean and non-archimedean worlds, and such an object is unique to the Berkovich world.
- (*iii*) If \mathfrak{a} is a closed ideal (e.g., a maximal ideal) of a Banach ring $(\mathscr{A}, \|\cdot\|)$, then $(\mathscr{A}/\mathfrak{a}, \text{residue norm})$ is a Banach ring.

(iv) If $\{(\mathscr{A}_i, \|\cdot\|)\}_{i \in I}$ are Banach rings, then we can consider

$$\prod_{i \in I} \mathscr{A}_i = \{(a_i)_{i \in I}, \ a_i \in \mathscr{A}_i \mid \text{there exists } C > 0 \text{ such that } \|a_i\| \le C \text{ for all } i\}$$

This forms a Banach ring with the " L^{∞} norm": $||(a_i)|| = \sup_i ||a_i||$.

(v) If $(\mathscr{A}, \|\cdot\|)$ is a Banach ring, and r > 0, then

$$\mathscr{A}\langle r^{-1}T\rangle \coloneqq \left\{ f = \sum_{i=0}^{\infty} a_i T^i \ \middle| \ a_i \in \mathscr{A}, \ \|f\| \coloneqq \sum_{i=0}^{\infty} \|a_i\| r^i < \infty \right\}$$

is a Banach ring with the norm ||f|| above. This is some sort of "relative disk." Note that the notation conflicts with that of [BGR84].

Note that in the examples above, β , γ are in fact valuation fields, which we define below, but δ is not since the norm is not multiplicative.

Definition 2.13. A valuation field is a Banach ring $(\mathscr{A}, \|\cdot\|)$ such that \mathscr{A} is a field, and $\|\cdot\|$ is a multiplicative norm. A non-archimedean (NA) field is a valuation field whose norm is non-archimedean: $\|f - g\| \leq \max\{\|f\|, \|g\|\}$.

2.1.3 Spectral radius and uniformization

Since Banach rings can have many norms that are equivalent, we can ask the following:

Question 2.14. Given a Banach ring $(\mathscr{A}, \|\cdot\|)$, does there exist a "nicest" norm $\|\cdot\|$ equivalent to $\|\cdot\|$?

Since we can identify equivalent norms from a topological point of view, the idea is that we want to define a norm that is as close to being multiplicative as possible. We can almost make this happen by constructing the *uniformization* of the original Banach ring.

Let $(\mathscr{A}, \|\cdot\|)$ be a Banach ring, and let $f \in \mathscr{A}$. Then, $n \mapsto \log \|f^n\|$ is subadditive, that is, $a_{m+n} \leq a_m + a_n$. There we have the following cute little result that has probably been discovered many times:

Fekete's Lemma 2.15. The limit $\rho(f) \coloneqq \lim_{n \to \infty} ||f^n||^{1/n}$ exists, and is equal to $\inf_n ||f^n||^{1/n}$.

Definition 2.16. The number $\rho(f)$ is called the spectral radius of f.

This is the radius of the "smallest disk" containing the spectrum of f.

Properties 2.17.

- $\rho(f^n) = \rho(f)^n;$
- $\rho(1) = 1;$
- $\rho(fg) \le \rho(f) \cdot \rho(g)$.

The first property is where the terminology "uniformization" comes from: a complex Banach algebra is *uniform* if its norm is power-multiplicative.

Exercise 2.18. $\rho(f-g) \leq \rho(f) + \rho(g)$. In fact, $\rho(f-g) \leq \max\{\rho(f), \rho(g)\}$ if $\|\cdot\|$ is NA.

This is similar to Proposition 1.12 from last time, but is harder/more annoying.

We can therefore define a new seminorm ρ from the norm $\|\cdot\|$ on \mathscr{A} , and moreover ρ does not change if we replace $\|\cdot\|$ by an equivalent norm. In general, however, $\rho: \mathscr{A} \to \mathbf{R}_{\geq 0}$ is only a seminorm, and not a norm (e.g., \mathscr{A} could have nilpotents).

Definition 2.19. The uniformization \mathscr{A}^u of \mathscr{A} is the separated completion of (\mathscr{A}, ρ) . Then, $\mathscr{A} \to \mathscr{A}^u$ is bounded, and satisfies a universal property: any bounded homomorphism $\mathscr{A} \to \mathscr{B}$ to a uniform ring \mathscr{B} factors through \mathscr{A}^u .

2.1.4 Seminormed modules

In commutative algebra, to study rings you also study modules over them; we do the same for seminormed rings. Let \mathscr{A} be a (semi)normed ring (not necessarily complete).

Definition 2.20. A (semi)normed \mathscr{A} -module is a pair $(M, \|\cdot\|)$, such that

- M is an \mathscr{A} -module;
- $\|\cdot\|$ is a (semi)norm on M as an additive group; and
- there exists a constant C > 0 such that $||fm|| \le C||f|| \cdot ||m||$ for all $f \in \mathcal{A}$, $m \in M$.

Exercise 2.21. We can (and will) replace the (semi)norm $\|\cdot\|$ by an equivalent (semi)norm $\|\cdot\|'$ for which we can take C = 1, that is, we can assume the inequality $\|fm\|' \le \|f\|' \cdot \|m\|'$ holds for all $f \in \mathscr{A}$, $m \in M$.

You can then add on a completeness condition: since a Banach ring is a seminormed ring which is complete, we can similarly define:

Definition 2.22. A Banach \mathscr{A} -module is a complete normed \mathscr{A} -module.

Note that in the special case where \mathscr{A} is a valuation field and M is a vector space, there is more structure, and therefore there are more things you can say.

2.1.5 Complete tensor products

Just as in commutative algebra, we would like to have cartesian products and pushouts in our categories of modules or rings. The correct notion (categorically) turns out to be the *complete tensor product*.

Let \mathscr{A} be a normed ring, which we will assume to be non-archimedean (Berkovich defines the complete tensor product in a more general setting, but the NA case is easier). Let M, N be seminormed NA \mathscr{A} -modules. We can then define the usual tensor product $M \otimes_{\mathscr{A}} N$, which satisfies a universal property in the category of \mathscr{A} -modules. We then define a seminorm on $M \otimes_{\mathscr{A}} N$ as follows:

$$\|v\| \coloneqq \inf\left\{\max_{i} \|m_i\| \cdot \|n_i\| \mid v = \sum m_i \otimes n_i\right\},\tag{1}$$

where the infimum runs over all representations of the form on the right hand side.

Definition 2.23. The complete tensor product $M \otimes_{\mathscr{A}} N$ is the separated completion of $M \otimes_{\mathscr{A}} N$ with respect to the seminorm (1). $M \otimes_{\mathscr{A}} N$ is a Banach \mathscr{A} -module, as well as a Banach $\widehat{\mathscr{A}}$ -module. $M \otimes_{\mathscr{A}} N$ satisfies the universal property (which you can write down in detail):



where P is a Banach \mathscr{A} -module. Note that $M \otimes_{\mathscr{A}} N$ can be much smaller than you expect it to be.

We now have had a "whirlwind tour" of normed commutative algebra. Later on, we'll be in a more specific situation, but it is nice to have a nice, general setup for the theory, even though our main focus will be a special case, just as it is useful to know scheme theory to study varieties in algebraic geometry.

2.2 The spectrum [Ber90, §1.2]

Let $(\mathscr{A}, \|\cdot\|)$ be a Banach ring. We want to cook up a space associated to this, so that \mathscr{A} is a ring of nice functions on the space.

Definition 2.24. The Berkovich spectrum $\mathcal{M}(\mathcal{A})$ is defined (as a set) as the set of bounded (nontrivial) multiplicative seminorms on \mathcal{A} , that is

$$\mathcal{M}(\mathscr{A}) \coloneqq \left\{ \begin{array}{c} \text{multiplicative seminorms} \\ |\cdot| \colon \mathscr{A} \to \mathbf{R}_+ \end{array} \middle| |f| \leq ||f|| \text{ for all } f \in \mathscr{A} \right\},$$

where by "nontrivial" we mean that $|1| = 1.^{1}$

The topology on $\mathscr{M}(\mathscr{A})$ is the weakest one such that the functions

$$\mathcal{M}(\mathscr{A}) \longrightarrow \mathbf{R}_{-}$$
$$|\cdot| \longmapsto |f|$$

are continuous for all $f \in \mathscr{A}$.

Example 2.25. If \mathscr{A} is the zero ring, then $\mathscr{M}(\mathscr{A}) = \emptyset$.

Theorem 2.26 [Ber90, Thm. 1.2.1]. If \mathscr{A} is a nonzero Banach ring, then $\mathscr{M}(\mathscr{A})$ is a nonempty, compact Hausdorff topological space.

We will give examples later. There are not too many more things you can say in general. We will use this Theorem a few times, so it's important that we give a proof.

Proof. We will show nonemptiness last, even though logically it should be proved first.

We first show that $\mathscr{M}(\mathscr{A})$ is Hausdorff. Pick two distinct points $|\cdot|_1$ and $|\cdot|_2$ in $\mathscr{M}(\mathscr{A})$. After possible relabeling, we can assume there exists $f \in \mathscr{A}$ such that $|f|_1 < |f|_2$. Now pick $t \in \mathbf{R}_+$ such that $|f|_1 < t < |f|_2$, and consider the two sets

$$U_1 = \{ |\cdot| \in \mathcal{M}(A) \mid |f| < t \}$$
 and $U_2 = \{ |\cdot| \in \mathcal{M}(A) \mid |f| > t \}.$

By definition these are open in $\mathcal{M}(\mathscr{A})$, but it is also clear that they are disjoint, and that $|\cdot|_i \in U_i$ for i = 1, 2.

We now show that $\mathscr{M}(\mathscr{A})$ is compact (this is slightly different from Berkovich's proof). We can embed

$$\mathcal{M}(\mathscr{A}) \hookrightarrow P \coloneqq \prod_{f \in \mathscr{A}} \left[0, \|f\| \right],$$

where the target factors [0, ||f||] have the subspace topology as intervals in **R**, and *P* has the product topology. Tychonoff's Theorem implies *P* is compact. The map above is in fact a closed embedding, and so $\mathcal{M}(\mathcal{A})$ is compact since the map realizes $\mathcal{M}(\mathcal{A})$ as a closed subset of a compact space.

We finally show that $\mathscr{M}(\mathscr{A})$ is nonempty; note that if the Banach ring has a multiplicative norm, then this would already give a point in $\mathscr{M}(\mathscr{A})$, so the proof is only nontrivial if the norm is not multiplicative.

We first make a series of reductions: we claim we may assume that

- (1) \mathscr{A} is a field;
- (2) The norm $\|\cdot\|$ on \mathscr{A} is minimal, that is, if $\|\cdot\|'$ is a seminorm (not necessarily multiplicative on \mathscr{A}) and $\|f\|' \leq \|f\|$ for all $f \in \mathscr{A}$, then $\|\cdot\|' = \|\cdot\|$;
- (3) The norm on \mathscr{A} is power-multiplicative.

For (1), pick a maximal ideal \mathfrak{m} of \mathscr{A} , using Zorn's Lemma. The ideal \mathfrak{m} is closed in \mathscr{A} , and so we can consider the Banach ring $(\mathscr{A}/\mathfrak{m}, \text{residue norm})$, where now \mathscr{A}/\mathfrak{m} is a field. There is a bounded map $(\mathscr{A}, \|\cdot\|) \to (\mathscr{A}/\mathfrak{m}, \text{residue norm})$, which induces a continuous map $\mathscr{M}(\mathscr{A}/\mathfrak{m}, \text{residue norm}) \to \mathscr{M}(\mathscr{A}, \|\cdot\|)$ by pulling back norms via the quotient map, and so it suffices to prove that $\mathscr{M}(\mathscr{A}/\mathfrak{m}) \neq \emptyset$.

For (2), we use Zorn's Lemma to produce a minimal seminorm $\|\cdot\|$, and take the separated completion with respect to it so that $\|\cdot\|$ is in fact a norm.

For (3), we replace \mathscr{A} with its uniformization \mathscr{A}^u , and note that we have a morphism $\mathscr{M}(\mathscr{A}^u) \to \mathscr{M}(\mathscr{A})$. Note that the proof of [Ber90, Thm. 1.2.1] shows that this reduction is unnecessary: the minimal norm produced in (2) is in fact already power-multiplicative.

$$|f|^n = |f^n| \le C \cdot ||f^n|| \le C \cdot ||f||^n$$

¹Note that categorically speaking, the right condition is that $|f| \leq C \cdot ||f||$ for some constant C > 0. Following [Ber90, p. 12], we show that this condition implies $|f| \leq ||f||$. Since $|\cdot|$ is multiplicative and $||\cdot||$ is submultiplicative, we have

for every integer $n \ge 1$. Taking *n*th roots, we have $|f| \le \sqrt[n]{C} \cdot ||f||$ for every integer $n \ge 1$, which implies $|f| \le ||f||$.

Claim 2.27. $\|\cdot\|$ is a multiplicative norm, so $\|\cdot\| \in \mathcal{M}(\mathcal{A})$.

Exercise 2.28. It suffices to prove $||f^{-1}|| = ||f||^{-1}$ for all $f \in A \setminus \{0\}$.

So suppose $||f^{-1}|| > ||f||^{-1}$. Set $r := ||f^{-1}||^{-1}$, so that ||f|| > r. Consider the ring $A' := A\langle r^{-1}T \rangle$. Then, you can check f - T is not invertible in A' (here, you need to use power-multiplicativity), and consider A'' := A'/(f - T), which is a normed ring. Then, we have homomorphisms $A \to A' \to A''$, and the norm $||\cdot||''$ on A'' induces a seminorm on A by pulling back through this composition. Then, $||\cdot||'' \le ||\cdot||$, and ||f||' = ||T||' = r < ||f||, which is a contradiction.

Next time, we will give more examples of Berkovich spaces and prove more properties about them.

3 September 13

The second homework is already posted. Feel free to ask Mattias questions, in particular you can come to office hours. There are also many, many exercises about normed commutative algebra on the course website.

We will continue with our presentation of the foundations of Berkovich spaces, following [Ber90] closely.

3.1 The spectrum (continued) [Ber90, §1.2]

Let $(\mathscr{A}, \|\cdot\|)$ be a Banach ring, and consider the Berkovich spectrum

$$\mathcal{M}(\mathscr{A}) \coloneqq \left\{ \begin{aligned} \text{multiplicative} \\ \text{seminorms on } \mathscr{A} \end{aligned} \right\} = \left\{ |\cdot| \colon \mathscr{A} \to \mathbf{R}_{\geq 0} \middle| \begin{array}{c} |f - g| \leq |f| + |g| \\ |fg| = |f| \cdot |g| \\ |1| = 1 \\ |f| \leq \|f\| \\ \forall f \in \mathscr{A} \end{array} \right\}$$

The topology on $\mathcal{M}(\mathscr{A})$ is the weakest one such that the map $|\cdot| \to |f|$ is continuous for all $f \in \mathscr{A}$.

We showed the following theorem last time:

Theorem 3.1. If \mathscr{A} is nonzero, then $\mathscr{M}(\mathscr{A})$ is a nonempty compact Hausdorff space.

The hard part of the proof was showing that $\mathcal{M}(\mathcal{A})$ is nonempty. The fact that it is nonempty is useful.

The assignment $\mathscr{A} \to \mathscr{M}(\mathscr{A})$ satisfies a functoriality property that we often used in the proof: if $\varphi \colon \mathscr{A} \to \mathscr{B}$ is a bounded morphism of Banach rings, then there is an induced pullback map

$$\begin{split} \varphi^* \colon \mathscr{M}(\mathscr{B}) &\longrightarrow \mathscr{M}(\mathscr{A}) \\ & |\cdot| \longmapsto \{f \mapsto |\varphi(f)|\} \end{split}$$

which is continuous. This defines a functor $\mathsf{BanRing} \to \mathsf{Top}$.

Exercise 3.2. What properties does this functor have? Is it "good"?

Here is one important example:

Example 3.3. Let \mathscr{A} be a valuation field. It already has a multiplicative norm $\|\cdot\|$. It is in fact the only multiplicative norm: the inequality $|f| \leq \|f\|$ holds by definition, and the inequality $|f^{-1}| \leq \|f^{-1}\|$ together with multiplicativity implies the converse inequality. Thus, $\mathscr{M}(\mathscr{A}) = \{\text{point}\}.$

This is similar to how in algebraic geometry, $\operatorname{Spec}(k)$ is a point for k a field. In this manner, valuation fields play the same role as fields do in algebraic geometry. Later on, we will define structure sheaves on the topological spaces $\mathscr{M}(\mathscr{A})$ so that the global sections on $\mathscr{M}(\mathscr{A})$ for the valuation field \mathscr{A} give back \mathscr{A} .

Definition 3.4. Let \mathscr{A} be a Banach ring. Then, if $x \in \mathscr{M}(\mathscr{A})$ is a point, we denote $|\cdot|_x$ to be the associated seminorm, and we define the ideal

$$\mathfrak{p}_x \coloneqq \ker(|\cdot|_x) = \{ f \in \mathscr{A} \mid |f|_x = 0 \} \subseteq \mathscr{A}.$$

The ideal \mathfrak{p}_x is prime since the norm is multiplicative. We can also define the residue field

$$\mathscr{K}(x) \coloneqq \operatorname{Frac}(\mathscr{A}/\mathfrak{p}_x),$$

which has the induced seminorm $|\cdot|_x$, which is now a norm since all elements with norm zero are now zero. The field $\mathscr{K}(x)$ is not complete in general, and so we define the *complete residue field*

$$\mathscr{H}(x) \coloneqq$$
 the completion of $\mathscr{K}(x)$ with respect to the norm $|\cdot|_x$.

You can then look at the homomorphism

$$\mathcal{A} \longrightarrow \mathcal{A}/\mathfrak{p}_x \longrightarrow \mathcal{K}(x) \longrightarrow \mathcal{H}(x)$$
$$f \longmapsto f(x)$$

and define the image of r under the composition to be f(x). We call f(x) the value of f at x, which lives in the completion of the residue field at x.

The completed residue field $\mathscr{H}(x)$ is an important gadget in Berkovich's theory, but is not as important in other theories for rigid geometry.

Remark 3.5. Since $|f(x)| = |f|_x$, we use the former notation, since it is "psychologically more satisfying."

In this way, the elements of \mathscr{A} become functions on the Berkovich spectrum $\mathscr{M}(\mathscr{A})$, so that their values live in completed residue fields.

Definition 3.6. We define the *Gelfand transform* to be the function

$$\mathscr{A} \longrightarrow \prod_{x \in \mathscr{M}(\mathscr{A})} \mathscr{H}(x)$$
$$f \longmapsto (f(x))_{x \in \mathscr{M}(\mathscr{A})}$$

We can get a lot of information from the Gelfand transform; we will come back to this in $\S3.3$.

3.2 Complex Banach algebras [Jon16, App. G]

Berkovich's theory is inspired not only by algebraic geometry, but also by Gelfand's theory of complex Banach algebras. It would be dishonest not to mention this source of inspiration, and so we give a short crash course on the theory.

Let \mathscr{A} be a complex Banach algebra, i.e., a Banach ring containing $(\mathbf{C}, |\cdot|_{\infty})$. An important first example is the following:

Example 3.7. The ring $\mathscr{A} = \mathcal{C}^0(X, \mathbb{C})$ of complex-valued continuous functions on a compact Hausdorff space X, together with the uniform or L^{∞} norm, forms a complex Banach algebra.

In the theory of complex Banach algebras, it's actually important to consider non-commutative rings, possibly without unit, but we will stick to our context where \mathscr{A} is always commutative with unit.

It is not true that all complex Banach algebras are of the form $\mathcal{C}^0(X, \mathbf{C})$, but we can ask the following:

Question 3.8. Given $\mathscr{A} = \mathcal{C}^0(X, \mathbb{C})$ that comes from a space X, can you recover X?

Definition 3.9. Let $f \in \mathscr{A}$. Then, the spectrum of f is defined as

$$\sigma(f) \coloneqq \{\lambda \in \mathbf{C} \mid \lambda - f \text{ not invertible in } \mathscr{A}\} \subseteq \mathbf{C}.$$

Exercise 3.10. In a general Banach ring \mathscr{A} , the element 1 - f is invertible if ||f|| < 1: the inverse must be of the form

$$(1-f)^{-1} = \sum_{j=0}^{\infty} f^j.$$

Theorem 3.11. The spectrum $\sigma(f)$ is nonempty and compact.

Sketch. Exercise 3.10 implies $\sigma(f)$ is closed in **C**. On the other hand, it is bounded since $\lambda \in \sigma(f)$ implies $|\lambda| \leq ||f||$ by Exercise 3.10 again. This implies $\sigma(f)$ is compact.

Now suppose $\sigma(f) = \emptyset$. Consider the function

$$\mathbf{C} \longrightarrow \mathscr{A}$$
$$\lambda \longmapsto (\lambda - f)^{-1}$$

This is defined on all of **C**, hence is an "entire" \mathscr{A} -valued function that is bounded by the construction of $(\lambda - f)^{-1}$. By a version of Liouville's theorem for functions with values in a complex Banach algebra, the function $(\lambda - f)^{-1}$ is constant as a function of λ , which is a contradiction.

From this, you can get the Gelfand–Mazur theorem, which we mentioned before.

Gelfand–Mazur Theorem 3.12. If \mathscr{A} is a complex Banach algebra field, then in fact $\mathscr{A} \cong \mathbb{C}$.

Proof. Pick $f \in \mathscr{A}$. Then, we know that $\sigma(f) \neq \emptyset$. This implies there exists $\lambda = \lambda(f)$ such that $\lambda - f$ is not invertible, and so $\lambda = f$, since zero is the only non-unit in \mathscr{A} . This gives a map $\mathscr{A} \to \mathbf{C}$ defined by $f \mapsto \lambda(f)$, which you can show is an isomorphism.

Corollary 3.13. If $\mathfrak{m} \subseteq \mathscr{A}$ is a maximal ideal in a complex Banach algebra, then $\mathscr{A}/\mathfrak{m} \cong \mathbb{C}$.

Proof. \mathscr{A}/\mathfrak{m} is a complex Banach field.

Theorem 3.14. If \mathscr{A} is a complex Banach algebra, then

$$\underbrace{\operatorname{Max} \mathscr{A}}_{\text{maximal}} \xrightarrow{\sim} \underbrace{\mathscr{M}(\mathscr{A})}_{\operatorname{Berkovich}}$$

$$\mathfrak{m} \longmapsto \text{residue norm on } \mathscr{A}/\mathfrak{m} \cong \mathbf{C}$$

is a bijection, with inverse given by $x \mapsto \mathfrak{p}_x$.

The map is in fact a homeomorphism, although we do not define the topology on the left-hand side.

Sketch. The main difficulty is surjectivity. Pick $x \in \mathcal{M}(\mathcal{A})$, and consider the prime ideal $\mathfrak{p}_x = \ker(|\cdot|_x) \subseteq \mathcal{A}$. If this is not a maximal ideal, then $\mathcal{H}(x)$ is a complex Banach field strictly containing \mathbb{C} , which contradicts the Gelfand-Mazur Theorem 3.12.

This crash course on complex Banach algebras was a bit sketchy, but we don't mind since we will not be using this later on. The philosophy is that the Berkovich spectrum is a generalization of Max \mathscr{A} , the spectrum in the sense of Banach algebras, and that many theorems in the Berkovich theory are analogues of facts that were already known about Max \mathscr{A} .

3.3 The Gelfand transform [Ber90, pp. 13–15]

We return to the general Berkovich theory, although now we mainly focus on the non-archimedean setting, even if the theory can be developed to encompass both non-archimedean and archimedean situations.

Let \mathscr{A} be a Banach ring, and consider the Berkovich spectrum $\mathscr{M}(\mathscr{A})$. Recall that we defined the *Gelfand* transform (Definition 3.6)

$$\mathscr{A} \longrightarrow \prod_{x \in \mathscr{M}(\mathscr{A})} \mathscr{H}(x)$$
$$f \longmapsto (f(x))_{x \in \mathscr{M}(\mathscr{A})}$$

This is a map of bounded Banach algebras: we define the supremum norm on $\prod \mathscr{H}(x)$ (Example 2.12(*iv*)), in which case

$$\left\| (f(x))_{x \in \mathscr{M}(\mathscr{A})} \right\| = \sup_{x \in \mathscr{M}(\mathscr{A})} |f(x)| \le \|f\| < \infty,$$

since all seminorms in $\mathcal{M}(\mathcal{A})$ are bounded by $\|\cdot\|$.

Corollary 3.15 [Ber90, Cor. 1.2.4]. $f \in \mathcal{A}$ is invertible if and only if $f(x) \neq 0$ for all $x \in \mathcal{M}(\mathcal{A})$.

Proof. " \Rightarrow " is clear: fg = 1 implies that for all x,

$$1 = |1| = |f(x)g(x)| = |f(x)| \cdot |g(x)|,$$

which implies $f(x) \neq 0$.

" \leftarrow " We show the contrapositive. Suppose $f \in \mathscr{A}$ is not invertible. Then, f lies in some maximal ideal \mathfrak{m} (using Zorn's Lemma), which is closed in \mathscr{A} . Then, \mathscr{A}/\mathfrak{m} is a Banach ring (in fact, it is a field, but not necessarily a valuation field, since the norm is not necessarily multiplicative). This implies $\mathcal{M}(\mathcal{A}/\mathfrak{m}) \neq \emptyset$, and so the map $\mathscr{A} \to \mathscr{A}/\mathfrak{m}$ induces a continuous map $\psi \colon \mathscr{M}(\mathscr{A}/\mathfrak{m}) \to \mathscr{M}(\mathscr{A})$. If x lies in the image of ψ , then f(x) = 0. \square

3.4 Properties of the spectrum [Ber90, §1.3]

We continue with more properties of the spectrum, and then we'll consider some more interesting examples than just valuation fields.

Let $f \in \mathcal{A}$, where \mathcal{A} is a Banach ring. We have already seen the spectral radius (Definition 2.16)

$$\rho(f) \coloneqq \lim_{n \to \infty} \|f^n\|^{1/n}$$

which is a power-multiplicative seminorm on \mathscr{A} . We can then relate this to the different valuations on $\mathscr{M}(\mathscr{A})$:

Theorem 3.16. The spectral radius can be computed as

$$\rho(f) = \max_{x \in \mathscr{M}(\mathscr{A})} |f(x)|.$$

In particular, the max is attained, since $|\cdot|_x \mapsto |f(x)|$ is continuous as a function of f, and the space $\mathcal{M}(\mathcal{A})$ is compact Hausdorff. This is a generalization of a result about complex Banach algebras, where $\rho(f)$ is the smallest radius of a ball containing $\sigma(f)$.

Proof. For all $x \in \mathcal{M}(\mathcal{A})$, then

$$|f(x)| = |f^n(x)|^{1/n} \le ||f^n||^{1/n} \to \rho(f)$$

For the other inequality, we use an auxiliary construction that we called a "relative disc" (Example 2.12(v)).

Suppose $r > \max_{x \in \mathcal{M}(\mathscr{A})} | f(x) |$. We want to show $\rho(f) \leq r$. We introduce the Banach algebra $\mathscr{B} := \mathscr{A} \langle rT \rangle$, which comes from a modification of the power series ring:

$$\mathscr{B} := \mathscr{A} \langle rT \rangle = \left\{ g = \sum_{i=0}^{\infty} a_i T^i \ \middle| \ a_i \in \mathscr{A}, \ \|g\| := \sum_{i=0}^{\infty} \|a_i\| r^{-i} < \infty \right\},$$

so that $||T|| = r^{-1}$ in \mathscr{B} ; you can think of this as a disc of radius 1/r relative to \mathscr{A} . Then, $|T(x)| \leq r^{-1}$ for all $x \in \mathcal{M}(\mathcal{B})$, so that |(fT)(x)| < 1 for all $x \in \mathcal{M}(\mathcal{B})$ (where we use implicitly that $\mathcal{A} \to \mathcal{B}$ induces a map $\mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{A})$. Thus, $|(1-fT)(x)| \neq 0$ for all $x \in \mathcal{M}(\mathcal{B})$, and Corollary 3.15 implies 1-fT is invertible in \mathcal{B} . Finally (Exercise),

$$\|(1 - fT)^{-1}\| = \sum_{i=0}^{\infty} \|f^i\| r^{-i} < \infty,$$

$$\rho(f) = \underline{\lim} \|f^i\|^{1/i} \le r.$$

which implies that

Note this is similar to the proof that the Berkovich spectrum is nonempty, in that we got information from constructing an auxiliary ring from \mathscr{A} .

Theorem 3.16 gives us a formula for the spectral radius for any element $f \in \mathcal{A}$, giving us the following: **Corollary 3.17.** f(x) = 0 for all $x \in \mathcal{M}(\mathcal{A})$ if and only if $\rho(f) = 0$, that is, f is quasinilpotent.

Note that quasinilpotence is weaker than nilpotence, although for most of the Banach rings we will study later, the notions will in fact be equivalent.

Remark 3.18. If \mathscr{A}^u is the uniformization of \mathscr{A} , which we recall (Definition 2.19) is the separated completion of (\mathscr{A}, ρ) , then there is a bounded map $\mathscr{A} \to \mathscr{A}^u$, inducing a map $\mathscr{M}(\mathscr{A}^u) \to \mathscr{M}(\mathscr{A})$, which is in fact a homeomorphism [Jon16, p. 11]. The spectral radius is therefore a canonical seminorm which gives a spectrum that is homeomorphic to the old one, that only depends on the equivalence class of the norm on \mathscr{A} .

3.4.1 Fibers [Jon16, pp. 13–14]

Suppose $\varphi \colon \mathscr{A} \to \mathscr{B}$ is a (bounded) map of Banach rings. This induces a map $f = \varphi^* \colon Y \to X$, where $Y = \mathscr{M}(\mathscr{B})$ and $X = \mathscr{M}(\mathscr{A})$, which is continuous. You can then ask what the fibers over a point in X are.

Exercise 3.19. Pick $x \in X = \mathscr{M}(\mathscr{A})$. Then, $f^{-1}(x) \cong \mathscr{M}(\mathscr{H}(x) \widehat{\otimes}_{\mathscr{A}} \mathscr{B})$ as topological spaces. In particular, if $f^{-1}(x)$ is empty, then $\mathscr{H}(x) \widehat{\otimes}_{\mathscr{A}} \mathscr{B} = 0$ (use universal properties of completed tensor products.) Note that Berkovich defines $\mathscr{H}(x) \widehat{\otimes}_{\mathscr{A}} \mathscr{B}$ differently for archimedean and non-archimedean rings, but this result holds regardless of which definition is chosen.

3.4.2 Ground field extension [Jon16, pp. 14–17]

You can also do more general fiber products, which we will not discuss, except for a special case: *ground field* extensions, mimicking what you would do in algebraic geometry.

Let k be a NA field, and let K be another NA field extending k. Then, \mathscr{A} is a (NA) Banach k-algebra. Consider $X = \mathscr{M}(\mathscr{A})$. You can then define $\mathscr{A}_K := \mathscr{A} \otimes_k K$, the complete tensor product, and then define $X_K := \mathscr{M}(\mathscr{A}_K)$.

Just by functoriality, you get a pair of commutative diagrams



Exercise 3.20. The fiber of $X_K \to X$ above $x \in X$ is $\mathscr{M}(\mathscr{H}(x) \widehat{\otimes}_k K)$.

Theorem 3.21 [Gru66, §3, Thm. 1(4)]. If V and W are (non-Archimedean) k-Banach spaces, then the canonical map $V \otimes_k W \to V \widehat{\otimes}_k W$ is injective.

We will not show this, but the idea of the proof is to reduce to the case of finite-dimensional k-vector spaces by using direct limits.

Gruson's Theorem 3.21 implies that, for all $x \in X$, we have an injective map $\mathscr{H}(x)\widehat{\otimes}_k K \leftrightarrow \mathscr{H}(x)\otimes_k K \neq 0$. By Theorem 2.26, this implies $\mathscr{M}(\mathscr{H}(x)\widehat{\otimes}_k K) \neq \emptyset$ for all $x \in X$, and so the map $X_K \to X$ is surjective.

We also consider another special case of the field extension construction: that of extension to the algebraic closure [Ber90, p. 16; Jon16, pp. 16–17]. Let k be a NA field, and consider the algebraic closure k^a .

Fact 3.22. The valuation on k extends uniquely to k^a , and so you can define $K = \widehat{k^a}$, the completion of k^a (which is actually algebraically closed).

The group $G := \operatorname{Gal}(k^a/k)$ then acts on $\widehat{k^a}$ by isometries. This implies G also acts on $X_{\widehat{k^a}}$, where $X = \mathscr{M}(\mathscr{A})$ as above.

Corollary 3.23 [Ber90, Cor. 1.3.6]. $X_{\widehat{ka}}/G \to X$ is a homeomorphism.

Berkovich does this very briefly, but the idea is to consider finite extensions, which factor into separable and purely inseparable extensions, and then to take the appropriate limits.

4 September 15

Today we will finally do some examples of this Berkovich spectrum. We did the Berkovich affine line during the first lecture (\S 1.4.1), but today we will do some more.

Let \mathscr{A} be a Banach ring, and let $\mathscr{M}(\mathscr{A})$ be the Berkovich spectrum, which recall is (as a set) the bounded multiplicative seminorms on \mathscr{A} .

4.1 $\mathcal{M}(Z)$ [Ber90, Ex. 1.4.1]

Let $\mathscr{A} = \mathbf{Z}$ with the usual Archimedan norm $|\cdot|_{\infty}$. You can think of the Euclidean norm $|\cdot|_{\infty}$ as the "largest possible norm" you can put on \mathscr{A} .

A point $x \in \mathcal{M}(\mathcal{A})$ corresponds to a seminorm $|\cdot|_x$ on \mathbb{Z} , which gives a prime ideal $\mathfrak{p}_x = \{|\cdot|_x = 0\} \subseteq \mathbb{Z}$. There are a couple of cases for what this can look like:

Case 1. For some p prime, we have $\mathbf{p}_x = p\mathbf{Z}$. Then, $|\cdot|_x$ induces a norm on $\mathbf{Z}/\mathbf{p}_x = \mathbf{F}_p$. It is easy to see (Example 3.3) that any multiplicative norm on \mathbf{F}_p is the trivial norm, and so $|\cdot|_x = |\cdot|_{p,0}$, where

$$|n|_{p,0} \coloneqq \begin{cases} 0 & \text{if } p \mid n, \\ 1 & \text{otherwise.} \end{cases}$$

Case 2. We have $\mathbf{p}_x = (0)$, the zero ideal. In this case, the valuation $|\cdot|_x$ can be extended to one on \mathbf{Q} by multiplicativity. This is the situation of the following theorem, which says that Examples 1.6–1.8 give all norms on \mathbf{Q} .

Ostrowski's Theorem 4.1. Any valuation on **Q** is one of the following: the trivial norm $|\cdot|_0$, the Euclidean norm $|\cdot|_{\infty}^{\epsilon}$ with scaling $\epsilon \in (0, 1)$, or the p-adic norm $|\cdot|_{p,c}$ with base $c \in (0, 1)$.

Proof. There are small estimates which are not worth doing on the board, so we give an outline.

Case 2a. $|\cdot|_x = |\cdot|_0$ is the trivial valuation on **Z**.

Case 2b. There exists n > 1 such that $|n|_x > 1$. Then, $|\cdot|_x$ is Archimedean by Proposition 1.12. Let $\epsilon \in (0, 1]$ such that $|n|_x = |n|_{\infty}^{\epsilon}$. Now you can show that in fact, $|n|_x = |\cdot|_{\infty}^{\epsilon}$ on all of **Z** (you can find a proof in various places; the idea is to expand powers of n in basis m, and play around with the triangle inequality).

Case 2c. The inequality $|n|_x \leq 1$ holds for all n. Then, $|\cdot|_x$ is non-Archimedean by Proposition 1.12, and moreover, since we are already not in Case 2a, there exists a prime p such that $|p|_x < 1$. Let $c := |p|_x \in (0, 1)$. You can check that $|\cdot|_x = |\cdot|_{p,c}$ is the p-adic norm with base c (Example 1.8), that is, if $n = p^{\ell}a$ where $p \nmid a$, then $|n|_{p,c} = c^{\ell}$.

You can then put this all together to get Figure 4.2.



Figure 4.2: A picture of $\mathcal{M}(\mathbf{Z})$.

There is a generalization where you take the ring of integers for any number field, but you get several archimedean branches corresponding to different embeddings into \mathbf{C} . Note there is also an emerging theory due to Poineau about Berkovich spaces over \mathbf{Z} .

4.2 $\mathcal{M}(\mathscr{A})$, where \mathscr{A} has the trivial norm [Ber90, Ex. 1.4.2]

Let \mathscr{A} be any ring with the trivial norm; this is a Banach ring. In this case, we define two maps relating $\mathscr{M}(\mathscr{A})$ to $\operatorname{Spec}(\mathscr{A})$:

Definition 4.3. The kernel map is the map

$$\ker\colon \mathscr{M}(\mathscr{A}) \longrightarrow \operatorname{Spec}(\mathscr{A})$$
$$x \longmapsto \ker(|\cdot|_x)$$

where the kernel is prime since every seminorm $|\cdot|_x$ is multiplicative. In the trivially valued case, this map has a canonical section

$$\operatorname{Spec}(\mathscr{A}) \longrightarrow \mathscr{M}(\mathscr{A})$$
$$\mathfrak{p} \longmapsto \begin{cases} \operatorname{residue norm with} \\ \operatorname{respect to} \mathscr{A} \to \mathscr{A}/\mathfrak{p} \end{cases}$$

where \mathscr{A}/\mathfrak{p} has the trivial norm. Thus, the residue norm is defined for $f \in \mathscr{A}$ by

$$|f|_{\mathfrak{p},0} = \begin{cases} 0 & \text{if } f \in \mathfrak{p}, \\ 1 & \text{otherwise.} \end{cases}$$
(2)

The reduction map is defined as

red:
$$\mathcal{M}(\mathscr{A}) \longrightarrow \operatorname{Spec}(\mathscr{A})$$

 $x \longmapsto \{|\cdot|_x < 1\}$

Note that the image of x is an ideal if \mathscr{A} has the trivial norm.

Factoid 4.4. ker is continuous, while red is anticontinuous, i.e., the preimage of open is closed.

You can visualize $\mathcal{M}(\mathcal{A})$ only in some cases.

Special Case 1. If \mathscr{A} is a field, then $\mathscr{M}(\mathscr{A})$ is a point.

Special Case 2. Suppose \mathscr{A} is a DVR, e.g., \mathbb{Z}_p or k[[T]]. This has a unique maximal ideal $\mathfrak{m} = (f)$, where f is the uniformizer. You can put the trivial norm on \mathscr{A} . Now pick $x \in \mathscr{M}(\mathscr{A})$. Then, $|g|_x = 1$ for all $g \in A \setminus \mathfrak{m}$ by the argument in Example 3.3, and $|f|_x \in [0, 1]$. Depending on this value, you get three different cases: Case 2a. $|f|_x = 1$, in which case $|\cdot|_x = |\cdot|_0$ is the trivial norm on \mathscr{A} .

Case 2b. $|f|_x = 0$, in which case $|\cdot|_x = |\cdot|_{\mathfrak{m},0}$ is the image of \mathfrak{m} under the canonical section of the kernel map. Case 2c. $0 < |f|_x < 1$, in which case $|\cdot|_x$ is uniquely determined by the number $r := |f|_x \in (0,1)$ by the formula $|g|_x = r^{\max\{\ell:f^\ell|g\}}$.

In this special case, you get Figure 4.5.



Figure 4.5: A picture of $\mathscr{M}(\mathscr{A})$ for \mathscr{A} a DVR.

Recall that in this situation, $\text{Spec}(\mathscr{A})$ has two points: the generic point 0, and the closed point \mathfrak{m} . The preimage of \mathfrak{m} under ker and red are

$$\ker^{-1}(\mathfrak{m}) = \{ |\cdot|_{\mathfrak{m},0} \}, \quad \operatorname{red}^{-1}(\mathfrak{m}) = \mathscr{M}(\mathscr{A}) \setminus \{ |\cdot|_0 \}.$$

Note the latter preimage $\operatorname{red}^{-1}(\mathfrak{m})$ is open.

Special Case 3. We can globalize Special Case 2. Let \mathscr{A} be a Dedekind ring (e.g., $\mathbf{Z}, k[T]$), still with the trivial absolute value. Pick $x \in \mathscr{M}(\mathscr{A})$, and set

$$\mathfrak{p} = \ker(x) = \{|f|_x = 0\}$$
 and $\mathfrak{q} = \operatorname{red}(x) = \{|f|_x < 1\},\$

in which case $0 \subseteq \mathfrak{p} \subseteq \mathfrak{q} \subseteq \mathscr{A}$. Since \mathscr{A} is a Dedekind domain, this chain cannot be very long, so there are only a couple of cases:

Case 3a. $\mathfrak{p} = \mathfrak{q} = 0$, in which case $|\cdot|_x = |\cdot|_0$ is the trivial norm on \mathscr{A} .

Case 3b. $\mathfrak{p} = \mathfrak{q} = \text{maximal}$, in which case $|\cdot|_x = |\cdot|_{\mathfrak{p},0}$, the residue seminorm for $\mathscr{A} \to \mathscr{A}/\mathfrak{p}$ as defined in (2). *Case* 3c. $0 = \mathfrak{p} \subsetneq \mathfrak{q} = \text{maximal}$, in which case, $|\cdot|_x$ is a norm (bounded by the trivial norm). This norm extends to the localization $\mathscr{A}_{\mathfrak{q}}$, which is a DVR. Now we are in Special Case 2, and so we get an open interval for each maximal ideal \mathfrak{q} as in Figure 4.5.

Combining this information, we get Figure 4.6. Note that $\operatorname{red}^{-1}(\mathfrak{p})$ is the whole leg from $|\cdot|_{\mathfrak{p},0}$ to $|\cdot|_0$, excluding the point $|\cdot|_0$.



Figure 4.6: A picture of $\mathcal{M}(\mathscr{A})$ for \mathscr{A} a Dedekind ring.

This is all going toward the Berkovich closed disc, which we will use often later on.

4.3 The general disc [Ber90, Ex. 1.4.3]

Let k be a valuation field, let r > 0, and let (Example 2.12(v))

$$\mathscr{A} = k \langle r^{-1}T \rangle = \left\{ f = \sum_{i=0}^{\infty} a_i T^i \ \left| \ a_i \in k, \ \|f\| \coloneqq \sum_{i=0}^{\infty} \|a_i\| r^i < \infty \right\} \right\}.$$

 \mathscr{A} is not uniform in general, that is, you do not necessarily get an equality in the inequality $||f^n|| \le ||f||^n$. Also note that even if you start with a non-Archimedean field, the norm on \mathscr{A} may no longer be non-Archimedean.

Exercise 4.7. In the non-Achimedean case, $\rho(f) = \max_i |a_i| r^i$, and

$$\mathscr{A}^{u} = k\{r^{-1}T\} = \left\{\sum_{i=0}^{\infty} a_{i}T^{i} \mid a_{i} \in k, \lim_{i \to \infty} |a_{i}|r^{i} = 0\right\},$$

with the norm $||f|| = \max_i |a_i| r^i$.

In general, $\mathscr{M}(\mathscr{A}^u) \xrightarrow{\sim} \mathscr{M}(\mathscr{A})$, so that $\mathscr{M}(\mathscr{A})$ is indeed homeomorphic to the Berkovich closed disc (which we will study next) if k is non-Archimedean.

4.4 The Berkovich closed disc [Ber90, Ex. 1.4.4]

We will now spend some time on the Berkovich closed disc. Berkovich treats this very minimally in one page; you can instead look at [BR10, Ch. 1; Jon15, §3.3].

Let k be a complete non-archimedean field, and let r > 0. Then, the ring $\mathscr{A} = k\{r^{-1}T\}$ from Exercise 4.7 consists of "power series which converge on the closed disc of radius r." Then, we use the following:

Gauss's Lemma 4.8. The norm $\|\cdot\|$ on $k\{r^{-1}T\}$ is multiplicative.

This is classical, but it does not hurt to prove it here.

Proof. Consider two power series and their product:

$$f = \sum_{i=0}^{\infty} a_i T^i, \quad g = \sum_{j=0}^{\infty} b_j T^j, \quad fg = \sum_{\ell=0}^{\infty} c_\ell T^\ell.$$

Since all seminorms are assumed to be submultiplicative (§2.1.2), to show multiplicativity it suffices to show $||fg|| \ge ||f|| \cdot ||g||$. Pick a minimal index I such that $||f|| = |a_I|r^I = \max_i |a_i|r^i$. Similarly, pick a minimal index J such that $||g|| = |b_J|r^J = \max_j |b_j|r^j$. Then, you can check that $|c_{I+J}| = |a_I| \cdot |b_J|$, in which case $||fg|| \ge |c_{I+J}|r^{I+J} = ||f|| \cdot ||g||$.

Note that this shows that $\mathcal{M}(k\{r^{-1}T\})$ is nonempty, since the norm we have is already multiplicative and hence gives a point in $\mathcal{M}(k\{r^{-1}T\})$.

Set $X := E(r) := \mathcal{M}(k\{r^{-1}T\})$. This is the Berkovich closed disc of radus r over k. For now, we will only study the topological structure of this space, and will define a structure sheaf on it later. We look at how the structure of $E(r) = E_k(r)$ depends on k. There are three different cases:

Case 1. k is trivially valued. In this case, you get the affine line (§1.4.1).

Case 2. k is non-trivially valued, and k is algebraically closed.

Case 3. k is non-trivially valued, and k is not algebraically closed.

Note that you can reduce Case 3 to Case 2 by using Corollary 3.23, but we will do this more carefully.

4.4.1 The trivially valued case [Jon15, §3.9.2]

In this situation, the structure and behavior of E(r) depends on r. While for $(\mathbf{C}, |\cdot|_{\infty})$, there is no big difference between closed discs of different radii because of scaling, in our situation this does not work since you cannot scale under the trivial norm. This is an example of how the Berkovich setting is interesting: you can still treat trivially valued rings, even if you have to treat them separately.

We claim that

$$k\{r^{-1}T\} = \left\{ f = \sum_{i=0}^{\infty} a_i T^i \ \middle| \ a_i \in k, \ \lim_{i \to \infty} |a_i| r^i = 0 \right\} = \begin{cases} k[[T]] & \text{if } r < 1\\ k[T] & \text{if } r \ge 1 \end{cases}$$

This is because if r < 1, then the limit $\lim_{i\to\infty} |a_i|r^i$ is always zero since $|a_i| = 0$ or 1; on the other hand, if $r \ge 1$, then the sequence a_i must eventually be zero.

If r = 1, we get what is called the Berkovich closed unit disc E(1). In this case, $||f|| = \max_i |a_i|r^i$ is the trivial norm on $\mathscr{A} = k[T]$, which is a Dedekind ring. We can therefore use our analysis from §4.2, Special Case 3 to get the picture in Figure 4.9.



Figure 4.9: A picture of the Berkovich closed unit disc E(1).

If k is algebraically closed, then $\operatorname{Max} \mathscr{A} \cong k$; in general, $\operatorname{Max} \mathscr{A} \cong k^a / \operatorname{Gal}(k^a/k)$.

Remark 4.10. For general k, if r < s, then you have a bounded map $k\{s^{-1}T\} \to k\{r^{-1}T\}$ (it is not admissible, though, but you can still admit it!), which induces a closed embedding $E(r) \hookrightarrow E(s)$ of topological spaces.

If 0 < r < 1, then $k\{r^{-1}T\} = k[[T]]$, with norm

$$\left\| f = \sum_{i=0}^{\infty} a_i T^i \right\| = \max_i |a_i| r^i = r^{\operatorname{ord}_0(f)},$$

where $\operatorname{ord}_0(f) = \min\{i \mid a_i \neq 0\}$. Pick $x \in \mathscr{M}(\mathscr{A})$, and set $t := |T(x)| = |T|_x \in [0, r]$. Then, x is determined uniquely by t by the formula $|f(x)| = t^{\operatorname{ord}_0(f)}$. Thus, E(r) is a segment sitting inside of E(1), as depicted in Figure 4.9.

If r > 1, then $k\{r^{-1}T\} = k[T]$ with norm

$$\left\| f = \sum_{i=0}^{\infty} a_i T^i \right\| = \max_i |a_i| r^i = r^{\deg f}$$

Pick $x \in \mathcal{M}(\mathcal{A})$, and set $t \coloneqq |T(x)| = |T|_x \in [0, r]$. If $t \leq 1$, then $|\cdot| \leq |\cdot|_0$, and so $x \in E(r) \subseteq E(1)$. So we only need to look at the case when $1 < t \leq r$. We essentially already did this for the Berkovich affine line: x is uniquely determined by t by the formula $|f(x)| = t^{\deg f}$ (§1.4.1, Case 2). So the picture you get for E(r) is the unit disc E(1) with an extra segment up top. This can be visualized as being part of $\mathbf{A}_k^{1,\mathrm{an}}$ as is done in Figure 4.11.



Figure 4.11: A picture of $E(r) \subseteq \mathbf{A}_{k}^{1,\mathrm{an}}$.

Next time we will talk about the non-trivially valued case, including the non-algebraically closed case, and classification of points into Types in these cases.

5 September 20

Homework 1 is due Thursday. Office hours this week are:

- Tuesday 1:30–2:10PM;
- Wednesday 1:30–2:30PM;
- Thursday 2:10–3:30PM (as usual).

5.1 The Berkovich closed disc (continued) [Ber90, Ex. 1.4.4]

Last time we did the Berkovich unit (or closed) disk over a trivially valued field. While nice, it would probably be more interesting with a non-trivially valued field.

Note it is more natural to work with closed discs than open discs, unlike in complex analysis.

5.1.1 The non-trivially valued, algebraically closed case [BR10, Ch. 1]

Let k be a (complete) NA field. Assume that the valuation is non-trivial, and that k is algebraically closed. Fix a radius r > 0, and consider the ring consisting of series that converge on the closed disc of radius r:

$$\mathscr{A} \coloneqq k\{r^{-1}T\} = \left\{ f = \sum_{i=0}^{\infty} c_i T^i \ \left| \ c_i \in k, \ \lim_{i \to \infty} |c_i| r^i = 0 \right. \right\}$$

with $||f|| \coloneqq \max_i |c_i| r^i$; recall that by Gauss's Lemma 4.8, this defines a multiplicative norm.

Definition 5.1. We define the Berkovich disc of radius r as

$$E(r) \coloneqq \mathscr{M}(k\{r^{-1}T\}),$$

and the rigid or classical disc as

$$\overline{\mathbf{D}}(r) \coloneqq \{ a \in k \mid |a| \le r \}.$$

Goal 5.2. We want to describe the points in E(r), and the structure on these points.

Remark 5.3. Since k is algebraically closed, a point $x \in E(r)$ is uniquely determined by $|(T-a)(x)| = |T-a|_x$ for $a \in k$. This is because any polynomial factors into linear polynomials, and $k[T] \subseteq k\{r^{-1}T\}$ is dense, so that the seminorm on k[T] determines it on $k\{r^{-1}T\}$.

We want to study E(r) because to study a theory of Berkovich spaces over some fixed field, we need some notion analogue of affine *n*-space or a polydisc to build spaces out of them.

There are four types of points in E(r).

Type 1 points Given $a \in \overline{\mathbf{D}}(r) \subseteq k$, define $|\cdot|_a \in E(r)$ by $|f|_a := |f(a)|$, where we note $f(a) \in k$, and |f(a)| uses the norm on k. This is a seminorm with kernel (T - a).

Definition 5.4. A point $x \in E(r)$ given by a seminorm of the form $|\cdot|_a$ above for $a \in \overline{\mathbf{D}}(r)$ is called a point of Type 1.

Exercise 5.5. Prove that $\overline{\mathbf{D}}(r) \to E(r)$ is a homeomorphism onto its image (which is not closed, and is in fact dense).

You can show that any seminorm with nonzero kernel must be of this form, since the kernel must be prime, and all primes are of the form (T - a). The remaining points in E(r) correspond to multiplicative *norms* on $k\{r^{-1}T\}$. We study them using *discs* in E(r) or $\overline{\mathbf{D}}(r)$. Types 2 and 3 are relatively easy to describe, but Type 4 points are a bit more difficult.

Definition 5.6 (Discs). Given $a \in \overline{\mathbf{D}}(r)$ and $0 < \rho \leq r$, define

$$\overline{\mathbf{D}}(a,\rho) \coloneqq \{b \in k \mid |b-a| \le \rho\} \subseteq \overline{\mathbf{D}}(r)$$
$$E(a,\rho) \coloneqq \{x \in E(r) \mid |(T-a)(x)| \le \rho\} \subseteq E(r)$$

Exercises 5.7.

- (1) $E(r) \cap \overline{\mathbf{D}}(a, \rho) = E(a, \rho) \cap \overline{\mathbf{D}}(r)$ via the inclusion $\overline{\mathbf{D}}(r) \subset E(r)$.
- (2) $\overline{\mathbf{D}}(a,\rho) = \overline{\mathbf{D}}(b,\rho)$ if and only if $|a-b| \leq \rho$. The same is true for E.
- (3) The radius $\rho(\overline{\mathbf{D}}(\cdot))$ is well-defined, that is, if $\overline{\mathbf{D}}(a,\rho) = \overline{\mathbf{D}}(b,\rho')$, then $\rho = \rho'$. Here, you need to use that k is non-trivially valued and algebraically closed.
- (4) Given $\overline{\mathbf{D}}_1, \overline{\mathbf{D}}_2$, either $\overline{\mathbf{D}}_i \subseteq \overline{\mathbf{D}}_j$, or $\overline{\mathbf{D}}_1 \cap \overline{\mathbf{D}}_2 = \emptyset$. The same is true for E_1, E_2 .
- (5) If $a \in \overline{\mathbf{D}}(0,r)$, then $T \mapsto T a$ is a bounded map of $k\{r^{-1}T\}$ inducing a homeomorphism of E(r), which sends $E(b,\rho)$ to $E(a+b,\rho)$.

Definition 5.8. The value group of k is $|k^*| = \{|a| \mid a \in k^* = k \setminus \{0\}\}$. This is a subgroup of \mathbf{R}_+^* .

So, by definition k being nontrivially valued is equivalent to $|k^*| \neq \{1\}$, and k being algebraically closed implies $|k^*|$ is divisible. These two facts imply that under our hypotheses on k, the value group $|k^*|$ is dense in \mathbf{R}^*_+ . This is the fact used in Exercise 5.7(3) above.

Remark 5.9. If $E_1, E_2 \subset E(r)$ are discs, then there exists an affine map $T \mapsto aT + b$ mapping E_1 onto E_2 if and only if $\rho(E_1)/\rho(E_2) \in |k^*|$. The same holds for $\overline{\mathbf{D}}_1, \overline{\mathbf{D}}_2$.

Of course, the value group may be all of \mathbf{R}^*_{\pm} , but in many interesting cases this is not the case.

Type 2 and 3 points Given a closed disc $E = E(a, \rho) \subseteq E(r)$, you can define a norm $|\cdot|_E$ on $k\{r^{-1}T\}$ by

$$|f|_E = \max_i |c_i| \rho^i$$
 where $f = \sum_{i=0}^{\infty} c_i (T-a)^i$.

Denote the corresponding point in E(r) by p(E).

Remark 5.10. $|T - b|_E = \max\{|a - b|, \rho\}.$

Exercise 5.11. $|\cdot|_E$ only depends on E, not on the center a in the description $E = E(a, \rho)$.

Set $\rho(E) \coloneqq \rho$ = radius of E. In particular, if $\rho = 0$, then this construction gives back a Type 1 point.

Definition 5.12. The corresponding point $p(E) \in E(r)$ is of Type 2 if $\rho(E) \in |k^*|$, and of Type 3 if $\rho(E) \in |k^*|$.

We now give a series of remarks (in the form of exercises).

Exercise 5.13. The point p(E) (corresponding to $|\cdot|_E$) lies in $E \subseteq E(r)$, and in fact, p(E) is the maximal point of E in the following sense: $|f(x)| \leq |f|_E$ for all $x \in E$ and for all $f \in k\{r^{-1}T\}$, where equality holds for x = p(E).

Exercise 5.14. $E \subset E'$ if and only if $|\cdot|_E \leq |\cdot|_{E'}$ on $k\{r^{-1}T\}$.

Exercise 5.15. Given $f \in k\{r^{-1}T\}$, $|f|_E = \sup\{|f(b)| \mid b \in E \cap \overline{\mathbf{D}}(r)\}$, where |f(b)| uses the norm on k. Moreover, if p(E) is of Type 2, then the supremum is a maximum, i.e., the supremum is attained.

Exercise 5.16. Given $a \in \overline{\mathbf{D}}(r)$, the map $[0, r] \to E(r)$ given by $\rho \mapsto p(E(a, \rho))$ is a homeomorphism onto its image. So the Berkovich disc contains many intervals.

We now sketch what the Berkovich disc E(r) looks like, or at least the subset of E(r) consisting of points we have seen so far, in Figure 5.17.



Figure 5.17: A picture of points of Types 1–3 in E(r).

We note that along any segment, Type 2 points are dense, since the value group is dense in \mathbf{R}^*_+ . Moreover, at any Type 2 point, there are infinitely many branches. On the other hand, there are some points along each segment where there is no branching: these are Type 3 points.

After drawing our picture, we can ask

Question 5.18. Have we found all the points in E(r)?

Sometimes, the answer is "yes," but often it is "no." These missing points are the Type 4 points, which come from decreasing families of discs.

Type 4 points

Definition 5.19. A collection \mathscr{E} of discs in E(r) is nested (Mattias's terminology) if

- (i) $E, E' \in \mathscr{E}, E \subseteq E'$ if and only if $\rho(E) \le \rho(E')$;
- (*ii*) If $E \in \mathscr{E}$ and $E \subseteq E' \subseteq E(r)$, then $E' \in \mathscr{E}$.

Then, $\rho: \mathscr{E} \to \mathbf{R}^*_+$ is a bijection of \mathscr{E} onto [s, r] or (s, r], where $0 \leq s \leq r$.

Note that the term "nested" is Mattias's terminology, and that (ii) is not strictly needed, but makes our discussion easier.

Given a nested collection \mathscr{E} as above, we define a multiplicative seminorm on $k\{r^{-1}T\}$ by

$$|f|_{\mathscr{E}} \coloneqq \inf_{E \in \mathscr{E}} |f|_E.$$

This infimum is kind of a limit as $\rho \searrow s$. Write $p(\mathscr{E})$ for the corresponding point in E(r), and let

$$\begin{split} \rho(\mathscr{E}) &\coloneqq \inf \rho(E) &\in [0,r] \\ \sigma(\mathscr{E}) &\coloneqq \bigcap_{E \in \mathscr{E}} \underbrace{E \cap \overline{\mathbf{D}}(r)}_{\text{disc in } k} \subseteq \overline{\mathbf{D}}(r) \end{split}$$

There are three different combinations for what $\rho(\mathscr{E})$ and $\sigma(\mathscr{E})$ can be:

Case 1. $\rho(\mathscr{E}) = 0$. In this case, since k is complete, we have that $\sigma(\mathscr{E})$ is a point in $\overline{\mathbf{D}}(r)$, and so $p(\mathscr{E})$ is a Type 1 point.

Case 2. $\rho(\mathscr{E}) > 0$, and $\sigma(\mathscr{E}) \neq \emptyset$. In this case (Exercise), $\sigma(\mathscr{E})$ is a closed disc in $\overline{\mathbf{D}}(r)$, and so $p(\mathscr{E})$ is a Type 2 or Type 3 point.

Case 3. $\rho(\mathscr{E}) > 0$, and $\sigma(\mathscr{E}) = \emptyset$. This is a new type of point.

Definition 5.20. A point $p(\mathscr{E})$ such that $\rho(\mathscr{E}) > 0$ and $\sigma(\mathscr{E}) = \emptyset$ is called a point of Type 4.

Now we can ask if these points of Type 4 actually exist: this depends on the ground field.

Remark 5.21. Essentially by definition, points of Type 4 exist if and only if k is not spherically complete (which says that decreasing intersections of discs are non-empty). There are other characterizations of this condition (e.g., the value group being maximally complete), but we will not discuss this now.

We give two interesting examples of fields which are not spherically closed:

Examples 5.22.

- The complex *p*-adic numbers $\mathbf{C}_p = \widehat{\mathbf{Q}_p^a}$ are not spherically complete.
- The completion $\widehat{\mathbf{C}((T))^a}$ of the field of Puiseux series is not spherically complete.

We now sketch a proof of the fact that we get all points in E(r) in the way we have described, even though Type 4 points may not exist.

Theorem 5.23. All points in E(r) are of Type 1, 2, 3, or 4.

Sketch. Given $x \in E(r)$, define

$$\mathscr{E}(x) \coloneqq \left\{ E(b, |(T-b)(x)|) \mid b \in \overline{\mathbf{D}}(r) \right\}.$$

Note that many of these discs may be the same. Then, $\mathscr{E}(x)$ is nested, and $p(\mathscr{E}(x)) = x$. In fact, you can check that $x \mapsto p(\mathscr{E})$ and $x \mapsto \mathscr{E}(x)$ are almost inverses.

We can now sketch the Berkovich disc E(r) again, but this time including the Type 4 points we were missing before.



Figure 5.24: A picture of $E(r) \supseteq E(a, |a|)$.

5.1.2 The tree structure on the Berkovich closed disc [BR10, §1.4]

As we saw in Figure 5.24, the Berkovich disc E(r) looks like a lot of real intervals, glued together without loops to form a tree. There is one way we can make this precise; this is especially useful to those interested in dynamics or potential theory.

We define a partial ordering on X = E(r) in the following way: $x \le y$ if and only if $|f(x)| \le |f(y)|$ for all $f \in k\{r^{-1}T\}$. You can then check, using our description of points so far, that we have the following description of E(r):

Theorem 5.25. The poset (X, \leq) satisfies the following properties:

- (1) There exists a maximal element x_0 (the given norm);
- (2) For all $x \in X \setminus \{x_0\}$, the subposet $[x, x_0] := \{y \ge x\}$ is isomorphic to $[0, 1] \subseteq \mathbf{R}$ (in particular, it is totally ordered);
- (3) For all $x, y \in X$, there exists a unique $x \lor y \in X$ such that $z \ge x$ and $z \ge y$ if and only if $z \ge x \lor y$;
- (4) Any totally ordered subset of X has an infimum in X.

You can develop a theory of such objects (called trees or **R**-trees, at least in the Berkovich literature). We sketch some of these elements into our picture of E(r) in Figure 5.26.



Figure 5.26: A picture of E(r), with some tree structure.

Note our end goal is that we want to understand not just the topology on the Berkovich space E(r), but also to have a good notion of analytic functions on them, forming a structure sheaf. Global sections should give back the original ring $k\{r^{-1}T\}$ we started from, but we don't know what analytic functions live on each open set. We will come back to this later.

5.1.3 Invariants of non-Archimedean fields [Jon15, §3.1.3]

We want to assign invariants to different types of points, both in E(r) and on Berkovich spectra in general. To do so, we will assign invariants to various fields attached to a point, and so we start with invariants of fields.

Definition 5.27. Let k be a NA field. The valuation ring of k is defined as

$$k^{\circ} \coloneqq \{a \in k \mid |a| \le 1\}.$$

Note it is the unit ball in k under the norm topology, and it is in fact, a valuation ring: if $a \in k^*$, then either $a \in k^\circ$ or $a^{-1} \in k^\circ$. The maximal ideal of k° is

$$k^{\circ\circ} \coloneqq \{a \in k \mid |a| < 1\},\$$

and the residue field of k is defined to be

$$\tilde{k} \coloneqq k^{\circ}/k^{\circ \circ}.$$

Example 5.28. If k is trivially valued, then $k^{\circ} = k$, $k^{\circ \circ} = 0$, and $\tilde{k} = k$.

Example 5.29. If $k = \tilde{k}((T))$ is the field of formal Laurent series over \tilde{k} , then $k^{\circ} = \tilde{k}[[T]]$, $k^{\circ\circ} = (T)$, and $\tilde{k} = \tilde{k}$.

Example 5.30. If $k = \mathbf{Q}_p$ are the *p*-adic numbers, then $k^{\circ} = \mathbf{Z}_p$, $k^{\circ \circ} = p\mathbf{Z}_p$, and $\tilde{k} = \mathbf{F}_p$.

This leads to a sort of "zoology" of fields, where we characterize valuation fields by properties of the field itself, and its residue field. We won't do too much that depends on this characterization, but certainly in some parts of the theory, the residue field (especially the characteristic) matter a lot.

Definition 5.31. Let k be a valuation field, and let \hat{k} be its residue field. Then, we say that

- 1. k has residue characteristic 0 if char $k = \text{char } \tilde{k} = 0$, e.g., $k = \mathbf{C}((T))$.
- 2. k is of (pure) characteristic p if char k = char k = p, e.g., $k = \mathbf{F}_p((T))$.
- 3. k is of mixed characteristic if char k = 0 but char $\tilde{k} = p$, e.g., $k = \mathbf{Q}_p$. In this case, $0 \neq p \in k$, but |p| < 1.

For a while, we won't be doing anything where this distinction makes a difference, but in Bhargav's course, for example, it will make a difference.

Another invariant we have associated to a valuation field is the value group $|k^*| \subseteq \mathbf{R}^*_+$ (Definition 5.8). We can combine the information of the value group and the residue field into one invariant first constructed by Temkin, which is sort of a graded reduction (later):

$$k \longmapsto \bigoplus_{r \in \mathbf{R}^*_+} \{|\cdot| \le r\} / \{|\cdot| < r\}$$

This is useful in some parts of the Berkovich theory.

5.1.4 Invariants of points [Ber90, p. 19]

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Suppose $x \in E(r) = \mathcal{M}(\mathcal{A})$ where $\mathcal{A} = k\{r^{-1}T\}$; we can also replace \mathcal{A} with an arbitrary Banach ring. Then, recall we have defined the following invariants attached to points (Definition 3.4):

$$\mathfrak{p}_x = \ker(|\cdot|_x)$$

the residue field $\mathscr{K}(x) = \operatorname{Frac}(\mathscr{A}/\mathfrak{p}_x)$
the complete field $\mathscr{H}(x) = \widehat{\mathscr{H}(x)}$

Note that $\mathscr{H}(x)$ is itself a valuation field, with value group $|\mathscr{H}(x)^*|$.

Definition 5.32. We define the "double" residue field to be

$$\mathscr{H}(x) \coloneqq \mathscr{H}(x)^{\circ} / \mathscr{H}(x)^{\circ \circ}.$$

We will study how different Types of points on E(r) change these invariants.

6 September 22

Homework 2 has been corrected online; it is due Tuesday.

The plan for today is to try to finish up the general discussion about the Berkovich closed disc. Next time we will talk about affinoid spaces, which are the building blocks for analytic spaces over a non-Archimedean ground field.

6.1 The Berkovich closed disc (continued) [Ber90, Ex. 1.4.4]

6.1.1 Invariants of points (continued) [Ber90, p. 19]

Let \mathscr{A} be a Banach ring, and consider the Berkovich spectrum $X = \mathscr{M}(\mathscr{A})$. Consider a point $x \in X$. Then, we have defined the following invariants associated to x:

the residue field
$$\mathscr{K}(x) = \operatorname{Frac}(\mathscr{A}/\ker(|\cdot|_x))$$

The norm on \mathscr{A} defines a norm on $\mathscr{K}(x)$, which may not be complete, and so we define

the complete residue field $\mathscr{H}(x) = \widetilde{\mathscr{K}(x)}$

The complete residue field has a norm with value group $|\mathscr{H}(x)^*|$, and so we can take the residue field of $\mathscr{H}(x)$ to obtain

the "double" residue field
$$\widetilde{\mathscr{H}(x)} = \mathscr{H}(x)^{\circ}/\mathscr{H}(x)^{\circ\circ}$$
.

Now consider the special case where $\mathscr{A} = k\{r^{-1}T\}, r > 0$, and $X = E(r) = \mathscr{M}(\mathscr{A})$. Then, $k \subset \mathscr{H}(x)$ (the extension can be quite large, in particular it is not necessarily finite), $|k^*| \subset |\mathscr{H}(x)^*|$, and $\tilde{k} \subset \mathscr{H}(x)$. Then, at least in the non-trivially valued, algebraically closed case, we have the following:

Theorem 6.1 [BR10, Prop. 2.3]. Assume k is non-trivially valued and that k is algebraically closed. For any $x \in E(r)$, we have that:

- If x is of Type 1, then $\mathscr{H}(x) = k$.
- If x is of Type 2, then $\mathscr{H}(x) \cong \tilde{k}(T)$ and $|\mathscr{H}(x)^*| = |k^*|$.
- If x is of Type 3, then $\mathcal{H}(x) \cong \tilde{k}$ and $|\mathcal{H}(x)^*|$ is freely generated by $|k^*|$ and $\rho(x)$.
- If x is of Type 4, then $\widetilde{\mathscr{H}(x)} \cong \tilde{k}$ and $|\mathscr{H}(x)^*| = |k^*|$ but $\mathscr{H}(x) \supseteq k$ (it is an immediate extension).

We won't prove all of this, but we will look at Type 3 points, since they will lead to a construction we will use often.

Complete residue field at Type 3 point. Let $x \in p(E(0,\rho))$ and $0 < \rho \leq r$, where $\rho \notin |k^*|$ (without loss of generality, you can move 0 to a center of the ball). Now letting $f = \sum_{i=0}^{\infty} a_i T^i$, we have $|f(x)| = \max_i |a_i| \rho^i$. Since $\rho \notin |k^*|$, all these values $|a_i|\rho^i$ are different, and so the max is uniquely attained, i.e., for every $f \neq 0$, there exists a unique *i* such that $|f(x)| = |a_i|\rho^i$. Then,

$$f = a_i T^i \left(1 + \sum_{j \neq i} \frac{a_j}{a_i} T^{j-i} \right),$$

where $|\frac{a_j}{a_i}T^{j-i}|_x < 1$, and in fact $|\frac{a_j}{a_i}T^{j-i}|_x \to 0$ as $j \to \infty$.

We get a similar description for $f \in \kappa(X) = \operatorname{Frac}(k\{r^{-1}T\})$, and eventually we arrive at the fact that

$$\mathscr{H}(x) = \left\{ f = \sum_{i=-\infty}^{\infty} a_i T^i \ \left| \ a_i \in k, \ \lim_{i \to \pm \infty} |a_i| \rho^i = 0 \right\} \right\}$$

with norm given by $|f(x)| = \max_i |a_i| \rho^i$, which again is uniquely attained if $f \neq 0$. We can now compute the value group and the residue field: it is clear that $|\mathscr{H}(x)^*|$ is generated by $|k^*|$ and ρ , and

- $f \in \mathscr{H}(x)^{\circ}$ if and only if $a_0 \in k^{\circ}$ and $|a_i|\rho^i < 1$ for $i \neq 0$; and
- $f \in \mathscr{H}(x)^{\circ\circ}$ if and only if $a_0 \in k^{\circ\circ}$ and $|a_i|\rho^i < 1$ for $i \neq 0$.

Thus, $\mathcal{H}(x) \cong \tilde{k} = k^{\circ}/k^{\circ \circ}$.

Remark 6.2. Field extensions such as the extension $k \subset \mathscr{H}(x)$ above will be crucial later: it shows how you can increase the value group by some $\rho \notin |k^*|$.

Remark 6.3. If $k \neq \overline{k}$, the correct condition for a Type 3 point that $\rho \notin \sqrt{|k^*|}$, the divisible value group, which is formed by taking roots of all elements in $|k^*|$.

6.1.2 The non-trivially valued, non-algebraically closed case [Ber90, §4.2]

The idea is to reduce to the algebraically closed case.

Let k^a be the algebraic closure of k, and consider the Galois group

$$G \coloneqq \operatorname{Gal}(k^a/k) \coloneqq \{\gamma \in \operatorname{Aut}(k^a) \mid \gamma|_k = \operatorname{id}\}$$

Facts 6.4. The following facts are standard, but not entirely trivial:

- (1) The valuation on k extends uniquely to k^a (every finite extension has a unique extension).
- (2) If $a \in k^a$ has minimal polynomial $T^n + b_1 T^{n-1} + \cdots + b_n$, then $|a| = |b_n|^{1/n}$ (although defining the valuation in this way makes it hard to check it is a valuation).
- (3) $|\gamma(a)| = |a|$ for all $a \in k^a$ and $\gamma \in G$, and so G acts by isometries.

Definition 6.5. $\widehat{k^a}$ is the completion of k^a .

Fact 6.6 [BGR84, Prop. 3.4.1/3]. $\widehat{k^a}$ is algebraically closed (use continuity of roots).

Example 6.7. $k = \mathbf{R}$ has algebraic closure $k^a = \widehat{k^a} = \mathbf{C}$.

Example 6.8 [BGR84, §3.4.3]. $k = \mathbf{Q}_p$ has algebraic closure $k^a \subseteq \widehat{k^a} \eqqcolon \mathbf{C}_p$.

Example 6.9. Consider the field of Laurent series over C:

$$k = \mathbf{C}((t)) = \left\{ a = \sum_{\beta \in B_a} a_{\beta} t^{\beta} \middle| \begin{array}{c} a_{\beta} \in \mathbf{C}^{*} \\ B_{a} \subseteq \mathbf{Z} \text{ bounded below} \end{array} \right\}$$

Then, by Puiseux's theorem the algebraic closure is the field of Puiseux series

$$k^{a} = \varinjlim \mathbf{C}((t^{1/N})) = \left\{ a = \sum_{\beta \in B_{a}} a_{\beta} t^{\beta} \middle| \begin{array}{l} a_{\beta} \in \mathbf{C}^{*} \\ B_{a} \subseteq \frac{1}{N} \mathbf{Z} \text{ for some } N \ge 1, \text{ bounded below} \end{array} \right\},$$

with Galois group $G = \lim_{N \to \infty} \mu_N$, where μ_N are the roots of unity. Then,

$$\widehat{k^{a}} = \left\{ a = \sum_{\beta \in B_{a}} a_{\beta} t^{\beta} \middle| \begin{array}{c} a_{\beta} \in \mathbf{C}^{*} \\ B_{a} \subseteq \mathbf{Q} \text{ well-ordered, discrete} \end{array} \right\}.$$

In all cases, $|a| = \epsilon^{\min B_a}$, where $0 < \epsilon < 1$ is fixed.

So far, we have only done the Berkovich disc, but in greater generality, if you have a space over k, you can consider it as a quotient of a space over $\widehat{k^a}$. In general, G acts on k^a by isometries, so G acts on $\widehat{k^a}$ by isometries as well.

Corollary 6.10 [Ber90, Cor. 1.3.6]. For any Banach k-algebra \mathscr{A} , G acts on $\mathscr{A} \otimes_k \widehat{k^a}$, hence on $\mathscr{M}(\mathscr{A} \otimes_k \widehat{k^a})$, and $\mathscr{M}(\mathscr{A} \otimes_k \widehat{k^a})/G \xrightarrow{\sim} \mathscr{M}(\mathscr{A})$.

The idea is to look at finite subextensions, which factor into separable and purely inseparable extensions, which you can treat separately, and then take limits. See [Jon16, pp. 15–17].

Now let us return to the special case when $\mathscr{A} = k\{r^{-1}T\}, r > 0$, so that $\mathscr{M}(\mathscr{A}) = E(r) = E_k(r)$. Then, by the universal property of the complete tensor product [Jon16, D.4.1], we have

$$\mathscr{A} \widehat{\otimes}_k \widehat{k^a} \cong \widehat{k^a} \{ r^{-1}T \}.$$

Then, G acts by

$$\gamma\left(\sum_{i=0}^{\infty}a_iT^i\right) = \sum_{i=0}^{\infty}\gamma(a)T^i,$$

and G acts on $E_{\widehat{k^a}}(r)$ by

$$|f(\gamma(x))| = |\gamma(f)(x)|.$$

You can then see how G acts on points of Type 1, 2, 3, and 4.

Exercise 6.11. Most of this is easy.

- (1) G maps Type 1 points to Type 1 points.
- (2) G maps discs to discs, preserving radii:

$$\gamma(E(a,\rho)) = E(\gamma(a),\rho) \text{ for all } a \in \widehat{k^a}, \ \rho \in (0,r].$$

Thus, G preserves Type 2 points, and preserves Type 3 points.

- (3) G preserves Type 4 points.
- $(4^*) \bullet$ Points of Type 2 and 3 have finite orbit.
 - Points of Type 1 have finite orbit if and only if they come from k^a , and not from $\widehat{k^a} \setminus k^a$.
 - Points of Type 4 may or may not have finite *G*-orbit (e.g. for complex Laurent series, the orbits are always infinite; in some cases, you may have some Type 4 points with either behavior).

We now draw a somewhat naïve picture of the Berkovich disc over a non-trivially valued, non-algebraically closed field in Figure 6.12. Note that the "Gauss norm" is the given norm on the algebra $k\{r^{-1}T\}$.



Figure 6.12: A picture of $E_{\widehat{k^a}}(r) \to E_k(r)$, with k non-trivially valued, non-algebraically closed.

The tree structure on $E_{\widehat{k^a}}(r)$ descends to one on $E_k(r)$:

Exercise 6.13. The partial ordering on $E_k(r)$ defined by $x \leq y$ if and only if $|f(x)| \leq |f(y)|$ for all $f \in k\{r^{-1}T\}$ satisfies the same "tree axioms" as before (Theorem 5.25).

Ground field extension You can still get maps of closed discs when passing to a non-Archimedean field extension, which is not the algebraic closure, which is useful and does not work in the rigid setting.

Fix K/k a non-Archimedean field extension. Then, $k \hookrightarrow K$ induces a map from $E_K(r) \twoheadrightarrow E_k(r)$.

Exercise 6.14. The map $E_K(r) \twoheadrightarrow E_k(r)$ preserves the tree structure.

Remark 6.15. Even if k, K are algebraically closed, Type 1 points may not map to Type 1 points!

When k is algebraically closed, there is a map in the other direction

$$E_k(r) \xrightarrow{id} E_K(r) \xrightarrow{id} E_k(r)$$

This map exists since discs map to discs. There is also a better explanation explanation for what is going on: the fiber of $x \in E_k(r)$ is $\mathscr{M}(\mathscr{H}(x) \otimes_k K)$ (this is not zero by the result by Theorem 3.21), and the norm on this Banach K-algebra $\mathscr{H}(x) \otimes_k K$ happens to be multiplicative (which is not completely trivial to see). See [Poi13, Cor. 3.14].

6.2 Potential theory [BR10]

Another direction that we could go in is potential theory, as discussed in [BR10]. We will do just a taste.

Potential theory over Berkovich spaces is the analogue of the study of subharmonic functions, the Laplace equation, etc. over **C**. Suppose we have a disc $\mathbf{D} \subset \mathbf{C}$, with a point $a \in \mathbf{D}$, as in Figure 6.16.



Figure 6.16: A complex disc.

Given a point $a \in \mathbf{D}$, we want to find a subharmonic function $\varphi \in \mathsf{SH}(\mathbf{D})$, satisfying two conditions:

$$\begin{cases} \Delta \varphi = \delta_a & \text{(fundamental solution)} \\ \varphi \equiv 0 \text{ on } \partial \mathbf{D} & \text{(boundary value)} \end{cases}$$

Note that here, we take the Laplacian in the sense of distributions. A solution to this problem is the function $\varphi = \log \left| \frac{T-a}{1-\overline{aT}} \right|.$

Now suppose k is a non-trivially valued, non-Archimedean field. You can define a Laplace operator on the Berkovich disc, but we bypass this and define φ directly. Given $a \in \overline{\mathbf{D}} = \overline{\mathbf{D}}(1) \subseteq k$, consider the function $\varphi = \log|T - a|$ on E = E(1), that is,

$$\varphi \colon E \longrightarrow \mathbf{R} \cup \{-\infty\}$$
$$x \longmapsto \log |T - a|_x = \log |(T - a)(x)|$$

Exercise 6.18. φ satisfies the following properties:

- $\varphi = 0$ at the Gauss point x_G , the given norm on $k\{T\}$.
- φ is locally constant outside $[a, x_G] \subseteq E$.
- On $[a, x_G]$, φ is an affine function of $\log \rho$ with slope 1.

So the Laplacian Δ on **C** corresponds to the real Laplacian on segments in *E*.



Figure 6.17: The potential function φ on E = E(1).

6.3 Further properties of the Berkovich disc [BR10, §§1.4–1.5]

Let $X = E(r) = \mathcal{M}(k\{r^{-1}T\}).$

Proposition 6.19. X is pathwise connected.

Proof. First assume k is non-trivially valued and algebraically closed. Then, the result is clear from the description of E(r) using discs.

Now assume k is a general non-Archimedean field. Pick K/k a non-Archimedean extension such that K is non-trivially valued, and K is algebraically closed (in the trivially valued case, choose e.g., the completed algebraic closure of the Laurent series field). Then, you get a continuous map $E_K(r) \twoheadrightarrow E_k(r)$, which is surjective, as we saw at the end of §6.1.2. Then, since $E_K(r)$ is pathwise connected, the same thing is true for $E_k(r)$.

Note that the trivial case this is pretty easy, but we wanted to point out how statements like these can be reduced to the non-trivial, algebraically closed case in a similar way.

6.3.1 Metrizability/embedding into affine space [BR10, §1.5]

We state some results about Berkovich spaces in the case of the Berkovich closed disc, without proof. We will not need them later.

Theorem 6.20 [HLP14, Thm. 1.1]. $E_k(r)$ embeds in some Euclidean space \mathbb{R}^n if and only if k has a countable dense subset.

Theorem 6.21 [Poi13, Thm. 5.14]. $E_k(r)$ is "angelic," i.e., $E_k(r)$ has certain countability properties, for any ground field k.

In algebraic geometry, you construct general schemes by first constructing affine spaces, then affine varieties, and finally glueing these together. Next time, we will start building general Berkovich spaces in an analogous way: by first building polydiscs, then affinoid spaces, and finally glueing them together.

7 September 27

Today, we will start discussing general Berkovich spaces over a non-Archimedean field k. References are [Ber90, Chs. 2–3; Ber93; Jon16, Ch. 2].

We start with some motivation. General Berkovich spaces are supposed to be the non-Archimedean analogue of the following constructions we are more familiar with:

1. Complex manifolds. These are formed by glueing together balls $B \subseteq \mathbb{C}^n$ to form a space X.

- 2. Complex analytic varieties. These are formed by glueing together complex analytic sets of the form $\{f_1 = \cdots = f_m = 0\} \subseteq B \subseteq \mathbb{C}^n$ for holomorphic functions f_1, \ldots, f_n . We remember this structure by putting a structure sheaf \mathcal{O}_X on the resulting space X.
- 3. Schemes (locally) of finite type over a field k. These are formed by taking affine schemes Spec A of finite type over k, where $A = k[T_1, \ldots, T_n]/I$ is a finitely generated k-algebra, and glueing them together to form a scheme X. Alternatively, you can describe A by saying there exists a surjection $k[T_1, \ldots, T_n] \twoheadrightarrow A$, corresponding to a closed embedding Spec $A \hookrightarrow \mathbf{A}_k^n$.

We briefly describe what our strategy for Berkovich spaces is, and then flesh out the details for the construction. Let $\mathcal{M}(\mathcal{A})$ be a h efficiency of a Berkovich space is a Berkovich space of a Berkovich space.

Let $\mathscr{M}(\mathscr{A})$ be a *k*-affinoid space, i.e., a Berkovich spectrum of a Banach ring \mathscr{A} , which is a *k*-affinoid algebra, which are the analogues of quotients of polynomial rings in the scheme case. More precisely, a *k*-affinoid algebra is a quotient of the ring of power series that converge on a polydisc:



where this surjection is admissible. Dually, $\mathscr{M}(\mathscr{A})$ embeds as a closed subset of the Berkovich closed polydisc E(r). After constructing these, we will glue them together to form a Berkovich space X.

Notice there is a lot of work to be done! We will begin by studying k-affinoid algebras in detail.

7.1 Affinoid algebras [Ber90, §2.1; Ber]

This material is originally due to Tate in 1962, as published in [Tat71], although there are some modifications due to Berkovich. We won't prove everything we need about affinoid algebras, but you can look at [Ber90, §2.1; BGR84; Ber] for more proofs.

Fix a non-Archimedean field k. We first define the polydisc algebra, which you can think of as the ring of power series that converge on the closed polydisc of some fixed polyradius r.

Definition 7.1. Given a polyradius $r = (r_1, \ldots, r_n) \in (\mathbf{R}^*_+)^n$, we define the polydisc algebra as

$$k\{r^{-1}T\} \coloneqq k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} = \left\{ f = \sum_{\nu \in \mathbf{Z}_+^n} a_{\nu}T^{\nu} \mid c_{\nu} \in k, \ \lim_{|\nu| \to \infty} |a_{\nu}|r^{\nu} = 0 \right\},$$

where $|\nu| = \nu_1 + \cdots + \nu_n$ and $r^{\nu} = r_1^{\nu_1} \cdots r_n^{\nu_n}$. This ring is a Banach *k*-algebra with the norm $||f|| = \max_{\nu} |a_{\nu}| r^{\nu}$, which is multiplicative by Gauss's Lemma 4.8.

When $r_i = 1$ for all *i*, then $\mathscr{T}_n := k\{T\}$ is called the Tate algebra.

We need different rings to describe power series that converge in polydiscs of different radii, since in the non-Archimedean world, you can't just scale discs like you can in the complex setting.

We will later define the Berkovich (poly)disc of (poly)radius r to be

$$E(r) = E_k(r) = E_k^n(r) \coloneqq \mathscr{M}(k\{r^{-1}T\}).$$

We have already seen the one-dimensional version of the following fact in $\S4.4.1$:

Exercise 7.2. If k is trivially valued, then

$$k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} = \begin{cases} k[T_1, \dots, T_n] & \text{if } r_i \ge 1 \text{ for all } i \\ k[[T_1, \dots, T_n]] & \text{if } r_i < 1 \text{ for all } i \end{cases}$$

Definition 7.3. A Banach k-algebra \mathscr{A} is k-affinoid if there exists an admissible epimorphism

$$\varphi \colon k\{r^{-1}T\} \twoheadrightarrow \mathscr{A} \tag{3}$$

for some polyradius $r \in (\mathbf{R}^*_+)^n$. We say \mathscr{A} is strictly k-affinoid if we can choose $r_i = 1$ for all i in the admissible epimorphism (3).

Note that the definition for a k-affinoid algebra is analogous to that of a finitely generated k-algebra.

The strictly k-affinoid algebras are the basic algebras that show up in Tate's theory [Tat71] of rigid analytic spaces and in [BGR84]. In the rigid analytic literature, what we call strictly k-affinoid algebras are simply called k-affinoid algebras [BGR84, 6.1.1/1]; for Berkovich spaces, though, it's better to have the flexibility of having polydiscs of arbitrary polyradii.

Remark 7.4. φ is an admissible epimorphism if

- φ is surjective; and
- the norm on \mathscr{A} is equivalent to the residue norm on $k\{r^{-1}T\}/\ker \varphi$ (which is equivalent to saying that φ is continuous or bounded if k is non-trivially valued [BGR84, 2.8.1]).

It is true that $\ker(\varphi)$ is always a closed ideal.

Definition 7.5. We say that $X := \mathscr{M}(\mathscr{A})$ is a *k*-affinoid space. If \mathscr{A} is strictly *k*-affinoid, then we say X is strictly *k*-affinoid.

Before discussing general properties of k-affinoid spaces, we look at some examples.

Examples 7.6. In the following, $X = \mathcal{M}(\mathcal{A})$.

- Polydiscs. Let $\mathscr{A} = k\{r^{-1}T\}$. Then $X \eqqcolon E(r)$ is the (Berkovich) polydisc of radius r.
- Annuli. Fix $0 < r_1 \le r_2 < \infty$. We let

$$\mathscr{A} = \frac{k\{r_1T_1, r_2^{-1}T_2\}}{(T_1T_2 - 1)} \tag{4}$$

with the residue norm. We think of this Banach algebra as consisting of functions that converge on $r_1 \leq |T| \leq r_2$. X is then the (Berkovich) annulus of radius r_1, r_2 . One way to think of this is as follows: the conditions on convergence are saying that $|T_1| \leq r_1^{-1}$ and $|T_2| \leq r_2$, and so letting $T_2 = T$ and $T_1 = T^{-1}$, we have that $r_1 \leq |T| \leq r_2$. You can get similar semialgebraic sets by imposing inequalities of this form. Note that if $r_2 = 0$, then $T_1T_2 - 1$ is the unit ideal, and so $\mathscr{A} = 0$.

• Circles. Let $r \in \mathbf{R}^*_+$ be a positive real number, and let $\mathscr{A} = k\{rT_1, r^{-1}T_2\}/(T_1T_2 - 1)$. Then, X is the (Berkovich) circle of radius r.

Exercise 7.7. Let $r \notin \sqrt{|k^*|}$, which means that $r^n \notin |k^*|$ for all $n \ge 0$. Consider the Banach ring

$$K_r := \left\{ f = \sum_{i=-\infty}^{\infty} a_i T^i \ \left| \ a_i \in k, \ \lim_{|i| \to \infty} |a_i| r^i = 0 \right\}, \right.$$

with norm $||f|| = \max_i |a_i|r^i$. Recall that $K_r = \mathscr{H}(x)$, where $x \in E(r)$ is a point of Type 3 with $\rho(x) = r$ (see §6.1.1). Then, K_r is a non-Archimedean field extension of K_r/k . To show it is a field, let $f \neq 0$, and write $f = a_i T^i(1+h)$, where $a_i T^i$ is the term maximizing $|a_i|r^i$. Then, ||h|| < 1, and so 1+h is invertible. In this case, $X = \mathscr{M}(\mathscr{A})$ is a point. Since K_r is of the form (4) for $r_1 = r_2 = r$, we can think of $\mathscr{M}(\mathscr{A})$ as a "circle with one point."

This last construction, and its higher-dimensional analogue below, will turn out to be very useful to use as a ground field extension.

Definition 7.8. We say an *n*-tuple $r = (r_1, \ldots, r_n) \in (\mathbf{R}^*_+)^n$ is k-free if

$$r_1^{\alpha_1} \cdots r_n^{\alpha_n} \in |k^*|, \qquad \alpha \in \mathbf{Z}^n$$

implies $\alpha_i = 0$ for all *i* (another way of saying this is that the images of the r_i in $\mathbf{R}^*_+/|k^*|$ are **Q**-linearly independent). If *r* is *k*-free, then we define

$$K_r \coloneqq \left\{ f = \sum_{\nu \in \mathbf{Z}^n} a_{\nu} T^{\nu} \mid a_{\nu} \in k, \ \lim_{|\nu| \to \infty} |a_{\nu}| r^{\nu} = 0 \right\},$$

which is a non-Archimedean field extension K_r/k , such that $|K_r^*|$ is generated by $|k^*|$ and r_1, \ldots, r_n .

Next, we will look at algebraic and metric properties of

- Tate algebras;
- strictly k-affinoid algebras; then
- general k-affinoid algebras.

The strategy is to extend results about simpler objects to the more complicated ones; in particular, the field extensions K_r/k are useful for reducing questions about general k-affinoid algebras to ones about strictly k-affinoid algebras.

7.1.1 Properties of the Tate algebra [Ber, §§2.1,2.3]

Recall that we defined the Tate algebra to be

$$\mathscr{T}_n \coloneqq k\{T_1, \dots, T_n\} = \left\{ f = \sum_{\nu \in \mathbf{Z}_+^n} a_\nu T^\nu \ \left| \ a_\nu \in k, \ \lim_{|\nu| \to \infty} |a_\nu| = 0 \right\},\right.$$

where $||f|| = \max_{\nu} |a_{\nu}|$. Then, you can define its valuation ring

$$\mathscr{T}_{n}^{\circ} \coloneqq k^{\circ}\{T_{1}, \dots, T_{n}\} \coloneqq \{f \mid ||f|| \leq 1\} = \{f \mid a_{\nu} \in k^{\circ}\},\$$

and we can define the ideal

$$\mathscr{T}_{n}^{\circ\circ} \coloneqq \{f \mid ||f|| < 1\} = \{f \mid a_{\nu} \in k^{\circ\circ}\}$$

which is prime in \mathscr{T}_n° , since the norm is multiplicative.

Exercise 7.9. The residue ring is

$$\widetilde{\mathscr{T}_n} = \mathscr{T}_n^{\circ} / \mathscr{T}_n^{\circ \circ} \cong \tilde{k}[T_1, \dots, T_n].$$

This result is one of the key tools for studying Tate algebras and strictly affinoid k-algebras: we will reduce to the residue ring $\widetilde{\mathscr{T}}_n$, which is a polynomial ring over \tilde{k} .

We now prove a few things about Tate algebras, to give an idea for what ingredients go into these proofs.

Easy Properties 7.10 [Ber, 2.1.1–2.1.4].

- (1) An element $f \in \mathscr{T}_n$ is invertible if and only if ||f f(0)|| < ||f|| = |f(0)|.
- (2) For all $f \in \mathscr{T}_n$, there exists $a \in k$ such that |a| = ||f|| and f + a is not invertible.
- (3) The zero ideal is the intersection of all maximal ideals in \mathscr{T}_n , i.e., the Jacobson radical $\mathfrak{R}(\mathscr{T}_n)$ is zero.
- (4) Every k-algebra homomorphism $\varphi \colon \mathscr{T}_n \to \mathscr{T}_m$ is a contraction: $\|\varphi(f)\| \leq \|f\|$ for all $f \in \mathscr{T}_n$.

The last property (4) is a Berkovich analogue of the Schwarz lemma in complex analysis.

The proofs of the first three properties are straightforward manipulations. We do the last one:

Proof of (4). Suppose there exists f such that $\|\varphi(f)\| > \|f\|$. By (2), there exists $a \in k$ such that $|a| = \|\varphi(f)\|$ and $\varphi(f) + a = \varphi(f + a)$ is not invertible. Thus, f + a is not invertible, contradicting (1) for f + a, which has a bigger norm, hence satisfies the condition in (1).

This means that any k-algebra homomorphism actually respects the norm. This is useful, since it means you can use "reduction machinery": $\varphi: \mathscr{T}_n \to \mathscr{T}_m$ satisfies

•
$$\varphi(\mathscr{T}^\circ) \subseteq \mathscr{T}_m^\circ$$

 $\bullet \ \varphi(\mathscr{T}^{\circ\circ}) \subseteq \mathscr{T}^{\circ\circ}_m$

and so we get a well-defined reduction



Properties of $\tilde{\varphi}$ will tell us things about φ .
(5) If $\varphi \colon \mathscr{T}_n \to \mathscr{T}_m$ is a k-algebra isomorphism, then m = n, and φ is an isometric isomorphism, that is, $\|\varphi(f)\| = \|f\|$ for all f.

Proof. $\tilde{\varphi}: \tilde{k}[T_1, \dots, T_n] \to \tilde{k}[T_1, \dots, T_m]$ is an isomorphism, and so m = n (look at Krull dimension). \Box

There is another thing which is more annoying to prove, which we skip for now:

(6) A k-algebra homomorphism $\varphi \colon \mathscr{T}_n \to \mathscr{T}_m$ is bijective if and only if $\widetilde{\varphi} \colon \widetilde{\mathscr{T}_n} \to \widetilde{\mathscr{T}_m}$ is bijective as well.

Maximum modulus principle Given a non-Archimedan field K/k, set

$$E^{n}(K) \coloneqq \overline{\mathbf{D}}_{K}^{n}(1) = \{ x \in K^{n} \mid |x_{i}| \le 1 \text{ for all } i \}.$$

Then, any $f = \sum_{\nu \in \mathbf{Z}_+^n} a_{\nu} T^{\nu} \in \mathscr{T}_n$ (over k) defines a continuous function $E^n(K) \to K$, which you can check converges and is continuous. By the non-Archimedean triangle inequality, $|f(x)| \leq ||f||$ for all $x \in E^n(K)$.

Theorem 7.11 (Maximum modulus principle [Ber, Lem. 2.1.5]). If \widetilde{K} is infinite, then there exists $x \in E^n(K)$ such that |f(x)| = ||f||, i.e., $||f|| = \max_{x \in E^n(K)} |f(x)|$.

Remark 7.12. This is similar to the general formula $\rho(f) = \max_{x \in \mathscr{M}(\mathscr{A})} |f(x)|$ in Theorem 3.16, where f is an element of a Banach ring \mathscr{A} . We will make this analogy more precise later. You can think of the maximum in the statement of the maximum modulus principle as only going over K-points in $\mathscr{M}(\mathscr{A})$.

Proof. After scaling f, we can assume without loss of generality that ||f|| = 1. Since \widetilde{K} is infinite, there exists a point $\widetilde{x} \in \widetilde{K}^n$ such that $\widetilde{f} \in \widetilde{k}[T_1, \ldots, T_n]$ is not zero at \widetilde{x} . Pick any $x \in E^N(K)$ such that the reduction of each coordinate x_i is \widetilde{x}_i for all i. Then, |f(x)| = 1 = ||f||.

So we see that the reduction machinery is extremely useful for proving things like this.

Further algebraic properties of \mathscr{T}_n . We will skip the proof of the next Theorem, since we won't be needing the techniques involved.

Theorem 7.13 [Ber, Cor. 2.3.3]. The Tate algebra \mathscr{T}_n is Noetherian, factorial (i.e., \mathscr{T}_n is a unique factorization domain), and Jacobson.

The proof is similar to how you show the ring of germs of holomorphic functions at a point is Noetherian: you use induction and what [BGR84, §5.2] calls "Weierstrass–Rückert theory," which takes the role of the Weierstrass preparation theorem.

Further metric properties of \mathscr{T}_n . In principle, we have so far allowed k to be trivially valued, in which case Theorem 7.13 says that the polynomial ring is noetherian and factorial.

For now, assume k is nontrivially valued, so that we get some information from the metric. Then, we have the following results from functional analysis:

Lemma 7.14. A k-linear map between normed k-vector spaces is bounded if and only if it is continuous.

The idea of the standard proof from functional analysis is that you essentially just scale balls, which works verbatim for an arbitrary non-trivially valued field. On the other hand, this does not work for trivially valued fields, since balls don't scale well.

Banach Open Mapping Theorem 7.15. A k-linear map of k-Banach spaces that is both continuous and surjective is open.

Again, this is not true over a trivially valued field. The usual proof uses the Baire category theorem, which is something that works for any complete metric space, hence the proof works in our setting.

Corollary 7.16. Any continuous surjective map between k-Banach spaces is admissible.

We will later emphasize admissibility in Berkovich's theory, but this Corollary says that in the non-trivially valued case, in some sense admissibility is automatic.

Next time, we will continue discussing algebraic and metric properties of Tate algebras and strictly k-affinoid algebras, starting with the fact that in a noetherian Banach k-algebra, all ideals are closed.

8 September 29

Today, we will continue proving some properties of k-affinoid algebras; it's good to prove some things, but we will have to black box many results.

8.1 Affinoid algebras (continued) [Ber90, §2.1; Ber]

Recall from last time that we will now assume k is a non-trivially valued non-Archimedean field (this is important, for example, since we need the Open Mapping Theorem 7.15). Recall that the Tate algebra is defined by

$$\mathscr{T}_n \coloneqq k\{T_1, \dots, T_n\} = \left\{ f = \sum_{\nu \in \mathbf{Z}_+^n} a_\nu T^\nu \ \left| \ a_\nu \in k, \ \lim_{|\nu| \to \infty} |a_\nu| = 0 \right\},$$

with the norm $||f|| = \max_{\nu} |a_{\nu}|$.

Last time, we mentioned some algebraic properties of \mathscr{T}_n : it is Noetherian, factorial, and Jacobson (Theorem 7.13). The proof of this result is by induction on n and "Weierstrass–Rückert theory." The fact that \mathscr{T}_n Noetherian is pretty strong already, and so we start with general properties about Noetherian Banach k-algebras.

8.1.1 Properties of Noetherian Banach k-algebras [Ber, §§2.3,2.5]

Let \mathscr{A} be a Noetherian Banach k-algebra, where k is non-trivially valued as before. For example, \mathscr{A} could be the Tate algebra \mathscr{T}_n . By scaling, k-linear maps are bounded if and only if they are continuous (Lemma 7.14); we also have the Banach Open Mapping Theorem 7.15.

Proposition 8.1 (cf. Proof of [Ber, Prop. 2.3.4]). Let \mathscr{A} be as above, and let M be a normed \mathscr{A} -module, such that the (separated) completion \widehat{M} is a finite \mathscr{A} -module. Then, $M = \widehat{M}$, i.e., M is complete.

Corollary 8.2 (cf. [Ber, Prop. 2.3.4(ii)]). Every ideal in \mathscr{A} is closed.

Proof that Proposition \Rightarrow Corollary. Let M = I be an ideal in \mathscr{A} .

Proof of Proposition. There exist $x_1, \ldots, x_n \in \widehat{M}$ such that

$$\mathscr{A}^n \xrightarrow{\pi} M$$
$$(a_1, \dots, a_n) \longmapsto \sum_{i=1}^n a_i x_i$$

is surjective (and bounded, hence admissible by Corollary 7.16). By the Banach Open Mapping Theorem 7.15, π is open, and so $\sum_{1}^{n} \mathscr{B}x_{j}$ is a neighborhood of $0 \in \widehat{M}$, where $\mathscr{B} = \{ \|\cdot\| < 1 \} \subseteq \mathscr{A}$. Since M is dense in \widehat{M} , we can approximate the x_{i} by y_{i} . Since $x_{i} \in M + \sum_{j=1}^{n} Bx_{j}$ for all i, there exist $f_{ij} \in \mathscr{B}$ such that $y_{i} \coloneqq x_{i} - \sum_{j=1}^{n} f_{ij}x_{j} \in M$. Now write $Y = (\mathbf{1} - F)X$. Since $\|f_{ij}\| < 1$, $\mathbf{1} - F$ is an invertible matrix, with coefficients in \mathscr{A} . This means $X = (\mathbf{1} - F)^{-1}Y \in M^{n}$, i.e., $x_{i} \in M$ for all i. This implies that $\widehat{M} = M$. \Box

The main ingredient here is the Banach Open Mapping Theorem 7.15, which requires that the field is nontrivially valued. For the trivially valued case, you want to reduce to the nontrivially valued case somehow.

There are many things you can say about finitely generated modules over a Banach ring \mathscr{A} ; the following is similar to the homework problem about finite-dimensional k-Banach spaces.

Proposition 8.3 (cf. [Ber90, Prop. 2.1.9; Ber, Prop. 2.3.4(i)]). If \mathscr{A} is as above, then the forgetful functor

 $\{finite Banach \mathscr{A}\text{-}modules\} \longrightarrow \{finite \mathscr{A}\text{-}modules\}$

is an equivalence of categories.

This is not all that different from the homework problem.

Proof Sketch. It is automatically faithful; we skip the proof that it is full. For essential surjectivity, we know that M is finite, so there exists a surjective \mathscr{A} -linear map $\pi \colon \mathscr{A}^n \to M$. Thus, $M \cong \mathscr{A}^n / \ker \pi$. Since \mathscr{A} is Noetherian, ker π is a closed ideal, and so you can equip M with the residue norm.

Corollary 8.4. An \mathscr{A} -module homomorphism of finite Banach \mathscr{A} -modules is in fact admissible.

This says that finite \mathscr{A} -modules really act like vector spaces over a valuation field.

Proof. Proposition 8.3 implies boundedness. We must prove the image is closed, in which case admissibility follows by Corollary 7.16. But this follows from Proposition 8.1, since the image is complete. \Box

These properties are a bit trickier over trivially valued fields!

Further properties

Proposition 8.5 (cf. [Ber90, Prop. 2.1.12; Ber, Cor. 2.5.4(i)]). The category of finite Banach \mathscr{A} -algebras is equivalent to the category of finite \mathscr{A} -algebras.

The proof is similar to before, but you have to work harder for essential surjectivity, since you need to check multiplicativity. One ingredient for the proof is that tensor products are the same on either side in Proposition 8.3.

Proposition 8.6 (cf. [Ber90, Prop. 2.1.10(*i*),(*ii*); Ber, Lem. 2.5.3]). Let \mathscr{A} be as above, and let M, N be finite Banach \mathscr{A} -modules, \mathscr{B} a finite Banach \mathscr{A} -algebra. Then, the canonical maps

$$\begin{split} M \otimes_{\mathscr{A}} N &\longrightarrow M \widehat{\otimes}_{\mathscr{A}} N \\ M \otimes_{\mathscr{A}} \mathscr{B} &\longrightarrow M \widehat{\otimes}_{\mathscr{A}} \mathscr{B} \end{split}$$

are bijective. Hence, $M \otimes_{\mathscr{A}} N$ is a finite Banach \mathscr{A} -module, and $M \otimes_{\mathscr{A}} \mathscr{B}$ is a finite Banach \mathscr{B} -module.

So \mathscr{A} acts a lot like a valuation field. These results are in [BGR84] or in [Ber]. You have to chase around some diagrams, which is not fun to do on the board.

At the end of the day, you don't want to be in such a general situation: we'll return to a situation closer to the classical Tate picture.

8.1.2 Properties of strictly *k*-affinoid algebras [Ber, §§2.4–2.5]

These are called k-affinoid algebras in [Tat71; BGR84]. Recall the definition:

Definition 8.7. A Banach k-algebra \mathscr{A} is strictly k-affinoid if there exists an admissible (or equivalently, bounded, by the Banach Open Mapping Theorem 7.15) epimorphism $\pi: \mathscr{T}_n \twoheadrightarrow \mathscr{A}$ of Banach k-algebras, for some $n \geq 0$, in which case $\mathscr{A} \cong \mathscr{T}_n / \ker \pi$.

Since \mathscr{T}_n is Noetherian and Jacobson (Theorem 7.13),

Corollary 8.8. If \mathscr{A} is strictly k-affinoid, then \mathscr{A} is automatically Noetherian and Jacobson.

Corollary 8.9. Every ideal of \mathscr{A} is closed.

There is one key result that would be good to prove, but it requires Weierstrass–Rückert theory, so we will leave it as a black box.

Noether Normalization Lemma 8.10 [Ber, Cor. 2.4.2]. For every strictly k-affinoid algebra \mathscr{A} , there exists $d \geq 0$ and a finite bounded monomorphism $\mathscr{T}_d \hookrightarrow \mathscr{A}$.

We skip the proof: it uses Weierstrass–Rückert theory. If k is trivially valued, it reduces to the ordinary Noether normalization lemma from algebraic geometry.

Geometrically, if $X = \mathscr{M}(\mathscr{A})$ and $E^n = \mathscr{M}(\mathscr{T}_n)$, the surjection $\mathscr{T}_n \twoheadrightarrow \mathscr{A}$ corresponds to an inclusion $X \hookrightarrow E^n$, and so the inclusion $\mathscr{T}_d \hookrightarrow \mathscr{A}$ corresponds to a finite map $X \to E^d$.

Corollary 8.11 [Ber, Cor. 2.4.3]. For every maximal ideal $\mathfrak{a} \subseteq \mathscr{A}$, $\dim_k(\mathscr{A}/\mathfrak{a}) < \infty$.

Proof. By Noether normalization, there exists $\varphi \colon T_d \hookrightarrow \mathscr{A}/\mathfrak{a}$, which is finite (bounded) injective, since \mathscr{A}/\mathfrak{a} is a field, unless \mathscr{T}_d is a field. This implies d = 0, and so $\mathscr{T}_d = k$, and φ is a finite field extension. \Box

We will return to the Berkovich spectrum in a while, but this already has consequences for the Berkovich spectrum: every maximal ideal in \mathscr{A} defines a point in the Berkovich spectrum $\mathscr{M}(\mathscr{A})$.

Corollary 8.12. For every maximal ideal \mathfrak{a} , the valuation on k extends uniquely to a valuation on \mathscr{A}/\mathfrak{a} , hence defines a point $x \in \mathscr{M}(\mathscr{A})$:

$$|f(x)| \coloneqq norm \ of \ image \ of \ f \ in \ \mathscr{A}/\mathfrak{a},$$

giving an injective map $\operatorname{Max} \mathscr{A} \hookrightarrow \mathscr{M}(\mathscr{A})$.

The proof boils down to a general fact about valuations extending uniquely to finite field extensions.

Remark 8.13. Max \mathscr{A} is the space associated to \mathscr{A} in rigid geometry. Giving Max \mathscr{A} the correct topology, this map Max $\mathscr{A} \hookrightarrow \mathscr{M}(\mathscr{A})$ is an embedding, at least when k is non-trivially valued [Ber90, Prop. 2.1.15; Ber, Prop. 2.5.6(*ii*)].

Proposition 8.14 [Ber, Prop. 2.5.6]. The image of Max \mathscr{A} is dense in $\mathscr{M}(\mathscr{A})$.

Example 8.15. Let $\mathscr{A} = \mathscr{T}_1$ be algebraically closed. Then, $\mathscr{M}(\mathscr{A}) = E$ and $\operatorname{Max} \mathscr{A} = k^\circ$.

Proof Sketch. Pick $x_0 \in \mathscr{M}(\mathscr{A})$ and $U \ni x_0$ an open neighborhood. We must show $(\operatorname{Max} \mathscr{A}) \cap U \neq \emptyset$. Without loss of generality,

$$U = \left\{ x \in \mathscr{M}(\mathscr{A}) \middle| \begin{array}{cc} |f_i(x)| < a_i & 1 \le i \le m \\ |g_j(x)| > b_j & 1 \le j \le m \end{array} \right\}$$

Pick $p_i, q_j \in \sqrt{|k^*|}$ such that $|f_i(x_0)| < p_i < a_i$ and $|g_j(x_0| > q_j > b_j)$. Replace f_i, g_j by powers, and so without loss of generality $p_i, q_j = 1$, since

$$|f_i(x)| < p_i = |c_i|^{1/n} \iff |(c_i^{-1}f_i^n)(x)| < 1.$$

Now set

$$\mathscr{B} \coloneqq \mathscr{A} \left\{ S_1, \dots, S_m, T_1, \dots, T_n \right\} / (f_i - S_i, g_j T_j - 1).$$

Then, you can check that \mathscr{B} is strictly k-affinoid, and $\mathscr{A} \to \mathscr{B}$ induces $\mathscr{M}(\mathscr{B}) \to \mathscr{M}(\mathscr{A})$ and $\operatorname{Max} \mathscr{B} \to \operatorname{Max} \mathscr{A}$. You can check that the image of $\mathscr{M}(\mathscr{B})$ in $\mathscr{M}(\mathscr{A})$ is the closed subset

 $\{x \in \mathscr{M}(\mathscr{A}) \mid |f_i(x)| \le 1, \ |g_j(x)| \ge 1, \ \forall i, j\},\$

which is contained in U. Since $\operatorname{Max} \mathscr{B} \neq \emptyset$, the image of $\operatorname{Max} \mathscr{B} \subset \mathscr{M}(\mathscr{B})$ under $\mathscr{M}(\mathscr{B}) \to \mathscr{M}(\mathscr{A})$ is contained in $(\operatorname{Max} \mathscr{A}) \cap U$.

We can visualize a Berkovich space as in Figure 8.16. From the picture, it doesn't seem like Max \mathscr{A} should be dense in $\mathscr{M}(\mathscr{A})$, but that is because the picture does not reflect the topology: any open set containing the Gauss point in E(r), for example, will contain all but finitely many points coming from Max $k\{r^{-1}T\}$.

Now we go back to the more algebraic part, about strictly k-affinoid algebras. Everything about Noetherian Banach k-algebras from before still hold, but there is an interesting new fact:

Fact 8.17 [Ber, Prop. 2.4.4]. Any k-algebra homomorphism between strictly k-affinoid algebras is bounded.

We won't use this, but we did say this for $\mathscr{T}_m \to \mathscr{T}_n$ (Easy Property 7.10(4)); it's a bit trickier.

8.1.3 Constructions on strictly *k*-affinoid algebras [Ber, §2.5]

Ground field extension If \mathscr{A} is strictly k-affinoid, and k'/k is a non-Archimedean field extension, then $\mathscr{A} \otimes_k k'$ is a strictly k'-affinoid algebra [Ber, Cor. 2.5.2(*ii*)].



Figure 8.16: Max \mathscr{A} is dense in $\mathscr{M}(\mathscr{A})$.

Cofiber coproducts Suppose $\mathscr{A}, \mathscr{B}, \mathscr{C}$ are strictly k-affinoid, and we have bounded morphisms



Proposition 8.18 (cf. [Ber, Cor. 2.5.2(i)]). $\mathscr{B} \widehat{\otimes}_{\mathscr{A}} \mathscr{C}$ is strictly k-affinoid.

We skip the proof.

8.1.4 The spectral radius [Ber, §§2.6–2.7]

Next up are some remarks about the spectral radius and the maximal modulus principle, which we showed for the Tate algebra; it is true for strictly k-affinoid algebras as well.

For any nonzero Banach ring recall we have defined the spectral radius (Definition 2.16) as

$$\rho(f) \coloneqq \lim_{n \to \infty} \|f^n\|^{1/n} = \max_{x \in \mathscr{M}(\mathscr{A})} |f(x)|,$$

for any $f \in \mathscr{A}$. Now assume that \mathscr{A} is strictly k-affinoid.

Theorem 8.19 (Maximum Modulus Principle [Ber, Prop. 2.6.1]). $\rho(f) = \max_{x \in Max(\mathscr{A})} |f(x)|$.

We saw this already for the Tate algebra $\mathscr{A} = \mathscr{T}_n$. We essentially want to reduce to his case, using Noether normalization (Lemma 8.10).

Corollary 8.20 [Ber, Cor. 2.6.3]. $\rho(f) \in \sqrt{|k^*|} \cup \{0\}$ for every $f \in \mathscr{A}$.

Using a similar technique as in the proof of the maximum modulus principle, you can prove:

Proposition 8.21 [Ber, Prop. 2.6.4]. Let $\varphi \colon \mathscr{A} \to \mathscr{B}$ be a finite homomorphism of strictly k-affinoid algebras. Then, for every $g \in \mathscr{B}$, there exists a polynomial $P(T) = T^n + f_1T^{n-1} + \cdots + f_n \in \mathscr{A}[T]$, such that P(g) = 0 and

$$\rho(g) = \max \rho(f_i)^{1/i}.$$

Corollary 8.22 [Ber, Prop. 2.6.5]. If \mathscr{A} is strictly k-affinoid and $f \in \mathscr{A}$, then the following are equivalent: (1) f is power bounded, i.e., $\sup \|f^n\| < \infty$. (2) $\rho(f) \leq 1$.

Sketch. (1) \Rightarrow (2) is clear. (2) \Rightarrow (1) is nontrivial: a priori you could have $||f^n|| \sim n^k$ or anything else that is subexponential. The proof uses Noether normalization (Lemma 8.10) and Proposition 8.21 (plus the result for \mathscr{T}_n , which is trivial since the norm on \mathscr{T}_n is already multiplicative).

There are some more things about the spectral radius for next time, and Berkovich has a trick to reduce the k-affinoid case to this strictly k-affinoid case.

9 October 4

First, some comments about Homework 2:

#3c: The correct assumption you need to make is that $1 + f_n$ is not a zero divisor

#4b: It suffices to show $\rho(n) \leq 1$ for all n since ρ is power multiplicative.

#5: The main point is the following: any k-linear map between finite-dimensional vector space is bounded, or equivalently, any two norms on a finite-dimensional vector space are equivalent.

9.1 Affinoid algebras (continued) [Ber90, §2.1; Ber]

The plan today is to finish up our discussion of affinoid algebras in [Ber90, §2.1]. The reason why we spent so long on one section is that Berkovich assumes background in Tate's rigid theory from [BGR84].

9.1.1 The spectral radius [Ber, §§2.6–2.7]

Fix a non-trivially valued non-Archimedean field k. We recall the following definition:

Definition 9.1. A Banach k-algebra \mathscr{A} is strictly k-affinoid if there exists an admissible epimorphism $\mathscr{T}_n = k\{T_1, \ldots, T_n\} \twoheadrightarrow \mathscr{A}.$

Recall that if k is non-trivially valued, then surjectivity implies admissibility (Corollary 7.16). We also recall the following important fact:

Theorem 8.19 (Maximum Modulus Principle). $\rho(f) = \max_{x \in Max(\mathscr{A})} |f(x)|$.

The interesting thing about this Theorem is that the maximum only goes over Max \mathscr{A} , not the entire Berkovich spectrum $\mathscr{M}(\mathscr{A})$.

We proved this for the Tate algebra in Theorem 7.11. For strictly k-affinoid algebras, the idea is to reduce to the case of Tate algebras using the Noether Normalization Lemma 8.10. A similar proof yields the following:

Corollary 8.22. If \mathscr{A} is strictly k-affinoid, and $f \in \mathscr{A}$, then the following are equivalent:

(1) f is power bounded: $\sup_n ||f^n|| < \infty$;

(2) $\rho(f) \le 1$.

We skip the proof, but the implication $(1) \Rightarrow (2)$ is trivial, and for $(2) \Rightarrow (1)$, you first prove it for the Tate algebra, and then use the Noether Normalization Lemma 8.10 to reduce to this case.

This Corollary has some interesting consequences. The main point is that they show that the spectral radius is well-behaved, even though it is defined as a limit.

Proposition 9.2 (cf. [Ber, Prop. 2.7.3(*ii*),(*iii*)]). If \mathscr{A} is strictly k-affinoid algebra, and $f \in \mathscr{A}$, then

(1) $\rho(f) = 0$ (i.e., f quasinilpotent) if and only if f is nilpotent.

(2) If f is not nilpotent, then there exists C > 0 such that $||f^n|| \le C \cdot \rho(f)^n$ for every $n \ge 1$.

Remark 9.3. In (2), the inequality $\rho(f)^n \leq ||f^n||$ always holds by definition of the spectral radius; the interesting part of this result is that a non-trivial upper bound also holds.

Proof. For (1), we note the following sequence of equivalent statements:

$$\rho(f) = 0 \iff f \in \bigcap_{\mathfrak{m} \in \operatorname{Max} \mathscr{A}} \mathfrak{m} \qquad \text{(by Theorem 8.19)}$$
$$\iff f \in \bigcap_{\mathfrak{p} \in \operatorname{Spec} \mathscr{A}} \mathfrak{p} \qquad \text{(since } \mathscr{A} \text{ is Jacobson, Corollary 8.8)}$$
$$\iff f \text{ nilpotent}$$

For (2), we know that $\rho(f) \in \sqrt{|k^*|} \cup \{0\}$ by the maximum modulus principle (Theorem 8.19), and we know $\rho(f) \neq 0$ by (1), since f is assumed to not be nilpotent. This means that there exists $a \in k^*$ and $m \geq 1$ such that $\rho(f)^m = |a|$, and so $\rho(a^{-1}f^m) = 1$. By Corollary 8.22, we have $\sup_n ||(a^{-1}f^m)^n|| < \infty$. This implies (Exercise) that you get an estimate of the form $||f^n|| \leq C\rho(f)^n$.

So we see that the spectral radius is very nicely behaved for strictly k-affinoid algebras, even though it may not be for general k-Banach algebras.

Before moving on to general k-affinoid algebras, we ask how to identify k-affinoid algebras among modified Tate algebras of the form

$$\mathscr{T}_{n,r} \coloneqq k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$$

Proposition 9.4 (cf. [Ber, Cor. 2.7.5]). The k-Banach algebra $\mathscr{T}_{n,r}$ is strictly k-affinoid if and only if $r_i \in \sqrt{|k^*|}$ for all *i*.

Proof Sketch. " \Rightarrow " is clear since $r_i = ||T_i|| = \rho(T_i) \in \sqrt{|k^*|}$ for all *i* by the maximum modulus principle (Theorem 8.19).

" \Leftarrow " We can prove this directly: $r_i^{m_i} = |a_i|$ for $m_i \ge 1$ and $a_i \in k^*$ for $1 \le i \le n$. Now you can construct an admissible epimorphism $\mathscr{T}_N \to \mathscr{T}_{n,r}$ for $N \gg 1$ using monomials, where $\mathscr{T}_N = k\{S_1, \ldots, S_N\}$.

9.1.2 Properties of k-affinoid algebras [Ber, §2.7]

Now we move on to k-affinoid algebras, which only appear in Berkovich's theory. Note that they are very useful, because they give you more flexibility, although you do not always have to consider them in applications.

Assume that k is any non-Archimedean field (possibly trivially valued).

Definition 9.5. A Banach k-algebra \mathscr{A} is k-affinoid if there exists an admissible epimorphism

$$k\{r^{-1}T\} = \mathscr{T}_{n,r} = k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \twoheadrightarrow \mathscr{A}$$

for some $r \in (\mathbf{R}^*_+)^n$ and $n \ge 1$.

Geometrically, such an epimorphism corresponds to a closed embedding $X := \mathscr{M}(\mathscr{A}) \hookrightarrow E(r)$.

If k is trivially valued, we do not have the Banach Open Mapping Theorem 7.15, and so proving analogues of properties for strictly k-affinoid algebras is tricky. The main idea is that we use a ground field extension by the field K_r from Definition 7.8 in order to reduce to the strictly k-affinoid case, where k will now be nontrivially valued.

Remark 9.6. Not all properties carry over to the k-affinoid case; for example, the maximum modulus principle (Theorem 8.19) is false!

We remind ourselves of the definition of K_r . Let k be a non-Archimedean field, and consider an n-tuple $r \in (\mathbf{R}^*_+)^n$ that is k-free, which we recall means that $r^{\nu} \in |k^*| \Rightarrow \nu = 0$, or equivalently, the r_i 's are **Q**-linearly independent in $\mathbf{R}^*_+/|k^*|$. Then, we defined

$$K_r \coloneqq \left\{ f = \sum_{\nu \in \mathbf{Z}^n} a_{\nu} T^{\nu} \mid a_{\nu} \in k, \lim_{|\nu| \to \infty} |a_{\nu}| r^{\nu} = 0 \right\},$$

where the norm is given by $||f|| = \max_{\nu} |a_{\nu}| r^{\nu}$. This is a k-affinoid algebra, since it is a sort of "polycircle of radius r":

$$K_r \cong \frac{k\{r_1S_1, \dots, r_nS_n, r_1^{-1}T_1, \dots, r_n^{-1}T_n\}}{(S_iT_i - 1)_{1 \le i \le n}}$$

Furthermore, K_r is a non-Archimedean field extension of k such that $|K_r^*|$ is generated by $|k^*|, r_1, \ldots, r_n$. This is where you need to use that r is $|k^*|$ -free.

Trick 9.7. If \mathscr{A} is k-affinoid, then $\mathscr{A} \otimes_k K_r$ is strictly K_r -affinoid, so that we can reduce the k-affinoid case to the strictly k-affinoid case.

We start with the following:

Lemma 9.8 [Ber, Lem. 2.7.1].

- (1) If V is a k-Banach space, then $V \to V \widehat{\otimes}_k K_r$ is an isometric embedding;
- (2) A sequence $V' \to V \to V''$ of bounded homomorphisms of k-Banach spaces is exact and admissible if and only if $V' \widehat{\otimes}_k K_r \to V \widehat{\otimes}_k K_r \to V'' \widehat{\otimes}_k K_r$ is exact and admissible.

Proof. Exercise. Use/prove that

$$V \widehat{\otimes}_k K_r \cong \left\{ \sum_{\nu \in \mathbf{Z}^n} v_{\nu} T^{\nu} \mid v_{\nu} \in V, \ \lim_{|\nu| \to \infty} \|v_{\nu}\| r^{\nu} = 0 \right\}$$

and let $V \to V \widehat{\otimes}_k K_r$ be the map putting a vector in V in the $\nu = 0$ slot.

Question 9.9. Is this Lemma true for general non-Archimedean extensions K/k?

Corollary 9.10. Given a k-affinoid algebra \mathscr{A} , there exists $r \in (\mathbf{R}^*_+)^n$ such that $\mathscr{A} \otimes_k K_r$ is strictly K_r -affinoid.

Proof. We know that there exist m, s and $\mathscr{T}_{m,s} \to \mathscr{A}$ which is an admissible epimorphism. Now pick $r \in (\mathbf{R}^*_+)^n$ that is $|k^*|$ -free, such that the s_j lie in the group generated by $|k^*|$ and the r_i . After tensoring, $\mathscr{T}_{m,s}$ becomes strictly k-affinoid, and now using Lemma 9.8, you are done.

Now we list some results that are true for k-affinoid algebras like they were in the strictly k-affinoid case.

Proposition 9.11 [Ber, Prop. 2.7.3(i)]. If \mathscr{A} is k-affinoid, then \mathscr{A} is Noetherian and all its ideals are closed.

We already saw that Noetherian implies all ideals are closed (over a non-trivially valued field), but we will show everything again by reducing to the strictly k-affinoid case, since now k is allowed to be trivially valued.

Proof. By Corollary 9.10, we can choose an *n*-tuple $r \in (\mathbf{R}^*_+)^n$ that is $|k^*|$ -free, such that $\mathscr{A} \otimes_k K_r$ is strictly K_r -affinoid. We know that $\mathscr{A} \otimes_k K_r$ is Noetherian, and that all its ideals are closed (Corollary 8.8). We want to deduce the corresponding properties for \mathscr{A} . Pick an ideal \mathfrak{a} an ideal of \mathscr{A} . We want to show it is closed and finitely generated.

Denote the extension of \mathfrak{a} in $\mathscr{A} \otimes_k K$ as $\mathfrak{a} \cdot \mathscr{A} \otimes_k K_r$, which by the previous paragraph is finitely generated by $f_1, \ldots, f_n \in \mathfrak{a} \cdot \mathscr{A} \otimes_k K$. You can show that the f_1, \ldots, f_n can be chosen to lie in \mathfrak{a} . Then, any $f \in \mathfrak{a}$ can be written $\sum_{j=0}^m f_j g_j$ for $g_j = \sum_{\nu} g_{j,\nu} T^{\nu}$, and so $f = \sum_0^m f_j g_{j,0}$. Thus, $\mathfrak{a} = \mathscr{A} \cap \{\mathfrak{a} \cdot \mathscr{A} \otimes_k K_r\}$. \Box

Showing that the f_1, \ldots, f_n can be chosen to lie in \mathfrak{a} seems non-trivial.

Proposition 9.12 [Ber, Prop. 2.7.3(*ii*),(*iii*)]. If \mathscr{A} is k-affinoid, and $f \in \mathscr{A}$, then

(1) $\rho(f) = 0$ (i.e., f quasinilpotent) if and only if f is nilpotent.

(2) If f is not nilpotent, then there exists C > 0 such that $||f^n|| \le C \cdot \rho(f)^n$ for every $n \ge 1$.

Proof. The isometric embedding $\mathscr{A} \hookrightarrow \mathscr{A} \widehat{\otimes}_k K_r$ from Lemma 9.8(1) implies the spectral radii are the same. Then, use the result for $\mathscr{A} \widehat{\otimes}_k K_r$.

With some more work, you can show:

Proposition 9.13. If \mathscr{A} is reduced, then ρ is a norm equivalent to $\|\cdot\|$, i.e., there exists C such that $\rho(f) \leq \|f\| \leq C \cdot \rho(f)$.

We skip the proof. The idea is to reduce to the strict case as we did in Proposition 9.12, although you need to use the Noether Normalization Lemma 8.10.

We have the following results, which are technically useful.

Proposition 9.14 [Ber, Cor. 2.7.4]. Let $\varphi: \mathscr{A} \to \mathscr{B}$ be a bounded homomorphism of k-affinoid algebras. Let $f_1, \ldots, f_n \in \mathscr{B}$ and $r_1, \ldots, r_n > 0$ such that $r_i \ge \rho(f_i)$ for all i. Then, there exists a unique bounded homomorphism $\Phi: \mathscr{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to \mathscr{B}$ such that $\Phi|_{\mathscr{A}} = \varphi$ and $\Phi(T_i) = f_i$.

This Proposition can be interpreted geometrically in terms of maps of Berkovich spaces as in Figure 9.15. *Proof.* Set

$$\Phi\bigg(\sum_{\nu\in \mathbf{Z}^n_+}a_\nu T^\nu\bigg)=\sum_\nu \varphi(a_\nu)f^\nu$$

We only need to check that the right-hand side of the equation above converges in \mathscr{B} . But this follows since

$$||f^{\nu}|| \le C \cdot \rho(f_1)^{\nu_1} \cdots \rho(f_n)^{\nu_n} \le r^{\nu}$$

by Proposition 9.12(2), and since $\|\varphi(a_i)\| \leq C \cdot \|a_i\|$ by the assumption that $\varphi \colon \mathscr{A} \to \mathscr{B}$ is bounded. \Box



Figure 9.15: A visualization of Proposition 9.14.

Corollary 9.16 [Ber, Cor. 2.7.5]. If \mathscr{A} is k-affinoid, then \mathscr{A} is strictly k-affinoid if and only if $\rho(f) \in \sqrt{|k^*|}$ for all $f \in \mathscr{A}$.

Proof. " \Rightarrow " follows from the maximum modulus principle (Theorem 8.19).

"\equiv "We have an admissible epimorphism $\varphi: k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \twoheadrightarrow \mathscr{A}; \text{ set } f_i = \varphi(T_i) \in \mathscr{A}, \text{ and set } s_i \coloneqq \rho(f_i) \leq \rho(T_i) = r_i.$ Then, by Proposition 9.14, there is an extension

$$\psi \colon k\{s_1^{-1}T_1, \dots, s_n^{-1}T_n\} \longrightarrow \mathscr{A}$$

of φ , such that $\psi(T_i) = f_i$, i.e., we have adjusted the radii so that they match the spectral radius. You can then check that ψ is surjective and admissible. Since $s_i = \rho(f_i) \in \sqrt{|k^*|}$, we have that $k\{s^{-1}T\}$ is strictly *k*-affinoid, which implies \mathscr{A} is strictly *k*-affinoid.

The step where you check that ψ is surjective and admissible seems non-trivial.

Next time, we will talk about finite maps of k-affinoid algebras. Most of the things about finite modules and algebras over k-Banach algebras will work in the same way as they did before. We will then start thinking about Berkovich spectra of k-affinoid rings, and how to put structure sheaves on them.

10 October 6

Homework 3 is due Monday, Homework 4 is due Thursday 10/27.

10.1 Affinoid algebras (continued) [Ber90, §2.1; Ber]

Let k be a (possibly trivially valued) non-Archimedean field. Today, our goal is to finish up our discussion of k-affinoid algebras from before, and move on to discussing affinoid spaces.

10.1.1 Properties of k-affinoid algebras (continued) [Ber, §2.7]

Recall our definitions from before:

Definition 10.1. A Banach k-algebra \mathscr{A} is k-affinoid if there exists an admissible epimorphism

$$k\{r_1^{-1}T_1,\ldots,r_n^{-1}T_n\} \twoheadrightarrow \mathscr{A},$$

for some $n \ge 0$ and $r_i \in \mathbf{R}^*_+$. We say that \mathscr{A} is strictly k-affinoid if we can take $r_i = 1$ in an epimorphism of the form above.

We also recall the "useful field extension" $k \hookrightarrow K_r$ that we have been repeatedly using to reduce proofs to the strictly k-affinoid case:

Definition 10.2. We say that $r \in (\mathbf{R}^*_+)^n$ is $|k^*|$ -free if $r^{\nu} \in |k^*| \Leftrightarrow \nu = 0$. In this case, set

$$K_r \coloneqq \left\{ f = \sum_{\nu \in \mathbf{Z}^n} a_{\nu} T^{\nu} \mid a_{\nu} \in k, \lim_{|\nu| \to \infty} |a_{\nu}| r^{\nu} = 0 \right\},$$

where we define the norm on K_r by $||f|| = \max_{\nu} |a_{\nu}| r^{\nu}$.

We have the following facts about K_r :

Properties 10.3.

- $k \hookrightarrow K_r$ is a non-Archimedean field extension;
- $|K_r^*|$ is generated by $|k^*|$ and r_1, \ldots, r_n ;
- K_r is k-affinoid; and
- There exists a bounded k-linear map

$$\pi \colon K_r \longrightarrow k$$
$$\sum_{\nu \in \mathbf{Z}^n} a_{\nu} T^{\nu} \longmapsto a_0$$

which is the identity on k. In other words, π is a section of the inclusion $k \hookrightarrow K_r$.

By Corollary 9.10, given any k-affinoid algebra \mathscr{A} , we can pick an n-tuple $r \in (\mathbb{R}^*_+)^n$ that is $|k^*|$ -free such that $\mathscr{A} \otimes_k K_r$ is a strictly K_r -affinoid algebra; note also that the ground field is now K_r , which is nontrivially valued. The natural map $\mathscr{A} \hookrightarrow \mathscr{A} \otimes_k K_r$ is an isometric embedding by Lemma 9.8, and by tensoring the section $\pi \colon K_r \to k$ by \mathscr{A} , we get a section $\pi \colon \mathscr{A} \otimes_k K_r \to \mathscr{A}$, which is \mathscr{A} -linear.

We clear up some of the proofs of results from last time.

Proposition 9.11. If \mathscr{A} is k-affinoid, then \mathscr{A} is Noetherian and all ideals are closed.

Proof. Pick r as in Corollary 9.10. Set $\mathscr{B} := \mathscr{A} \otimes_k K_r$. The inclusion $\mathscr{A} \subset \mathscr{B}$ has a section $\pi : \mathscr{B} \to \mathscr{A}$, which is \mathscr{A} -linear and bounded. We know already that \mathscr{B} is Noetherian and all ideals of \mathscr{B} are closed, since \mathscr{B} is strictly K_r -affinoid (Corollary 8.8). We claim the following:

Claim 10.4. For any $\mathfrak{a} \subset \mathscr{A}$, $\mathfrak{a} = (\mathscr{B} \cdot \mathfrak{a}) \cap \mathscr{A}$.

"⊂" is clear, so we want to show "⊃". Let $f \in \mathscr{B} \cdot \mathfrak{a} \cap \mathscr{A}$. Choose a finite number of generators h_j of $\mathscr{B} \cdot \mathfrak{a}$, and write

$$h_j = \sum_{s=1}^{n_j} h_{js} f_s, \qquad h_{js} \in \mathscr{B}, \ f_s \in \mathfrak{a}$$

Then, any $f \in \mathscr{B} \cdot \mathfrak{a} \cap \mathscr{A}$ can be written

$$f = \sum_{j=1}^{m} t_j h_j = \sum_{j,s} t_j h_{js} f_s, \qquad t_j h_{js} \in \mathscr{B}, \ f_s \in \mathfrak{a}.$$

Now $f, f_s \in \mathscr{A}$, and so

$$f = \pi(f) = \sum_{j,s} \pi(t_j h_{js}) f_s \in \mathfrak{a},$$

since $\pi(t_j h_{js}) \in \mathscr{A}$. Finally, $\mathscr{B} \cdot \mathfrak{a}$ is closed in \mathscr{B} , and \mathscr{A} is closed in \mathscr{B} , and so $\mathfrak{a} = \mathscr{B} \cdot \mathfrak{a} \cap \mathscr{A}$ is closed in \mathscr{A} .

Note that the proof above showed that \mathfrak{a} was finitely generated by using the elements $f_s \in \mathfrak{a}$ as generators. Alternatively, we can show \mathscr{A} is Noetherian as follows. Suppose we have an ascending chain

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$$

of ideals in \mathscr{A} . Then, their extensions stabilize in \mathscr{B} , i.e., $\mathscr{B} \cdot \mathfrak{a}_{m+1} = \mathscr{B} \cdot \mathfrak{a}_m$ for all $m \gg 0$, since \mathscr{B} is Noetherian. Now intersecting back with \mathscr{A} , we see that $\mathfrak{a}_{m+1} = \mathfrak{a}_m$ for all $m \gg 0$ in \mathscr{A} .

We also restate the following result from last time:

Corollary 9.16. If \mathscr{A} is k-affinoid, then \mathscr{A} is strictly k-affinoid if and only if $\rho(f) \in \sqrt{|k^*|} \cup \{0\}$ for all $f \in \mathscr{A}$.

Even though the proof last time was sketchy, we won't reprove this result since we don't need it later.

10.1.2 Finite modules and algebras over k-affinoid algebras; finite maps [Ber, §2.7]

Let \mathscr{A} be a k-affinoid algebra. Since we have shown \mathscr{A} is Noetherian, our previous general results in §8.1.1 concerning Noetherian Banach k-algebras and modules/algebras over them carry over, unless $|k^*| = \{1\}$. To recover results from the strictly k-affinoid case, we want to use the same base change trick as before. For example, this works for the equivalences of categories in Propositions 8.3 and 8.5, and for our results about tensor products in Proposition 8.6; see [Ber, Props. 2.7.4, 2.7.5].

On the other hand, there are some results which do not hold. For example, it is true that any homomorphism of Noetherian k-affinoid algebras is bounded [BGR84, Thm. 6.1.3/1]; we mentioned this for strictly k-affinoid algebras in Fact 8.17. This is not true for k-affinoid algebras, where k is now trivially valued:

Example 10.5 [Ber90, Rem. 2.1.13]. Let k be trivially valued, and choose $r, s \in \mathbf{R}$ such that 0 < r < s < 1. Consider the k-affinoid algebras $\mathscr{A} = k\{s^{-1}T\}$, $\mathscr{B} = k\{r^{-1}T\}$; note they are not strictly k-affinoid. As k-algebras, we know that $\mathscr{A} = \mathscr{B} = k[[T]]$ (see §4.4.1). The map $\mathscr{B} \to \mathscr{A}$ is k-linear, but is not bounded, since $||T^n||_{\mathscr{A}} = s^n$, whereas $||T^n||_{\mathscr{B}} = r^n$. The map $\mathscr{A} \to \mathscr{B}$ in the other direction is bounded, but not admissible. This shows how the Banach open mapping theorem 7.15 does not work.

Moreover, \mathscr{B} is a Banach \mathscr{A} -algebra, since $||ab||_{\mathscr{B}} \leq ||a||_{\mathscr{A}} \cdot ||b||_{\mathscr{B}}$, and is finite as an \mathscr{A} -algebra. But \mathscr{B} is not finite as a Banach \mathscr{A} -algebra (there is no admissible epimorphism $\mathscr{A}^n \to \mathscr{B}$): the main point is that if you take a base change $\mathscr{B} \otimes_k K_{\rho}$, this cannot be finite as a $\mathscr{A} \otimes_k K_{\rho}$ -algebra.

Upshot 10.6. When working over a trivially valued field k, you must include norms in all definitions in the form of admissibility.

We won't use the following result on finite maps right away, but we will soon enough: just as in algebraic geometry, finite maps play an interesting role.

Exercise 10.7. Let $\mathscr{A} \hookrightarrow \mathscr{B}$ be a finite injective ring homomorphism. Then,

- (1) $f: \operatorname{Spec} \mathscr{B} \to \operatorname{Spec} \mathscr{A}$ is surjective with finite fibers.
- (2) $f^{-1}(\operatorname{Max} \mathscr{A}) = \operatorname{Max} \mathscr{B}.$

We ask if there is a version of this in Berkovich's theory, or even possibly for Banach rings in general. We can say things for k-affinoid algebras, but Berkovich doesn't know if the above statements have analogues for general Banach rings.

Exercise 10.8. Let $\mathscr{A} \hookrightarrow \mathscr{B}$ be as above with \mathscr{A} a field. Then, $\mathscr{B}_{red} \cong K_1 \times \cdots \times K_m$, where K_i is a finite field extension of \mathscr{A} .

Proposition 10.9. Let \mathscr{A} be k-affinoid, and let \mathscr{B} be a finite Banach \mathscr{A} -algebra (this implies that \mathscr{B} is k-affinoid) such that the canonical map $\mathscr{A} \hookrightarrow \mathscr{B}$ is injective. Then, the induced map $f: \mathscr{M}(\mathscr{B}) \to \mathscr{M}(\mathscr{A})$ is surjective with finite fibers.

Proof. The inverse image under f of $x \in \mathcal{M}(\mathcal{A})$ is isomorphic to $\mathcal{M}(\mathcal{H}(x) \otimes_{\mathcal{A}} \mathcal{B})$. We have to show this is nonempty, but still finite. Note that you have to be careful with separated completion: you an easily make things become zero.

For finitude, since $\mathscr{A} \to \mathscr{B}$ is finite, we have that the base change

$$\mathscr{H}(x) = \mathscr{H}(x) \otimes_{\mathscr{A}} \mathscr{A} \longrightarrow \mathscr{H}(x) \otimes_{\mathscr{A}} \mathscr{B}$$

is also finite. This implies that $\mathscr{H}(x) \otimes_{\mathscr{A}} \mathscr{B} = \mathscr{H}(x) \otimes_{\mathscr{A}} \mathscr{B}$ by the analogue of Proposition 8.6 [Ber, Prop. 2.7.4(*iv*)]. Now, Exercise 10.8 implies that

$$(\mathscr{H}(x)\widehat{\otimes}_{\mathscr{A}}\mathscr{B})_{\mathrm{red}}=K_1\times\cdots\times K_m,$$

where K_i are finite field extensions of $\mathscr{H}(x)$. Thus,

$$f^{-1}(x) = \mathscr{M}((\mathscr{H}(x) \widehat{\otimes}_{\mathscr{A}} \mathscr{B})_{\mathrm{red}}) = \mathscr{M}(K_1 \times \cdots \times K_m)$$

is a finite set, where we note that modding out by the nilradical or the quasinilradical don't affect the Berkovich spectrum as a topological space. For nonemptiness, consider $f^{-1}(x) = \mathscr{M}(\mathscr{H}(x) \widehat{\otimes}_{\mathscr{A}} \mathscr{B})$. Pick $r \in (\mathbf{R}^*_+)^n$ which is $|k^*|$ -free such that $\mathscr{A} \widehat{\otimes}_k K_r$ and $\mathscr{B} \widehat{\otimes}_k K_r$ are strictly K_r -affinoid, and $|K_r^*| \neq \{1\}$ (use Corollary 9.10). We then have the commutative diagram

where the vertical arrows are surjective by Gruson's Theorem 3.21. We can therefore assume without loss of generality that \mathscr{A}, \mathscr{B} are strictly k-affinoid. Now by Exercise 10.7, the map $f: \operatorname{Max} \mathscr{B} \to \operatorname{Max} \mathscr{A}$ is surjective. Since $\operatorname{Max} \mathscr{A}$ is dense in $\mathscr{M}(\mathscr{A})$ (Proposition 8.14), and so $f: \mathscr{M}(\mathscr{B}) \to \mathscr{M}(\mathscr{A})$ is surjective. \Box

We remark that historically, the way that Berkovich presented this theory is he assumed all of Tate's theory. There aren't any sources that present the material logically from scratch for Berkovich spaces, so we will probably run into instances where we do things in the wrong logical order.

10.2 Weierstrass, Laurent, and rational domains [Ber90, §2.2]

We will mostly do motivation for the rest of class.

Let $X = \mathcal{M}(\mathcal{A})$, where \mathcal{A} is a k-affinoid algebra. We want to develop an analogue for affine schemes of finite type over a field, so that \mathcal{A} becomes the ring of global analytic functions on X.

Goal 10.10. Define a structure sheaf \mathcal{O}_X on X, so that for suitable $V \subset X$, $A_V \coloneqq \Gamma(V, \mathcal{O}_X)$ is the ring of analytic functions on V.

We will (surprisingly) take V to be closed sets. This makes sense because we should keep track of norms (which restrict well to closed sets), and because the building blocks we have so far are k-affinoid algebras, which are like power series that converge on closed polydiscs.

Idea 10.11. Use $V \subset X$ closed defined by (non-strict) inequalities, so that \mathscr{A}_V is a K-affinoid algebra.

This already gives rise to three different classes of such domains V. The reason why they are called "domains" is because if you intersect them with Max \mathscr{A} and look at them in the rigid topology, they are open.

Definition 10.12 (Weierstrass domains). Given $f_1, \ldots, f_n \in \mathscr{A}$ and $p_1, \ldots, p_n \in \mathbb{R}^*_+$, we define

$$V \coloneqq X(p^{-1}f) = \left\{ x \in X \mid |f_i(x)| \le p_i \ 1 \le i \le n \right\}$$

which is a closed subset of X, analogous to a polydisc. The analytic functions on it are

$$\mathscr{A}_{V} \coloneqq \frac{\mathscr{A}\{p_{1}^{-1}T_{1}, \dots, p_{n}^{-1}T_{n}\}}{(f_{i} - f_{i})} \quad \text{with residue norm}$$
$$\cong \left\{ g = \sum_{\nu \in \mathbf{Z}_{+}^{n}} a_{\nu}f^{\nu} \mid a_{\nu} \in \mathscr{A}, \ \lim_{|\nu| \to \infty} ||a_{\nu}|| p^{\nu} = 0 \right\}$$

Definition 10.13 (Laurent domains). Given $f_1, \ldots, f_n, g_1, \ldots, g_m \in \mathscr{A}$ and $p_1, \ldots, p_n, q_1, \ldots, q_m \in \mathbb{R}^*_+$, we define

$$V \coloneqq X(p^{-1}f, qg^{-1}) = \left\{ x \in X \mid |f_i(x)| \le p_i, |g_j(x)| \ge q_j \; \forall i, j \right\}$$

which is a closed subset of X, analogous to an annulus. The analytic functions on it are

$$\mathscr{A}_{V} \coloneqq \frac{\mathscr{A}\{p_{1}^{-1}T_{1}, \dots, p_{n}^{-1}T_{n}, q_{1}S_{1}, \dots, q_{m}S_{m}\}}{(f_{i} - T_{i}, g_{j}S_{j} - 1)} \quad \text{with residue norm}$$
$$\cong \left\{ h = \sum_{\mu,\nu \in \mathbf{Z}_{+}^{n+m}} a_{\mu\nu} f^{\mu}g^{-\nu} \middle| a_{\mu\nu} \in \mathscr{A}, \lim_{|\mu| + |\nu| \to \infty} ||a_{\mu\nu}|| p^{\mu}q^{-\nu} = 0 \right\}$$

Remark 10.14. Given $x \in X$, the family of all Laurent domains containing x form a basis of closed neighborhoods of x: if $U \ni x$ is open, then $U = \{y \in X \mid |f_i(y)| < t_i, |g_j(y)| > s_j\}$ form an open neighborhood basis, and so $V = X(p^{-1}f, qg^{-1}) \subseteq U$, where $|f_i(x)| < p_i < t_i$ and $|g_j(x)| > q_j > s_j$.

Definition 10.15 (*Rational domains*). Given $f_1, \ldots, f_n, g \in \mathscr{A}$ without common zero in X, and $p_1, \ldots, p_n \in \mathbb{R}^*_+$, we define

$$V \coloneqq X(p^{-1}\frac{f}{g}) = \left\{ x \in X \mid |f_i(x)| \le p_i |g(x)| \; \forall i \right\}$$

which is a closed subset of X. The analytic functions on it are

$$\begin{split} \mathscr{A}_{V} &\coloneqq \mathscr{A}\{p^{-1}\frac{f}{g}\} \\ &= \frac{\mathscr{A}\{p_{1}^{-1}T_{1}, \dots, p_{n}^{-1}T_{n}\}}{(gT_{i} - f_{i})} \quad \text{with residue norm} \\ &\cong \left\{h = \sum_{\nu \in \mathbf{Z}^{n}} a_{\nu} \left(\frac{f}{g}\right)^{\nu} \middle| a_{\nu} \in \mathscr{A}, \ \lim_{|\nu| \to \infty} \|a_{\nu}\| p^{\nu} = 0\right\} \end{split}$$

A priori, it is not obvious that you only need one denominator; using a trick you can show intersections of rational domains are rational domains.

Exercise 10.16.

- (1) V_1, \ldots, V_n Weierstrass $\implies V_1 \cap \cdots \cap V_n$ Weierstrass.
- (2) V_1, \ldots, V_n Laurent $\implies V_1 \cap \cdots \cap V_n$ Laurent.
- (3) V_1, \ldots, V_n rational $\implies V_1 \cap \cdots \cap V_n$ rational.
- (4) Weierstrass \implies Laurent \implies rational.
- (5) Let $\mathscr{A} \to \mathscr{B}$ be a bounded homomorphism of k-affinoid algebras, and consider the induced map $f: Y \to X$ on the Berkovich spectra $X = \mathscr{M}(\mathscr{A})$ and $Y = \mathscr{M}(\mathscr{B})$. Then, if $V \subseteq X$ is Weierstrass/Laurent/rational, then its preimage $f^{-1}(V)$ is Weierstrass/Laurent/rational.

Tate basically did this, but there is one more class of domains (affinoid domains) defined by universal properties that incorporate all of this.

11 October 11

Homework 4 is due Thursday, October 27. There will be no class next week.

We finished talking about affinoid algebras last time; today we will continue talking about affinoid spaces. Let k be a non-Archimedean field. Recall the following definition:

Definition 11.1. A Banach k-algebra \mathscr{A} is k-affinoid if there exists an admissible epimorphism $k\{r^{-1}T\} \to \mathscr{A}$ for some $r \in (\mathbf{R}^*_+)^n$, where $n \ge 1$. The algebra \mathscr{A} is strictly k-affinoid if we can pick $r_i = 1$ for all i.

As was the case in §§8–9, we will mostly consider strictly k-affinoid algebras when k is not trivially valued, since these appear naturally when we reduce from general k-affinoid algebras to strictly k-affinoid ones by using the field extension $k \hookrightarrow K_r$.

11.1 The category of k-affinoid spaces [Tem05, §3]

We think of k-affinoid algebras as the algebra of global functions on Berkovich spectra. We want to ultimately make this precise by putting structure sheaves on Berkovich spectra. Our first step is to define a suitable category of spaces to work with; our notation follows [Tem05, §3].

Definition 11.2. The category k-Aff of k-affinoid spaces has objects $X = \mathscr{M}(\mathscr{A})$, where \mathscr{A} k-affinoid algebra, and morphisms $\mathscr{M}(\mathscr{B}) \to \mathscr{M}(\mathscr{A})$ induced by bounded k-linear morphisms $\mathscr{A} \to \mathscr{B}$ of k-affinoid algebras. Similarly, you can define st-k-Aff to be the category of strictly k-affinoid spaces.

The categories k-Aff and st-k-Aff admit fiber products, since they are (by definition) the opposite categories to {k-affinoid algebras} and {strictly k-affinoid algebras}, respectively:



You can check that $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{C}$ is k-affinoid if \mathscr{B} and \mathscr{C} are, by constructing an adissible epimorphism from a polydisc algebra with different combinations of radii coming from \mathscr{B} and \mathscr{C} .

Remark 11.3. If $f: Y \to X$ is a morphism of k-affinoid spaces, and $y \in Y$ is a point, then letting x := f(y), there exists an isometric embedding $\mathscr{H}(x) \hookrightarrow \mathscr{H}(y)$ between completed residue fields.

11.2 Weierstrass, Laurent, and rational domains (continued) [Ber90, §2.2]

We will now work toward defining structure sheaves on affinoid spaces. For suitable closed subsets of a k-affinoid space, we already have a nice definition for analytic functions on that closed subset. We will later ask how these functions can be restricted to smaller subsets.

Fix a k-affinoid space $X = \mathscr{M}(\mathscr{A})$. We want a "large" collection of pairs (V, \mathscr{A}_V) , where $V \subseteq X$ is a closed subset, and \mathscr{A}_V is a k-affinoid algebra, which we think of as the algebra of analytic functions on V. There exists a bounded map $\mathscr{A} \to \mathscr{A}_V$, which can be thought of as restriction of functions; these restriction maps will satisfy suitable sheafy properties.

There are different kinds of domains. We want to get to affinoid domains, which are the analogue of affine subschemes of a scheme in algebraic geometry. But first, we recall that we have already defined three natural classes of domains (V, \mathscr{A}_V) :

Definition 11.4 (Weierstrass domains). Given $f_1, \ldots, f_n \in \mathscr{A}$ and $p_1, \ldots, p_n \in \mathbf{R}^*_+$, we define

$$V \coloneqq X(p^{-1}f) \coloneqq \left\{ x \in X \mid |f_i(x)| \le p_i, \ 1 \le i \le n \right\}$$

which is a closed subset of X, analogous to a polydisc. The analytic functions on it form the "relative polydisc algebra"

$$\mathscr{A}_V \coloneqq \frac{\mathscr{A}\{p_1^{-1}T_1, \dots, p_n^{-1}T_n\}}{(f_i - T_i)}$$
 with residue norm

The following are more general:

Definition 11.5 (Laurent domains). Given $f_1, \ldots, f_n, g_1, \ldots, g_m \in \mathscr{A}$ and $p_1, \ldots, p_n, q_1, \ldots, q_m \in \mathbb{R}^*_+$, we define

$$V \coloneqq X(p^{-1}f, qg^{-1}) \coloneqq \{ x \in X \mid |f_i(x)| \le p_i, \ |g_j(x)| \ge q_j, \ \forall i, j \}$$

which is a closed subset of X, analogous to an annulus. The analytic functions on it are

$$\mathscr{A}_V := \frac{\mathscr{A}\{p_1^{-1}T_1, \dots, p_n^{-1}T_n, q_1S_1, \dots, q_mS_m\}}{(f_i - T_i, g_jS_j - 1)} \quad \text{with residue norm}$$

The following are a further, more flexible generalization:

Definition 11.6 (*Rational domains*). Given $f_1, \ldots, f_n, g \in \mathscr{A}$ without common zero in X, and $p_1, \ldots, p_n \in \mathbb{R}^*_+$, we define

$$V \coloneqq X(p^{-1}\frac{f}{g}) \coloneqq \left\{ x \in X \mid |f_i(x)| \le p_i |g(x)|, \ \forall i \right\}$$

which is a closed subset of X. The analytic functions on it are

$$\mathscr{A}_V \coloneqq \frac{\mathscr{A}\{p_1^{-1}T_1, \dots, p_n^{-1}T_n\}}{(gT_i - f_i)} \quad \text{with residue norm}$$

We go through some easy properties of these:

Elementary Properties 11.7.

- (1) Invariance under pullback: If $f: Y \to X$ is a morphism of k-affinoid spaces, and if $V \subseteq X$ is a Weierstrass/Laurent/rational domain, then its preimage $f^{-1}(V) \subseteq Y$ is Weierstrass/Laurent/rational.
- (2) Invariance under finite intersections: If $V_1, \ldots, V_n \subset X$ is Weierstrass/Laurent/rational, then $V_1 \cap \cdots \cap V_n$ is Weierstrass/Laurent/rational.
- (3) Weierstrass \implies Laurent \implies rational.

Note that in (1), if \mathscr{B} denotes the k-affinoid algebra associated to Y, the algebra $\mathscr{B}_{f^{-1}(V)}$ can be described as $\mathscr{B}_{f^{-1}(V)} \cong \mathscr{B} \widehat{\otimes}_{\mathscr{A}} \mathscr{A}_V$. For (2), the hardest case is showing that intersections of rational domains are still rational. For (3), the hardest implication is that Laurent domains are rational, but the trick is that you can reduce to the case where there is only one denominator q defining the Laurent domain by using (2).

There are examples showing these domains are different. It is relatively easy to find examples of Laurent domains that are not Weierstrass, and there is an example of a rational domain that is not Laurent; see [Jon16, p. 38] and [BGR84, 7.2.4]. Last time, we noted that there are enough of these domains to give closed neighborhood bases around any point (Remark 10.14). Note that one reason we want rational domains that Weierstrass domains satisfy a transitivity property but Laurent domains do not; see [BGR84, 7.2.4].

We will now deviate a bit from [Ber90, $\S2.2$].

11.3The Tate acyclicity theorem for rational domains [BGR84, Ch. 8]

We now come to one of the main theorems that really gets everything off the ground. This theorem is due to Tate in the rigid analytic setting [Tat71, Thm. 8.2]; we will show an analogue of his result in the setting of Berkovich spaces. The motivation for the theorem is that it allows some sort of sheaf theory to be developed.

We will first prove a special case for rational domains.

Let $X = \mathcal{M}(\mathcal{A})$ be a k-affinoid space, and consider a finite number $V_1, \ldots, V_n \subset X$ of rational domains such that $X = \bigcup_{i=1}^{n} V_i$. Each intersection $V_i \cap V_j$ is a rational domain, and so we get restriction maps $\mathscr{A} \to \mathscr{A}_{V_i} \to \mathscr{A}_{V_i \cap V_i}$, etc., between the algebras of analytic functions on finite intersections of members of our cover. These restriction maps form a complex:

$$0 \longrightarrow \mathscr{A} \longrightarrow \prod_{i} \mathscr{A}_{V_{i}} \longrightarrow \prod_{i,j} \mathscr{A}_{V_{i} \cap V_{j}} \longrightarrow \cdots$$

$$f \longmapsto (f|_{V_{i}})_{i}$$

$$(f_{i}) \longmapsto ((f_{i} - f_{j})|_{V_{ij}})_{i,j}$$

$$(5)$$

The theorem is that this complex is exact.

Tate Acyclicity Theorem 11.8 (for rational domains). This complex (5) is exact and admissible.

Remark 11.9. This says that $V \mapsto \mathscr{A}_V$ is a sheaf on a suitable G-topology (a special kind of Grothendieck topology) on X: exactness at $\prod_i \mathscr{A}_{V_i}$ says that a function that is zero on a cover is zero on X, and exactness at $\prod_{i,j} \mathscr{A}_{V_i \cap V_i}$ says that functions can be glued if they match on overlaps. However, rational domains do not generate a nice topology in the standard sense.

Proof Sketch. Tate only worked with strictly k-affinoid algebras, where k is nontrivially valued, and so the first step is to reduce to that case:

Step 1. Reduce to the case when \mathscr{A} is a strictly k-affinoid algebra, where k is nontrivially valued, and the V_i are defined using constants $p_{i\ell} = 1$.

Pick $r \in (\mathbf{R}^*_+)^m$ that is $|k^*|$ -free, such that $\mathscr{A} \otimes_k K_r$ is strictly K_r -affinoid, $|K^*_r| \neq \{1\}$, and $p_{i\ell} \in \sqrt{|K^*_r|}$. Recall the following:

Key Fact 11.10 (Lemma 9.8(2)). A sequence $V' \to V \to V''$ of bounded homomorphisms of k-Banach spaces is exact and admissible if and only if $V' \widehat{\otimes}_k K_r \to V \widehat{\otimes}_k K_r \to V'' \widehat{\otimes}_k K_r$ is exact and admissible.

The direction \Leftarrow essentially completes Step 1.

Remark 11.11. The admissibility condition is automatic if $|k^*| \neq \{1\}$ by Corollary 7.16, which says bounded k-linear maps are admissible. But the correct statement in the trivially valued case includes admissibility.

What is left is to show that we can assume $p_{i\ell} = 1$ for all ℓ . Write

$$V_i = X(p_i^{-1} \frac{f_i}{g_i}) = \left\{ x \in X \mid |f_{i\ell}(x)| \le p_{i\ell} |g(x)| \; \forall \ell \right\} \quad \text{where} \quad p_{i\ell} \in \sqrt{|k^*|}.$$

Then, writing $p_{i\ell} = |a_{i\ell}|^{1/n}$ for $a_{i\ell} \in k^*$, the inequalities are equivalent to $|f_{i\ell}^n| \le |a_{i\ell}g_i^n|$. After Step 1, the rest of the proof follows Tate's original proof.

Step 2. Reduce to the case of a special covering

$$X = X(f) \cup X(\frac{1}{f})$$

where $X(f) = \{|f| \le 1\}$ and $X(\frac{1}{f}) = \{|f| \ge 1\}$, which are Weierstrass and Laurent domains, respectively.

The proof of this step is a diagram chase, mimicking Čech cohomology-style arguments from [Ser55]. See [BGR84, 8.1.4] for details.

Step 3. End of proof.

After the reduction in the previous step, we want to show that the complex

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{A}\{f\} \times \mathscr{A}\{f^{-1}\} \longrightarrow \mathscr{A}\{f, f^{-1}\} \longrightarrow 0$$

is exact. Here, we are being a bit informal with our notation; more precisely, the objects in our complex are

$$\mathscr{A}\lbrace f\rbrace = \left\{ \sum_{i=0}^{\infty} a_i f^i \mid \|a_i\| \to 0 \text{ as } i \to +\infty \right\}$$
$$\mathscr{A}\lbrace f^{-1}\rbrace = \left\{ \sum_{i=-\infty}^{0} a_i f^i \mid \|a_i\| \to 0 \text{ as } i \to -\infty \right\}$$
$$\mathscr{A}\lbrace f, f^{-1}\rbrace = \left\{ \sum_{i=-\infty}^{\infty} a_i f^i \mid \|a_i\| \to 0 \text{ as } i \to \pm\infty \right\}$$

In this case, exactness is clear; see [BGR84, 8.2.3] for details.

So now we have something to start with: the G-topology has a well-defined notion of analytic functions on rational domains.

11.4 Affinoid domains [Tem05, §3]

This is a bit of a detour, but it ties up well with affine schemes. We will follow the approach in [Tem05]. Let $X = \mathscr{M}(\mathscr{A})$ be a k-affinoid space.

Definition 11.12. An affinoid domain of X is a pair (V, \mathscr{A}_V) , where

• $V \subset X$ is a closed subset and

• \mathscr{A}_V is a k-affinoid algebra with bounded map $\mathscr{A} \to \mathscr{A}_V$,

satisfying the following properties:

(1) The image of $\mathscr{M}(\mathscr{A}_V) \to \mathscr{M}(\mathscr{A}) = X$ is exactly V,

(2) Any morphism $Y \to X$ with image contained in V factors uniquely through $\mathcal{M}(\mathcal{A}_V) \to V$:

Working in the category st-k-Aff gives the notion for a strictly affinoid domain of a strictly k-affinoid space X.

Remark 11.13. This definition is not the definition in [Ber90, Def. 2.2.1], but is (non-trivially) equivalent to it by [Tem05, Cor. 3.2]. The differences are that in (1), [Ber90, Def. 2.2.1] only requires that the image of $\mathcal{M}(\mathcal{A}_V) \to \mathcal{M}(\mathcal{A})$ is contained in V, and in (2), the universal property is a bit stronger.

We show that affinoid domains are a generalization of rational domains.

Proposition 11.14. Every rational domain is an affinoid domain.

Proof. Write $X = \mathcal{M}(\mathscr{A})$, and $V = X(p^{-1}\frac{f}{g}) = \{|f_i| \leq p_i|g|\}$, where the functions $f_1, \ldots, f_n, g \in \mathscr{A}$ have no common zero. Consider the following algebra

$$\mathscr{A}_V = \frac{\mathscr{A}\{p_1^{-1}T_1, \dots, p_n^{-1}T_n\}}{(gT_i - f_i)}$$

where the map $\mathscr{A} \to \mathscr{A}_V$ is the obvious one.

We first check (1). Pick $y \in \mathcal{M}(\mathcal{A}_V)$. Then, $|T_i(y)| \leq p_i$ for all *i*. This implies that $|f_i(y)| \leq p_i |g(y)|$ for all *i*, and implies that the image of *y* under $\mathcal{M}(\mathcal{A}_V) \to \mathcal{M}(\mathcal{A}) = X$ lies in *V*. Conversely, every point in *V* is the image of such a point.

Now we have to check the universal property (2). Consider a bounded map $\varphi : \mathscr{A} \to \mathscr{B}$, and suppose $\varphi^* \colon Y \to X$ satisfies $\varphi^*(Y) \subseteq V$. Then, for all $y \in Y$ and for all i, we have $|\varphi(f_i)(y)| \leq p_i |\varphi(g)(y)|$. Since f_1, \ldots, f_n, g have no common zero, $\varphi(g)(y) \neq 0$ for every $y \in Y$. This implies $\varphi(g)$ is invertible in \mathscr{B} by Corollary 3.15. Now set $h_i \coloneqq \varphi(f_i)/\varphi(g) \in \mathscr{B}$. For every $y \in Y$, we have that $|h_i(y)| \leq p_i$, and so $\rho(h_i) \leq p_i$ for all i by Theorem 3.16. Now define

$$\varphi_V \colon \mathscr{A}\{p_1^{-1}T_1, \dots, p_n^{-1}T_n\} \longrightarrow \mathscr{B}$$
$$T_i \longmapsto h_i$$

Since

$$\varphi_V(gT_i - f_i) = \varphi(g)h_i - \varphi(f_i) = 0,$$

this map φ_V descends to a map $\mathscr{A}_V \to \mathscr{B}$ that makes the diagram on the right of (6) commute. Uniqueness is left as an exercise.

Our next goal is to work towards proving basic properties of affinoid domains, and ultimately prove the Tate acyclicity theorem for affinoid domains.

11.5 The Gerritzen–Grauert theorem [Tem $05, \S3$]

The next Theorem is a structure result for affinoid domains, which says that in some sense, affinoid domains are not that much more general than rational domains. This is the Gerrizten–Grauert Theorem, which we state below:

Theorem 11.15 (Gerritzen–Grauert; Temkin). Every affinoid domain is a finite union of rational domains.

We mention Temkin's name since the proof we will present is due to him. Note that the converse of Theorem 11.15 is not true!

Remark 11.16. The full version of the Gerritzen–Grauert Theorem [Tem05, Thm. 3.1] says more, and concerns monomorphisms $\mathscr{M}(\mathscr{A}_V) \to X$.

Exercise 11.17. Every affinoid domain embedding $\mathscr{M}(\mathscr{A}_V) \to X$ is a monomorphism in the category of k-Aff of k-affinoid spaces.

Next time, we will prove part of Theorem 11.15, and go on to studying the structure sheaf on $\mathcal{M}(\mathcal{A})$ in more detail.

12 October 13

Homework 4 is due Thursday, October 25. There is no class next week.

Today, we will try to finish up our discussion of affinoid domains of affinoid spaces.

12.1 Affinoid domains (continued) [Tem05, §3]

Let k be a non-Archimedean field, and let \mathscr{A} be a k-affinoid algebra; this means that there exists an admissible epimorphism $k\{r^{-1}T\} \rightarrow \mathscr{A}$. Recall that these form a category with bounded maps as morphisms. We then defined the category k-Aff of k-affinoid spaces, whose objects are of the form $X = \mathscr{M}(\mathscr{A})$ for \mathscr{A} as above, and whose morphisms are induced by morphisms of k-affinoid algebras (Definition 11.2). This category admits fiber products (§11.1), and you can define a similar category st-k-Aff with strictly affinoid k-algebras.

We also discussed a bit about how we could define the structure sheaf on a k-affinoid space, or at least what its values should be for Weierstrass, Laurent, and rational domains; later we will define the structure sheaf for all affinoid domains.

Affinoid domains are the analogue of affine subschemes. We give the definition again, following [Tem05]. Note that this notion was actually present in Tate's work [Tat71, §7].

Definition 11.12. An affinoid domain of X is a pair (V, \mathscr{A}_V) , where

• $V \subset X$ is a closed subset and

• \mathscr{A}_V is a k-affinoid algebra with a bounded map $\mathscr{A} \to \mathscr{A}_V$, turning \mathscr{A}_V into a \mathscr{A} -Banach algebra, satisfying the following properties:

- (1) The induced map $\mathscr{M}(\mathscr{A}_V) \to \mathscr{M}(\mathscr{A}) = X$ has image exactly V,
- (2) Any morphism $Y \to X$ with image *contained* in V factors uniquely through $\mathcal{M}(\mathcal{A}_V) \to V$:



Working in the category st-k-Aff gives the notion for a strictly affinoid domain in a strictly k-affinoid space X.

We saw the following fact last time:

Example 12.1 (Proposition 11.14). Any rational domain $V = X(p^{-1}\frac{f}{g}) = \{|f_i| \leq p_i|g|\}$ is an affinoid domain, where $f_i, g \in \mathscr{A}$ have no common zero, $p_i \in \mathbf{R}_+$. Similarly, if we have a rational domain such that $p_i \in \sqrt{|k^*|}$ inside of a strictly k-affinoid space, we then V is a strictly affinoid domain. Note that in this strictly k-affinoid case, we can assume that $p_i = 1$ for all i.

We say that map $\mathscr{M}(\mathscr{A}_V) \to X$ is an affinoid domain embedding. We want these to satisfy some properties that open subschemes do. In particular, we start off with the analogue of the fact that open embeddings of schemes are monomorphisms in the category of schemes.

Definition 12.2. A morphism $Y \to X$ in a category is a monomorphism if for every object Z the induced map $\operatorname{Hom}(Z, Y) \to \operatorname{Hom}(Z, X)$ is injective.

Exercise 12.3. An affinoid domain embedding is a monomorphism in k-Aff.

Other examples of monomorphisms will include closed immersions $Y \hookrightarrow X$, defined by an admissible epimorphism $\mathscr{A} \twoheadrightarrow \mathscr{B}$.

Remark 12.4. If a category admits fiber products, then $Y \to X$ is a monomorphism if and only if the diagonal map $Y \to Y \times_X Y$ is an isomorphism (use universal properties).

Now we want to prove some statements about affinoid domain embeddings. The first result is not much more difficult to prove for all monomorphisms, so we work in this more general setting.

Proposition 12.5. Suppose $Y \to X$ is a monomorphism in k-Aff. Pick $y \in Y$ and set x = f(y). Then, the canonical map $\mathscr{H}(x) \hookrightarrow \mathscr{H}(y)$ (which is always an isometric embedding) is an (isometric) isomorphism, and moreover, $f^{-1}(x) = \{y\}$.

This shows that at least, monomorphisms are injective on underlying sets. Moreover, since Y is compact, and X is Hausdorff, we have

Corollary 12.6. $f: Y \to X$ is a topological embedding, that is, f is a homeomorphism onto its image.

We now prove the Proposition; the trick is to use complete tensor products.

Proof of Proposition 12.5. Let \mathscr{A}, \mathscr{B} such that $X = \mathscr{M}(\mathscr{A}), Y = \mathscr{M}(\mathscr{B})$. Recall that in general, the preimage $f^{-1}(x)$ of x is isomorphic to $\mathscr{M}(\mathscr{B} \widehat{\otimes}_{\mathscr{A}} \mathscr{H}(x))$. It now suffices to show that the canonical map

$$\mathscr{H}(x) \longrightarrow \mathscr{B} \widehat{\otimes}_{\mathscr{A}} \mathscr{H}(x) \tag{7}$$

is an isomorphism. By Remark 12.4, $f: Y \to X$ is a monomorphism if and only if $\Delta: Y \to Y \times_X Y$ is an isomorphism. Since k-Aff is the opposite category to the category of k-affinoid algebras, the diagonal $\Delta: Y \to Y \times_X Y$ is an isomorphism if and only if the codiagonal $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{B} \to \mathscr{B}$ is an isomorphism. Now set $K := \mathscr{H}(x)$, which is a non-Archimedean extension of k. Applying $- \otimes_{\mathscr{A}} K$ to the codiagonal, we have an isomorphism

$$\mathscr{B}_K \widehat{\otimes}_K \mathscr{B}_K \xrightarrow{\sim} \mathscr{B}_K \coloneqq \mathscr{B} \widehat{\otimes}_{\mathscr{A}} K_{\mathcal{A}}$$

where we note that \mathscr{B}_K is a finite-dimensional K-vector space. By Gruson's Theorem 3.21, we have an injection $\mathscr{B}_K \otimes_K \mathscr{B}_K \hookrightarrow \mathscr{B}_K \widehat{\otimes}_K \mathscr{B}_K$. Thus, $\mathscr{B}_K \otimes_K \mathscr{B}_K \hookrightarrow \mathscr{B}_K$ is an injection of finite-dimensional vector spaces; by counting dimensions, this implies that $\dim_K \mathscr{B}_K = 1$. Thus, the morphism (7) is an isomorphism.

We now give an example as to why the "exactly" condition is necessary in Definition 11.12(1).

Example 12.7. Let $X = E(1) = \mathscr{M}(k\{T\})$ be the unit disc, and let $V = \{x_0\} \cup E(r)$, where x_0 is the Gauss point, and 0 < r < 1. This is not affinoid (which we can't prove now; the point is that there is no reasonable choice for \mathscr{A}_V). On the other hand, suppose that $Y \to X$ has image contained in V. Then, you can show that the image already lies in E(r), and so the point $\{x_0\}$ should be taken out of V to get an actual affinoid domain.

This issue doesn't come up for rigid analytic space, since in Tate's theory, you are only considering the rational points in Berkovich spaces; you can in fact show that in Tate's theory, containment in Definition 11.12(1) automatically implies equality, using the universal property in Definition 11.12(2). Note that Berkovich's definition [Ber90, Def. 2.2.1] is different: \mathscr{B} is allowed to be any K-affinoid algebra, where K is a non-Archimedean extension of k.

We now state one of the main Theorems, which we won't prove now.

Gerritzen–Grauert Theorem 11.15. Every affinoid domain of a k-affinoid space is a finite union of rational domains.

The converse of Theorem 11.15 is not true, however. These sorts of properties are well-behaved under finite intersections, but not unions.

The proof uses the theory of reduction [Ber90, $\S2.4$]: defining

$$\widetilde{\mathscr{A}} \coloneqq \{\rho \le 1\} / \{\rho < 1\},\$$

we get a map $\mathcal{M}(\mathscr{A}) \to \operatorname{Spec}(\widetilde{\mathscr{A}})$. There is a nice proof of the Gerritzen–Grauert Theorem 11.15 in [Tem05], which uses a generalized version of reduction.

Remark 12.8. The full Gerritzen–Grauert theorem (as formulated in [Tem05, Thm. 1.1]) gives a similar statement for all monomorphisms, and shows that any monomorphism is a special kind of closed immersion, called a *Runge immersion*.

We now give an alternative definition for affinoid domains. There is also a strict version of this as well, and monomorphisms in this category end up being locally closed immersions.

Definition 12.9 [Ber90, Def. 2.2.1]. (V, \mathscr{A}_V) is an affinoid domain in $X = \mathscr{M}(\mathscr{A})$ if $V \subset X$ is a closed subset satisfying the following properties:

- (1) The induced map $\mathcal{M}(\mathscr{A}_V) \to \mathcal{M}(\mathscr{A}) = X$ has image contained in V, and
- (2) For all bounded k-linear maps $\mathscr{A} \to \mathscr{B}$, where \mathscr{B} is a K-affinoid algebra for some non-Archimedean extension K/k, such that the image of $\mathscr{M}(\mathscr{B}) \to \mathscr{M}(\mathscr{A})$ is contained in V, the map $\mathscr{A} \to \mathscr{B}$ factors uniquely through $\mathscr{A} \to \mathscr{A}_V$:



This is a bit technical, but we want to say something about why these two definitions are equivalent.

Proposition 12.10. The two definitions of affinoid domains are equivalent.

Sketch. Berkovich \Rightarrow Temkin. We must prove that the image of $\mathscr{M}(\mathscr{A}_V) \to X$ is all of V. Pick any $x \in V$, and set $\mathscr{B} = K = \mathscr{H}(x)$. Then, we have morphisms



and the induced map $\mathscr{A}_V \to \mathscr{H}(x)$ induces a bounded multiplicative seminorm on $\mathscr{M}(\mathscr{A}_V)$, which is just a point $x' \in \mathscr{M}(\mathscr{A}_V)$. Now you check that the image of x' under the $\mathscr{M}(\mathscr{A}_V) \to \mathscr{M}(\mathscr{A})$ is x.

Temkin \Rightarrow Berkovich. Given a non-Archimedean field extension K/k, a K-affinoid algebra \mathscr{B} , and a k-linear bounded map $\mathscr{A} \to \mathscr{B}$ such that the image of $\mathscr{M}(\mathscr{B}) \to \mathscr{M}(\mathscr{A})$ is contained in V, we must show that $\mathscr{A} \to \mathscr{B}$ factors as



The rest of the proof, at least the proof in [Tem05, Cor. 3.2], uses the Gerritzen–Grauert Theorem 11.15. \Box

13 October 25

Homework 4 is due Thursday. We will hopefully finish up our discussion of affinoid domains today.

13.1 Affinoid domains (continued) [Tem05, §3]

Let k be a non-Archimedean field, and denote by $X = \mathscr{M}(\mathscr{A})$ a k-affinoid space, where \mathscr{A} is a k-affinoid algebra. Recall the convention that k-Aff is the opposite category to the category of k-affinoid algebras, with bounded maps between them.

We state the two possible definitions of affinoid domains in parallel:

Definition 13.1 (Temkin (resp. Berkovich)). An affinoid domain in $X = \mathcal{M}(\mathcal{A})$ is a pair (V, \mathcal{A}_V) consisting of a *closed* subset $V \subseteq X$ and a *k*-affinoid algebra \mathcal{A}_V with a bounded *k*-linear map $\mathcal{A} \to \mathcal{A}_V$ satisfying: (1) The induced map $\mathcal{M}(\mathcal{A}_V) \to X$ has image equal to (resp. contained in) V, and

- (1) The induced map $\mathcal{M}(\mathcal{M}) \to \mathcal{N}$ has image equal to (resp. contained in) V, and (2) For $K = l_{1}$ (non-any non-Archimedean systemation K/l_{2}) and any l_{1} linear bounded
- (2) For K = k (resp. any non-Archimedean extension K/k) and any k-linear bounded map $\varphi \colon \mathscr{A} \to \mathscr{B}$, where \mathscr{B} is a K-affinoid algebra such that $\varphi^* \colon (\mathscr{M}(\mathscr{B})) \subset V$, there exists a unique $\varphi_V \colon \mathscr{A}_V \to \mathscr{B}$ such that the diagrams below commute:



where $Y = \mathcal{M}(\mathcal{B})$.

We started proving the following proposition last time:

Proposition 12.10. The two definitions of affinoid domains are equivalent.

Proof. The direction Berkovich \Rightarrow Temkin is easier; it suffices to show that the image of $\mathscr{M}(\mathscr{A}_V) \to \mathscr{M}(\mathscr{A}) = X$ is all of V. Given $x \in X$, use $\mathscr{B} = K = \mathscr{H}(x)$.

For the other direction Temkin \Rightarrow Berkovich, we use the Gerritzen–Grauert theorem. Since this technique will come up later, we will prove this carefully. Given \mathscr{B} a K-affinoid algebra with a bounded map $\varphi \colon \mathscr{A} \to \mathscr{B}$, we need to produce $\varphi_V \colon \mathscr{A}_V \to \mathscr{B}$. This is easy if V is a rational domain; see the proof that rational domains are affinoid (in the sense of Temkin; essentially the same proof shows that they are affinoid in the sense of Berkovich) in Proposition 11.14. In general, the Gerritzen–Grauert theorem 11.15 says that $V = \bigcup_{i \in I} V_i$, where V_i are rational domains, and I is finite. This implies we can pullback this covering to $Y = \mathscr{M}(\mathscr{B})$; call W_i the preimage of these V_i . Note they cover Y, and are rational domains, defined by the images under φ of their defining equations. Then, let \mathscr{B}_{W_i} be the algebra $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{A}_{V_i}$. We get the following commutative diagram of bounded maps

$$\begin{array}{cccc} 0 & \longrightarrow & \mathscr{A}_{V} & \longrightarrow & \prod_{i \in I} \mathscr{A}_{V_{i}} & \longrightarrow & \prod_{i,j} \mathscr{A}_{V_{i} \cap V_{j}} \\ & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathscr{B} & \longrightarrow & \prod_{i \in I} \mathscr{B}_{W_{i}} & \longrightarrow & \prod_{i,j} \mathscr{B}_{W_{i} \cap W_{j}} \end{array}$$

By Tate's acyclicity theorem 11.8 for rational coverings, you can show that the rows are exact and admissible. In particular, there is a map $\mathscr{A}_v \to \mathscr{B}$, since a function $f \in \mathscr{A}_V$ gets sent to $(f_i)_{i \in I} \in \ker(\prod \mathscr{A}_{V_i} \to \prod \mathscr{A}_{V_i \cap V_j})$ in the top row, which maps to $(g_i)_{i \in I} \in \ker(\prod \mathscr{B}_{W_i} \to \prod \mathscr{B}_{W_i \cap W_j}) = \operatorname{im}(\mathscr{B} \to \prod \mathscr{B}_{W_i})$ in the bottom row, giving rise to a unique $g \in \mathscr{B}$ which maps to $(g_i)_{i \in I}$. Defining $\varphi_V(f) \coloneqq g$ gives a well-defined map making the diagram commute, which is bounded by admissibility of the injection in the top row. \Box

This idea of reducing to rational domains and applying Tate's acyclicity theorem will come up again. We now state some properties of affinoid domains.

Properties 13.2 (of affinoid domains).

- (1) If $V \subset X$ is an affinoid domain, and $f: Y \to X$ is a morphism where $Y = \mathscr{M}(\mathscr{B})$ and $X = \mathscr{M}(\mathscr{A})$, then $f^{-1}(V) \subset Y$ is an affinoid domain with algebra $\mathscr{B} \widehat{\otimes}_{\mathscr{A}} \mathscr{A}_V$.
- (2) If $U, V \subset X$ are affinoid domains, then $U \cap V$ is an affinoid domain, and $\mathcal{A}_{U \cap V} = \mathcal{A}_U \otimes_{\mathscr{A}} \mathcal{A}_V$.
- (3) [Transitivity] If $V \subset X$ is an affinoid domain, and $U \subset V$ is an affinoid domain, then $U \subset X$ is an affinoid domain via the map $\mathscr{A} \to \mathscr{A}_V \to \mathscr{A}_U$.

The following is done in [BGR84, 8.2.2]:

Tate Acyclicity Theorem 13.3 (for affinoid domains). The complex

$$0 \longrightarrow \mathscr{A} \longrightarrow \prod_{i} \mathscr{A}_{V_{i}} \longrightarrow \prod_{i,j} \mathscr{A}_{V_{i} \cap V_{j}} \longrightarrow \cdots$$
(8)

is exact and admissible, where $X = \bigcup_i V_i$ is a covering of a k-affinoid space $X = \mathcal{M}(\mathcal{A})$ by finitely many affinoid domains.

Proof. Use the Gerritzen–Grauert theorem 11.15 and some Čech cohomology arguments to reduce to the case of a rational cover.

13.2 Strictly affinoid domains [Tem05, §3]

Now assume that $|k^*| \neq \{1\}$, and that X is strictly k-affinoid (that is, there is an admissible epimorphism $\mathscr{T}_n \twoheadrightarrow \mathscr{A}$). We can then define strictly affinoid domains in X using Temkin's definition: it is a pair (V, \mathscr{A}_V) where \mathscr{A}_V is a strictly k-affinoid algebra, whose image is equal to V, satisfying a universal property in the category st-k-Aff. Since $\overline{\text{Max}} \mathscr{A} = \mathscr{M}(\mathscr{A})$ under our assumptions (Proposition 8.14), you can check that this definition is equivalent to Tate's definition [Tat71, §7] of (strictly) affinoid domains using the maximal ideal

spectrum, so that closures of (strictly) affinoid domains in Tate's sense will be (strictly) affinoid domains in Temkin's sense. Note that the relevant diagram Tate's definition calls for a diagram



In this setting, you still have a version of the Gerritsen–Grauert theorem [Tem05, Thm. 1.1], where now the rational domains in the cover can be chosen with coefficients all equal to 1. There is also a version of the Tate acyclicity theorem.

It would be nice to be able to prove everything about affinoid algebras, domains, and spaces directly, but it seems hard to avoid passing to strictly affinoid analogues. So we need some results about how the notions are related.

Proposition 13.4. Let X be a strictly k-affinoid space, where k is nontrivially valued.

- (i) Every strictly affinoid domain in X is an affinoid domain.
- (ii) If $V \subset X$ is an affinoid domain, then V is a strictly affinoid domain if and only if \mathscr{A}_V is strictly k-affinoid.

Sketch. (i) This is similar to the proof that "Temkin \Rightarrow Berkovich" (and so it uses the Gerritzen–Grauert theorem 11.15). For (ii), " \Rightarrow " is clear. For " \Leftarrow " you check that the universal property still holds: this is clear since st-k-Aff \rightarrow k-Aff is fully faithful (they are all just bounded maps).

13.2.1 Ground field extension

We now say something about ground field extension. Let k be an arbitrary non-Archimedean field, and let $X = \mathscr{M}(\mathscr{A})$ be k-affinoid. Then, given K/k a non-Archimedean field extension, set $\mathscr{A}_K := \mathscr{A} \otimes_k K$, and $X_K = \mathscr{M}(\mathscr{A}_K)$. There is a continuous, surjective map $\pi \colon X_K \to X$ (you have to be careful with complete tensor products, but Gruson's Theorem 3.21 guarantees it is surjective).

Proposition 13.5. If V is an affinoid domain in X, then $\pi^{-1}(V)$ is an affinoid domain in X_K with algebra $\mathscr{A}_V \otimes_k K$.

Sketch. This is trivial if V is rational: the same inequalities will be able to define $\pi^{-1}(V)$, giving a rational domain. In general, you use the Gerritzen–Grauert theorem 11.15 and the technique from the proof of Proposition 12.10.

Remark 13.6. Note that Proposition 13.5 does not just follow from Property 13.2(1), since X_K is not k-affinoid (K can be a very large field extension of k); this is true in general in the theory, where ground field extensions have to be treated differently.

Remark 13.7. There is also a version of Proposition 13.5 where you replace affinoid domains with strictly affinoid domains, where you assume that k is nontrivially valued, and that V is a strictly affinoid domain.

The following allows reductions to strictly k-affinoid spaces and domains.

Corollary 13.8. If V is an affinoid domain in X, then there exists a ground field extension K/k such that X_K is strictly K-affinoid and $V_K = \pi^{-1}(V)$ is a strictly affinoid domain.

Proof. Pick $K = K_r$, where $r \in (\mathbf{R}^*_+)^n$ is $|k^*|$ -free, such that \mathscr{A}_K and \mathscr{A}_{V_K} are strictly k-affinoid. We are done by the Proposition 13.5.

There are a few more remarks to give about (strictly) affinoud domains, but we will do this a bit later.

13.3 *G*-topologies [BGR84, §9.1]

The interpretation of Tate's acyclicity theorem 13.3 is that you can define a structure sheaf for affinoid spaces using what is called a G-topology. This is a special case of the general notion of a Grothendieck topology, which you would use in étale cohomology, for example. The idea behind a G-topology is that you have some notion of an open set that only satisfies a subset of the axioms for a topology, with the added data of a notion of coverings.

13.3.1 The weak G-topology [Ber90, p. 30]

Let $X = \mathcal{M}(\mathcal{A})$ be a k-affinoid space.

Definition 13.9. The weak G-topology on X is the G-topology on X, where

- admissible "open" sets are affinoid domains $V \subset X$; and
- admissible coverings of an affinoid domain are *finite* coverings by other affinoid domains.

We won't write down the general axioms for a G-topology. Note that in the most generality, admissible opens don't have to be subsets of X, but only have to be maps to X. The point is that this is still enough to have sheaf theory.

Note that these admissible open sets actually restrict to open subsets in Tate's version of Max \mathscr{A} .

The Tate acylicity theorem implies that $V \mapsto \mathscr{A}_V$ is a sheaf in the weak G-topology:

$$0 \longrightarrow \mathscr{A}_V \longrightarrow \prod_i \mathscr{A}_{V_i} \longrightarrow \prod_{i,j} \mathscr{A}_{V_i \cap V_j}$$

is exact, and exactness at \mathscr{A}_V says that if $f \in \mathscr{A}_V$ and $f|_{V_i} = 0$ for all i, then f = 0; exactness at $\prod_i \mathscr{A}_{V_i}$ says that if $f_i \in \mathscr{A}_{V_i}$ and $f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}$, then $f_i = f|_{V_i}$.

Our goal will be to define a structure sheaf on the usual topology on X, since after all, Berkovich spaces already come equipped with a good topology, so that we can say what the ring of analytic functions $\mathcal{O}_X(\mathscr{U})$ on an open set $\mathscr{U} \subset X$ should look like. As an intermediate step, we define a slightly stronger *G*-topology on X, and the structure sheaf on this *G*-topology.

13.3.2 The "special" G-topology [Ber90, p. 30]

Let $X = \mathscr{M}(\mathscr{A})$ be a k-affinoid space.

Definition 13.10. The "special" G-topology on X is the G-topology on X, where

- admissible "open" sets are finite unions of affinoid domains, which by the Gerritzen–Grauert theorem 11.15 are also finite unions of rational domains. We call these sets special subsets.
- admissible coverings are *finite* coverings by other special subsets.

We want to extend the structure sheaf $V \mapsto \mathscr{A}_V$ to the special *G*-topology. Given $V = \bigcup_{i \in I} V_i$ where V_i are affinoid and *I* is finite, we *define*

$$\mathscr{A}_V \coloneqq \mathrm{ker} \Bigl(\prod_i \mathscr{A}_{V_i} \longrightarrow \prod_{i,j} \mathscr{A}_{V_i \cap V_j} \Bigr).$$

By Tate's acyclicity theorem 13.3, this matches the previous definition if V is actually affinoid, and so

$$0 \longrightarrow \mathscr{A}_V \longrightarrow \prod_i \mathscr{A}_{V_i} \longrightarrow \prod_{i,j} \mathscr{A}_{V_i \cap V_j}$$

is exact for a special subset V with the particular finite affinoid cover $\bigcup_i V_i$. Note that this definition depends a priori on the choice of finite affinoid cover, and so we should check that this definition actually defines a sheaf, and does not depend on the choice of finite cover.

Proposition 13.11 [Ber90, Cor. 2.2.6].

- (i) \mathscr{A}_V is independent of the choice of covering $V = \bigcup V_i$.
- (ii) $V \mapsto \mathscr{A}_V$ is a sheaf in the special G-topology (so the complex for V_i special is also exact).
- (iii) V is an affinoid domain if and only if \mathscr{A}_V is k-affinoid, and the image of the map $\mathscr{M}(\mathscr{A}_V) \to \mathscr{M}(\mathscr{A}) = X$ is equal to V.

Sketch. (i)-(ii): Use the yoga of Čech cohomology.

For (*iii*), " \Rightarrow " is clear. For " \Leftarrow " you need to check the universal property in Temkin's definition. Given a k-affinoid algebra \mathscr{B} with a map $\varphi \colon \mathscr{A} \to \mathscr{B}$ bounded such that $\varphi^*(\mathscr{M}(\mathscr{B})) \subset V$ must find a unique homomorphism $\varphi_V \colon \mathscr{A}_V \to \mathscr{B}$ such that



commutes. Write $V = \bigcup_i V_i$ as a finite union of affinoid domains. Set $W_i := (\varphi^*)^{-1}(V_i) \subset Y = \mathscr{M}(\mathscr{B})$, with corresponding affinoid algebras $\mathscr{B}_{W_i} := \mathscr{B} \otimes_{\mathscr{A}} \mathscr{A}_{V_i}$. Then, these W_i are affinoid domains in Y, and they cover $Y = \bigcup_i W_i$. We then get the following commutative diagram:



Now if $f \in \mathscr{A}_V$, then by chasing the diagram as before in the proof of Proposition 12.10, exactness at $\prod \mathscr{A}_{V_i}$ and $\prod \mathscr{B}_{W_i}$ defines a unique map $\varphi_V \colon \mathscr{A}_V \to \mathscr{B}$ which is bounded by admissibility of the rows.

There are a couple more things to say about affinoid domains, but we will just move on to what is next in [Ber90]: the discussion of the structure sheaf on the actual Berkovich topology.

14 October 27

Homework 4 is due today.

There are a few more things to say about affinoid domains, but we will move on.

14.1 The structure sheaf on a k-affinoid space [Ber90, $\S2.3$]

We will now look at the structure sheaf on a k-affinoid space. We already defined it for the various G-topologies, but in the Berkovich setting, since the spaces in question are compact Hausdorff spaces, you want to define a structure sheaf on that topology.

Let \mathscr{A} be a k-affinoid algebra, so it is a quotient of a polydisc algebra. Let $X = \mathscr{M}(\mathscr{A})$ be a k-affinoid space. Given an open subset in X, we want to define the analytic functions on it; for example, if the open subset is X, then the global analytic functions on X should just be \mathscr{A} .

Given a special subset $V \subseteq X$, i.e., $V = \bigcup_{i \in I} V_i$ where I is finite and the V_i are affinoid, we defined

$$\mathscr{A}_V\coloneqq \mathrm{ker}\Bigl(\prod_i \mathscr{A}_{V_i} \longrightarrow \prod_{i,j} \mathscr{A}_{V_i \cap V_j}\Bigr),$$

which has the structure of a k-Banach algebra (it won't be a k-affinoid algebra in general). This gives a definition for the structure sheaf in the special G-topology.

For the Berkovich topology, we make the following definition:

Definition 14.1. Given an open subset $\mathscr{U} \subset X$ in the Berkovich topology, define

$$\mathcal{O}_X(\mathscr{U}) \coloneqq \lim_{\substack{V \subseteq \mathscr{U} \\ \text{special}}} \mathscr{A}_V,$$

where the limit is in the category of k-algebras (with no norm).

The idea is that we can exhaust \mathscr{U} by special subsets V_i , and so analytic functions on \mathscr{U} should consist of compatible functions on V_i . We shouldn't expect there to be a norm on $\mathcal{O}_X(\mathscr{U})$, though, since analytic functions can be unbounded as you go to the boundary of an open set. Remark 14.2. The collection of special subsets $V \subset \mathscr{U}$ is directed under inclusion (this does not work for affinoid domains, since they are not closed under finite unions in general). Also, given a special subset V, and a point $x \in \mathscr{U}$, there exists some special subset V' such that $\{x\} \cup V \subset V \subset \mathscr{U}$: you can choose V' to be the union of V with a suitable Laurent domain. Thus, this inverse limit exhausts \mathscr{U} .

Presumably, although there is no norm on $\mathcal{O}_X(\mathcal{U})$, it still has a limit of norms, and so there could be some sort of Fréchet algebra structure.

Exercise 14.3. This assignment $\mathscr{U} \mapsto \mathcal{O}_X(\mathscr{U})$ is a sheaf of k-algebras in the Berkovich topology on X (Use the Tate acyclicity theorem 13.3).

You can interpret any $f \in \mathcal{O}_X(\mathscr{U})$ as a function on \mathscr{U} , by which we mean

$$\begin{aligned} f \colon \mathscr{U} & \longrightarrow & \bigsqcup_{x \in \mathscr{U}} \mathscr{H}(x) \\ x & \longmapsto & \left(f(x) \right)_{x \in \mathscr{U}} \end{aligned}$$

In particular, you get a continuous function $|f|: \mathscr{U} \to \mathbf{R}_+$, which is a priori not bounded from above.

For this to make sense, recall that $\mathscr{H}(x)$ was defined by $\operatorname{Frac}(\mathscr{A}/\mathfrak{p}_x)^{\wedge}$. But this seems to depend on how we compute $\mathscr{H}(x)$: we could also try to compute it as $\mathscr{H}_V(x) := \operatorname{Frac}(\mathscr{A}_V/\mathfrak{p}_x)^{\wedge}$.

14.2 Local rings and residue fields [Ber93, §2.1]

Now that we have a structure sheaf, we want it to tell us something about the space. We therefore define local rings for a k-affinoid space $X = \mathcal{M}(\mathcal{A})$, since just as for schemes, they should reflect information about the structure of the scheme.

Let $X = \mathcal{M}(\mathcal{A})$ as before. Then, consider the stalk

$$\mathcal{O}_{X,x} \coloneqq \varinjlim_{\mathscr{U} \ni x} \mathcal{O}_X(\mathscr{U})$$

of \mathcal{O}_X at $x \in X$. This looks fairly complicated since it is a direct limit of a bunch of inverse limits, but in practice, it is not too bad:

Exercise 14.4.

- (1) $\mathcal{O}_{X,x} \cong \varinjlim_V \mathscr{A}_V$, where V varies over V either affinoid or special neighborhoods of x.
- (2) $\mathcal{O}_{X,x}$ is a local ring with unique maximal ideal

$$\mathfrak{m}_x \coloneqq \{ f \in \mathcal{O}_{X,x} \mid f(x) = 0 \in \mathscr{H}(x) \}.$$

Definition 14.5. The residue field of X at x is

$$\kappa(x) \coloneqq \mathcal{O}_{X,x}/\mathfrak{m}_x.$$

Remark 14.6. We earlier set $\mathscr{K}(x) = \operatorname{Frac}(\mathscr{A}/\ker(|\cdot|_x))$ (Definition 3.4), but this actually depends on \mathscr{A} (that is, $\mathscr{K}_X(x) \neq \mathscr{K}_V(x) \coloneqq \operatorname{Frac}(\mathscr{A}_V/\ker(|\cdot|_x))$ for $x \in V \subseteq X$, where V is an affinoid neighborhood of x in X, although they have the same completion). This new definition $\kappa(x)$ is the more canonical object; see Example 14.9.

Note $\kappa(x) \subset \mathscr{H}(x)$. The following Exercise implies that $\mathscr{H}(x) = \mathscr{H}_V(x)$ for all affinoid neighborhoods $V \ni x$.

Exercise 14.7. $\kappa(x)$ is dense in $\mathscr{H}(x)$, so the completion of $\kappa(x)$ is exactly $\mathscr{H}(x)$.

Definition 14.8. $\kappa(x)$ is the residue field of X at x, and $\mathscr{H}(x)$ is the complete residue field of X at x.

We now do some examples. In general, however, it is difficult to describe them explicitly. We go back to our favorite example of the unit disc. **Example 14.9.** Let $X = E(1) = \mathcal{M}(k\{T\})$, where k is algebraically closed. Then, we consider two cases: (a) x a Type 1 point;

(b) x a Type 3 point.

For (a), without loss of generality we may assume x = 0 (otherwise you can move it by an automorphism). There is a countable neighborhood basis of closed sets around x: p(E(0,r)) for $0 < r \le 1$, where $E(0,r) = E(r) = \mathscr{M}(k\{r^{-1}T\})$. See Figure 14.10.

•
$$p(E(1))$$

• $p(E(0,r))$
• $0 = x$

Figure 14.10: A countable neighborhood basis around a Type 1 point.

Since we have

$$\mathcal{O}_{X,x} = \varinjlim_{r} k\{r^{-1}T\} \cong \left\{ f = \sum_{i=0}^{\infty} a_i T^i \ \middle| \ a_i \in k, \ \varlimsup_{i \to \infty} |a_i|^{1/i} < \infty \right\},$$

which is what you would expect from the complex analysis story: we are taking the limit of rings $k\{r^{-1}T\}$ as $r \to 0$. Then, $\mathfrak{m}_x = \{f \mid a_0 = 0\}$, so that $\kappa(x) = k = \mathscr{H}(x)$. In the rigid theory, these are the only points you look at, since they are the points where the residue field is a finite extension of k.

For (b), without loss of generality we may assume $x = p(E(\rho))$, where $0 < \rho < 1$, and $\rho \notin |k^*|$. In this case, we have a basis of closed neighborhoods of x in X = E(1): $V_{r,s} = \{r \leq |T| \leq s\}$, where $0 < r < \rho < s \leq 1$ (this is an annulus, hence a Laurent domain; see Figure 14.11). Note that we could define the same neighborhoods



Figure 14.11: A countable neighborhood basis around a Type 3 point.

for a Type 2 point, but if we only take a countable number, their intersection won't just be x. Now let

$$\mathscr{A}_{r,s} \coloneqq \mathscr{A}_{V_{r,s}} \cong \left\{ f = \sum_{-\infty}^{\infty} a_i T^i \ \left| \ \lim_{i \to \infty} |a_i| s^i = \lim_{i \to -\infty} |a_i| r^i = 0 \right\} \right\}$$

We want to take the limit (or "union") of these, to get

$$\mathcal{O}_{X,x} = \left\{ f = \sum_{-\infty}^{\infty} a_i T^i \mid \overline{\lim_{i \to \infty}} |a_i|^{1/i} < \rho^{-1}, \ \overline{\lim_{i \to -\infty}} |a_i|^{1/i} < \rho \right\}$$

Then,

$$\mathscr{H}(x) \cong \left\{ f = \sum_{-\infty}^{\infty} a_i T^i \ \bigg| \ \lim_{i \to \pm \infty} |a_i| \rho^i = 0 \right\}$$

In this case $\mathfrak{m}_x = (0)$, so $\mathcal{O}_{X,x} = \kappa(x)$ is a field. This also shows that $\mathcal{O}_{X,x} \hookrightarrow \mathscr{H}(x)$ is injective. Note that $\operatorname{Frac}(k\{T\}) \neq \kappa(x)$, and so this is a concrete case in which $\mathscr{H}(x)$ is not a good definition.

14.2.1 The kernel map

We now want to study various properties of these special local rings, for example they are Noetherian (which is a nontrivial fact in complex analysis), and Henselian (which we probably won't prove). Moreover, there is a sort of comparison using the kernel map (Mattias's terminology). We have already seen this map, but the result we want to state was hard to state before.

Let $X = \mathscr{M}(\mathscr{A})$, where \mathscr{A} is a k-affinoid algebra; we have the Berkovich topology on X with the structure sheaf \mathcal{O}_X . On the other hand, we can consider the space $\mathscr{X} = \operatorname{Spec} \mathscr{A}$, which has the Zariski topology. We then have a map

$$\ker\colon X \longrightarrow \mathscr{X}$$
$$x \longmapsto \ker(|\cdot|_x$$

)

which is continuous (recall the reduction map is anticontinuous). You can define this map even for general k-Banach algebras, but in this case not much is known. In the setting we are in, where we are doing geometry over a base field, we have the following:

Theorem 14.12 [Ber93, Prop. 2.1.1]. The kernel map ker: $X \to \mathscr{X}$ is surjective.

The proof illustrates a "typical" proof for various things in the theory; there are a lot of reduction steps, that reduce to the case of a strictly affinoid space, after which you use Noether normalization to reduce to the case of the unit polydisc, in which case it is "obvious."

Proof. Reduction 1 (may not be necessary): Without loss of generality, \mathscr{A} is reduced. This is clear since $\operatorname{Spec} \mathscr{A}_{\operatorname{red}} = \operatorname{Spec} \mathscr{A}$ and $\mathscr{M}(\mathscr{A}_{\operatorname{red}}) = \mathscr{M}(\mathscr{A}_{\operatorname{red}})$ as topological spaces.

Reduction 2: Without loss of generality, \mathscr{A} is an integral domain and the target point is equal to the generic point of \mathscr{X} . Pick any $\xi \in \mathscr{X}$, with corresponding prime ideal $\mathfrak{p}_{\xi} \subset \mathscr{A}$. Then, we want to find $x \in X$ such that $\ker(|\cdot|_x) = \mathfrak{p}_{\xi}$. But now set $\mathscr{B} = \mathscr{A}/\mathfrak{p}_{\xi}$, which is an integral domain, and suppose there exists $y \in Y = \mathscr{M}(\mathscr{B})$ whose corresponding kernel $\ker(|\cdot|_y) = (0) \subseteq \mathscr{B}$. Then, the image $x \in X$ via $Y \to X$ induced by $\mathscr{B} \to \mathscr{A}$ is mapped to ξ by $X \to \mathscr{X}$ by the commutativity of the diagram



Now given \mathscr{A} an integral domain that is k-affinoid, we want to find $x \in X = \mathscr{M}(\mathscr{A})$ such that $\ker(|\cdot|_x)$ is the generic point of $\mathscr{X} = \operatorname{Spec} \mathscr{A}$, i.e., $|\cdot|_x$ is a norm on \mathscr{A} .

First suppose \mathscr{A} is strictly k-affinoid. In this case, we have Noether normalization, which says there exists a finite injective homomorphism $\mathscr{B} = k\{T_1, \ldots, T_d\} \hookrightarrow \mathscr{A}$ for some $d \ge 0$. In this setting, we have seen that the map $X = \mathscr{M}(\mathscr{A}) \to \mathscr{M}(\mathscr{B}) = Y$ is *surjective* (with finite fibers). The commutative diagram



and so it suffices to pick a point whose kernel is the generic point of $\operatorname{Spec} \mathscr{B}$, which lifts to the generic point in $\operatorname{Spec} \mathscr{A}$, since \mathscr{B} is an integral domain.

So it suffices to find a point $y \in Y$ which corresponds to a multiplicative *norm* on \mathscr{B} . Any preimage $x \in X$ of y then has a kernel mapping to the generic point of \mathscr{Y} , hence by finitude, must be the generic point of \mathscr{X} . Now we can pick y to be the point corresponding to the norm on $\mathscr{B} = k\{T_1, \ldots, T_d\}$. This is a typical application of Noether normalization.

Now we want to prove the general case, where \mathscr{A} is k-affinoid, which is not necessarily strict. By using a ground field extension $K = K_r/k$, consider $\mathscr{A}' \coloneqq \mathscr{A} \otimes_k K$, and $X' \coloneqq \mathscr{M}(\mathscr{A}')$, and $\mathscr{X}' \coloneqq \operatorname{Spec} \mathscr{A}'$. We may

assume that \mathscr{A}' is strictly K-affinoid. We then have a new diagram

$$\begin{array}{ccc} X' \xrightarrow{\ker'} \mathscr{X}' \\ \pi & & \downarrow \\ X \xrightarrow{\ker} \mathscr{X} \end{array}$$

Then, $\pi: X' \to X$ is surjective (by Gruson's theorem 3.21). By assumption, $X' \to \mathscr{X}'$ is surjective. It then suffices to prove $\mathscr{X}' \to \mathscr{X}$ is surjective, which is not completely obvious since you are using the *complete* tensor product. We then use a Lemma essentially due to Gruson:

Lemma 14.13 [Ber93, Lem. 2.1.2]. The algebra \mathscr{A}' is faithfully flat (in the usual sense) over \mathscr{A} .

This implies $\mathscr{X}' \to \mathscr{X}$ is surjective.

Proof. Use the following result by Gruson:

Theorem 14.14 [Gru66, §3, Thm. 1(1),(4)]. Let k be a general non-Archimedean field (which is non-trivially valued, but you can reduce to this case using the standard field extension).

(1) For k-Banach spaces B, M, the map

$$M \otimes_k B \longrightarrow M \widehat{\otimes}_k B$$

is injective;

(2) ("normed flatness") If

 $0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$

is an exact admissible sequence of k-Banach spaces, then the sequence

$$0 \longrightarrow M \widehat{\otimes}_k B \longrightarrow N \widehat{\otimes}_k B \longrightarrow P \widehat{\otimes}_k B \longrightarrow 0$$

is also exact and admissible. The converse holds if B is a non-Archimedean field extension of k.

Now we can prove the Lemma: we want to show that $\mathscr{A}' = \mathscr{A} \otimes_k K$ is faithfully flat over \mathscr{A} . It suffices to prove that if $M \to N$ is an homomorphism of *finite* \mathscr{A} -modules, then $M \to N$ is injective if and only if $M \otimes_{\mathscr{A}} \mathscr{A}' \to N \otimes_{\mathscr{A}} \mathscr{A}'$ is injective. As a preparatory step, we may equip M and N with a (unique) structure of a finite Banach \mathscr{A} -module by Proposition 8.3. But in this case, M, N are finite Banach \mathscr{A} -modules, and \mathscr{A}' is Noetherian (Corollary 8.8), so that $M \otimes_{\mathscr{A}} \mathscr{A}' \to M \otimes_{\mathscr{A}} \mathscr{A}'$ and $N \otimes_{\mathscr{A}} \mathscr{A}' \to N \otimes_{\mathscr{A}} \mathscr{A}'$ by Proposition 8.6. Then, using Gruson's theorem,

$$M \otimes_{\mathscr{A}} \mathscr{A}' = M \widehat{\otimes}_{\mathscr{A}} \mathscr{A}' = M \widehat{\otimes}_{\mathscr{A}} (\mathscr{A} \widehat{\otimes}_k K) = M \widehat{\otimes}_k K$$
$$N \otimes_{\mathscr{A}} \mathscr{A}' = N \widehat{\otimes}_{\mathscr{A}} \mathscr{A}' = N \widehat{\otimes}_{\mathscr{A}} (\mathscr{A} \widehat{\otimes}_k K) = N \widehat{\otimes}_k K$$

Gruson's Theorem 14.14(2) implies $0 \to M \to N$ is exact and admissible if and only if $M \otimes_k K \to N \otimes_k K$ is exact and admissible.

This finishes the proof of Theorem 14.12, since faithfully flat homomorphisms of rings induce surjective maps on prime spectra by [Mat89, Thm. 7.3(i)]. \Box

We will prove that local rings are Noetherian next time.

15 November 1

Homework 3, Problem 8(b) said that if k'/k is a non-Archimedean field extension, where both k, k' are algebraically closed, then the map $E_{k'}(r) \to E_k(r)$ cannot decrease the type of point. This is false, so don't worry if you weren't able to show it.

Today we will prove that local rings of affinoid spaces are noetherian, and some other related facts. We start by recalling what we did last time. **Recall 15.1.** Let $X = \mathcal{M}(\mathcal{A})$ be a k-affinoid space. We then defined a structure sheaf \mathcal{O}_X , which had stalks

$$\mathcal{O}_{X,x} = \varinjlim_{x \in \operatorname{int} V} \mathscr{A}_V$$

at each $x \in X$, called the *local ring* at x. This ring $\mathcal{O}_{X,x}$ is indeed a local ring, with maximal ideal $\mathfrak{m}_x = \{f \in \mathcal{O}_{X,x} \mid f(x) = 0\}$ and residue field $\kappa(x) \coloneqq \mathcal{O}_{X,x}/\mathfrak{m}_x$. We also defined the complete residue field $\mathscr{H}(x)$ to be the completion of $\kappa(x)$, and defined the kernel map

$$\ker\colon X\longrightarrow\mathscr{X}\coloneqq\operatorname{Spec}\mathscr{A}.$$

Theorem 14.12. The kernel map ker: $X \to \mathscr{X}$ is surjective.

Sketch. Pick a sufficiently large non-Archimedean extension K/k, such that $\mathscr{A}' := \mathscr{A} \otimes_k K$ is a strictly K-affinoid algebra. Set $X' := \mathscr{M}(\mathscr{A}')$, and $\mathscr{X}' := \operatorname{Spec} \mathscr{A}'$. We then use the following:

Lemma 14.13. $\mathscr{A} \to \mathscr{A}'$ is faithfully flat.

The idea of the proof of this lemma is to go back and forth between (ordinary) tensor products and complete tensor products, which we are allowed to do by the results in Theorem 14.14 of Gruson on Banach spaces over fields, and also by results from before about finite Banach modules over We then have the following diagram:



Then, $X' \to \mathscr{X}'$ is surjective by Noether normalization, and $\mathscr{X}' \to \mathscr{X}$ is surjective since $\mathscr{A} \to \mathscr{A}'$ was faithfully flat. By the commutative diagram above, this implies that $X \to \mathscr{X}$ is surjective. Note that $X' \to X$ is also surjective by Gruson's theorem 3.21, but this is not necessary in the proof.

15.1 Noetherianity of local rings of affinoid spaces [Ber93, §2.1]

We are now ready to state the main theorem for today.

Theorem 15.2 [Ber93, Thm. 2.1.4]. For any $x \in X$, the local ring $\mathcal{O}_{X,x}$ is Noetherian and faithfully flat over the corresponding local ring $\mathcal{O}_{\mathscr{X},\xi}$, where $\xi = \ker(x) \in \mathscr{X}$.

Example 15.3. Let $X = E(1) = \mathcal{M}(k\{T\})$, and x = 0. Then,

$$\mathcal{O}_{X,x} = \left\{ f = \sum_{i=0}^{\infty} a_i T^i \mid \overline{\lim}_{i \to \infty} |a_i|^{1/i} < \infty \right\}$$
$$\mathcal{O}_{\mathscr{X},x} = k\{T\}_{(T)}$$

In complex analytic geometry you have something similar [Ser56, §2, n^o 6, Cor. 1], although in that setting the analytic space only contains the closed points of the prime spectrum. The content of Theorem 15.2 is that even though the local rings are different when computed on the Berkovich space or on the prime spectrum, local rings are nonetheless close enough to have a decent relationship.

It would be nice to have a direct proof of Theorem 15.2, but we will have to go back to the strict case and go back using commutative algebraic facts about flatness and faithful flatness.

15.1.1 Proof of Noetherianity in the strict case [BGR84, 7.3.2]

Let $X = \mathscr{M}(\mathscr{A})$, where \mathscr{A} is a *strictly* k-affinoid algebra and k is nontrivially valued. Let $x \in \operatorname{Max} \mathscr{A} \subset \mathscr{M}(\mathscr{A})$, and let \mathfrak{m} be the corresponding maximal ideals. In this setting, the local ring is

$$\mathcal{O}_{X,x} = \varinjlim_{x \in V} \mathscr{A}_V, \quad V \subset X \text{ strictly affinoid.}$$

Thus, if $x \in V \subseteq X$, then $\varphi \colon \mathscr{A} \to \mathscr{A}_V$ induces a morphism $\mathscr{M}(\mathscr{A}_V) \to \mathscr{M}(\mathscr{A}) = X$, which is a homeomorphism onto its image.

Before we prove Noetherianity, we prove some preliminary results.

Lemma 15.4 [BGR84, Prop. 7.2.2/1(*ii*)]. For all $n \ge 1$, φ induces an isomorphism $\mathscr{A}/\mathfrak{m}^n \to \mathscr{A}_V/\mathfrak{m}^n \mathscr{A}_V$. Sketch. Using the universal property for $\mathscr{A} \to \mathscr{B} = \mathscr{A}/\mathfrak{m}^n$, we obtain the commutative diagram:



Now consider the following commutative diagram:



Both π' and $\sigma \circ \alpha$ complete the diagram uniquely, so $\pi' = \sigma \circ \alpha$. Surjectivity of π' implies surjectivity of σ , and surjectivity of π implies surjectivity of α . Moreover, ker $\pi' = \mathfrak{m}^n \mathscr{A}_V \subset \ker \alpha$ implies σ is injective. \Box

Corollary 15.5 [BGR84, Prop. 7.3.2/3]. The canonical map $\mathscr{A} \to \mathcal{O}_{X,x}$ factors through the localization $\mathscr{A}_{\mathfrak{m}}$ and for every $n \geq 1$, we have the chain of isomorphisms

$$\mathscr{A}/\mathfrak{m}^n \stackrel{\sim}{\longrightarrow} \mathscr{A}_\mathfrak{m}/\mathfrak{m}^n \mathscr{A}_\mathfrak{m} \stackrel{\sim}{\longrightarrow} \mathcal{O}_{X,x}/\mathfrak{m}^n \mathcal{O}_{X,x}.$$

In particular, we have isomorphisms

$$\widehat{\mathscr{A}} \xrightarrow{\sim} \widehat{\mathscr{A}}_{\mathfrak{m}} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x},$$

where the completions are with respect to the \mathfrak{m} -adic topology.

The following corollary is something that can be proved more elementarily; see the second proof in [BGR84, Cor. 7.3.2/6].

Corollary 15.6 [BGR84, Cor. 7.3.2/6]. $\mathscr{A} \to \mathscr{A}_V$ is flat.

Proof. It suffices to prove that $\mathscr{A}_{\mathfrak{m}} \to (\mathscr{A}_V)_{\mathfrak{m}\mathscr{A}_V}$ is flat for all \mathfrak{m} , and moreover it suffices to show $\widehat{\mathscr{A}_{\mathfrak{m}}} \to (\widehat{\mathscr{A}_V})_{\mathfrak{m}\mathscr{A}_V}$ is flat. But the last map is an isomorphism

$$\widehat{\mathscr{A}}_{\mathfrak{m}} \cong \widehat{\mathscr{A}} \cong (\widehat{\mathscr{A}}_{V})_{\mathfrak{m}}_{\mathfrak{M}_{V}},$$

since this last one is isomorphic to $\widehat{\mathcal{O}}_{X,x}$, regardless of choice of V.

We are now ready to prove the Noetherianity statement in Theorem 15.2.

Proposition 15.7 [BGR84, Prop. 7.3.2/7]. $\mathcal{O}_{X,x}$ is Noetherian for all $x \in \operatorname{Max} \mathscr{A} \subset \mathscr{M}(\mathscr{A})$.

Sketch. The proof comprises of a few steps.

Step 1. $\widehat{\mathcal{O}}_{X,x}$ is Noetherian.

Since \mathscr{A} is Noetherian, the ring $\widehat{\mathscr{A}}$ is also Noetherian. But then, $\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathscr{A}}$, so we are done.

Step 2. $\mathcal{O}_{X,x} \hookrightarrow \widehat{\mathcal{O}}_{X,x}$ is injective.

Suppose $f \in \bigcap_{n\geq 1} \mathfrak{m}^n \cdot \mathcal{O}_{X,x}$. Without loss of generality (after shrinking X), we may assume $f \in \mathscr{A}$. But now, $\mathscr{A}/\mathfrak{m}^n \cong \mathcal{O}_{X,x}/\mathfrak{m}^n \cdot \mathcal{O}_{X,x}$ implies $f \in \bigcap_{n\geq 1} \mathfrak{m}^n$. This means the image of f in $\mathscr{A}_{\mathfrak{m}}$ is zero by the Krull intersection theorem [Mat89, Thm. 8.10], and so f = 0 in $\mathcal{O}_{X,x}$.

Step 3. Every finitely generated ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,x}$ is \mathfrak{m} -adically closed in $\mathcal{O}_{X,x}$.

This is a bit handwavy. Without loss of generality, $\mathfrak{a} \subseteq \mathscr{A}$ after possibly shrinking X. We must show that $\mathcal{O}_{X,x}/\mathfrak{a} \cdot \mathcal{O}_{X,x}$ is Hausdorff, that is, that the residue seminorm is a norm. But $\mathcal{O}_{X,x}/\mathfrak{a} \cdot \mathcal{O}_{X,x} \cong \mathcal{O}_{Y,y}$, where $Y = \mathscr{M}(\mathscr{A}/\mathfrak{a})$ and \mathscr{A}/\mathfrak{a} is a strictly k-affinoid algebra, and where $y \in \operatorname{Max}(\mathscr{A}/\mathfrak{a})$. By Step 2, we have that the seminorm on $\mathcal{O}_{Y,y}$ is a norm.

Conclusion. Consider an ascending chain of finitely generated ideals

$$\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots \subset \mathcal{O}_{X,x}$$

In the completion, we get an ascending chain of their extensions

$$\widehat{\mathfrak{a}}_1 \subset \widehat{\mathfrak{a}}_2 \subset \cdots \subset \mathcal{O}_{X,x}$$

Since $\widehat{\mathcal{O}}_{X,x}$ is Noetherian, $\widehat{\mathfrak{a}}_{n+1} = \widehat{\mathfrak{a}}_n$ for all n large enough. Now since $\mathcal{O}_{X,x} \hookrightarrow \widehat{\mathcal{O}}_{X,x}$ and $\mathfrak{a}_n \subset \mathcal{O}_{X,x}$ is closed, we have that $\mathfrak{a}_{n+1} = \mathfrak{a}_n$ for all $n \gg 0$.

Remark 15.8 (added by notetaker). The faithful flatness statement was not explicitly proved in the strictly k-affinoid case. This follows by using the commutative diagram

$$\begin{array}{ccc} \mathscr{A}_{\mathfrak{m}} & \longrightarrow & \mathcal{O}_{X,x} \\ & & & \downarrow \\ & & & \downarrow \\ \widehat{\mathscr{A}_{\mathfrak{m}}} & \stackrel{\sim}{\longrightarrow} & \widehat{\mathcal{O}}_{X,x} \end{array}$$

where both vertical arrows are faithfully flat by [Mat89, Thm. 8.14]. Faithful flatness of $\mathscr{A}_{\mathfrak{m}} \to \mathscr{O}_{X,x}$ then follows by a transitivity property of faithful flatness [Mat89, p. 46].

15.1.2 Proof of Noetherianity in the general case [BGR84, 7.3.2]

Now let $X = \mathcal{M}(\mathcal{A})$, where \mathcal{A} is k-affinoid.

Proposition 15.9. For any $V \subseteq X$ an affinoid domain, the map $\mathscr{A} \to \mathscr{A}_V$ is flat.

Proof. We know this is true when X is strictly k-affinoid, $|k^*| \neq \{1\}$, and $V \subset X$ is strictly affinoid. Pick a large enough non-Archimedean extension K/k. Then, $\mathscr{A}' \coloneqq \mathscr{A} \widehat{\otimes}_k K$ is strictly K-affinoid, and $V' \coloneqq \pi^{-1}(V) \subset X'$ is a strictly affinoid domain, where $X' = \mathscr{M}(\mathscr{A}') \xrightarrow{\pi} X$. Also, $\mathscr{A}'_{V'} \coloneqq \mathscr{A}_V \widehat{\otimes}_k K$ is a strictly K-affinoid algebra.

We first give a slick proof suggested by Bhargav. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathscr{A} & \longrightarrow & \mathscr{A}_{V} \\ \text{faithfully} & & & & & & \\ \text{flat} & & & & & & \\ \mathscr{A}' & \xrightarrow{\text{flat}} & & \mathscr{A}'_{V'} \end{array}$$

where faithful flatness of the vertical arrows is by Lemma 14.13, and flatness of the bottom arrow is Corollary 15.6. By a transitivity property for flatness [Mat89, p. 46], this shows that the top arrow is also flat.

We give another, more elementary proof. Let $M \stackrel{\varphi}{\hookrightarrow} N$ be an injective \mathscr{A} -linear map, where M, N are finite \mathscr{A} -modules. We must prove that $M \otimes_{\mathscr{A}} \mathscr{A}_V \to N \otimes_{\mathscr{A}} \mathscr{A}_V$ is injective. We will turn this into a normed algebraic statement, and then go back. First, endow M, N with the unique structure of a finite *Banach* \mathscr{A} -module (Proposition 8.3). Then, $\varphi: M \to N$ is injective and admissible, i.e.,

$$0 \longrightarrow M \longrightarrow N$$

is admissible and exact. By Gruson's theorem 14.14(2), we have that

$$0 \longrightarrow M \widehat{\otimes}_k K \longrightarrow N \widehat{\otimes}_k K$$

is admissible and exact, where we note $M' \coloneqq M \widehat{\otimes}_k K$ and $N' \coloneqq N \widehat{\otimes}_k K$ are finite Banach \mathscr{A}' -modules. By the strict case (Corollary 15.6),

$$0 \longrightarrow M' \otimes_{\mathscr{A}'} \mathscr{A}'_{V'} \longrightarrow N' \otimes_{\mathscr{A}'} \mathscr{A}'_{V'}$$

is exact. Since M', N' are finite Banach \mathscr{A}' -modules, and $\mathscr{A}'_{V'}$ is K-affinoid, Proposition 8.6 implies that we have isomorphisms $M' \otimes_{\mathscr{A}'} \mathscr{A}'_{V'} \xrightarrow{\sim} M' \widehat{\otimes}_{\mathscr{A}'} \mathscr{A}'_{V'}$ and $N' \otimes_{\mathscr{A}'} \mathscr{A}'_{V'} \xrightarrow{\sim} N' \widehat{\otimes}_{\mathscr{A}'} \mathscr{A}'_{V'}$. Thus,

$$M' \widehat{\otimes}_{\mathscr{A}'} \mathscr{A}'_{V'} = (M \widehat{\otimes}_k K) \widehat{\otimes}_{\mathscr{A}'} (\mathscr{A}_V \widehat{\otimes}_k K) = (M \widehat{\otimes}_{\mathscr{A}} \mathscr{A}_V) \widehat{\otimes}_k K = (M \otimes_{\mathscr{A}} \mathscr{A}_V) \widehat{\otimes}_k K$$

and similarly for $N' \widehat{\otimes}_{\mathscr{A}'} \mathscr{A}'_{V'}$. This implies

$$0 \longrightarrow (M \otimes_{\mathscr{A}} \mathscr{A}_V) \widehat{\otimes}_k K \longrightarrow (N \otimes_{\mathscr{A}} \mathscr{A}_V) \widehat{\otimes}_k K$$

is exact. We also have that $M \otimes_{\mathscr{A}} \mathscr{A}_V \to N \otimes_{\mathscr{A}} \mathscr{A}_V$ is admissible by appealing to Gruson's theorem 14.14(2) again. Using the converse statement in Gruson's theorem 14.14(2) (or the explicit calculation when $K = K_r$) then implies that $0 \to M \otimes_{\mathscr{A}} \mathscr{A}_V \to N \otimes_{\mathscr{A}} \mathscr{A}_V$ is exact and admissible.

Now pick $x \in X$. Pick K a large non-Archimedean extension of k such that \mathscr{A}' is strictly K-affinoid, and such that there exists $x' \in \operatorname{Max} \mathscr{A}' \subset X' := \mathscr{M}(\mathscr{A}')$ such that $\pi(x') = x$, where $\pi \colon X' \to X$ (i.e., choose K to be an extension of $\mathscr{H}(x)$). Now set $\xi = \ker x \in \mathscr{X}$, and $\xi' = \ker(x') \in \mathscr{X}' = \operatorname{Spec} \mathscr{A}'$. We then have a diagram

Lemma 15.10. $\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'}$ is faithfully flat.

Sketch. By definition, $\mathcal{O}_{X,x} = \varinjlim \mathscr{A}_V, \mathcal{O}_{X',x'} = \varinjlim \mathscr{A}'_{V'}$. We know that $\mathscr{A}_V \to \mathscr{A}'_{\pi^{-1}(V)}$ is faithfully flat, and that if $V' \subset V''$, then $\mathscr{A}_{V'} \to \mathscr{A}_{V'}$ is flat. Since flatness works well with direct limits, you get flatness, and faithful flatness follows since the homomorphism is local [Mat89, p. 48].

We are now ready to prove the general case of Theorem 15.2.

Proof of Theorem 15.2. In the commutative diagram (9), all maps except possibly $\mathcal{O}_{\mathscr{X},\xi} \to \mathcal{O}_{X,x}$ are faithfully flat, and so by $\mathcal{O}_{\mathscr{X},\xi} \to \mathcal{O}_{X,x}$ must be faithfully flat as well. To prove that $\mathcal{O}_{X,x}$ is Noetherian, we know that $\mathcal{O}_{X',x'}$ is Noetherian. Now using the fact that $\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'}$ is faithfully flat, we get that for any ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,x}$, we have that $\mathfrak{a} = \mathfrak{a} \mathcal{O}_{X',x'} \cap \mathcal{O}_{X,x}$. Now for an ascending chain

$$\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots \subset \mathcal{O}_{X,x}$$

we know they stabilize in $\mathcal{O}_{X',x'}$: for all $n \gg 0$, $\mathfrak{a}_{n+1}\mathcal{O}_{X',x'} = \mathfrak{a}_n\mathcal{O}_{X',x'}$. By the intersection property above, this gives $\mathfrak{a}_{n+1} = \mathfrak{a}_n$ for all $n \gg 0$.

Local rings are also Henselian (which is easy; see [Ber93, Thm. 2.1.5]) and excellent (which is hard; see [Duc09, Thm. 2.13]), but we won't show these statements.

16 November 3

Homework 4 is not yet graded, and Homework 5 has not yet been posted.

Today, we will do more material on structure sheaves.

Recall 16.1. For $X = \mathscr{M}(\mathscr{A})$ a k-affinoid space, where \mathscr{A} is a k-affinoid algebra, we defined $\mathscr{X} = \operatorname{Spec} \mathscr{A}$.

In this case, we have shown

Proposition 16.2 (Theorem 14.12). The kernel map ker: $X \to \mathscr{X}$ is surjective.

We then showed the following:

Theorem 16.3 (Theorem 15.2). If $x \in X$ and $\xi := \ker(x)$, then

- $\mathcal{O}_{X,x}$ is Noetherian;
- $\mathcal{O}_{X,\xi} \hookrightarrow \mathcal{O}_{X,x}$ is faithfully flat.

We proved these by using the yoga of reduction to the strict case; we are not aware of any direct proofs of these results.

With similar techniques, you can also prove the following:

Theorem 16.4 [Ber93, Thm. 2.1.5]. $\mathcal{O}_{X,x}$ is Henselian.

We will not prove this.

16.1 The Zariski topology on k-affinoid spaces [Ber93, p. 28]

There are many other properties local rings can have, and to discuss how they behave as a function of points on a k-affinoid space X, we introduce a new topology on X, induced by the Zariski topology on \mathscr{X} .

Definition 16.5. The Zariski topology on X is the weakest topology such that the kernel map ker: $X \to \mathscr{X}$ is continuous, i.e., the topology has open sets that are preimages of opens in \mathscr{X} .

This is weaker than the Berkovich topology. This is similar to how in complex analysis (e.g., on $\mathbf{A}_{\mathbf{C}}^{1}$), there are many closed sets (e.g., the closed disc) that are not Zariski closed.

Remark 16.6 [Ber93, Rem. 2.2.9]. If $V \subset X$ is an affinoid domain, then the Zariski topology on V may be strictly stronger than the induced Zariski topology from X, since there are more analytic functions on V than on X. This is in contrast to the scheme-theoretic setting.

Remark 16.7. If X is reduced and irreducible (so \mathscr{A} is an integral domain), then any Zariski closed subset of X is nowhere dense in the Berkovich topology.

16.2 Further properties of local rings [Ber93, §2.2]

This material is only a paragraph in [Ber90, $\S2.3$], but we will follow [Ber93, $\S2.2$] instead, which fleshes out the details.

Definition 16.8. We define

 $\operatorname{Reg}(X) \coloneqq \{x \in X \mid \mathcal{O}_{X,x} \text{ is a regular local ring}\} \subset X$ $\operatorname{Reg}(\mathscr{X}) \coloneqq \{\xi \in \mathscr{X} \mid \mathcal{O}_{\mathscr{X},\xi} \text{ is a regular local ring}\} \subset \mathscr{X}$

Similarly, we can define $P(X), P(\mathscr{X})$ for P one of the following properties:

- Nor (normal);
- Red (reduced);
- CM (Cohen–Macaulay);
- Gor (Gorenstein);
- CI (complete intersection).

These properties are stable under localization.

Theorem 16.9 [Ber93, Thm. 2.2.1]. For any property P as above,

- P(X) is Zariski open in X.
- $P(X) = \ker^{-1}(P(\mathscr{X})).$

Remark 16.10. Ducros's results (see [Duc09, p. 1444]) imply that in fact, $P(\mathscr{X})$ is Zariski open, although you do not need Ducros's result for strictly k-affinoid algebras.

We will focus on P = Reg. Most of the proof we give will work for other P, except for some details.

Corollary 16.11. If X is reduced and irreducible (i.e., \mathscr{A} is an integral domain), then $\operatorname{Reg}(X) \subset X$ is Zariski open, and nonempty (in particular, the generic point of \mathscr{X} is in $\operatorname{Reg}(\mathscr{X})$). This implies that the complement is nowhere dense in the Berkovich topology.

The same method of proof to be given will show the following:

Corollary 16.12. If $V \subset X$ is an affinoid subdomain, then $\operatorname{Reg}(V) = \operatorname{Reg}(X) \cap V$.

This is not a tautology: local rings were defined as a direct limit over affinoid domains whose *relative* interior contains x.

16.2.1 Proof of Theorem 16.9 for P = Reg

Lemma 16.13. Given a point $x \in X$, set $\xi = \ker(x) \in \mathscr{X}$. Recall $\mathcal{O}_{\mathscr{X},\xi} \hookrightarrow \mathcal{O}_{X,x}$ is faithfully flat. Thus,

- (i) If $\mathcal{O}_{X,x}$ is regular, then $\mathcal{O}_{\mathscr{X},\xi}$ is regular.
- (ii) If $\mathcal{O}_{\mathscr{X},\xi}$ is regular, and $\mathfrak{m}_{\mathscr{X},\xi} \cdot \mathcal{O}_{X,x} = \mathfrak{m}_{X,x}$ (\Leftrightarrow unramified, at least under finite type assumptions), then $\mathcal{O}_{X,x}$ is regular.

Note in (ii), the inclusion $\mathfrak{m}_{\mathscr{X},\xi} \cdot \mathcal{O}_{X,x} \subset \mathfrak{m}_{X,x}$ holds.

Proof. This is a general fact about faithfully flat maps of local rings.

Remark 16.14. Unfortunately, we may have $\mathfrak{m}_{\mathscr{X},\xi} \cdot \mathcal{O}_{X,x} \subsetneq \mathfrak{m}_{X,x}$. In the rigid world, this does not happen. Berkovich gives an example of this in [Ber93, Rem. 2.2.9]: the reason why is that there are many more analytic functions than algebraic functions.

Theorem 16.15 [Kie69, Satz 3.3]. Any strictly k-affinoid algebra is excellent.

We do not need the full power of this (e.g., two of the conditions for excellence are that the ring is caternary and that excellence is stable under completion). This was later generalized by Ducros:

Theorem 16.16 [Duc09, Thm. 2.13]. Any k-affinoid algebra is excellent.

This is not a trivial reduction; it is harder than others. We will not use this.

Excellence automatically implies $\operatorname{Reg}(X)$ is open (this is property J_1 : universal Japaneseness), as long as we show that $\operatorname{Reg}(X) = \ker^{-1}(\operatorname{Reg}(\mathscr{X}))$. By definition of excellence:

Corollary 16.17. For $\mathscr{X} = \operatorname{Spec} \mathscr{A}$, where \mathscr{A} is strictly k-affinoid, $\operatorname{Reg}(\mathscr{X}) \subset \mathscr{X}$ is Zariski open.

Given these results, to prove Theorem 16.9, what we do is prove the strict case first, and then base change.

Proof of Theorem 16.9 for P = Reg. Lemma 16.13(i) implies $\operatorname{Reg}(X) \subset \pi^{-1}(\operatorname{Reg}(\mathscr{X}))$. Conversely, given $x \in X$ such that $\xi := \ker(x) \in \operatorname{Reg}(\mathscr{X})$, we must prove $x \in \operatorname{Reg}(X)$.

First consider the strict case, where \mathscr{A} is strictly k-affinoid, and $|k^*| \neq \{1\}$. By Corollary 16.17, the locus $\operatorname{Reg}(\mathscr{X}) \subset \mathscr{X}$ is open. We will use the following general fact (about Noetherian rings):

Fact 16.18. $\mathcal{O}_{\mathscr{X},\xi}$ is regular if and only if $\widehat{\mathcal{O}}_{\mathscr{X},\xi}$ is regular.

First assume $x \in \operatorname{Max} \mathscr{A} \subset \mathscr{M}(\mathscr{A})$. In this case, we have shown $\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{\mathscr{X},\xi}$ is regular, and so $\mathcal{O}_{X,x}$ is regular. This implies $x \in \operatorname{Reg}(X)$. Now consider any $x \in X = \mathscr{M}(\mathscr{A})$. Then, $\operatorname{Reg}(\mathscr{X}) \subset \mathscr{X}$ is open, so its preimage $\ker^{-1}(\operatorname{Reg}(\mathscr{X})) \subset X$ is open. Thus, for any sufficiently small affinoid neighborhood $V \ni x$, we have that $V \subset \ker^{-1}(\operatorname{Reg}(\mathscr{X}))$. If $y \in \operatorname{Max}(\mathscr{A}_V) \subset \operatorname{Max}(\mathscr{A})$, then $\mathcal{O}_{\mathscr{X},\eta}$ is regular, where $\eta = \ker(y)$. Then, $\mathcal{O}_{X,y}$ is regular by what we showed above. Thus, $\mathcal{O}_{V,y} \cong \mathcal{O}_{X,y}$ is a regular local ring for all $y \in \operatorname{Max}(\mathscr{A}_V)$. Now set $\mathscr{V} = \operatorname{Spec} \mathscr{A}_V$. Lemma 16.13(*i*) implies $\mathcal{O}_{\mathscr{V},\eta}$ is regular for all $\eta \in \operatorname{Max}(\mathscr{A}_V) \subseteq \mathscr{V}$. This means that \mathscr{A}_V is regular, so $\operatorname{Reg}(\mathscr{V}) = \mathscr{V}$, and $\mathcal{O}_{\mathscr{V},\xi}$ is regular for all $\xi \in \mathscr{V}$. Now pick a small enough neighborhood $V \ni x$ such that

$$\mathfrak{m}_{V,x} = \mathfrak{m}_{\mathscr{V},\xi} \cdot \mathcal{O}_{V,x} = \mathfrak{m}_{\mathscr{V},\xi} \cdot \mathcal{O}_{X,x}.$$

Then, Lemma 16.13(*ii*) implies that $\mathcal{O}_{V,x} \cong \mathcal{O}_{X,x}$ is regular, i.e., $x \in \text{Reg}(X)$.

In the general case, we will use the usual field extension, but there is a new idea that we haven't seen. Let X be a general k-affinoid space, where k can now be trivially valued. Pick $K = K_r$ such that $|K^*| \neq \{1\}$, and $\mathscr{A}' \coloneqq \mathscr{A} \otimes_k K$ strictly *K*-affinoid. There is a canonical surjective map $\pi \colon X' \to X$ (Gruson's theorem 3.21), where $X' = \mathscr{M}(\mathscr{A}')$, and we can concretely describe the fibers as follows. First,

$$\mathscr{A}' \cong \left\{ f = \sum_{\nu} a_{\nu} T^{\nu} \mid a_{\nu} \in \mathscr{A}, \ \lim_{|\nu| \to \infty} ||a_{\nu}|| r^{\nu} = 0 \right\},$$

and the norm is given by $||f|| = \max_{\nu} ||a_{\nu}|| r^{\nu}$. The fiber $\pi^{-1}(x) \cong \mathscr{M}(\mathscr{H}(x) \widehat{\otimes}_{\mathscr{A}} \mathscr{A}') \cong \mathscr{M}(\mathscr{H}(x) \widehat{\otimes}_k K)$, and

$$\mathscr{H}(x)\widehat{\otimes}_{k}K \cong \left\{ f = \sum_{\nu} a_{\nu}T^{\nu} \mid a_{\nu} \in \mathscr{H}(x), \lim_{|\nu| \to \infty} ||a_{\nu}||r^{\nu} = 0 \right\}$$

is a Banach K-algebra with multiplicative norm. This gives rise to the special point ("Shilov boundary" of the fiber) $x' \in \pi^{-1}(x) \subseteq X'$ satisfying $|f(x')| = \max_{\nu} |a_{\nu}(x)| r^{\nu}$ for $f \in \mathscr{A}'$. This gives a section $\sigma \colon X \to X'$, $x \mapsto x'$ of $\pi \colon X' \to X$. We then use the following:

Lemma 16.19 [Ber93, Lem. 2.2.5].

- (i) $\sigma: X \to X'$ is continuous (in the Berkovich topology).
- (ii) $\mathfrak{m}_{\mathscr{X}',\xi'} = \mathfrak{m}_{\mathscr{X},\xi} \cdot \mathcal{O}_{\mathscr{X}',\xi'}$ where $\xi = \ker(x) \in \mathscr{X}$ and $\xi' = \ker(x') \in \mathscr{X}'$ via the map $\mathcal{O}_{\mathscr{X},\xi} \hookrightarrow \mathcal{O}_{\mathscr{X}',\xi'}$.
- (iii) If $Y' \subset X'$ is Zariski closed, then $\sigma^{-1}(Y') \subset X$ is Zariski closed (so σ is continuous in the Zariski topology).

The idea of the proof is to use Noetherianity, and the yoga of finite modules (finitely generated ideals in this case) which have an essentially unique finite Banach module structure.

Now we have a diagram:

$$\begin{array}{ccc} \mathcal{O}_{X',x'} & \longleftrightarrow & \mathcal{O}_{X,x} \\ \uparrow & & \uparrow \\ \mathcal{O}_{\mathscr{X}',\xi'} & \longleftrightarrow & \mathcal{O}_{\mathscr{X},\xi} \end{array}$$

where all maps are faithfully flat. By assumption, $\xi \in \operatorname{Reg}(\mathscr{X})$, and so $\mathcal{O}_{\mathscr{X},\xi}$ is regular. Lemma 16.13(*ii*) and Lemma 16.19(*ii*) implies $\mathcal{O}_{\mathscr{X}',\xi'}$ is regular. The strict case implies that $\mathcal{O}_{X',x'}$ is regular. The easy direction of a version of Lemma 16.13(*i*) implies that $\mathcal{O}_{X,x}$ is regular. Thus, the regular locus $\operatorname{Reg}(X) = \ker^{-1}(\operatorname{Reg}(\mathscr{X}))$. Also, $\operatorname{Reg}(X) = \sigma^{-1}(\operatorname{Reg}(X'))$, where $\sigma \colon X \to X'$. We know that $\operatorname{Reg}(X')$ is Zariski open, so Lemma 16.19(*iii*) implies $\operatorname{Reg}(X)$ is Zariski open. \Box

Note that this last step can be avoided by using Ducros's theorem 16.16.

We won't say much more about local rings. We will move on to talking about coherent sheaves and globalzing the Berkovich space construction.

17 November 8

17.1 The reduction map $[Ber90, \S2.4]$

Today, we will discuss the reduction map from [Ber90, §2.4]. This should strictly speaking be done earlier, since it is used in showing the Gerritzen–Grauert theorem. It is another way to relate schemes and Berkovich spaces.

Recall 17.1. If \mathscr{A} is a non-Archimedean Banach ring, and $X = \mathscr{M}(\mathscr{A}), \ \mathscr{X} := \operatorname{Spec} \mathscr{A}$, then we considered the map ker: $X \to \mathscr{X}$, which is surjective if X is k-affinoid.

Now let $\mathscr{A}^{\circ} = \{\rho \leq 1\}$, which is a ring since \mathscr{A} is non-Archimedean. We have an ideal $\mathscr{A}^{\circ\circ} = \{\rho < 1\}$, and we can define the reduction ring $\widetilde{\mathscr{A}} := \{\rho \leq 1\}/\{\rho < 1\}$, which has no norm. We also define $\widetilde{\mathscr{X}} := \operatorname{Spec} \widetilde{\mathscr{A}}$.

Example 17.2. If $\mathscr{A} = k\{T\}$, then $\mathscr{A}^{\circ} = k^{\circ}\{T\}$, and $\widetilde{\mathscr{A}} = \widetilde{k}[T]$.

This is functorial (Exercise): if $\mathscr{A} \to \mathscr{B}$ is bounded, then it induces a map $\widetilde{\mathscr{A}} \to \widetilde{\mathscr{B}}$.

The following is the analogue of the kernel map:

Definition 17.3. We define the reduction map red: $X \to \widetilde{\mathscr{X}}$ as follows: Given a point $x \in X$, consider the character (multiplicative map) $\chi_x \colon \mathscr{A} \to \mathscr{H}(x)$. Since it is bounded, this induces a map $\widetilde{\chi}_x \colon \widetilde{\mathscr{A}} \to \widetilde{\mathscr{H}}(x)$, where the latter is a field. We can then take the kernel of $\widetilde{\chi}_x$, which is a prime ideal of $\widetilde{\mathscr{A}}$, giving a point $\operatorname{red}(x) \in \widetilde{\mathscr{X}}$. This is functorial, and gives the following commutative square:



Example 17.4. Consider the special case where \mathscr{A} is trivially valued. We have $\widetilde{\mathscr{A}} = \mathscr{A}$ and $\widetilde{\mathscr{K}} = \mathscr{X}$. Then, $\operatorname{red}(x) \in \mathscr{X}$ corresponds to the prime ideal $\{|f(x)| < 1\}$ of \mathscr{A} , whereas $\ker(x) \in \mathscr{X}$ corresponds to the prime ideal $\{|f(x)| = 0\}$ of \mathscr{A} . Thus, $\operatorname{red}(x)$ is a specialization of $\ker(x)$.

Example 17.5. Let $\mathscr{A} = k\{T\}$, and $\widetilde{\mathscr{A}} = \widetilde{k}[T]$. Let $X = E_k$, and so $\widetilde{\mathscr{X}} = \mathbf{A}_{\widetilde{k}}^1$. Then, $X \to \widetilde{\mathscr{X}}$ is such that $\operatorname{red}^{-1}(\widetilde{x}) = \{x_0\}$, where \widetilde{x} is the generic point. For any $\xi \in \widetilde{\mathscr{X}} \setminus \{\widetilde{x}\}$, $\operatorname{red}^{-1}(\xi)$ is the connected component of $\mathscr{X} \setminus \{x_G\}$, which is a "limb" of \mathscr{X} attached to x_G .

There isn't too much you can say for general Banach rings, but we will begin in this generality nonetheless. We will specialize to strictly k-affinoid rings afterward, where we can say much more.

Let \mathscr{A} be a general Banach ring, and consider the Zariski topology on $\mathscr{X} = \operatorname{Spec} \mathscr{A}$. We want to describe what the open sets and the closed sets are. Let $\tilde{f} \in \mathscr{A}$, which gives $D(\tilde{f}) = \{\widetilde{x} \in \mathscr{\widetilde{X}} \mid \widetilde{f}(\widetilde{x}) \neq 0\}$ a basic open set. The condition is equivalent to $\tilde{f} \notin \widetilde{\mathfrak{p}}_{\widetilde{x}}$. An ideal $\widetilde{\mathfrak{a}} \subseteq \mathscr{\widetilde{A}}$ gives a closed set $V(\widetilde{\mathfrak{a}}) = \{\widetilde{x} \in \mathscr{\widetilde{X}} \mid \widetilde{f}(\widetilde{x}) = 0 \, \forall \widetilde{f} \in \widetilde{\mathfrak{a}}\}$.

What is nice about Berkovich spaces is that the reduction map is globally defined; in Tate's theory, the reduction map is harder to describe. They are compatible in the sense that restricting the restriction map to maximal ideals in the Berkovich space gives back Tate's definition for the reduction map.

The following Lemma appears in Berkovich's book without proof.

Lemma 17.6 [Ber90, Lem. 2.4.1].

(i) For $f \in \mathscr{A}^{\circ}$, then $\operatorname{red}^{-1}(D(\widetilde{f})) = \{x \in X \mid |f(x)| = 1\}$, which is compact or empty.

- (ii) red⁻¹($D(\tilde{f})$) = \emptyset if and only if $f \in \mathscr{A}^{\circ\circ} = \{\rho < 1\}$.
- (iii) If $\tilde{\mathfrak{a}}$ is generated by $\tilde{f}_i \in \mathscr{A}^\circ$ for some index set $i \in I$, then $\operatorname{red}^{-1}(V(\tilde{\mathfrak{a}})) = \{x \in X \mid |f_i(x)| < 1 \; \forall i \in I\}$.

Proof. This follows by essentially unwinding definitions, and using the "generalized maximum modulus principle" (Theorem 3.16)

$$\rho(f) = \max_{x \in Y} |f(x)|.$$

Thus, the reduction map is anticontinuous: it switches the role of open and closed subsets.

The following Corollary appears in Berkovich's book, but with a typo:

Corollary 17.7 (cf. [Ber90, Cor. 2.4.2]). If $\widetilde{\mathscr{A}}$ is Noetherian, then red is anticontinuous.

It's hard to tell in general whether \mathscr{A} Noetherian implies $\widetilde{\mathscr{A}}$ is Noetherian, which is why we need to correct Berkovich's statement.

The following statement will be strengthened later, under some extra conditions.

Corollary 17.8 [Ber90, Cor. 2.4.3]. If $\tilde{x} \in \tilde{X}$ is a generic point (so is a minimal prime ideal), then $\operatorname{red}^{-1}(\tilde{x})$ is nonempty, and compact.

Proof. If $\widetilde{x} \in \widetilde{X}$ corresponds to a prime ideal $\widetilde{\mathfrak{p}} \subset \widetilde{A}$, then

$$\{\widetilde{x}\} = \{\widetilde{y} \in \mathscr{X} \mid \widetilde{f}(\widetilde{y}) = 0 \,\,\forall f \in \widetilde{\mathfrak{p}}, \,\, \text{but} \,\, \widetilde{f}(\widetilde{y}) \neq 0 \,\,\forall \widetilde{f} \in \mathscr{A} \setminus \widetilde{\mathfrak{p}}\}.$$

Since $\tilde{\mathfrak{p}}$ is minimal,

$$\{\widetilde{x}\} = \{\widetilde{y} \in \widetilde{\mathscr{X}} \mid \widetilde{f}(\widetilde{y}) \neq 0 \; \forall f \notin \widetilde{\mathfrak{p}}\} = \bigcap_{\widetilde{f} \notin \widetilde{\mathfrak{p}}} D(\widetilde{f}).$$
This implies $\operatorname{red}^{-1}(\widetilde{x}) = \bigcap_{\widetilde{f}\notin\widetilde{\mathfrak{p}}} \operatorname{red}^{-1}(D(\widetilde{f}))$. These $\operatorname{red}^{-1}(D(\widetilde{f}))$ are compact and nonempty. This implies $\operatorname{red}^{-1}(\widetilde{x})$ is nonempty, by using the "finite intersection property": $D(\widetilde{f}_1) \cap D(\widetilde{f}_2) = D(\widetilde{f}_1\widetilde{f}_2)$ if $\widetilde{f}_1\widetilde{f}_2 \notin \widetilde{\mathfrak{p}}$ (you use the general topological fact that a family of compact subsets of a compact space have nonempty intersection if any finite subfamily does).

This is all you can say in general. Note that it is not clear that the reduction map is surjective; all we know is that it hits all the generic points.

17.1.1 Reduction for strictly k-affinoid algebras [BGR84, §6.3]

Let \mathscr{A} be a strictly k-affinoid, so that $\mathscr{T}_n \twoheadrightarrow \mathscr{A}$. If k is trivially valued, then $\widetilde{\mathscr{A}} = \mathscr{A}$, and you can treat this case separately; we will assume that k is non-trivially valued.

The following would take time to prove, and requires a lot of commutative algebra.

Theorem 17.9 [BGR84, Thm. 6.3.5/1]. If $\varphi \colon \mathscr{A} \to \mathscr{B}$ is a (bounded) homomorphism of strictly k-affinoid algebras, then the following are equivalent:

- (i) $\varphi \colon \mathscr{A} \to \mathscr{B}$ is finite;
- (ii) $\varphi \colon \mathscr{A} \to \mathscr{B}$ is integral;
- (iii) $\varphi^{\circ} \colon \mathscr{A}^{\circ} \to \mathscr{B}^{\circ}$ is integral;
- (iv) $\widetilde{\varphi} \colon \widetilde{\mathscr{A}} \to \widetilde{\mathscr{B}}$ is integral;
- (v) $\widetilde{\varphi} \colon \widetilde{\mathscr{A}} \to \widetilde{\mathscr{B}}$ is finite.

It is not necessarily true that these are equivalent to $\varphi^{\circ} \colon \mathscr{A}^{\circ} \to \mathscr{B}^{\circ}$ being finite; this depends on the ground field. See [BGR84, §6.4]. One example is as follows:

Example 17.10 (Proposed by Bhargav Bhatt and proved by Matt Stevenson). Let $\mathscr{A} = k$ and let $\mathscr{B} = k[\epsilon]$ with $\epsilon^2 = 0$. Then, an element of \mathscr{B} can be written as $a + b\epsilon$ for $a, b \in k$. Define a norm $|\cdot|$ on \mathscr{B} by

$$|a+b\epsilon| \coloneqq \max\{|a|, |b|\}.$$

This is a non-Archimedean norm that is submultiplicative (but not multiplicative) and such that the spectral seminorm is given by $\rho(a + b\epsilon) = |a|$. In particular, $\mathscr{B}^{\circ} = k^{\circ} \oplus k\epsilon$.

We will mostly just use the equivalence $(i) \Leftrightarrow (v)$ in Theorem 17.9. The dependencies required are a bit crazy and long, and so we will not prove the Theorem.

One useful Corollary of Theorem 17.9 is the following:

Corollary 17.11 [BGR84, Cor. 6.3.4/3]. $\widetilde{\mathscr{A}}$ is a finitely generated \widetilde{k} -algebra if \mathscr{A} is a strictly k-affinoid.

Proof. By Noether normalization (Lemma 8.10), there exists a finite map $\varphi \colon \mathscr{T}_d \hookrightarrow \mathscr{A}$. This implies $\widetilde{\varphi} \colon \widetilde{\mathscr{T}_d} \to \widetilde{\mathscr{A}}$ is finite, and so we are done since $\widetilde{\mathscr{T}_d} = \widetilde{k}[T_1, \ldots, T_d]$. With more work, you can show that this map is injective, and that the two rings have the same Krull dimension; see [BGR84, p. 248, Rem.].

17.1.2 The reduction map for strictly *k*-affinoid spaces

Let k be a nontrivially valued field, \mathscr{A} a strictly k-affinoid algebra, and $X = \mathscr{M}(\mathscr{A}), \ \mathscr{X} = \operatorname{Spec} \widetilde{\mathscr{A}}$. We have the reduction map red: $X \to \widetilde{\mathscr{X}}$, where $\widetilde{\mathscr{X}}$ is now an affine scheme of finite type over $\operatorname{Spec} \widetilde{k}$.

Example 17.12. We saw in Example 17.2 that if $\mathscr{A} = k\{T\}$, then $\widetilde{\mathscr{A}} = \widetilde{k}[T]$.

Example 17.13. Let $\mathscr{A} = k\{S, r^{-1}T\}/(ST-1)$, and for simplicity let $1 < r \in |k^*|$. Then, X is an annulus with radius 1, r, and $\widetilde{\mathscr{A}} \cong \widetilde{k}[\widetilde{S}, \widetilde{T}]/(\widetilde{S}\widetilde{T})$, where $S \mapsto \widetilde{S}$ and $a^{-1}T \mapsto \widetilde{T}$, and |a| = r. Then, \widetilde{X} has two generic points, which are the images of p(E(r)) and p(E(1)) (and there are no other preimages).

This is sort of typical of what happens here:

Theorem 17.14. Let k be a nontrivially valued field, \mathscr{A} a strictly k-affinoid algebra, and $X = \mathscr{M}(\mathscr{A})$, $\mathscr{X} = \operatorname{Spec} \widetilde{\mathscr{A}}$.

(i) red: $X \to \widetilde{\mathscr{X}}$ is surjective;

(ii) If $\widetilde{x} \in \widetilde{\mathscr{X}}$ is a generic point, then $\operatorname{red}^{-1}(\widetilde{x})$ is a singleton.

There are a couple of remarks: we will explain the first, and not really the second.

Remark 17.15. red⁻¹(\mathscr{X}_{gen}), where \mathscr{X}_{gen} is the (finite) set of generic points, is the Shilov boundary of \mathscr{A} (or X). We will talk about boundaries next time. This is a concept from function algebras.

Remark 17.16. This statement is due to Berkovich, but Tate proved something similar: namely, $\operatorname{red}(\operatorname{Max} \mathscr{A}) = \operatorname{Max} \widetilde{\mathscr{A}} \subset \widetilde{\mathscr{X}}$, where $\operatorname{Max} \mathscr{A} \subset X$.

Sketch of (i). Berkovich says that you can prove (i) in the same way as Tate's result [BGR84, Thm. 7.1.5/4] mentioned above, but this is not entirely true: there is one step which is different. We will proceed in steps.

Step 1. $X = E_k^n = \mathscr{M}(\mathscr{T}_n)$ is the closed unit polydisc.

If n = 1, we sort of did this already. If k is algebraically closed, it is not too difficult (look at Type 1 points), and the non-algebraically closed case is not much harder. In general, we use induction on n. Let n > 1. Then, pick a projection $\pi: X = E_k^n \to E_k^1 = Y$, e.g., onto the first coordinate. This also gives a map $\widetilde{\mathscr{X}} \to \widetilde{\mathscr{Y}}$ and a square



Pick $\widetilde{x} \in \widetilde{\mathscr{X}}$. We must find $x \in X$ such that $\operatorname{red}_X(x) = \widetilde{x}$. Set $\widetilde{y} = \widetilde{\pi}(\widetilde{x}) \in \widetilde{Y}$. By induction (the case n = 1), there exists $y \in Y$ such that $\operatorname{red}_Y(y) = \widetilde{y}$. Then, $\pi^{-1}(y) \cong E^{n-1}_{\mathscr{H}(y)}$, and similarly $\widetilde{\pi}^{-1}(\widetilde{y}) \cong \mathbf{A}^{n-1}_{\kappa(\widetilde{y})}$. So it suffices to prove that

$$E^{n-1}_{\mathscr{H}(y)} \xrightarrow{\operatorname{red}_X} \mathbf{A}^{n-1}_{\kappa(\widetilde{y})}$$

is surjective. This is "almost" the inductive hypothesis, but there is a twist: $\kappa(\tilde{y})$ is not the residue field of $\mathscr{H}(y)$. On the other hand, we have an embedding $\kappa(\tilde{y}) \hookrightarrow \mathscr{H}(y)$, which comes as part of the construction of the reduction map. This gives a map $\mathbf{A}_{\mathscr{H}(y)}^{n-1} \to \mathbf{A}_{\kappa(y)}^{n-1}$, which is surjective. This map fits in the diagram below



By induction, the first map is surjective, so red_X is surjective.

Note that this is somewhat easier if you only care about maximal ideals: you don't need to use induction. To do the general case, we want to use Noether normalization. We need an intermediate special case, first.

Step 2. $X = \mathcal{M}(\mathcal{A})$, where \mathcal{A} is an integral domain.

Noether normalization gives a finite injection $\mathscr{T}_d \hookrightarrow \mathscr{A}$, giving a map $X \to E_k^d$, which is surjective and has finite fibers. By Theorem 17.9, $\widetilde{\mathscr{T}}_d \hookrightarrow \widetilde{\mathscr{A}}$ is also a finite injection (the spectral radius is in general contracting, but is actually an isometry in this case). Then, $\widetilde{\mathscr{X}} \to \mathbf{A}_{\widetilde{k}}^d$ is surjective, and has finite fibers. This gives the diagram



It doesn't immediately follow formally that $X \to \widetilde{\mathscr{X}}$ is surjective. Instead, you have to work with $\operatorname{Frac}(\mathscr{A})$ to get spaces $Z, \widetilde{\mathscr{X}}$ which are slightly bigger, and use Galois actions (we will skip this step). This is the only part where the fact that \mathscr{A} is an integral domain is important. See the proof of [BGR84, Thm. 7.1.5/4].

Step 3. \mathscr{A} is reduced.

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal prime ideals in \mathscr{A} , and define $\mathscr{A}_i \coloneqq \mathscr{A}/\mathfrak{p}_i, X_i = \mathscr{M}(\mathscr{A}_i), \widetilde{\mathscr{X}_i} = \operatorname{Spec} \widetilde{\mathscr{A}_i}$. Then, $\mathscr{A} \hookrightarrow \prod_i \mathscr{A}_i$, and so there is a map $\bigsqcup_i X_i \to X$ that is surjective. This gives a map $\widetilde{\mathscr{A}} \hookrightarrow \prod_i \widetilde{\mathscr{A}_i}$, and so $\bigsqcup_i \widetilde{\mathscr{X}_i} \to \widetilde{\mathscr{X}}$ is surjective, and we have a commutative diagram



which implies $X \to \widetilde{\mathscr{X}}$ is surjective.

Step 4. Let \mathscr{A} be general, and $\mathscr{A}' := \mathscr{A}/\operatorname{rad}(\mathscr{A})$.

Then,
$$\mathcal{M}(\mathcal{A}') = \mathcal{M}(\mathcal{A})$$
, and $\widetilde{\mathcal{A}'} = \widetilde{\mathcal{A}}$.

The proof in [BGR84, Thm. 7.1.5/4] is almost the same, except they don't have to treat steps 3 and 4 separately.

We now want to say something brief about general k-affinoid algebras. We would like to use Berkovich's usual reduction to the strict case using the ground field extension K_r/k , but it is not clear how it would help here, as we will see in the following example.

Consider a k-affinoid algebra \mathscr{A} , and let $X = \mathscr{M}(\mathscr{A})$.

Example 17.17. $\mathscr{A} = k\{r^{-1}T\}$, and $r \notin \sqrt{|k^*|}$, $\widetilde{\mathscr{A}} \cong \widetilde{k}$. Thus, $\widetilde{\mathscr{X}}$ is just a point, which is not so well-behaved.

Temkin's construction in $[\text{Tem}04, \S3]$ is to use another ring:

$$\widetilde{\mathscr{A}} \coloneqq \bigoplus_{r \in \mathbf{R}^*_+} \{ \rho \le r \} / \{ \rho < r \},$$

which is an \mathbf{R}^*_+ -graded ring. You can then define a "reduction" map $\mathscr{A} \to \widetilde{\mathscr{A}}$. You can then develop all of commutative algebra over graded rings that look like this, and define $\widetilde{\mathscr{X}} := \operatorname{Spec} \widetilde{\mathscr{A}}$ to consist of homogeneous prime ideals.

Theorem 17.18 [Tem04, Prop. 3.3(*i*),(*ii*)]. This reduction map red: $X \to \widetilde{\mathscr{X}}$ is surjective, and if $\widetilde{x} \in \widetilde{\mathscr{X}}$ is a generic point, then $\# \operatorname{red}^{-1}(\widetilde{x}) = 1$.

Temkins' reduction map is a "better" definition than the classical one, and can be used to prove a more general Gerritzen–Grauert theorem [Tem04, Prop. 3.5]. It is also useful in studying when the functor from strict k-analytic spaces to k-analytic spaces is fully faithful in the global Berkovich theory [Tem04, Cor. 4.10].

We will talk about relative interiors and boundaries next time.

18 November 10

Today we will discuss coherent sheaves and the global theory of Berkovich spaces. We will also talk about relative interior and boundary in Berkovich theory, which are unique to this setting.

18.1 Coherent sheaves [Ber90, §2.3]

Let $X = \mathscr{M}(\mathscr{A})$ is a k-affinoid space. To this, we have associated a structure sheaf \mathcal{O}_X , which is just a sheaf of k-algebras, where

$$\mathcal{O}_X(U) \coloneqq \lim_{\substack{V \subseteq U \\ \text{special}}} \mathscr{A}_V,$$

Recall \mathscr{A}_V makes sense for affinoid domains, and for special sets, we defined them using Tate's acyclicity theorem. The inverse limit then gives a k-algebra. More structure can be kept when passing to the inverse limit, but we won't use this.

Definition 18.1. An \mathcal{O}_X -module \mathscr{F} is coherent if

- (i) \mathscr{F} is locally finitely generated, i.e., for all $x \in X$, there exists a neighborhood $x \in U \subset X$ and a surjective map $\mathcal{O}_X(U)^n \to \mathscr{F}(U)$ for some $n \ge 1$ (there is also a *G*-topology type of condition for this).
- (*ii*) For every surjection $(\mathcal{O}_X|_U)^n \xrightarrow{\psi} \mathscr{F}|_U$, the kernel is locally finitely generated.

There is supposedly no good notion of quasi-coherent sheaves.

The following is fairly trivial in algebraic geometry because of how you define coherence, but is a deep theorem in complex geometry (cf. Oka's coherence theorem). The Berkovich analogue is also non-trivial:

Proposition 18.2 [Duc09, Lem. 0.1]. The direct sums \mathcal{O}_X^n of the structure sheaf are coherent for every $n \geq 1$.

The proof is a bit easier than Oka's by using affinoids in a nice way.

We now try to construct examples of coherent sheaves.

Definition 18.3. Given a finite \mathscr{A} -module M, you can define a coherent \mathcal{O}_X -module $\mathcal{O}_X(M)$ by

$$\Gamma(U, \mathcal{O}_X(M)) \coloneqq \lim_{\substack{V \subseteq U \\ \text{special}}} M \otimes_{\mathscr{A}} \mathscr{A}_V.$$

Note that you can use a complete tensor product, but this doesn't matter since the \mathscr{A}_V are Noetherian, and M is finite over \mathscr{A} .

To check this defines a sheaf, use that $\mathscr{A} \to \mathscr{A}_V$ is flat when $V \subseteq X$ is an affinoid domain. Then, applying the functor $M \otimes_{\mathscr{A}} -$ to the exact sequence (8) in the statement of Tate's acyclicity theorem 13.3 results in a sequence whose exactness implies that the definition above indeed results in a sheaf.

The following was proved by Kiehl for some G-topologies on rigid analytic spaces.

Theorem 18.4 (Kiehl, Berkovich [Ber90, Prop. 2.3.1]). Every coherent \mathcal{O}_X -module is of the form $\mathcal{O}_X(M)$.

You can say much more about coherent sheaves, of course, but we want to do one example where you can look at properties of maps and interpret them in terms of coherent sheaves.

Definition 18.5. A morphism $f: Y = \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{A}) = X$ of k-affinoid spaces is

- (i) finite if \mathscr{B} is a finite Banach \mathscr{A} -algebra (so there exists an admissible epimorphism $\mathscr{A}^n \twoheadrightarrow \mathscr{B}$).
- (*ii*) closed immersion if $\mathscr{A} \to \mathscr{B}$ is an admissible epimorphism.

Proposition 18.6 [Ber90, Cor. 2.3.2]. With notation as in the Definition above:

- (i) f is finite if and only if $f_*\mathcal{O}_Y$ is a coherent \mathcal{O}_X -module.
- (ii) f is a closed immersion if and only if f is finite, and $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective.

The proof boils down to moving back and forth between finite algebras and finite Banach algebras.

Sketch proof of (i). " \Rightarrow " If f is finite, then \mathscr{B} is finite (Banach) \mathscr{A} -module, so you need to show that $f_*\mathcal{O}_Y = \mathcal{O}_X(\mathscr{B})$.

" \Leftarrow " Assume that $f_*\mathcal{O}_Y$ is coherent. Set $\mathscr{B}' \coloneqq \Gamma(X, f_*\mathcal{O}_Y)$ as an \mathscr{A} -module, which is a finite \mathscr{A} -module, and is equal to \mathscr{B} as an \mathscr{A} -module (but a priori, \mathscr{B} is not a *finite* Banach \mathscr{A} -module). We therefore need to

equip \mathscr{B}' with the unique structure of a finite *Banach* \mathscr{A} -module (Proposition 8.3). You then get a canonical bounded map

$$\mathscr{A}^{n} \xrightarrow{\text{admissible}} \mathscr{B}' \xrightarrow{\sim} \mathscr{B}$$

You must show $\mathscr{B}' \to \mathscr{B}$ is admissible (and so you get an isomorphism of Banach \mathscr{A} -modules). This is automatic by the Banach Open Mapping Theorem 7.15 if $|k^*| \neq \{1\}$. In the trivially valued case, this needs more work. Instead of passing to a larger ground field, however, the idea is to add in variables until you get an admissible epimorphism $\mathscr{B}'\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \twoheadrightarrow \mathscr{B}$, and then prove that adding the extra variables was redundant.

We can say more about the global theory, but we will skip that for now.

18.2 The relative interior and boundary [Ber90, §2.5]

These are unique to Berkovich spaces: they do not exist in the rigid setting, and (probably) not in the adic setting, either.

Definition 18.7. Given a morphism $Y \to X$ of k-affinoid space, define the relative interior

$$\operatorname{Int}(Y|X) \subset Y,$$

and the relative boundary

$$\partial(Y/X) \subset Y$$

If $X = \mathcal{M}(k) = \text{pt}$, then write Int Y and ∂Y , respectively.

The reason we would want to think about interiors and boundaries is that affinoid subsets are closed (hence compact), hence should have a boundary. This is similar to how in complex anlysis, you often don't use closed discs to define things; there should be a way to talk about the interior. We list some desirable features of these definitions explicitly:

- $\operatorname{Int}(Y|X) \subset Y$ should be open;
- $\partial(Y/X) = Y \setminus \operatorname{Int}(Y/X)$ should be closed;
- When $Y \to X$ is a closed immersion, or even just finite, we should have $\partial(Y/X) = \emptyset$;
- If $Y \xrightarrow{f} Y$ is an automorphism, then we should have

$$f^{-1}(\operatorname{Int} Y) = \operatorname{Int} Y$$
 and $f^{-1}(\partial Y) = \partial Y$.

We will see where demanding such properties leads. We won't axiomatically define boundaries and interiors as "things satisfying the four properties above," but we use these properties as motivation.

Example 18.8. Let $Y = E_k = \mathcal{M}(k\{T\})$ be the closed unit disc, and let $X = \mathcal{M}(k)$. Then, recall we said $Y = \{|T| \leq 1\} \subseteq \mathbf{A}_k^{1,\mathrm{an}}$, and Y had image equal to the closed unit disc. Thus, the guess would be that $\partial Y = \{|T| = 1\}$. But this is not very invariant: it depends on coordinates, and is not invariant under automorphisms $T \mapsto T + a$, where $a \in k^\circ$. Instead,

$$\partial Y = \{ \text{Gauss point} \} = \{ \text{given norm on } k\{T\} \}.$$

Trying to guess what boundaries and interiors of morphisms is hard: they are sometimes non-intuitive. But we can give a definition nonetheless:

Definition 18.9. Consider $Y \xrightarrow{\varphi} X$, which is induced by a bounded map $\mathscr{A} \to \mathscr{B}$. Then, $y \in \operatorname{Int}(Y/X)$ if and only if there exists an admissible epimorphism $\mathscr{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \twoheadrightarrow \mathscr{B}$ extending $\mathscr{A} \to \mathscr{B}$, such that $|T_i(\psi(y))| < r_i$ for $i = 1, 2, \ldots, n$.

It is clear that $\operatorname{Int}(Y/X)$ is open; set $\partial(Y/X) = Y \setminus \operatorname{Int}(Y/X)$, which is closed.

This is a fairly original definition by Berkovich, which does not seem to come from related notions in rigid analytic spaces. We can visualize the definition as in Figure 18.10.



Figure 18.10: A visualization of relative interior and relative boundary.

Exercise 18.11. Prove that if $Y = E_k$ is a closed unit disc, then with this definition, $\partial Y = \{\text{Gauss point}\}$.

To actually prove things using this notion, you need to work a bit. First, there are alternate descriptions of relative interior:

Proposition 18.12 [Ber90, Prop. 2.5.2]. Consider a morphism $\psi: Y \to X$ and let $y \in Y$. Then, the following are equivalent:

(i) $y \in \operatorname{Int}(Y|X);$

(ii) For any bounded \mathscr{A} -linear homomorphism $\mathscr{A}\{r^{-1}T\} \to \mathscr{B}$ (where $r^{-1}T$ is just one variable), there exists a polynomial

$$P(T) = T^m + a_1 T^{m-1} + \dots + a_m \in \mathscr{A}[T]$$

such that $\sigma(P) \coloneqq \max_i \rho(a_i)^{1/i} \leq r$, and $|P(\psi(y))| < r^m$. If X and Y are strictly k-affinoid (and $|k^*| \neq \{1\}$), then (i) and (ii) are also equivalent to (iii) The ring $\tilde{\chi}_y(\widetilde{\mathscr{B}})$ is finite over $\tilde{\chi}_x(\widetilde{\mathscr{A}})$, where $x = \psi(y)$.

The map in (iii) is obtained as follows. Consider the commutative diagram

where $\mathscr{H}(x) \hookrightarrow \mathscr{H}(y)$ is an isometric immersion. Tildefying gives

Remark 18.13. In the special case where $Y = E_k(1)$, and $X = \mathscr{M}(k)$ is a point, the statement *(ii)* is saying that the image of y under the map $Y \to E_k(r)$ is not the Gauss point. In this way, the condition to lie in the interior allows us to characterize the Gauss point. Precisely, $\mathscr{M}(k\{T\}) \ni x$ is not the Gauss point if and only if there exists $a \in k$ such that $|T - a|_x < 1$, if k is algebraically closed. In general, the characterization is that there exists a polynomial $P \in k^{\circ}[T]$ such that |P(x)| < 1. This is what the Gauss norm is basically defined to do. Proposition 18.12 is then just a spruced up version of this Remark.

Proof of $(iii) \Rightarrow (ii)$. Assume for simplicity that r = 1. Then, there is a bounded map $\mathscr{A}\{T\} \xrightarrow{\pi} \mathscr{B}$, which induces a map $\widetilde{\mathscr{A}}[T] \xrightarrow{\tilde{\pi}} \widetilde{\mathscr{B}}$. (*iii*) then implies $\widetilde{\chi_y}(\widetilde{\mathscr{B}})$ is integral over $\widetilde{\chi_x}(\widetilde{\mathscr{A}})$. This means there exists a polynomial $P = T^m + a_1 T^{m-1} + \cdots + a_m \in \mathscr{A}^{\circ}[T]$ (by using integrality for the element $\widetilde{\chi_y}(\widetilde{\pi}(T))$), such that $\widetilde{\chi_y}(\widetilde{\pi}(\widetilde{P})) = 0$. Unwinding this, you get (*ii*).

For the other properties here, you have to use Theorem 17.9, which says that a map of strictly k-affinoid algebras is finite if and only if its reduction is. Berkovich does a bit more: he defines things called "inner morphisms" [Ber90, Def. 2.5.1].

Corollary 18.14. If Y is strictly k-affinoid, then $y \in Int(Y)$ if and only if red(y) is a closed point in $\widetilde{\mathscr{Y}} = \operatorname{Spec} \widetilde{\mathscr{B}}$, where $red: Y \to \widetilde{\mathscr{Y}}$ is the reduction map.

This follows from (iii).

We end with a couple more properties of relative interiors and boundaries, stated without proof.

Further Properties 18.15.

- (1) $\operatorname{Int}(Y/X) = Y$ if and only if $Y \to X$ is finite [Ber90, Cor. 2.5.13(i)].
- (2) If $Y \hookrightarrow X$ is an affinoid domain embedding, then $\partial(Y/X)$ is the relative topological boundary, and $\operatorname{Int}(Y/X)$ is the relative topological interior [Ber90, Cor. 2.5.13(*ii*)]. So while relative boundary and interior are analytic properties, if Y is an affinoid subset, then it can be deflected topologically. For example, if $Y = E(r) \hookrightarrow E(1) = X$, then $\partial(Y/X)$ is the Gauss point of Y.
- (3) If $Z \xrightarrow{\psi} Y \to X$, then

$$\operatorname{Int}(Z/X) = \operatorname{Int}(Z/Y) \cap \psi^{-1}(\operatorname{Int}(Y/X))$$

by [Ber90, Prop. 2.5.8(iii)].

(4) Relative interior behaves well under ground field extension [Ber90, Prop. 2.5.8(ii)].

Eventually, we will see that relative interiors and boundaries are what is needed to define the correct notion of properness for Berkovich spaces.

Next time, we will talk about how to build global Berkovich spaces.

19 November 15

Today we will talk about global Berkovich spaces, although we won't do too many things in detail, since they get technical. Towards the end of the course, we will focus on analytifications of algebraic varieties, where these global Berkovich spaces show up.

19.1 Global Berkovich spaces [Ber93, §1]

We have so far spent a lot of time talking about k-affinoid spaces, which form a category k-Aff with objects $X = \mathscr{M}(\mathscr{A})$, where \mathscr{A} is a k-affinoid algebra, and morphisms $\mathscr{M}(\mathscr{B}) = Y \to X = \mathscr{M}(\mathscr{A})$, which are induced by bounded homomorphisms $\mathscr{A} \to \mathscr{B}$. We also defined the structure sheaf $\mathcal{O}_X(V)$, which is defined for affinoid domains $V \subseteq X$ as $\mathcal{O}_X(V) = \mathscr{A}_V$. If $V = \bigcup_{i=1}^n V_i$ is a special subset, we defined $\mathcal{O}_X(V) = \mathscr{A}_V := \ker(\prod \mathscr{A}_{V_i} \to \prod \mathscr{A}_{V_i \cap V_j})$. Finally, if $V \subset X$ is open, then we defined

$$\mathcal{O}_X(V) = \varprojlim_{\substack{W \subset V \\ \text{special}}} \mathscr{A}_W.$$

There is another G-topology containing both special subsets and open subsets of X, which appears in [BGR84]; we will come back to this soon.

In this category k-Aff, we have fiber products

$$Y \times_X Z = \mathscr{M}(\mathscr{B} \widehat{\otimes}_{\mathscr{A}} \mathscr{C}),$$

and we have a good notion of ground field extensions

$$X_K \coloneqq \mathscr{M}(\mathscr{A} \widehat{\otimes}_k M) \twoheadrightarrow X$$

for a non-Archimedean extension K/k.

There are two approaches to global Berkovich spaces:

1. [Ber90, §3] Locally ringed spaces: (X, \mathcal{O}_X) is locally isomorphic to $(\mathcal{M}(\mathcal{A}), \mathcal{O}_{\mathcal{M}(\mathcal{A})})$.

2. [Ber93, §1] Atlases (closer to differential geometry): this is subtle, because charts are *closed*.

The following example illustrates the difference between the two notions. It is easy to write down, but is hard to prove is an example.

Example 19.2. Let $X = \{|T_1| = 1\} \cup \{|T_2| = 1\} \subseteq E^2(1)$. The two closed components are both Weierstrass domains, and their intersection is also a Weierstrass domain. However, one can show that X is a Berkovich space in the sense of 2, but not in the sense of 1. See Figure 19.1.

So construction 1 does not cover all "reasonable" rigid analytic spaces, and construction 2 is necessary to get a reasonable functor between rigid analytic spaces and Berkovich spaces. Also, all Berkovich spaces of form 1 are of form 2, and there is a way to check whether a space of form 2 comes from one of form 1.

19.1.1 Quasinets and nets [Ber93, §1.1]

Let X be a locally Hausdorff topological space. Recall that this allows non-Hausdorff spaces like the unit interval with two points at 1, as in Figure 19.3.

The following is a bit more restrictive than Berkovich's definition:

Definition 19.4. A quasinet on X is a collection τ of compact (Hausdorff) subsets $V \subseteq X$ such that for every $x \in X$, there exist $V_1, \ldots, V_n \in \tau$ such that $x \in V_1 \cap \cdots \cap V_n$, and $V_1 \cup \cdots \cup V_n$ is a neighborhood of x in X.

This is a way of saying there are "many" compact subsets, which "glue" in some way.

Example 19.5. The octants in $[-1, 1]^n$ form a quasinet (Figure 19.6).

Definition 19.7. A quasinet on X is a *net* if for all $U, V \in \tau$, the collection $\tau|_{U \cap V}$ of sets $W \in \tau$ such that $W \subset U \cap V$ also forms a quasi-net on $U \cap V$.

Example 19.8. Example 19.5 is not a net, but will become one after adding more sets.

19.1.2 k-analytic spaces and morphisms of k-analytic spaces [Ber93, §1.2]

Now fix a non-Archimedean field k. We want to do analytic geometry over k, and so we need a class of sets for which analytic functions are well-defined. These will exactly be elements of a net.

Definition 19.9. A k-affinoid atlas on X is a net τ together with:

- For every $V \in \tau$, a k-affinoid algebra \mathscr{A}_V and a homeomorphism $V \xrightarrow{\sim} \mathscr{M}(\mathscr{A}_V)$ ("affinoid sets");
- For every $U, V \in \tau$ with $U \subseteq V$, a bounded k-linear "restriction" homomorphism

 $\alpha_{V/U} \colon \mathscr{A}_V \longrightarrow \mathscr{A}_U$

that identifies (U, \mathscr{A}_U) with an affinoid domain in (V, \mathscr{A}_V) . We say that the triple (X, \mathscr{A}, τ) is a *k*-analytic space or a Berkovich space.

You can say that \mathscr{A}_{-} is a functor from the category associated to the net to k-affinoid algebras: given a composition $U \subset V \subset W$, you can get $\alpha_{W/U} = \alpha_{V/U} \circ \alpha_{W/V}$ by working through the data given above.

There are two natural examples of nets on an affinoid space:



Figure 19.1: A k-analytic space that is not good.



Figure 19.3: The "bug-eyed" line segment, which is a locally Hausdorff space that is not Hausdorff.



Figure 19.6: Octants in $[-1, 1]^n$ form a quasinet.

Example 19.10. Let $X = \mathcal{M}(\mathcal{A})$, where \mathcal{A} is a k-affinoid algebra. Let $\tau = \{\text{affinoid domains in } X\}$, where \mathcal{A}_V is the corresponding k-affinoid algebra for $V \in \tau$. You can check this forms a net.

Example 19.11. $X = \mathcal{M}(\mathcal{A})$, where $\tau = \{X\}$.

Of course, you would want to say that these are isomorphic. This is a bit technical to work out. What you need to do is define morphisms so that these become isomorphic. There is a natural definition that is "strong": that elements of a net pullback to elements of a net. But this doesn't let these k-analytic spaces to be isomorphic. Berkovich therefore defines morphisms in this category by working in a quotient category by a calculus of right fractions. We will instead define morphisms by using *maximal* atlases, which is how Berkovich characterizes morphisms in this quotient category eventually, since this is how you work with Berkovich spaces in practice.

Maximal atlases are constructed in two steps:

$$(X, \mathscr{A}, \tau) \xrightarrow{} (X, \overline{\mathscr{A}}, \overline{\tau}) \xrightarrow{} (X, \widetilde{\mathscr{A}}, \widehat{\tau})$$

We now describe these steps.

Step 1. Define a new atlas

 $\overline{\tau} = \{ W \subset X \text{ compact} \mid W \text{ affinoid domain in some } V \in \tau \},\$

to form a new k-analytic space $(X, \overline{\mathscr{A}}, \overline{\tau})$. You should check this makes sense, but this connects Examples 19.10 and 19.11.

It looks like this is enough, but apparently it is not, and we have to do the following:

Step 2. Define a new atlas

$$\widehat{\tau} = \begin{cases} \text{compact, special subsets} \\ W = W_1 \cup \dots \cup W_n \end{cases} \begin{vmatrix} W_i \cap W_j \in \overline{\tau} \ \forall i, j \\ \mathscr{A}_W = \ker(\prod \mathscr{A}_{W_i} \to \prod \mathscr{A}_{W_i \cap W_j}) \text{ is } k\text{-affinoid} \\ \text{(some more conditions)} \end{vmatrix}$$

The exact conditions are on [Ber93, pp. 20–21].

For affinoid algebras and affinoid spaces, Step 2 doesn't add anything new: in this case, W is already an affinoid domain.

With this definition, one can show $\hat{\tau} = \hat{\tau}$.

Definition 19.12. $(\widehat{\mathscr{A}}, \widehat{\tau})$ is called the maximal atlas. Elements of $\widehat{\tau}$ are called affinoid domains.

You can define/characterize morphisms in the category of Berkovich spaces, using maximal atlases.

Definition 19.13. A morphism

$$(X, \mathscr{A}, \tau) \xrightarrow{\varphi} (X', \mathscr{A}', \tau')$$

where the atlases are maximal, is the following data:

- A continuous map $\varphi \colon X \to X'$, such that for all $x \in X$, there exist affinoid domains $V_1, \ldots, V_n \subset X$ and $V'_1, \ldots, V'_n \subset X'$ with $\varphi(V_i) \subset V'_i$ for all i, such that
 - $-x \in V_1 \cap \cdots \cap V_n$ and $V_1 \cup \cdots \cup V_n$ is a neighborhood of x,
 - $-\varphi(x) \in V'_1 \cap \cdots \cap V'_n$ and $V'_1 \cup \cdots \cup V'_n$ is a neighborhood of $\varphi(x)$.
- A system of compatible morphisms $\varphi_{V/V'}: (V, \mathscr{A}_V) \to (V', \mathscr{A}_{V'})$ for each pair $V \in \tau, V' \in \tau'$ such that $\varphi(V) \subset V'$.

Furthermore, φ is an isomorphism if $\varphi \colon X \xrightarrow{\sim} X'$ is a homeomorphism mapping one maximal atlas to the other, i.e., $\varphi(\hat{\tau}) = \hat{\tau}'$, and if the compatible maps $\varphi_{V/V'}$ for $V' = \varphi(V)$, are isomorphisms $(V, \mathscr{A}_V) \xrightarrow{\sim} (V', \mathscr{A}_{V'})$ of affinoid spaces.

This forms the category k-An of k-analytic spaces

Remark 19.14. The functor

$$k\text{-Aff} \longrightarrow k\text{-An}$$
$$(X, \mathscr{A}) \longmapsto (X, \mathscr{A}, \{X\})$$

is fully faithful.

Definition 19.15. A k-analytic space X is good if every point $x \in X$ has an affinoid neighborhood.

You can show that all spaces obtained from the construction from [Ber90, §3] are good.

Example 19.16. The space $X = \{\max_{i=1,2} | T_i | = 1\} \subseteq E^2(1)$ from Example 19.2 is not good.

19.1.3 Analytic domains [Ber93, §1.3]

We want to next define what the structure sheaf should do for open subsets of a k-analytic space. Using the approach with nets, you can define analytic functions for a wider class of subsets, called analytic domains.

Definition 19.17. A subset Y of a k-analytic space X is an analytic domain if for all $y \in Y$, there exist affinoid domains V_1, \ldots, V_n in X such that $V_i \subset Y$ for all $i, y \in V_1 \cap \cdots \cap V_n$, and $V_1 \cup \cdots \cup V_n$ is a neighborhood of y in Y ("G-covering"). This condition can be phrased as requiring that $\hat{\tau}|_Y$ is a net on Y.

Analytic domains are not necessarily open nor closed.

Remark 19.18.

- (1) Every affinoid domain is analytic;
- (2) Every open subset is also analytic;
- (3) If $Y_1, Y_2 \subset X$ is analytic, then $Y_1 \cap Y_2$ is also analytic (nontrivial).
- (4) If $\varphi: X' \to X$ is a morphism and $Y \subset X$ is analytic, then $\varphi^{-1}(Y) \subset X'$ is analytic.
- (5) Every analytic domain $Y \subset X$ is an analytic space, and $Y \hookrightarrow X$ is a morphism.
- (6) If $\varphi: Z \to X$ and $\varphi(Z) \subset Y$ for Y an analytic domain, then φ factors through $Y \hookrightarrow X$.

Warning 19.19. The intersection of affinoids may not be affinoid (unless the space is separated).

The bug-eyed line segment (Figure 19.3) is an example of this.

Example 19.20. $Y = \{r < |T| \le s\} \subseteq E(1)$ for $0 < r < s \le 1$ is neither open nor affinoid.

Analytic domains are the subsets for which it is reasonable to define analytic functions. Topologically, however, they can be fairly complicated.

One good source of examples are analytifications of algebraic varieties. Consider affine space, which we defined in $\S1$ as

 $\mathbf{A}_{k}^{n} \coloneqq \{ \text{multiplicative seminorms on } k[T_{1}, \dots, T_{n}] \text{ whose restriction to } k \text{ is the given norm} \},$

with the usual topology. We can define a net $\tau = \{E(r) \mid r \in (\mathbf{R}^*_+)^n\}$ is formed by polydics (this is not maximal; you need to throw in affinoid domains, etc.). This space is good: $x \in \text{Int } E(r)$ for $r_i > |T_i(x)|$. Spaces you get from any reasonable schemes (in particular, affine schemes), are good. Spaces from other sources will not necessarily be good.

19.1.4 The structure sheaf on a k-analytic space [Ber93, §1.3]

An elegant way to now define analytic functions is to define them to be the set of morphisms from that analytic domain to \mathbf{A}_k^1 . The following facts are stated on [Ber93, p. 25] without proof.

Fact 19.21. \mathbf{A}_k^1 is a ring object in k-An.

Fact 19.22. Hom $(\mathscr{M}(\mathscr{A}), \mathbf{A}^1_k) = \mathscr{A}$ for \mathscr{A} a k-affinoid algebra.

We are now ready to define the structure sheaf for the G-topology defined by affinoid domains.

Definition 19.23. Let X be a k-analytic space. For $Y \subseteq X$ an analytic domain, we set

$$\mathcal{O}_{X_G}(Y) \coloneqq \operatorname{Hom}(Y, \mathbf{A}_k^1).$$

This defines a structure sheaf \mathcal{O}_{X_G} in the *G*-topology defined by analytic domains, where admissible open sets are analytic domains, and admissible coverings correspond to the quasinet property.

If X is good, then we define the structure sheaf \mathcal{O}_X by restricting \mathcal{O}_{X_G} , that is,

$$\mathcal{O}_X(U) \coloneqq \operatorname{Hom}(U, \mathbf{A}_k^1)$$

for $U \subseteq X$ open.

Note that the definition of \mathcal{O}_X makes sense even for bad spaces, but it is not well-behaved. For example, properties of local rings we had before (Noetherian, etc.) carry over to good k-analytic spaces, but for bad ones, these nice properties no longer carry over.

19.1.5 Further constructions/objects [Ber93, §§1.3–1.4]

Gluing [Ber93, Prop. 1.3.3] Under suitable conditions, you can glue k-analytic spaces $(X_i)_{i \in I}$ along analytic subdomains X_{ij} . This works, for example, when all X_{ij} are open in X_i . If the X_{ij} are closed, then you need to impose some finiteness conditions for this to work.

Fiber products [Ber93, Prop. 1.4.1] Given $Y \to X$, $X' \to X$, there exists $Y \times_X X'$. This is not completely trivial, since the glueing is a bit annoying and you have to consider different cases. The main difficulty is that paracompactness, which usually holds in the analytic setting, is not automatic for us.

There is also a ground field extension functor, and you can define complete residue fields for arbitrary k-analytic spaces; see [Ber93, p. 30].

We will talk about these things and various classes of morphisms (e.g., separated, proper) next.

20 November 17

Homework 5 (to be posted) will be due Thursday, December 8th. This will be the last homework set.

20.1 Global Berkovich spaces [Ber93, §1]

Recall 20.1. A *k*-analytic space is a triple (X, \mathscr{A}, τ) , where X is a locally Hausdorff topological space, τ is a net of compact (Hausdorff) subsets of X, and \mathscr{A} is an atlas, which is a functor

$$\tau \longrightarrow k\text{-Aff}$$
$$V \longmapsto \mathscr{A}_V$$

which induces an isomorphism $V \xrightarrow{\sim} \mathcal{M}(\mathcal{A}_V)$. There are also some compatibility conditions.

You can extend the atlas to a maximal atlas $(X, \widehat{\mathscr{A}}, \widehat{\tau})$ by adding in elements to the net. Elements of $\widehat{\tau}$ are called *affinoid domains*, and X is good if every point has an affinoid neighborhood.

Morphisms of k-analytic spaces are given by a continuous map $\varphi \colon X \to X'$, and a morphism $\mathscr{A}_{V'} \to \mathscr{A}_{V}$ of k-affinoid algebras for every pair of affinoid domains $V \subset X$ and $V' \subset X'$ such that $\varphi(V) \subset V'$. There are some further properties as well.

A subset $Y \subset X$ is an *analytic domain* if $\hat{\tau}|_Y$ is a net, i.e., any point $x \in Y$ can be "*G*-covered" by affinoids contained in Y. This can be described explicitly by saying that for any $x \in Y$, there exist $V_1, \ldots, V_n \subset Y$ such that $x \in \bigcap V_i$ and $\bigcup V_i$ is a neighborhood of x. For example, Y could be a special subset, or an open subset in X.

The G-topology on X defined by analytic domains and G-coverings is denoted X_G , and we defined a structure sheaf \mathcal{O}_{X_G} by

$$\mathcal{O}_{X_G}(Y) = \operatorname{Hom}(Y, \mathbf{A}^1_k).$$

Restricting to open subsets of Y gives a sheaf \mathcal{O}_X in the "usual" topology on X, given by

$$\mathcal{O}_X(U) = \operatorname{Hom}(U, \mathbf{A}_k^1).$$

In practice, we only use this sheaf \mathcal{O}_X when X is good, since this is the only case where local rings would have nice properties.

20.1.1 Gluing [Ber93, Prop. 1.3.3]

We said last time that you can glue spaces. We won't go through the precise construction, but we will illustrate a few examples of when you can glue.

You can glue k-analytic spaces X_i along analytic domains $X_{ij} \subset X_i$ via $X_{ij} \xrightarrow{\sim} X_{ji}$, under some assumptions. We list two of these cases:

- (a) $X_{ij} \subset X_i$ is open for all i, j;
- (b) $X_{ij} \subset X_i$ is closed for all i, j, and for all $i, \#\{j \mid X_{ij} \neq \emptyset\} < \infty$ (which is a local finiteness property).

Example 20.2 (The projective line I). We can form this in two different ways.

For construction (b), let $X_0 = \mathcal{M}(k\{S_1\})$ and $X_1 = \mathcal{M}(k\{S_0\})$, which are both copies of the closed unit disc. You can think of $S_1 = \frac{T_1}{T_0}$, and $S_0 = \frac{T_0}{T_1} = S_1^{-1}$. We want to glue them along their intersection, which is the unit circle:

$$X_{01} = \{ |S_1| = 1 \} \cong \mathscr{M}(k\{S_1, S_1^{-1}\}) \subset X_0$$

$$X_{01} = \{ |S_0| = 1 \} \cong \mathscr{M}(k\{S_0, S_0^{-1}\}) \subset X_1$$

This is the same idea as in complex geometry, where you glue together two unit discs along their boundary to form the Riemann sphere. The map $S_0 \mapsto S_1^{-1}$ induces an isomorphism $X_{01} \xrightarrow{\sim} X_{10}$. You can glue X_0, X_1 along the affinoid domains $X_{01} \cong X_{10}$. You then get $X = \mathbf{P}_k^1$.

We will soon talk about analytifications of varieties; this will end up being the analytification of the projective line in algebraic geometry, justifying our choice of notation. Note that gluing along $S_0 = S_1$ gives the same topological space, but it is not separated as a k-analytic space.

Example 20.3 (The projective line II). For construction (a), let

 $X_0 = \mathbf{A}_k^1 = \{ \text{multiplicative seminorms on } k[S_1] \text{ extending the norm on } k \}$

 $X_1 = \mathbf{A}_k^1 = \{ \text{multiplicative seminorms on } k[S_0] \text{ extending the norm on } k \}$

 $X_{01} = \{x \in X_0 \mid S_1(x) \neq 0\} \cong \{\text{multiplicative seminorms on } k[S_1, S_1^{-1}] \text{ extending the norm on } k\}$

 $X_{10} = \{x \in X_0 \mid S_0(x) \neq 0\} \cong \{\text{multiplicative seminorms on } k[S_0, S_0^{-1}] \text{ extending the norm on } k\}$

The map $S_0 \mapsto S_1^{-1}$ induces an isomorphism $X_{01} \xrightarrow{\sim} X_{10}$, and we get \mathbf{P}_k^1 as the glueing of X_0 and X_1 along $X_{01} \cong X_{10}$.

You can also construct \mathbf{P}_k^n similarly using either method.

20.1.2 Futher constructions [Ber93, §1.4]

We also have other constructions we mentioned last time.

Fiber products [Ber93, Prop. 1.4.1] Given two morphisms $X' \to X$ and $Y \to X$, there exists a fiber product $Y \times_X X'$. This is a bit painful to do.

Example 20.4. $\mathbf{A}_k^m \times_{\mathcal{M}(k)} \mathbf{A}_k^n = \mathbf{A}_k^{m+n}$.

The fiber product is constructed using gluing and the affinoid case, where the fiber product corresponds to the completed tensor product.

Ground field extension [Ber93, p. 30] If X is k-analytic, and K/k is a non-Archimedean extension, then X_K is K-analytic, and there is a map $X_K \to X$ of topological spaces, which is surjective and continuous, by Gruson's theorem 3.21. This operation gives a functor k-An $\to K$ -An, but note that this does *not* give an endofunctor of k-An. In this way, this operation has to be treated differently than the fiber products above.

Complete residue field [Ber93, p. 30] Let $x \in X$ be a point. Then, you can define $\mathscr{H}(x) = \mathscr{H}_V(x)$ for $V \subset X$ an affinoid containing x. This is a non-Archimedean extension of k.

Fibers If $f: Y \to X$ is a morphism, and $x \in X$ is a point, then you can view $f^{-1}(x)$ as an $\mathscr{H}(x)$ -analytic space via the identification

$$f^{-1}(x) \cong Y_{\mathscr{H}(x)} \times_{X_{\mathscr{H}(x)}} \mathscr{M}(\mathscr{H}(x)).$$

Note that we have to extend our ground fields everywhere to make the fiber product make sense.

20.2 Analytifications of schemes [Ber90, §3.4]

It is nice to have this category of k-analytic spaces, but the main examples we are interested in are easier to describe. There are a couple classes of these:

- 1. Analytifications of algebraic varieties; and
- 2. Generic fibers of formal schemes.

We will now discuss the former.

Given a scheme X which is locally of finite type over k, we want to define an analytification functor, which associates to X two things:

- A good k-analytic space X^{an} , and
- A continuous map ker: $X^{\mathrm{an}} \to X$.

A more elegant way of defining this uses the language of representable functors: the space X^{an} together with the kernel map represent the functor

$$k\text{-}\mathsf{An} \longrightarrow \mathsf{Set}$$
$$\mathcal{X} \longmapsto \operatorname{Hom}_{k\text{-}\mathsf{LRS}}(\mathcal{X}, X)$$

where X is a scheme locally of finite type over k, as before, and k-RS denotes the category of k-locally ringed spaces.

If we only want to know what X^{an} looks like as a topological space, this is rather easy. On the other hand, making X^{an} into a k-analytic space is harder.

There are three steps:

Step 1. $X = \mathbf{A}_k^n = \operatorname{Spec}(k[T_1, \dots, T_n])$ (the scheme-theoretic affine *n*-space).

In this case,

 $X^{\mathrm{an}} = \{ \text{multiplicative seminorms on } k[T_1, \dots, T_n] \text{ which extend the given norm on } k \}$

The topology is given by pointwise convergence, and the kernel map is

ker:
$$X^{\operatorname{an}} \longrightarrow X$$

 $x \longmapsto \{f \in k[T_1, \dots, T_n] \mid f(x) = 0\}$

The atlas is given by

$$\tau = \{ E(r) \mid r \in (\mathbf{R}^*_+)^n \} \quad \mathscr{A}_{E(r)} = k\{r^{-1}T\}.$$

 X^{an} is locally compact since $X^{\text{an}} = \bigcup_r E(r)$, each E(r) is compact, and $E(r) \in E(s)$ if $r_i < s_i$ for all *i*.

Step 2. An affine scheme $X = \operatorname{Spec} A$, where A is a finitely generated k-algebra.

In this case,

$$X^{\mathrm{an}} = \{ \text{multiplicative seminorms on } A \text{ which extend the given norm on } k \}$$

The topology and the continuous map ker: $X^{an} \to X$ are as before. As a topological space, we have a closed embedding $X \hookrightarrow \mathbf{A}_k^n$, which induces a closed embedding $X^{an} \hookrightarrow \mathbf{A}_k^{n,an}$ as topological spaces. This gives an atlas on X^{an} by pulling back the atlas on $\mathbf{A}_k^{n,an}$. Note this embedding is not canonical.

This is also functorial: maps of affine schemes also analytify, in a way that preserves some properties.

Step 3. X general.

In this case, X is obtained by gluing open affine subschemes X_i of finite type over k along open subschemes $X_{ij} \xrightarrow{\sim} X_{ji}$. We then define X^{an} as the gluing of the corresponding X_i^{an} along X_{ij}^{an} . Note that we do not need any finiteness assumptions to make this gluing possible.

The kernel map ker: $X^{an} \to X$ is also defined as the gluing of the corresponding kernel maps on X_i^{an} . Remark 20.5. The kernel map ker: $X^{an} \to X$ satisfies:

$$\ker^{-1}(\xi) = \begin{cases} \text{valuations (multiplicative norms) on the residue field} \\ \kappa(\xi) \text{ at } \xi \text{ extending the valuation on } k \end{cases}$$

This means we can describe the analytification as a topological space as follows:

 $X^{\mathrm{an}} = \{(\xi, |\cdot|) \mid \xi \in X, |\cdot| \text{ valuation on } \kappa(\xi) \text{ extending the valuation on } k\}$

This is a refinement of a scheme: it has the same points as the scheme, but with extra structure given by the valuation. Then, $\ker(\xi, |\cdot|) = \xi$, and the topology is the weakest one such that the kernel map ker: $X^{\mathrm{an}} \to X$ is continuous, and for every open affine $U \subset X$ and for every $f \in \mathcal{O}_X(U)$, the function

$$\ker^{-1}(U) \longmapsto \mathbf{R}^*_+$$
$$(\xi, |\cdot|_x) \longmapsto |f(\xi)|_x$$

is continuous.

Example 20.6. Let $X = \mathbf{P}_k^1$. If ξ is a closed point of X, then $\# \ker^{-1}(\xi) = 1$, since maximal ideals give a unique point in the Berkovich space. On the other hand, "most" points of X^{an} map to the generic point of X under the kernel map ker.

The picture is more difficult in higher dimensions. Arcwise connectedness and other topological properties still hold, however.

20.3 Connectedness of Berkovich spaces [Ber90, §3.2]

There are many things you can say. We will state one theorem, and only describe one part of the proof.

Recall 20.7. A topological space X is pathwise connected if for every two points $x, y \in X$, there exists a continuous map $f: [0,1] \to X$ such that f(0) = x, and f(1) = y. It is arcwise connected if, for $x \neq y$, f can be chosen as a homeomorphism onto its image.

Fact 20.8 [Bou16, III, $\S2$, Prop. 18]. If X is pathwise connected, and X is Hausdorff, then X is arcwise connected.

The hard part of this Fact is making a path injective. The homeomorphism part is then clear since $[0,1] \rightarrow X$ is a map from a compact Hausdorff space to another Hausdorff space.

Theorem 20.9 [Ber90, Thm. 3.2.1]. If X is a k-analytic space, then X is locally arcwise connected, i.e., every point $x \in X$ admits a basis of arcwise connected neighborhoods (which can be chosen to be open).

We will say something about GAGA theorems later, in which case a connected scheme in the Zariski topology will have an arcwise connected analytification (we might need to assume the scheme is separated).

Sketch. For simplicity, assume X is good. Then, locally X is k-affinoid, and so we can assume without loss of generality that X is a k-affinoid space. We can also assume without loss of generality that X is strictly k-affinoid, by picking a non-Archimedean extension K/k such that X_K is strictly K-affinoid. Since $X_K \to X$ is surjective, and since we only need to check pathwise connectedness, we only need to connect two preimages of the points $x, y \in X$.

Now we use the Noether normalization lemma 8.10; after some work, we can assume without loss of generality that $X = E_k^n$ is the unit polydisc, since there is a finite map onto the E_k^n .

Proposition 20.10. E_k^n is pathwise, hence arcwise, connected.

We prove this by induction on n. If n = 1, we are okay since E_k is a tree, which is in fact uniquely arcwise connected. This is easy to see if k is algebraically closed, and needs a bit more work for non-algebraically closed ground fields.

If n > 1, we use a projection map $\pi: E_k^n \to E_k^{n-1}$. The fiber over $x \in E_k^{n-1}$ is isomorphic to $E_{\mathscr{H}(x)}^1$. Now we want to use the inductive hypothesis. Pick two points $y_0, y_1 \in E_k^n$. Set $x_i = \pi(y_i) \in E_k^{n-1}$. First, join y_i to y'_i , which is the rigid point corresponding to the maximal ideal (T) in the fiber $E_{\mathscr{H}(x_i)}^1 = \mathscr{M}(\mathscr{H}(x_i)\{T\})$ (you can also choose the Gauss point). Then, connect x_0 and x_1 via a path γ by using the inductive hypothesis. We then use the continuous section $\sigma: E_k^{n-1} \to E_k^n$ of π , where x maps to the maximal ideal (T) in $E_{\mathscr{H}(x)}^1$. Then, $\sigma \circ \gamma$ joins y'_0 and y'_1 .

Actually deducing the whole theorem is more work; Berkovich does it quickly.

Next time, we will mention some properties of different maps, for example separated and proper morphisms. Later, we will talk about GAGA and how analytifications can be computed more explicitly as limits of simplicial complexes in special cases, and how Berkovich spaces arise as generic fibers of formal schemes.

21 November 22

Homework 5 is due Thursday, December 8. There is no class Thursday because of Thanksgiving.

Recall 21.1. A k-analytic space is a triple (X, \mathscr{A}, τ) , where X is a locally Hausdorff topological space (written as |X|), τ is a net of compact subsets of X, and \mathscr{A} is an atlas, which maps $V \in \tau$ to a k-affinoid algebra \mathscr{A}_V , with some compatibility conditions.

Morphisms of k-analytic spaces were defined by replacing (X, \mathscr{A}, τ) with the corresponding k-analytic space $(X, \widehat{\mathscr{A}}, \widehat{\tau})$, where the atlas is replaced by the maximal atlas, where elements of $\widehat{\tau}$ are affinoid domains.

We say that $Y \subset X$ is an analytic subset if $\hat{\tau}|_Y \coloneqq \{V \in \tau \mid V \subset Y\}$ is a net on Y. For example, affinoid domains and open sets are analytic subsets.

We defined a structure sheaf \mathcal{O}_X on the usual topology on X by

$$\mathcal{O}_X(U) = \operatorname{Hom}(U, \mathbf{A}_k^1) = \lim_{\substack{V \subset U \\ \text{special}}} \mathscr{A}_V.$$

We mainly use this when X is good, i.e., every point has an affinoid neighborhood. We also have a G-topology on X, which is given by analytic domains, on which we defined a structure sheaf \mathcal{O}_{X_G} by

$$\mathcal{O}_{X_G}(Y) = \operatorname{Hom}(Y, \mathbf{A}_k^1).$$

21.1 Coherent modules [Ber93, pp. 25–26]

We can define coherent \mathcal{O}_X -modules and coherent \mathcal{O}_{X_G} -modules as follows:

Definition 21.2. A \mathcal{O}_X -module (resp. \mathcal{O}_{X_G} -module) is *coherent* if it is locally the cokernel of a morphism of finite type free \mathcal{O}_X -modules (resp. \mathcal{O}_{X_G} -modules).

There is something to be shown here: it is not automatically true that \mathcal{O}_X and \mathcal{O}_{X_G} are coherent in the usual sense (from, e.g., complex analytic geometry).

We recall Kiehl's theorem:

Theorem 18.4 (Kiehl). If $X = \mathscr{M}(\mathscr{A})$ is a k-affinoid space, then every coherent \mathcal{O}_{X_G} -module is of the form $\mathcal{O}_{X_G}(M)$ for some finite (Banach) \mathscr{A} -module M, in which case

$$\Gamma(V, \mathcal{O}_{X_G}(M)) = M \widehat{\otimes}_{\mathscr{A}} \mathscr{A}_V = M \otimes_{\mathscr{A}} \mathscr{A}_V.$$

The last isomorphism is by the fact that tensor products and completed tensor products coincide for finite type \mathscr{A} -modules (Proposition 8.6).

To state the next theorem about coherent sheaves, we note that there exists a morphism of G-topological spaces $\pi: X_G \to X$, where $\mathcal{O}_X = \pi_* \mathcal{O}_{X_G}$ via restriction.

Proposition 21.3 [Ber93, Prop. 1.3.4]. If X is good, then there is an equivalence of categories

$$\mathsf{Coh}(X) \xrightarrow{\sim} \mathsf{Coh}(X_G)$$
$$F \longmapsto F_G \coloneqq \pi^* F \otimes_{\pi^* \mathcal{O}_X} \mathcal{O}_{X_G}$$

where

$$(\pi^*F)(V) = \varinjlim_{\substack{U \supset V\\open}} F(U).$$

21.2 Finite morphisms and closed immersions [Ber93, pp. 27–28]

Just like in algebraic geometry, there are many classes and properties of morphisms one can define. Some of them are quite difficult. We start with finite morphisms and closed immersions. We have seen these notions before, but only for k-affinoid spaces.

Recall 21.4. A morphism $Y = \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{A}) = X$ of k-affinoid spaces is

- (i) a finite morphism (fm) if \mathscr{B} is a finite Banach \mathscr{A} -algebra, that is, there exists an admissible epimorphism $\mathscr{A}^n \twoheadrightarrow \mathscr{B}$;
- (*ii*) a closed immersion (*ci*) if $\mathscr{A} \to \mathscr{B}$ is an admissible epimorphism.

In this case, finite morphisms have finite fibers, and closed immersion are topologically closed embeddings.

We now want to globalize these notions. Essentially, what you do is you G-localize on the base.

Definition 21.5. A morphism $\varphi: Y \to X$ of k-analytic spaces is a finite morphism (resp. closed immersion) if it is "G-locally so," i.e., for all $x \in X$, there exists affinoid domains $V_1, \ldots, V_n \subset X$ such that

- (i) $x \in V_1 \cap \cdots \cap V_n$ and $V_1 \cup \cdots \cup V_n$ is a neighborhood of x in X; and
- (ii) for all $i, \varphi^{-1}(V_i) \to V_i$ is a finite morphism (resp. closed immersion) of k-affinoid spaces (in particular, $\varphi^{-1}(V_i)$ must be k-affinoid).

Just like in algebraic geometry, we want to characterize this in a stronger way.

Proposition 21.6 [Ber93, Lem. 1.3.7]. If $\varphi: Y \to X$ is a finite morphism/closed immersion, then for every affinoid $V \subset X$, the morphism $\varphi^{-1}(V) \to V$ is a finite morphism/closed immersion of k-affinoid spaces.

We will prove/sketch this proof, to show how to work with nets.

Warning 21.7. The intersection of two affinoid domains may not be affinoid; this would be true if X were separated. For example, in the "bug-eyed" interval (Figure 19.3), the intersection of the two copies of the interval is not affinoid.

Sketch of Proposition 21.6. Let σ be the collection of affinoid domains $V \subset X$ such that $\varphi^{-1}(V) \xrightarrow{\varphi} V$ is a finite morphism/closed immersion of k-affinoid spaces. We want to show that $\sigma = \tau = \hat{\tau}$ (i.e., all affinoid domains).

Lemma 21.8. If $V \in \sigma$ and $W \subset V$ is an affinoid domain, then $W \in \sigma$.

Proof of Lemma 21.8 for closed immersions. $\varphi^{-1}(W)$ is an affinoid domain in $\varphi^{-1}(V)$. Let \mathscr{B} be the k-affinoid algebra associated to $\varphi^{-1}(V)$, \mathscr{A} that for V, \mathscr{A}_W that for W, and $\mathscr{B}_{\varphi^{-1}(W)}$ that for $\varphi^{-1}(W)$. Then, $\mathscr{B}_{\varphi^{-1}(W)} = \mathscr{B} \otimes_{\mathscr{A}} \mathscr{A}_V$. The observation is that $\mathscr{A} \to \mathscr{B}$ is an admissible epimorphism, and so

$$\mathscr{A}_V = \mathscr{A} \widehat{\otimes}_{\mathscr{A}} \mathscr{A}_V \longrightarrow \mathscr{B} \widehat{\otimes}_{\mathscr{A}} \mathscr{A}_V = \mathscr{B}_{\varphi^{-1}(W)}$$

is also an admissible epimorphism. Note that it is true that if $M \twoheadrightarrow N$ and $M' \twoheadrightarrow N'$ are admissible epimorphisms, then $M \otimes_{\mathscr{A}} M' \to N \otimes_{\mathscr{A}} N'$ is an admissible epimorphism. \Box

Lemma 21.9. σ is a net. Further, $\sigma|_V$ is also a net on V for every affinoid $V \subset X$, i.e., for every $V \in \tau$.

Sketch of Lemma 21.9. By assumption, we know that σ is a quasinet. We will prove that $\sigma|_V$ is a net for all $V \in \tau$. Given $x \in V$, we can pick $V_1, \ldots, V_n \in \sigma$ such that $x \in V_1 \cap \cdots \cap V_n$ and $V_1 \cup \cdots \cup V_n$ is a neighborhood of x in X. For every $i, \tau|_{V \cap V_i}$ is a quasinet (since τ is a net). Further, there exist W_{i1}, \ldots, W_{in} affinoid such that $x \in \bigcap_j W_{ij}$ and $\bigcup_j W_{ij}$ is a neighborhood of x in $V \cap V_i$. In particular, $W_{ij} \subset V_i \in \sigma$, which implies $W_{ij} \in \sigma$ by Lemma 21.8. Now $x \in \bigcap_{i,j} W_{ij}$ and $\bigcup_{i,j} W_{ij}$ is a neighborhood of x in V. Thus, $\sigma|_V$ is a quasinet. A similar argument shows that $\sigma|_V$ is a net.

Now pick any affinoid $V \in \tau$. By Lemma 21.9 and compactness of V, we may write $V = V_1 \cup \cdots \cup V_m$, where $V_i \in \sigma$ for all i. Set $W_i \coloneqq \varphi^{-1}(V_i) \subset W \coloneqq \varphi^{-1}(V)$, where we note the W_i are affinoid. Now set \mathscr{B}_{W_i} to be the corresponding k-affinoid algebra. We want to show that W is a k-affinoid domain, and that $\mathscr{A}_V \twoheadrightarrow \mathscr{B}_W$ is an admissible epimorphism. We start by defining what \mathscr{B}_W should be:

$$\mathscr{B}_W \coloneqq \ker(\prod \mathscr{B}_{W_i} \to \prod \mathscr{B}_{W_i \cap W_j})$$

This gives a diagram

$$\begin{array}{cccc} \mathscr{A}_{V} & \longrightarrow & \prod \mathscr{A}_{V_{i}} & \longrightarrow & \prod \mathscr{A}_{V_{i} \cap V_{j}} \\ & & & \downarrow & & \downarrow \\ \mathscr{B}_{W} & \longrightarrow & \prod \mathscr{B}_{W_{i}} & \longrightarrow & \prod \mathscr{B}_{W_{i} \cap W_{j}} \end{array}$$

For closed immersions, since $\mathscr{A}_{V_i} \twoheadrightarrow \mathscr{B}_{W_i}$ is an admissible epimorphism, we have that $\mathscr{A}_V \to \mathscr{B}_W$ is an admissible epimorphism by doing a diagram chase. For finite morphisms, you need to use Kiehl's theorem 18.4.

21.3 Separated morphisms [Ber93, pp. 30–31]

There are other classes of morphisms. We next discuss separated morphisms.

In algebraic geometry, separated morphisms try to mimic the definition of Hausdorffness in the category of schemes. We do something similar for k-analytic spaces.

Definition 21.10. A morphism $Y \to X$ is separated if the diagonal map $Y \to Y \times_X Y$ is a closed immersion in the sense from before.

Definition 21.11. A k-analytic space Y is separated if $Y \to \mathcal{M}(k)$ is separated.

For schemes, the underlying topology of a scheme is not very nice, which is why you introduce these notions. For Berkovich spaces, you might imagine separated morphisms are unnecessary, since we already have a nice topology that is locally Hausdorff on them. But the relationship is a bit subtle:

Fact 21.12. If $Y \to X$ is separated, then the image of $|Y| \to |Y| \times_{|X|} |Y|$ is closed; in particular, if Y is separated, then |Y| is Hausdorff. But the converse does not necessarily hold; see [Ber93, Rem. 1.4.3(i)].

This is how Berkovich states the implication, but it is unclear if the diagonal morphism on topological spaces is actually a closed embedding.

Facts 21.13 (analogous to schemes).

- The composition of two separated morphisms is separated.
- Separated morphisms are closed under base change and ground field extension: if the diagram



is cartesian, then separatedness of $Y \to X$ implies $Y' \to X'$ is separated; separatedness of $Y \to X$ implies $Y_K \to X_K$ is separated for all non-Archimedean extensions K/k.

Note that we do not know if there is a valuative criterion for separatedness in the Berkovich setting.

21.4 The relative interior and proper morphisms [Ber90, p. 50; Ber93, p. 34]

The next analogue of scheme morphisms is proper morphisms. This is a bit of a mess. In rigid analytic geometry, there is a notion that is useful for some things, but not others. In particular, it is hard to prove that the composition of proper morphisms is proper; see the discussion in [Tem15, $\S4.2.4$].

We instead give Berkovich's definition, which uses the relative interior. Note that the two possible definitions can be shown to be equivalent.

Recall 21.14. If $Y \to X$ is a morphism of k-affinoid spaces, then $y \in \text{Int}(Y/X)$ if and only if we have the picture as in Figure 18.10.

Globalizing this definition, we define:

Definition 21.15. The relative interior Int(Y/X) of a morphism $f: Y \to X$ of k-analytic spaces is the set of points $y \in Y$ such that for every affinoid domain $U \subset X$ such that $x = f(y) \in U$, there exists an affinoid domain $V \subset Y$ containing y such that

• V is a neighborhood of y in $f^{-1}(U)$;

• $y \in \operatorname{Int}(V/U)$.

Set $\partial(Y/X) = Y \setminus \text{Int}(Y/X)$ to be the relative boundary. We say f is boundaryless (or closed) if $\partial(Y/X) = \emptyset$. For $X = \mathscr{M}(k)$, write ∂Y , Int Y.

It is not trivial to check that this gives back the definition for morphisms of k-affinoid spaces.

Definition 21.16. We say $f: Y \to X$ is proper if

- (i) f is "compact" as a map of topological spaces: $f^{-1}(\text{compact}) = \text{compact}$ (i.e., proper in the sense of topological spaces); and
- (*ii*) f is boundaryless: $\partial(Y/X) = \emptyset$.

Example 21.17. \mathbf{P}_k^n is proper over $\mathcal{M}(k)$.

Example 21.18. \mathbf{A}_k^n and E_k^n are not proper over $\mathcal{M}(k)$ for $n \ge 1$: the former is not compact, and the latter has a boundary.

Fact 21.19. Any finite morphism is proper.

One ingredient for the proof of this Fact is that if you have a map $f: Y \to X$ of k-affinoid spaces, then $\partial(Y/X) = \emptyset$ if and only if f is finite.

This definition for proper morphisms is good for some purposes, but there are some difficulties. We mention some difficult results. Recall that this is not the definition in rigid analytic geometry, since boundaries and relative interiors are not natural notions in that setting.

Non-trivial Results 21.20.

- (1) If $Y \subset X$ is an analytic domain, then Int(Y/X) is the topological interior of Y in X (this follows from [Ber90, Cor. 2.5.13]).
- (2) Boundaries are G-local on the base: If $(X_i)_{i \in I}$ is a quasinet of affinoid domains in $X, f: Y \to X$ is a morphism, and we set $Y_i = f^{-1}(X_i)$, then $\partial(Y/X) = \bigcup_i \partial(Y_i/X_i)$.
- (3) The class of proper maps is stable under composition, base change, and ground field extension.
- (4) Kiehl's theorem on direct images (the analogue of Grauert's theorem in the complex setting): If $f: Y \to X$ is proper, and \mathscr{F} is a coherent \mathcal{O}_{Y_G} -module, then the \mathcal{O}_{X_G} -module $\mathbf{R}^i f_* \mathscr{F}$ is a coherent \mathcal{O}_{X_G} -module for every $i \ge 0$.

Kiehl showed this last theorem in the rigid analytic setting. The idea is to reduce to the strictly affinoid case. However, Kiehl used a different definition for properness which made proving the theorem easier, but harder to show that the composition of proper morphisms is proper.

Next time we will talk about GAGA and properties of the analytification functor. We will also talk about topological properties of certain Berkovich spaces. We will not do much more about global k-analytic spaces.

22 November 29

Today we will talk more about analytification and GAGA statements. We will then talk about other approaches to Berkovich spaces, in particular their topology.

22.1 The analytification functor [Ber90, §§3.4–3.5; Ber93, §2.6]

Note that [Ber93, §2.6] fleshes out some details compared to [Ber90, §§3.4–3.5], and also generalizes the analytification functor to a relative setting.

We mentioned before how the analytification functor can be viewed as a representable functor. We will make this precise. Let k be a non-Archimedean field, and consider the following category:

Definition 22.1. The category GAn_k of good analytic spaces over k has objects being all K-analytic spaces for some K/k, which in particular are locally ringed spaces (Y, \mathcal{O}_Y) with local sections $\Gamma(U, \mathcal{O}_Y)$ forming a k-algebra for each open set $U \subset Y$. Morphisms are k-linear.

Now fix a scheme \mathscr{X} locally of finite type over k. We want to define \mathscr{X}^{an} as a good k-analytic space, in such a way that it represents some functor. Consider the functor

$$\Phi \colon \mathsf{GAn}_k \longrightarrow \mathsf{Set}$$
$$Y \longmapsto \operatorname{Hom}_k(Y, \mathscr{X})$$

where $\operatorname{Hom}_k(-, -)$ denotes morphisms of k-ringed spaces in Berkovich's terminology (which are in particular morphisms of locally ringed spaces).

Theorem 22.2. The functor Φ is represented by a good k-analytic space $X \coloneqq \mathscr{X}^{\mathrm{an}}$ and a morphism ker: $X \to \mathscr{X}$ of k-ringed spaces.

To get functoriality, you use the map

$$\operatorname{Hom}(Y,X) \xrightarrow{\operatorname{ker}} \operatorname{Hom}_k(Y,\mathscr{X}) = \Phi(Y).$$

We will say a few more things, but note we've already done most of this, except for the statement that the analytic space we constructed actually represents this functor. It is convenient to phrase the analytification result in this way, since you get lots of uniqueness statements for free.

Sketch. We recall the proof from before, and point out where representability comes in.

Step 1. $\mathscr{X} = \mathbf{A}_k^n = \operatorname{Spec} k[T_1, \dots, T_n].$

You must verify that $X := \mathscr{X}^{\mathrm{an}} := \mathbf{A}_k^{n,\mathrm{an}} = \{$ multiplicative seminorms on $k[T_1, \ldots, T_n]$ extending the valuation on $k\}$ together with ker: $X \to \mathscr{X}$ represents the functor Φ . This amounts to showing that, for any good analytic space Y over k,

$$\operatorname{Hom}(Y, \mathbf{A}_k^{n, \operatorname{an}}) \xrightarrow{\sim} \Gamma(Y, \mathcal{O}_Y)^n \xleftarrow{\sim} \operatorname{Hom}_k(Y, \mathbf{A}_k^n).$$

The isomorphism on the left is essentially clear by patching together morphisms on affines.

Step 2. If the theorem is true for \mathscr{X} , then it is true for any open/closed subscheme $\mathscr{Y} \subset \mathscr{X}$.

This is not too difficult, since you know what the analytification should do: for open subschemes, you consider restriction of morphisms, and for closed subschemes, you use the analytification of the ideal sheaf defining the closed subscheme.

Step 3. Gluing.

This follows by using uniqueness in Step 2, and the fact that schemes locally of finite type can be obtained by gluing together finite type affine schemes along open subschemes. \Box

22.1.1 Further properties

Corollary 22.3. This functor $\mathscr{X} \mapsto \mathscr{X}^{an}$ commutes with fiber products and ground field extension.

This might not be hard to prove directly, but it is definitely convenient to prove it using the language of representable functors.

Closed and rigid points For a scheme \mathscr{X} that is locally of finite type over k, let

$$\mathscr{X}_0 \coloneqq \{\xi \in \mathscr{X} \mid [k(\xi) : k] < \infty\}.$$

These are the closed points of \mathscr{X} . For a k-analytic space X, let

$$X_0 \coloneqq \{x \in X \mid [\mathscr{H}(x) : k] < \infty\}$$

These are the rigid points of X.

These points correspond to each other, in the following manner:

Proposition 22.4. The map ker: $X = \mathscr{X}^{an} \to \mathscr{X}$ satisfies:

(1) ker is continuous and surjective.

(2) For all $x \in X$, $\mathcal{O}_{\mathcal{X},\xi} \to \mathcal{O}_{X,x}$ is flat, where $\xi = \ker(x)$.

(3) ker: $X_0 \xrightarrow{\sim} \mathscr{X}_0$ as sets, and if $x \in X_0$, then $\widehat{\mathcal{O}}_{\mathscr{X},\xi} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}$.

Note that (1) and (2) together say that ker is faithfully flat.

Skip Proof. The proof is similar to the study of ker: $\mathscr{M}(\mathscr{A}) \to \operatorname{Spec} \mathscr{A}$ for a k-affinoid algebra \mathscr{A} .

Unfortunately, it is hard to give very concrete examples. We already saw the example of $\mathscr{X} = \mathbf{P}_k^1$ and $X = \mathbf{P}_k^{1,\mathrm{an}}$. If $k = k^a$ and char k = 0, then letting $x \in X_0$, we have that $\widehat{\mathcal{O}}_{X,x} \cong k[[T]]$.

22.1.2 GAGA (Géométrie algébrique et géométrie analytique)

The title refers to the paper [Ser56] of Serre, who proved an analogous statement for varieties over the complex numbers.

Proposition 22.5. Let $f: \mathscr{Y} \to \mathscr{X}$ be a morphism of schemes locally of finite type over k, and let $f^{an}: Y := \mathscr{Y}^{an} \to \mathscr{X}^{an} =: X$ be the analytification of the morphism f. Let P be one of the following properties:

- (1) flat (in the locally ringed sense);
- (2) unramified (if a fiber $Y_x \neq \emptyset$, then $Y_x = \mathscr{M}(K)$ for a finite separable $\mathscr{H}(x)$ -algebra K);
- (3) étale (unramified and flat);
- (4) smooth (flat and non-empty fibers have dimension n);

(5) separated $(Y \to Y \times_X Y \text{ is a closed immersion})$;

- (6) injective;
- (7) surjective;
- (8) open immersion;
- (9) *isomorphism*;
- (10) monomorphism (in the categorical sense).

Then, f satisfies P if and only if f^{an} satisfies P.

Remark 22.6. The proof uses the following description of the fibers of f and f^{an} :

$$Y_x = \left(\mathscr{Y}_{\xi} \times_{k(\xi)} \mathscr{H}(x)\right)^{\mathrm{ar}}$$

where $\xi = \ker(x)$. In this case, there is an embedding $k(\xi) \hookrightarrow \mathscr{H}(x)$ of residue fields. This implies the surjectivity statement (7). The flatness statement (1) follows from the fact that flatness can be tested on closed points, the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathscr{X},\xi} & \longrightarrow & \mathcal{O}_{X,x} \\ & & \downarrow \\ & & \downarrow \\ \mathcal{O}_{\mathscr{Y},\eta} & \longrightarrow & \mathcal{O}_{Y,y} \end{array}$$

the fact that flatness is "invariant under completion," and $\widehat{\mathcal{O}}_{\mathscr{X},xi} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}$ for $\xi \in \mathscr{X}_0$.

For properness, you need finite type hypotheses.

Proposition 22.7. Suppose $f: \mathscr{Y} \to \mathscr{X}$ is of finite type. Then, f is dominant/closed immersion/proper/finite if and only if $f^{an}: Y \to X$ is.

Theorem 22.8. Consider $\mathscr{X} \to \operatorname{Spec} k$, and its analytification $X \to \mathscr{M}(k)$.

- (1) \mathscr{X} is separated if and only if |X| is Hausdorff.
- (2) \mathscr{X} is proper if and only if |X| is compact (and Hausdorff).
- (3) \mathscr{X} is connected if and only if |X| is pathwise connected.

This is related to what we stated above, and what we proved before. For (3), the direction \Leftarrow is obvious since \mathscr{X} is the image of a connected space; for \Rightarrow , we already showed that all analytic spaces are locally arcwise connected.

Cohomology comparison theorems

Proposition 22.9. If \mathscr{X} is a proper scheme over k, then the functor

$$\operatorname{Coh}(\mathscr{X}) \longrightarrow \operatorname{Coh}(X)$$

is an equivalence of categories.

Corollary 22.10. The functor $\mathscr{X} \mapsto X$ where \mathscr{X} is proper, is fully faithful: a morphism of proper k-analytic spaces which are analytifications is the analytification of a map between the schemes they come from.

GAGA in the trivially valued case When Berkovich stated the GAGA theorems, he divided it up into two parts: the trivially and non-trivially valued case. Let k be a trivially valued field. In this case, many statements are still true for *non-proper* schemes.

Example 22.11. Let $\mathscr{X} = \mathbf{A}_k^n = \operatorname{Spec} k[T_1, \ldots, T_n]$. Then, $X = \mathbf{A}_k^n = \{$ multiplicative seminorms on $k[T_1, \ldots, T_n] \ldots \}$. Let V = E(r), where $r \in (\mathbf{R}_+^*)^n$. Then, if $r_i \ge 1$ for all i, we have that $\Gamma(V, \mathcal{O}_X) = k[T_1, \ldots, T_n]$.

This sort of feature appears a lot, where statements that ordinarily only hold for proper maps in fact holds for non-proper maps.

Exercise 22.12. Show that any $f: \mathbf{A}^{1,\mathrm{an}} \to \mathbf{A}^{1,\mathrm{an}}$ is polynomial.

22.2 Analysis of Berkovich spaces via models

We now switch gears to do something different, which gives more concrete descriptions of Berkovich spaces, or at least of analytifications of schemes. This is not directly in Berkovich's book or in his paper, but is in the literature. We will do a baby case, and discuss the general setup later.

Let k be a non-Archimedean field. Assume that k is discretely valued, so $|k^*| = r^2$, where $r \in (0, 1)$, and assume that char $\tilde{k} = 0$.

Example 22.13. $k = \mathbf{C}((t)), |t| = r.$

Let X/k be a smooth projective variety, so that $X \hookrightarrow \mathbf{P}_k^N$. Let $X^{\mathrm{an}}/\mathscr{M}(k)$ be its analytification, which is also smooth projective, so that $X^{\mathrm{an}} \hookrightarrow \mathbf{P}_k^{N,\mathrm{an}}$.

Goal 22.14. Understand the structure of X^{an} (as a topological space).

Recall that the cheapest way of defining Berkovich spaces as a topological space is as follows:

 $X^{\mathrm{an}} = \{(\xi, |\cdot|) \mid \xi \in X, |\cdot| \text{ valuation on } k(\xi) \text{ extending the valuation on } k\},\$

in which case ker: $X^{\text{an}} \to X$ maps $(\xi, |\cdot|) \mapsto \xi$. The topology is the weakest such that ker is continuous, and for every open affine $U \subset X$, and for all $f \in \Gamma(U, \mathcal{O}_X)$, we have that

$$X^{\mathrm{an}} \supset \ker^{-1}(U) \ni x \longmapsto |f(x)| \in \mathbf{R}_+$$

is continuous.

Definition 22.15. A model of X (over the valuation ring k°) is a flat, projective, normal scheme \mathscr{X}/k° together with an isomorphism of the generic fiber of \mathscr{X} and X.

We have $\operatorname{Spec} k^{\circ} = \{0, \eta\}$, where 0 is the closed or special point corresponding to $\operatorname{Spec} \widetilde{k}$, and η is the generic point corresponding to $\operatorname{Spec} k$.

Remark 22.16. Models exist! Given $X \hookrightarrow \mathbf{P}_k^N \hookrightarrow \mathbf{P}_{k^\circ}^N$, set \mathscr{X}' to be the closure of X in $\mathbf{P}_{k^\circ}^N$, and set \mathscr{X} to be the normalization of \mathscr{X}' .

There is a partial ordering on the set of (isomorphism classes of) models: if $\mathscr{X}, \mathscr{X}'$ are models of X, then there is a birational map $\mathscr{X}' \dashrightarrow \mathscr{X}$ via $\mathscr{X}'_{\eta} \xrightarrow{\sim} X \xleftarrow{\sim} \mathscr{X}_{\eta}$. We say $\mathscr{X}' \ge \mathscr{X}$ if this is a morphism.

Lemma 22.17. This is a directed set: any two models $\mathscr{X}, \mathscr{X}'$ can be dominated by a third \mathscr{X}'' .

Proof. Have $\mathscr{X}' \dashrightarrow \mathscr{X}$ birational. Take \mathscr{X}'' to be the normalization of the graph.

Remark 22.18. Under our assumptions on k, any model can be dominated by a simple normal crossings model, that is, \mathscr{X}_0 is a simple normal crossings divisor (by Hironaka's resolution of singularities).

This connects to Berkovich spaces as follows: the inverse limit over all dual graphs of \mathscr{X}_0 ends up being homeomorphic to the Berkovich analytification.

23 December 1

Homework 5 is due Thursday, December 8. There is no class Tuesday.

23.1 Berkovich spaces through models $[BFJ16, \S2]$

Our presentation follows [BFJ16], in that notation will be compatible with what follows, although the material itself predates that paper.

Let k be a non-Archimedean field, and assume that k is discretely valued, so that $|k^*| = r^{\mathbf{Z}}$ for some fixed number $r \in (0, 1)$. Assume that char $\tilde{k} = 0$, so that $k \cong \tilde{k}((t))$ (non-canonically).

Let X/k be a smooth projective variety, and consider X^{an} the analytification of X, which is a compact (and Hausdorff) space.

Goal 23.1. Understand X^{an} as a topological space.

Recall that you can view the points of this space as pairs

$$X^{\mathrm{an}} = \left\{ (\xi, |\cdot|) \middle| \begin{array}{l} \xi \in X \\ |\cdot| \text{ valuation on } k(\xi) \text{ extending the valuation on } k \end{array} \right\}$$

The kernel map ker: $X^{\mathrm{an}} \to X$ is defined by $(\xi, |\cdot|) \mapsto \xi$.

Definition 23.2. A model of X (over k°) is a normal, flat, projective scheme \mathscr{X}/k° together with an isomorphism $\mathscr{X}_{\eta} \xrightarrow{\sim} X$, where \mathscr{X}_{η} is the generic fiber. The special fiber is denoted \mathscr{X}_{0} .

Facts 23.3.

(1) Models exist: a particular model can be constructed by letting $\mathscr X$ be the normalization of the closure of the image of the composition

$$X \hookrightarrow \mathbf{P}_k^n \hookrightarrow \mathbf{P}_{k^\circ}^N$$

Note that flatness comes from [Har77, Prop. III.9.8].

(2) Any two models can be dominated by a third, where domination is defined by saying that in the diagram



the dashed rational map is a morphism.

Example 23.4. If $X = \mathbf{P}_k^n$, then $\mathscr{X} = \mathbf{P}_{k^\circ}^n$.

Example 23.5. If $\mathscr{X} = \{xyz + t(x^3 + y^3 + z^3) = 0\} \subseteq \mathbf{P}^n_{\mathbf{C}[[t]]}$, then X is an elliptic curve over $\mathbf{C}((t))$, but $\mathscr{X}_0 = \{xyz = 0\} \subseteq \mathbf{P}^2_{\mathbf{C}}$.

Once you have a model, you can blowup ideals in the special fiber to get new models. In this way, there are "lots of models."

23.1.1 Divisorial points

We want to study Berkovich spaces using these. These models give divisorial points in the Berkovich space.

Given a model \mathscr{X} of X, write $\mathscr{X}_0 = \sum b_E E$, where E ranges over irreducible components of \mathscr{X}_0 , where $b_E \in \mathbb{Z}_{>0}$. Note that E is not necessarily Q-Cartier, but E is a Cartier divisor at its generic point since \mathscr{X} is normal. Then, E defines a divisorial valuation

$$\operatorname{ord}_E \colon k(\mathscr{X}) = k(X) \longrightarrow \mathbf{Z}$$

(in additive notation). The integer $\operatorname{ord}_E(f)$ is the order of vanishing of the rational function f along E, which we describe concretely as follows: at the generic point ξ of E, the local ring $\mathcal{O}_{\mathscr{X},\xi}$ is a dvr with uniformizer π . Then, for $f \in \mathcal{O}_{\mathscr{X},\xi}$,

$$\operatorname{ord}_{E}(f) = \max\{j \mid f \in (\pi^{j})\},\$$

and $k(X) = k(\mathscr{X}) = \operatorname{Frac}(\mathcal{O}_{\mathscr{X},\xi}).$

Note 23.6. If t is a uniformizer of k° , then $\operatorname{ord}_{E}(t) = b_{E}$.

Now we get a point $x_E \in X^{\mathrm{an}}$ by setting

$$|f(x_E)| \coloneqq r^{\operatorname{ord}_E(f)/b_E} \qquad f \in k(X).$$

Such points are called *divisorial* points. Note that $\ker(x_E)$ is the generic point of X. For curves, divisorial points roughly correspond to Type 2 points. You get many divisorial points by varying the model \mathscr{X} of X. Our goal is to show the following:

Theorem 23.7 [Gub98, Thm. 7.12; YZ10, Lem. 2.4]. The set X^{div} if divisorial points is dense in X^{an} .

The idea is to study *continuous functions* on X^{an} .

Remark 23.8. In principle, you can recover X^{an} from $C^0(X^{an}; \mathbb{C})$ (this is a complex Banach algebra, and $X^{an} = \mathscr{M}(C^0(X^{an}; \mathbb{C})))$. In fact, we consider a dense subset of $C^0(X^{an}; \mathbb{R})$, called model functions.

We need to introduce some other things, first.

Reduction map Given a model \mathscr{X} of X, there exists a reduction map (also called the center or specialization map), which globalizes the reduction map from before, defined by

$$\operatorname{red}_{\mathscr{X}}: X^{\operatorname{an}} \longrightarrow \mathscr{X}_0$$

where \mathscr{X}_0 is the special fiber, defined by using the valuative criterion as follows: Given $x \in X^{\mathrm{an}}$, set $\xi = \ker(x) \in X \cong X_\eta \hookrightarrow \mathscr{X}$, and let R_{ξ} be the valuation ring in $k(\xi)$ (induced by x). There is a map $\operatorname{Spec} R_{\xi} \to \operatorname{Spec} k^{\circ}$ (from the map $k^{\circ} \to R_{\xi}$ on rings). The valuative criterion of properness implies that there is a unique lift in the diagram below



such that the generic point maps to ξ .

Definition 23.9. red $\mathscr{X}(x) :=$ image of the closed point in \mathscr{X}_0 .

Remark 23.10. This is a "global" version of red: $\mathcal{M}(\mathscr{A}) \to \operatorname{Spec} \widetilde{\mathscr{A}}$ for \mathscr{A} a strictly k-affinoid algebra.

Remark 23.11. If ker(x) is the generic point of X, i.e., x defines a valuation on $k(X) = k(\mathscr{X})$, then $\xi := \operatorname{red}_{\mathscr{X}}(x) \in \mathscr{X}_0$ is characterized by

$$\begin{cases} |f(x)| < 1 & \text{for } f \in \mathfrak{m}_{\xi} = \text{maximal ideal of } \mathcal{O}_{\mathscr{X},\xi} \\ |f(x)| \leq 1 & \text{for } f \in \mathcal{O}_{\mathscr{X},\xi} \end{cases}$$

This is called the *center* of the valuation in Zariski's terminology from valuation theory.

The goal of this theorem is to prove that the divisorial points are dense, so that we can understand the topology of Berkovich spaces. We first have to understand models and model functions, first.

Definition 23.12 (Vertical ideals and divisors). Fix a model \mathscr{X} of X.

- A vertical ideal \mathfrak{a} on \mathscr{X} is a coherent ideal sheaf on \mathscr{X} co-supported on the special fiber \mathscr{X}_0 (that is, $\mathfrak{a}|_{\mathscr{X}_n} = \mathcal{O}_{\mathscr{X}_n}$).
- A vertical fractional ideal is a coherent O_X-submodule of k(X) such that t^m ⋅ a is a vertical ideal for m ≫ 0.
- A vertical divisor on \mathscr{X} is a Cartier divisor D on \mathscr{X} supported on \mathscr{X}_0 (so $\mathcal{O}_{\mathscr{X}}(D)$ is a vertical fractional ideal on \mathscr{X}). These form sets

$$\operatorname{Div}_{0}(\mathscr{X}) = \operatorname{Div}_{0}(\mathscr{X})_{\mathbf{Z}} \qquad \operatorname{Div}_{0}(\mathscr{X})_{\mathbf{Q}} = \operatorname{Div}_{0}(\mathscr{X}) \otimes_{\mathbf{Z}} \mathbf{Q}$$

Definition 23.13 (Model functions). Given a fractional ideal \mathfrak{a} on a model \mathscr{X} , define

$$\begin{split} \log |\mathfrak{a}| \colon X^{\mathrm{an}} &\longrightarrow \mathbf{R} \\ x &\longmapsto \max \big\{ \log_r |f(x)| \ \big| \ f \in \mathfrak{a}_{\mathrm{red}_{\mathscr{X}}(x)} \big\}. \end{split}$$

This is essentially the max over the generators of $\mathfrak{a}_{\operatorname{red}_{\mathscr{X}}(x)}$, and so is a finite max. Then, $\log|\mathfrak{a}| \in C^0(X^{\operatorname{an}}; \mathbf{R})$ essentially by definition of the topology on X^{an} .

Given $D \in \text{Div}_0(\mathscr{X})$ a vertical divisor, set $\varphi_D \coloneqq \log |\mathcal{O}_{\mathscr{X}}(D)|$.

Facts 23.14.

- $\varphi_{\mathscr{X}_0} \equiv -\log r = 1$ (if you interpret logarithms as being taken with base r) since $\mathscr{X}_0 = \operatorname{div}(t)$ (recall $k \cong \widetilde{k}(t)$) and $|t| = r \in (0, 1)$).
- $\varphi_D \ge 0 \iff D$ effective.
- $D \mapsto \varphi_D$ is **Z**-linear.

We can now define φ_D for $D \in \text{Div}_0(\mathscr{X})_{\mathbf{Q}}$ a vertical **Q**-Cartier **Q**-divisor by linearity.

Definition 23.15. A model function is a (continuous) function on X^{an} of the form φ_D where $D \in \operatorname{Div}_0(\mathscr{X})_{\mathbf{Q}}$ for some model \mathscr{X} . Set $\mathscr{D}(X^{\operatorname{an}}) := \mathscr{D}(X^{\operatorname{an}})_{\mathbf{Q}} := \{ \operatorname{model functions} \} \subseteq C^0(X^{\operatorname{an}})$. Can also define $\mathscr{D}(X^{\operatorname{an}})_{\mathbf{Z}} \subset \mathscr{D}(X^{\operatorname{an}})$.

We think of the functions in $\mathscr{D}(X^{\mathrm{an}})$ as the analogues of, say, **Q**-piecewise affine functions on a simplex.

Proposition 23.16. For a given model \mathscr{X} of X, the subgroup of $C^0(X^{\mathrm{an}})$ spanned by functions of the form $\log|\mathfrak{a}|$, where \mathfrak{a} is a vertical fractional ideal, coincides with $\mathscr{D}(X^{\mathrm{an}})_{\mathbf{Z}}$, the integral model functions on X^{an} . Further, this subgroup is stable under "max" and separates points in X^{an} .

We will prove this next time.

Corollary 23.17. The **Q**-vector space $\mathscr{D}(X^{\mathrm{an}}) \coloneqq \mathscr{D}(X^{\mathrm{an}})_{\mathbf{Q}} \subseteq C^{0}(X^{\mathrm{an}})$ is stable under max, separates points, and contains the constant function $1 = \varphi_{\mathscr{X}_{0}}$.

The reason why we care about this is that we would like to work with divisors on models, which can be taken to be really nice, but it is not the case that the max of model functions for divisors correspond to taking maxima of coefficients on these divisors. On the other hand, maxima work better for ideals.

Corollary 23.18. $\mathscr{D}(X^{\mathrm{an}})$ is dense in $C^0(X^{\mathrm{an}})$.

Proof. This is a "boolean" version of the Stone–Weierstrass Theorem.

Recall our goal was to prove that divisorial points are dense.

Corollary 23.19. $X^{\text{div}} \subset X^{\text{an}}$ is dense.

Proof. X^{an} is compact (and Hausdorff), and so Urysohn's lemma applies. Thus, it suffices to show the following: if $\varphi \in C^0(X^{\mathrm{an}})$ vanishes on X^{div} then $\varphi \equiv 0$. Given $\varepsilon > 0$, pick a model function $\varphi_D \in \mathscr{D}(X^{\mathrm{an}})$ for $D \in \mathrm{Div}_0(\mathscr{X})_{\mathbf{Q}}$ such that $|\varphi - \varphi_0| \leq \varepsilon$ on X^{an} . Then, $|\varphi_D| \leq \varepsilon$ on X^{div} . But φ_D is uniquely determined by its values on X^{div} , since if $D = \sum r_E E$ and $\mathscr{X}_0 = \sum b_E E$, then $\varphi_D(x_E) = \frac{r_E}{b_E}$. Thus, $-\varepsilon \mathscr{X}_0 \leq D \leq \varepsilon \mathscr{X}_0$, so $|\varphi_D| \leq \varepsilon$ on all of X^{an} , so that $|\varphi| \leq 2\varepsilon$ on X^{an} , we have that $\varphi \equiv 0$.

Next time we will show the Proposition above. We will get more structure on these spaces by looking at simple normal crossings models.

24 December 8

Homework 5 is due Monday (if you need extra time).

24.1 Berkovich spaces through models

We continue our discussion of Berkovich spaces (which are analytifications) through models. Let k be a non-Archimedean field, $|k^*| = r^{\mathbf{Z}}$ for some 0 < r < 1 (implicitly we use $r = e^{-1}$ to make the logarithms work out; it should work if you think of all logarithms as base r^{-1}), and char $\tilde{k} = 0$. This implies that (non-canonically) $k \cong \tilde{k}((t))$. Let X/k be a smooth projective variety, and let X^{an} be its analytification. Let ker: $X^{\mathrm{an}} \to X$ be the kernel map. The goal is to understand X^{an} as a topological space.

Definition 24.1. A normal scheme \mathscr{X} which is flat and projective over Spec k° is a model of X if $\mathscr{X}_{\eta} \xrightarrow{\sim} X$.

There are plenty of models and form a direct system. Using all possible models gives information about the Berkovich spaces.

Divisorial points Let $\mathscr{X}_0 = \sum b_E E$, where *E* defines a *divisorial point* $x_E \in X^{\mathrm{an}}$ with kernel equal to the generic point of *X*. This defines a valuation

$$|f(x_E)| = r^{\operatorname{ord}_E(f)/b_E},$$

where $f \in k(X) \cong k(\mathscr{X})$.

Reduction map There is a map $\operatorname{red}_{\mathscr{X}}: X^{\operatorname{an}} \to \mathscr{X}_0$ which is anticontinuous.

Example 24.2. red $\mathcal{X}(x_E)$ = generic point of E (and is the only point of X^{an} with this property).

Last time we saw that divisorial dense were dense; the way we showed this was by using "model functions." These come from certain subschemes of \mathscr{X} .

Vertical ideals $\mathfrak{a} \subseteq \mathcal{O}_{\mathscr{X}}$ where $\mathfrak{a}|_{\mathscr{X}_{\eta}} = \mathcal{O}_{\mathscr{X}_{\eta}}$. Vertical fractional ideals are subsheaves $\mathfrak{a} \subseteq \mathscr{K}(\mathscr{X})$ such that $t^m\mathfrak{a}$ is a vertical ideal for $m \gg 0$.

Example 24.3. $\mathfrak{a} = \mathcal{O}_{\mathscr{X}}(D)$ for D a vertical Cartier divisor on \mathscr{X} .

24.1.1 Functions on X^{an}

Let $\log|\mathfrak{a}|(x) = \max\{\log|f(x)| \mid f \in \mathfrak{a}_{red(x)}\}\)$, where red(x) is a point on \mathscr{X}_0 . As long as \mathfrak{a} is a vertical ideal, this value is ≤ 1 . Then,

$$\varphi_D \coloneqq \log |\mathcal{O}_{\mathscr{X}}(D)|.$$

For example, $\varphi_{\mathscr{X}_0} \equiv 1$. These give rise to model functions coming from some model of X:

 $D(X^{\mathrm{an}})_{\mathbf{Z}} = \{\varphi_D \mid D \text{ vertical Cartier divisor on some model } \mathscr{X}\} \subseteq D(X^{\mathrm{an}})_{\mathbf{Q}} \subset C^0(X^{\mathrm{an}})$

We then had the following Proposition last time which said something about these functions. We will prove the portion we skipped the proof for last time.

Proposition 24.4.

- (i) $\log|\mathfrak{a}| \in D(X^{\mathrm{an}})_{\mathbb{Z}}$ for every vertical fractional ideal \mathfrak{a} on any model \mathscr{X} of X.
- (ii) Every $\varphi \in D(X^{\mathrm{an}})_{\mathbf{Z}}$ is of the form $\varphi = \log |\mathfrak{a}| \log |\mathfrak{b}|$, where $\mathfrak{a}, \mathfrak{b}$ are vertical ideals on some fixed model \mathscr{X} .
- (iii) $D(X^{\mathrm{an}})_{\mathbf{Z}}$ is stable under max.
- (iv) $D(X^{an})_{\mathbf{Z}}$ separates points.

Corollary 24.5. $D(X^{an})$ is dense in continuous functions $C^0(X^{an})$

Corollary 24.6. $X^{\text{div}} = \{ \text{divisorial points} \}$ is dense in X^{an} .

Sketch. Without loss of generality, we may assume \mathfrak{a} is an ideal, since $\log|\mathfrak{a}| = \log|t^m\mathfrak{a}| + m$.

For (i), pick $\pi: \mathscr{X}' \to \mathscr{X}$ to be the normalized blowup of \mathfrak{a} , so that $\mathfrak{a} \cdot \mathcal{O}_{\mathscr{X}'} = \mathcal{O}_{\mathscr{X}'}(D)$, where D is a vertical Cartier divisor on \mathscr{X}' . You can then check that $\log|\mathfrak{a}| = \varphi_D \in D(X^{\mathrm{an}})_{\mathbf{Z}}$.

For (*ii*), we first remark that if D is a vertical divisor on \mathscr{X} , and $\pi: \mathscr{X}' \to \mathscr{X}$ is a morphism of models, then $\varphi_{\pi^*D} = \varphi_D$, where π^*D is a vertical divisor on \mathscr{X}' (since rational functions are independent of the model). We know that there exists a model \mathscr{X}' with a vertical divisor D on \mathscr{X}' such that $\varphi = \varphi_D$. Without loss of generality, we may assume that $\mathscr{X}' \geq \mathscr{X}$ by pulling back to a higher model, and for the same reason that $\pi: \mathscr{X}' \to \mathscr{X}$ admits a relatively ample divisor A. Set $\mathfrak{a} := \pi_* \mathcal{O}_{\mathscr{X}'}(mA + D)$ and $\mathfrak{b} = \pi_* \mathcal{O}_{\mathscr{X}'}(mA)$ for $m \gg 0$. Then, $\varphi = \log|\mathfrak{a}| - \log|\mathfrak{b}|$.

For (*iii*), you can check that if D, D' are vertical ideals on the same model \mathscr{X} , then $\max\{\varphi_D, \varphi_{D'}\} = \log|\mathcal{O}_{\mathscr{X}}(D) + \mathcal{O}_{\mathscr{X}}(D')|$ (essentially for the reason that (x) + (y) = (x, y)).

For (iv), suppose $x, y \in X^{\mathrm{an}}$, $x \neq y$, and $\xi = \operatorname{red}_{\mathscr{X}}(x) = \operatorname{red}_{\mathscr{X}}(y) \in \mathscr{X}_0$. Pick $f \in \mathcal{O}_{\mathscr{X},\xi} \subset \mathscr{K}(X)$ such that $|f(x)| \neq |f(y)|$. This implies that $\log|(f) + (t)^m|(x) \neq \log|(f) + (t)^m|(y)$ for $m \gg 0$, since the left-hand side is $\max\{\log|f(x)|, -m\}$ and similarly for the right-hand side, and both ideals involved are vertical ideals.

This proposition gives information about continuous functions on X^{an} with a nice subalgebra with nice properties (it has a **Z**-structure, for example) and the statements show that the divisorial points are dense in X^{an} .

Note that what we said above can mostly be generalized to more general ground fields. On the other hand, we now want to understand the finer structure of X^{an} as a topological space using these models; here is where we really need our assumptions on the ground field.

Suppose \mathscr{X} is a model, where $\mathscr{X}_0 = \sum_{i \in I} b_i E_i$, and $b_i \in \mathbf{Z}_{>0}$.

Definition 24.7. \mathscr{X} is an *snc model* (simple or strict normal crossings) if " \mathscr{X}_0 locally (analytically/étale) looks like coordinate hyperplanes." More precisely,

- \mathscr{X} is regular;
- For every point $\xi \in \mathscr{X}_0$, let $I_{\xi} \subset I$ be the indices $i \in I$ such that $E_i \ni \xi$. For every $i \in I_{\xi}$, let $z_i \in \mathcal{O}_{\mathscr{X},\xi}$ be a function defining E_i at ξ . Then, $(z_i)_{i \in I_{\xi}}$ can be completed to a regular system of parameters of $\mathcal{O}_{\mathscr{X},\xi}$ (i.e., they form part of a coordinate system).

Locally at ξ , writing $k \cong \tilde{k}((t))$, you can write $t = \prod_{i \in I_{\xi}} z_i^{b_i}$. We allow divisors to intersect possibly more than once, but not something like the nodal cubic. This is because our model \mathscr{X} is normal, and so it is regular in codimension 1, and the irreducible components of the special fiber \mathscr{X}_0 (which are prime divisors on \mathscr{X}) are regular, i.e., smooth over the residue field \tilde{k} .

So, while the irreducible components may intersect to give singularities, the irreducible components themselves are nice. In particular, we cannot have a nodal cubic as an irreducible component.

Fact 24.8. By Hironaka, snc models are cofinal in the set of all models: any model can be dominated by an snc model.

It would be nice to do things over fields like \mathbf{Q}_p , since we do not have resolutions of singularities in characteristic p.

Now given an snc model \mathscr{X} , we can define a *dual complex* $\Delta_{\mathscr{X}}$. This is a (slightly generalized) simplicial complex with vertices e_i corresponding to irreducible components E_i of \mathscr{X}_0 , and simplexes spanned by e_i for all i in some subset $J \subseteq I$ corresponding to connected components of $E_J := \bigcap_{i \in J} E_i \ (\neq \emptyset)$.

Example 24.9. Three coordinate planes meeting gives a solid triangle.

Now the question is what they have to do with Berkovich spaces.

24.1.2 Dual complexes and the Berkovich space

1. There is an embedding $\Delta_{\mathscr{X}} \hookrightarrow X^{\operatorname{an}}$ for every snc model \mathscr{X} . A vertex $e_i \in \Delta_{\mathscr{X}}$ maps to a divisorial point $x_i \in X^{\operatorname{an}}$. Now look at a simplex σ spanned by e_0, \ldots, e_p , where $p \ge 0$. This corresponds to a stratum of \mathscr{X}_0 , which is a connected component of $E_0 \cap \cdots \cap E_p$, with generic point ξ . We have local equations $E_i = \{z_i = 0\}$ where $0 \le i \le p$ at ξ . Now specify a point $w \in \sigma$ using barycentric coordinates: $w = (w_0, \ldots, w_p) \in \mathbf{R}^{p+1}_+$ such that $\sum_{i=0}^p b_i w_i = 1$. To w we associate a monomial valuation (additively) v_w on $\mathcal{O}_{\mathscr{X},\xi}$, where $v_w(z_i) = w_i$, and v_w is the minimal valuation with this property. Essentially by Cohen's structure theorem, $\widehat{\mathcal{O}}_{\mathscr{X},\xi} \cong k(\xi)[[z_0, \ldots, z_p, z_{p+1}, \ldots, z_q]]$, and so any $f \in \mathcal{O}_{\mathscr{X},\xi}$ can be written

$$f = \sum_{\nu \in \mathbf{Z}_{+}^{q+1}} a_{\nu} z^{\nu} \qquad v_w(f) = \min\{\nu_0 w_0 + \dots + \nu_p w_p \mid a_{\nu} \neq 0\}.$$

This gives a point $x_w \in X^{\text{an}}$ by $x_w = r^{v_w}$ (multiplicative). You can check that this is well-defined (independent of choices), and $\Delta_{\mathscr{X}} \to X^{\text{an}}$ is an embedding.

2. Retraction: there is a map $r_{\mathscr{X}}: X^{\mathrm{an}} \to \Delta_{\mathscr{X}} \subset X^{\mathrm{an}}$. Pick $x \in X^{\mathrm{an}}$, and set $\xi := \mathrm{red}_{\mathscr{X}}(x) \in \mathscr{X}_0$. Let E_0, \ldots, E_p be the irreducible components of \mathscr{X}_0 containing ξ , and write $E_i = \{z_i = 0\}$. Define weights $w_i := \log|z_i(x)| / \log r > 0$. Since $t = z_0^{b_0} \cdots z_p^{b_p}$, this implies $\sum_{i=0}^p b_i w_i = 1$, and so $w = (w_0, \ldots, w_p)$ defines a point in the interior of the simplex spanned by the vertices e_0, \ldots, e_p , which can be viewed as a point $w \in \Delta_{\mathscr{X}}$. Set $r_{\mathscr{X}}(x) = w$ (the monomial point we defined above). You can now check that

•
$$r_{\mathscr{X}}(x) \in \Delta_{\mathscr{X}}$$
,

- $r_{\mathscr{X}} = \mathrm{id} \text{ on } \Delta_{\mathscr{X}},$
- $r_{\mathscr{X}}: X^{\mathrm{an}} \to \Delta_{\mathscr{X}}$ is continuous,
- If $\mathscr{X}' \geq \mathscr{X}$, then $r_{\mathscr{X}} \circ r_{\mathscr{X}'} = r_{\mathscr{X}}$.
- If $\mathscr{X}' \geq \mathscr{X}$, then $\Delta_{\mathscr{X}'} \supseteq \Delta_{\mathscr{X}}$ (harder).

Definition 24.10. The points in X^{an} that lie in *some* dual complex $\Delta_{\mathscr{X}}$ are called *quasimonomial*, forming a set $X^{\text{qm}} \subset X^{\text{an}}$, which also contains X^{div} .

We have the following structure theorem for Berkovich spaces:

Theorem 24.11. The retraction maps $r_{\mathscr{X}}: X^{\mathrm{an}} \to \Delta_{\mathscr{X}} \subset X^{\mathrm{an}}$ define a homeomorphism

$$X^{\mathrm{an}} \xrightarrow{\sim} \varprojlim_{\mathscr{X} \mathrm{snc}} \Delta_{\mathscr{X}}$$

We may sketch this next time. For now, we sketch what happens for an elliptic curve.

Example 24.12 (Tate). Consider the elliptic curve $X = \{xyz + t(x^3 + y^3 + z^3) = 0\} \subseteq \mathbf{P}^2_{\mathbf{C}((t))}$, where $k = \mathbf{C}((t))$. Then, $\mathscr{X} = \{xyz + t(x^3 + y^3 + z^3) = 0\} \subseteq \mathbf{P}^2_{\mathbf{C}[[t]]}$, since $k^\circ = \mathbf{C}[[t]]$. Then, $\mathscr{X}_0 = E_0 + E_1 + E_2$, and $\Delta_{\mathscr{X}}$ is topologically a circle (it is a triangle). You can get more models by taking blowups.

We will talk more about this, including when Berkovich spaces are contractible.

25 December 13

25.1 Graded reduction and the Riemann–Zariski spaces

We first want to talk about the homework, especially the graded reduction/Riemann–Zariski space problem. Let k be a non-Archimedean field, and let $X = \mathscr{M}(\mathscr{A})$ be a k-affinoid space. For $x \in X$, we want to think

of a germ (X, x) of X at x. This gives a Riemann–Zariski space (X, x).

Let $\widetilde{\mathscr{A}} = \bigoplus_r \widetilde{\mathscr{A}}_r$ be the graded reduction of \mathscr{A} , and $\widetilde{\mathscr{H}}(x) = \bigoplus_r \widetilde{\mathscr{H}}(x)_r$ the graded reduction of $\mathscr{H}(x)$. The character $\chi_x \colon \mathscr{A} \to \mathscr{H}(x)$ descends to a character $\widetilde{\chi}_x \colon \widetilde{\mathscr{A}} \to \widetilde{\mathscr{H}}(x)$. We can then define the Riemann–Zariski space

$$\mathbf{P}_{\widetilde{\mathscr{H}}(x)/\widetilde{k}} = \left\{ \text{graded valuation rings } \mathcal{O} \subset \widetilde{\mathscr{H}}(x) \mid \widetilde{k} \subseteq \mathcal{O}, \text{ Frac}(\mathcal{O}) = \widetilde{\mathscr{H}}(x) \right\}$$

Temkin [Tem04, §2] associates to (X, x) the space:

$$\widetilde{(X,x)} \coloneqq \mathbf{P}_{\widetilde{\mathscr{H}}(x)/\widetilde{k}}(\widetilde{\chi}_x(\widetilde{\mathscr{A}})) = \left\{ \mathcal{O} \in \mathbf{P}_{\widetilde{\mathscr{H}}(x)/\widetilde{k}} \mid \mathcal{O} \supseteq \widetilde{\chi}_x(\widetilde{\mathscr{A}}) \right\}.$$

Special case Let $k = k^a$, let $X = E_k(1)$ be the closed unit disc, and let x = p(E(r)) for $0 \le r \le 1$.

If r = 0 (i.e., Type 1), then $\mathscr{H}(x) = k$, so that $\widetilde{\mathscr{H}}(x) = \widetilde{k}$, and $\widetilde{\chi}_x(\mathscr{A}) = \widetilde{k}$. Thus, (X, x) is a singleton. If 0 < r < 1, and $r \notin \sqrt{|k^*|}$ so x is of Type 3, then $\widetilde{\mathscr{H}}(x) = \widetilde{k}[r^{-1}T, rT^{-1}]$, where T has weight r. Then, $\widetilde{\chi}_x(\widetilde{\mathscr{A}}) = \widetilde{k}$, since $\rho(T) = 1$ and |T(x)| = r < 1 imply $\widetilde{\chi}_x(T) = 0$. Then, (X, x) has three points:

$$\widetilde{(X,x)} = \{\widetilde{k}[r^{-1}T, rT^{-1}], \widetilde{k}[r^{-1}T], \widetilde{k}[rT^{-1}]\}$$

If 0 < r < 1, and $r \in \sqrt{|k^*|}$ so x is of Type 2, then $\widetilde{\chi}_x(\widetilde{\mathscr{A}}) = \widetilde{k}$, and $\widetilde{(X, x)} \cong \mathbf{P}^1_{\widetilde{k}_1}$. If r = 1, so x is still of Type 2, then $\widetilde{\chi}_x(\widetilde{\mathscr{A}}) = \widetilde{k}[T] = \widetilde{\mathscr{A}}$. Thus,

$$\widetilde{(X,x)} \cong \mathbf{A}^1_{\widetilde{k}_1}.$$

25.2 Berkovich spaces through models

If k is discretely valued, char $\tilde{k} = 0$, and X/k is a smooth projective variety, then we had

Theorem 25.1. $X^{\mathrm{an}} \xrightarrow{\sim} \varprojlim_{\mathscr{X} \mathrm{snc models}} \Delta_{\mathscr{X}}$, where $\Delta_{\mathscr{X}}$ are the dual complexes for those models.

Example 25.2. Let $X = \mathbf{P}_k^1$. We can pick $\mathscr{X} = \mathbf{P}_{k^{\circ}}^1$, so that $\mathscr{X}_0 = \mathbf{P}_{\widetilde{k}}^1$, and so $\Delta_{\mathscr{X}} = \{\text{pt}\}$. You can get more models by blowing up points in the special fiber. X^{an} is a tree.

Example 25.3. Let $X = \{txyz + (x^3 + y^3 + z^3) = 0\} \subseteq \mathbf{P}_k^2$, where $k = \mathbf{C}((t))$. Then, $\mathscr{X} = \{txyz + (x^3 + y^3 + z^3) = 0\} \subseteq \mathbf{P}_{k^\circ}^2$, where $k^\circ = \mathbf{C}[[t]]$. Thus, $\Delta_{\mathscr{X}} = \text{pt}$, so that X^{an} is a tree.

Example 25.4. Let $X = \{xyz + t(x^3 + y^3 + z^3) = 0\} \subseteq \mathbf{P}_k^2$, and let $\mathscr{X} = \{xyz + t(x^3 + y^3 + z^3) = 0\} \subseteq \mathbf{P}_{k^\circ}^2$. Then, $\Delta_{\mathscr{X}}$ is a circle. Blowing up points in the special fiber give more branches coming off from the circle, so that X^{an} is an "arboretum."

25.2.1 Further results

We will list some more difficult results.

Theorem 25.5 [Ber90, Thms. 6.1.4–5]. The polydisc $E_k^n(r)$ and \mathbf{P}_k^n are contractible.

We will indicate the proof for $E_k^n(r)$ later.

Theorem 25.6 [Ber99, Thm. 9.1]. If $|k^*| \neq \{1\}$, then any smooth k-analytic space is locally contractible.

He actually proves a bit more. This is an analogue of the fact that complex manifolds locally look like polydiscs, so they are locally contractible.

The proof in the k-analytic setting is more difficult: it uses techniques from the previous theorem and alterations (de Jong).

The proof of the following similar result uses model theory (in the sense of logic):

Theorem 25.7 [HL16, Ch. 14]. The analytification of any quasi-projective variety is locally contractible.

There are further results by

- Thuiller on the homotopy type of Berkovich spaces [Thu07]; and
- Brown and Foster on rational connectivity and its relation to contractibility of analytifications, whose proof relies on the minimal model program [BF16].

Groups and group actions [Ber90, §§5–6] We now aim to prove that $E_k^n(1)$ is contractible. The tool is to use group actions/convolutions.

Let G be a k-analytic group, and let X be a k-analytic space. This means there is a multiplication map $G \times G \xrightarrow{m} G$ and an inverse map $G \to G$ satisfying a categorical version of the group axioms; then, we say that G acts on X if there is a map $G \times X \to X$ satisfying some axioms.

Remark 25.8. A k-analytic group is not a group! It is a group object in the category of k-analytic spaces. This is related to the fact that $|G \times G| \neq |G| \times |G|$, and $|G \times X| \neq |G| \times |X|$.

Example 25.9. Let $G = X = E_k^n(1) = \mathcal{M}(k\{T_1, \ldots, T_n\})$ is a k-analytic group under addition, so we have a "multiplication" map

$$E^n \times E^n \xrightarrow{\text{add}} E^n$$

$$k\{S_1, \dots, S_n, T_1, \dots, T_n\} \longleftarrow k\{U_1, \dots, U_n\}$$

$$S_i + T_i \longleftarrow U_i$$

an inverse map $E^n \to E^n$ given by $T_i \mapsto -T_i$, and an identity map $\mathscr{M}(k) \to E^n$ given by $T_i \mapsto 0$. We will think of the action of E^n on itself via addition $(X = G = E^n)$.

Let $p = (p_1, p_2)$: $|E^n \times E^n| \to |E^n| \times |E^n|$, which is the restriction of multiplicative seminorm on $k\{S, T\}$ to $k\{S\}$ and to $k\{T\}$. Now given $x, y \in E^n$, the preimage $p^{-1}(x, y) \cong \mathscr{M}(\mathscr{H}(x) \widehat{\otimes}_k \mathscr{H}(y))$ is nonempty by Gruson's theorem 3.21, but not just a single point, either. But it does have a special point.

Definition 25.10. Say $x \in E^n$ is universal (Poineau) or peaked (Berkovich) if for every valuation field K/k, the norm on $\mathscr{H}(x) \widehat{\otimes}_k K$ is multiplicative (the fact that it is a norm is by Gruson's theorem).

When x or y is universal, where $x, y \in E^n$, then we can define $x * y \in E^n$ as the image under the addition map $E^n \times E^n \to E^n$ of the point in $p^{-1}(x, y)$ corresponding to the norm on $\mathscr{H}(x) \widehat{\otimes}_k \mathscr{H}(y)$.

Fact 25.11. Suppose $x = x_r := p(E(r))$, where $0 \le r_i \le 1$ for all $1 \le i \le n$. Then, x is universal. For E^1 , these are the points on the segment connecting 0 to x_G .

Sketch. We want to prove that the norm on $\mathscr{H}(x) \widehat{\otimes}_k \mathscr{H}(y)$ is multiplicative. This norm is induced by the norm $k\{S_1, \ldots, S_n, T_1, \ldots, T_n\}$, which is defined by saying that if

$$f(S,T) = \sum_{\nu \in \mathbf{Z}_{\perp}^n} g_{\nu}(T) S^n$$

then

$$||f|| = \max_{\nu} |g_{\nu}(y)| r^{\nu}.$$

You can check that this is multiplicative (something like a relative version of the Gauss Lemma 4.8).

Convolution We want to describe $x_r * y \in E^n$ for $0 \le r_i \le 1$, and $y \in E^n$. As before, let

$$E^n \times E^n \xrightarrow{\text{add}} E^n$$

$$k\{S,T\} \longleftrightarrow k\{U\}$$

$$S_i + T_i \longleftrightarrow U_i$$

The image of $f = \sum_{\nu} a_{\nu} U^{\nu}$ is $\sum_{\nu} a_{\nu} (S+T)^{\nu}$. We need to expand this to apply the norm from above:

$$\sum_{\nu} T^{\nu} \left(\sum_{\mu} \binom{\mu + \nu}{\mu} a_{\mu + \nu} S^{\mu} \right).$$

Then,

$$|f(x_r * y)| = \max_{\nu} |(\partial_{\nu} f)(y)| r^{\nu}, \quad \text{where} \quad \partial_{\nu} = \frac{1}{\nu!} \frac{\partial^{\nu}}{\partial S^{\nu}}.$$
(10)

Special cases

1. If $r_i = 0$ for all *i* ("convolution with the Dirac mass"), then

$$|f(x_r * y)| = \max_{\nu} |(\partial_{\nu} f)(y)| r^{\nu} = |f(y)|,$$

which implies $x_0 * y = y$.

2. If $r_i = 1$ for all *i* ("spreading the wealth around the entire Berkovich polydisc"), then

$$|f(x_r * y)| = \max_{\nu} |(\partial_{\nu} f)(y)|$$

Claim 25.12. This is equal to $||f|| = \max_{\nu} |a_{\nu}|$.

This is because letting $f = \sum a_{\nu} S^{\nu}$, we have

$$\partial_{\nu}f = a_{\nu} + \sum_{\mu \neq 0} a_{\mu+\nu} \binom{\mu+\nu}{\nu} S^{\mu},$$

in which case

$$\partial_{\nu} f(y) | \le \max \left\{ |a_{\nu}|, \max_{\mu \neq \nu} |a_{\mu+\nu}| | \underbrace{|S(y)^{\nu}|}_{<1} \right\} \le ||f||.$$

Choosing a particular ν gives the inequality in the other direction. Thus, $x_1 * y = x_1$.

Theorem 25.13. E_k^n is contractible.

Proof. Define $H: [0,1] \times E_k^n \to E_k^n$ by $H(t,y) = x_{(t,t)} * y$. It follows from (10) that H is continuous. Also, H(0,y) = y for all $y \in E_k^n$, and $H(1,y) = x_1$ (constant) for all $y \in E_k^n$.

In dimension 1, this function H takes points and moves them toward x_G along the limbs coming out of x_G . For \mathbf{P}^n , you use a similar argument, and you work instead with a multiplicative action (an analytic torus acting on it). There is some tropical geometry that shows up.

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