F-SINGULARITIES FOR NON-F-FINITE RINGS

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We review some classes of singularities defined using the Frobenius morphism, taking care to avoid F-finiteness assumptions. Most of this material is well-known, but some of the implications in Theorem 8 are new, at least in the non-F-finite case. We recommend [TW18] for a survey of F-singularities (mostly in the F-finite setting), and [DS16, §6] and [Has10, §3] as references for the material on strong F-regularity in the non-F-finite setting.

To define different versions of F-rationality, we will need the following:

Definition 1 [HH90, Def. 2.1]**.** Let R be a noetherian ring. A sequence of elements $x_1, x_2, \ldots, x_n \in R$ is a *sequence of parameters* if for every prime ideal $\mathfrak{p}$ containing $(x_1, x_2, \ldots, x_n)$, the images of $x_1, x_2, \ldots, x_n$ in $R_\mathfrak{p}$ are part of a system of parameters in $R_\mathfrak{p}$.

We now begin defining different classes of singularities. We start with F-singularities defined using tight closure. Recall that if R is a ring, then R° is the complement of the union of the minimal primes of R.

Definition 2 [HH90, Def. 8.2]**.** Let R be a ring of characteristic $p > 0$, and let $\iota\colon N \hookrightarrow M$ be an inclusion of R-modules. The *tight closure* of N in M is the R-module

$$N^*_M \coloneqq \left\{ x \in M \;\middle|\; \begin{array}{c} \text{there exists } c \in R^\circ \text{ such that for all } e \gg 0, \\ c \otimes x \in \operatorname{im}\bigl(\mathrm{id} \otimes \iota\colon F^e_* R \otimes_R N \to F^e_* R \otimes_R M\bigr) \end{array} \right\}.$$

We say that N is *tightly closed* in M if $N^*_M = N$.

Definition 3 (F-singularities via tight closure)**.** Let R be a noetherian ring of characteristic $p > 0$. We say that

- (a) R is *strongly F-regular* if N is tightly closed in M for every inclusion $N \hookrightarrow M$ of R-modules [Hoc07, Def. on p. 166];
- (b) R is *weakly F-regular* if I is tightly closed in R for every ideal $I \subseteq R$ [HH90, Def. 4.5];
- (c) R is *F-regular* if $R_\mathfrak{p}$ is weakly F-regular for every prime ideal $\mathfrak{p} \subseteq R$ [HH90, Def. 4.5]; and
- (d) R is *F-rational* if I is tightly closed in R for every ideal I generated by a sequence of parameters in R [FW89, Def. 1.10].

We note that (a) is not the usual definition of strong F-regularity, although it coincides with the usual definition (Definition 6(a)) for F-finite rings; see Theorem 8. We also note that the original definition of F-regularity asserted that localizations at every multiplicative set are weakly F-regular, but the definition using prime ideals is equivalent by [HH90, Cor. 4.15].

Next, we define F-singularities via purity of homomorphisms involving the Frobenius. We recall that a ring homomorphism $\varphi\colon R \to S$ is *pure* if the homomorphism

$$\varphi \otimes \mathrm{id}\colon R \otimes_R M \longrightarrow S \otimes_R M$$

is injective for every R-module M. To simplify notation, we fix the following:

Notation 4. Let R be a noetherian ring of characteristic $p > 0$. For every $c \in R$ and every integer $e > 0$, we denote by λ^e_c the composition

$$R \xrightarrow{F^e} F^e_* R \xrightarrow{F^e_*(-\cdot c)} F^e_* R.$$

Date: August 5, 2018. Updated November 8, 2018.

Definition 5 (F-singularities via purity)**.** Let R be a noetherian ring of characteristic $p > 0$. For $c \in R$, we say that R is *F-pure along c* if λ_c^e is pure for some $e > 0$. Moreover, we say that

(*a*) R is *F-pure regular* if it is F-pure along every $c \in R^\circ$ [HH94, Rem. 5.3];
(*b*) R is *F-pure* if it is F-pure along $1 \in R$ [HR76, p. 121]; and
(*c*) R is *strongly F-rational* if for every $c \in R^\circ$, there exists $e_0 > 0$ such that for all $e \geq e_0$, the homomorphism $\lambda_c^e \otimes R/I$ is injective for every ideal $I \subseteq R$ generated by a sequence of parameters in R [Vél95, Def. 1.2].

The terminology *F-pure regular* is from [DS16, Def. 6.1.1] to distinguish it from the definition using tight closure (Definition 3(*a*)). F-pure regular rings are also called *very strongly F-regular* [Has10, Def. 3.4].

We note that F-purity is a local condition [DS16, Lem. 6.1.4(*e*)]. Strong F-regularity is a local condition [Has10, Lem. 3.6], and while it is equivalent to F-pure regularity in the local case [Has10, Lem. 3.6], F-pure regularity is not known to be a local condition [DS16, Rem. 6.3.3].

Next, we define F-singularities via splitting of homomorphisms involving the Frobenius. We use the same notation as for F-singularities defined using purity (Notation 4).

Definition 6 (F-singularities via splitting)**.** Let R be a noetherian ring of characteristic $p > 0$. For $c \in R$, we say that R is *F-split along c* if λ_c^e splits as an R-module homomorphism for some $e > 0$. Moreover, we say that

(*a*) R is *split F-regular* if it is F-split along every $c \in R^\circ$ [HH94, Def. 5.1]; and
(*b*) R is *F-split* if it is F-split along $1 \in R$ [MR85, Def. 2].

The terminology *split F-regular* is from [DS16, Def. 6.6.1]. Split F-regularity is usually known as strong F-regularity in the literature.

Finally, we define F-injective singularities.

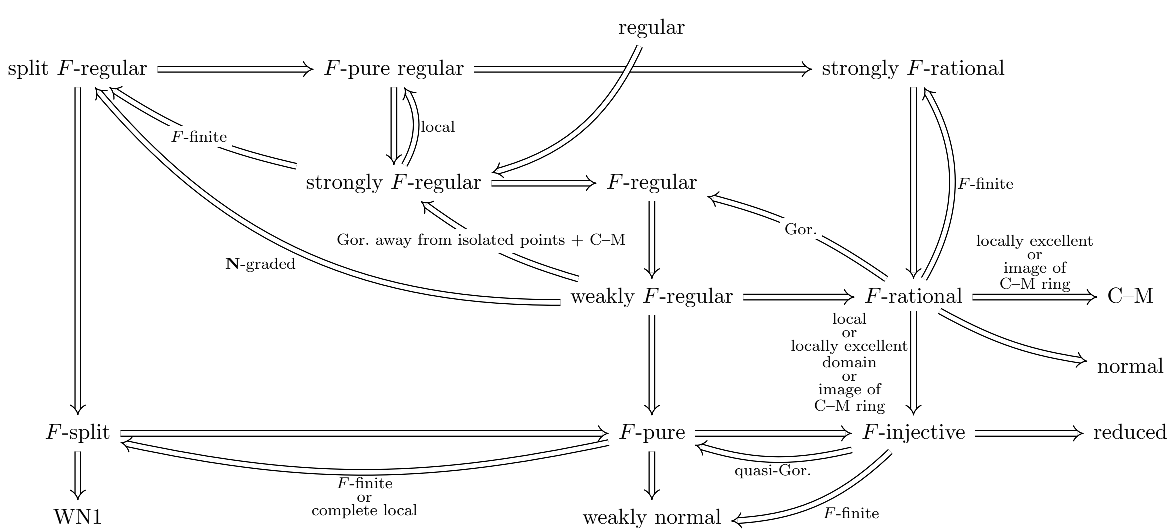
Definition 7 [Fed83, Def. on p. 473]**.** A noetherian local ring $(R, \mathfrak{m})$ of characteristic $p > 0$ is *F-injective* if the R-module homomorphism

$$H^i_{\mathfrak{m}}(F) \colon H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(F_*R)$$

induced by Frobenius is injective for all i. An arbitrary noetherian ring R of characteristic $p > 0$ is *F-injective* if $R_{\mathfrak{m}}$ is F-injective for every maximal ideal $\mathfrak{m} \subseteq R$.

The relationship between these classes of singularities can be summarized as follows:

Theorem 8. *Let R be a noetherian ring of characteristic $p > 0$. We have the following diagram of implications of properties of R:*



where "C–M" (resp. "Gor.") is an abbreviation for Cohen–Macaulay (resp. Gorenstein).

Proof. We first list the implications that are easy or appear in the literature.

Implication			Proof
split F-regular	$\Longrightarrow$	F-split	Definition
F-regular	$\Longrightarrow$	weakly F-regular	Definition
weakly F-regular	$\Longrightarrow$	F-rational	Definition
split F-regular	$\Longrightarrow$	F-pure regular	split maps are pure
F-split	$\Longrightarrow$	F-pure	split maps are pure
F-split	$\Longrightarrow$	WN1	[SZ13, Thm. 7.3]
regular	$\Longrightarrow$	strongly F-regular	[DS16, Thm. 6.2.1]
F-pure regular	$\Longrightarrow$	strongly F-regular	[Has10, Lem. 3.8]
F-pure regular	$\Longrightarrow$	strongly F-rational	[DS16, Rem. 6.1.5]
strongly F-regular	$\Longrightarrow$	F-regular	[Has10, Cor. 3.7]
weakly F-regular	$\Longrightarrow$	F-pure	[FW89, Rem. 1.6]
F-pure	$\Longrightarrow$	F-injective	[Fed83, Lem. 3.3]
strongly F-rational	$\Longrightarrow$	F-rational	[Vél95, Prop. 1.4]
F-rational	$\Longrightarrow$	normal	[HH94, Thm. 4.2(b)]
F-rational + locally excellent	$\Longrightarrow$	Cohen–Macaulay	[Vél95, Prop. 0.10]
F-rational + image of C–M ring	$\Longrightarrow$	Cohen–Macaulay	[HH94, Thm. 4.2(c)]
F-rational + local	$\Longrightarrow$	F-injective	[QS17, Thm. 3.7]
F-rational + locally excellent domain	$\Longrightarrow$	F-injective	[Smi94, Thm. 5.1] [QS17, Thm. 3.7]
F-rational + image of C–M ring	$\Longrightarrow$	F-injective	[HH94, Thm. 4.2(e)] [QS17, Thm. 3.7]
F-injective	$\Longrightarrow$	reduced	[QS17, Lem. 3.11]
F-injective + F-finite	$\Longrightarrow$	weakly normal	[Sch09, Thm. 4.7]
strongly F-regular + F-finite	$\Longrightarrow$	split F-regular	[Has10, Lem. 3.9]
strongly F-regular + local	$\Longrightarrow$	F-pure regular	[Has10, Lem. 3.6]
F-pure + F-finite	$\Longrightarrow$	F-split	[HR76, Cor. 5.3]
F-pure + complete local	$\Longrightarrow$	F-split	[Fed83, Lem. 1.2]
F-rational + Gorenstein	$\Longrightarrow$	F-regular	[HH94, Cor. 4.7(a)]
F-injective + quasi-Gorenstein	$\Longrightarrow$	F-pure	[EH08, Rem. 3.8]

We now show the remaining implications, for which we could not find a reference.

F-pure $\Rightarrow$ weakly normal. We adapt the proof of [Sch09, Thm. 4.7]. It suffices to show that if R is F-pure, then $R_\mathfrak{p}$ is weakly normal for every prime ideal $\mathfrak{p} \subseteq R$ by [Man80, Cor. IV.4]. Suppose not, and choose a prime ideal $\mathfrak{p} \subseteq R$ of minimal height such that $R_\mathfrak{p}$ is not weakly normal. The local ring $R_\mathfrak{p}$ is F-pure by [DS16, Lem. 6.1.4(e)] hence F-injective and reduced. Moreover, the punctured spectrum $\operatorname{Spec}(R_\mathfrak{p}) \smallsetminus \{\mathfrak{p}R_\mathfrak{p}\}$ is weakly normal by the minimality of $\mathfrak{p}$, hence [Sch09, Lem. 4.6] implies $R_\mathfrak{p}$ is weakly normal, a contradiction.

Weakly F-regular + Gorenstein away from isolated points + Cohen–Macaulay $\Rightarrow$ strongly F-regular. Let R be the weakly F-regular ring that is Cohen–Macaulay, and also Gorenstein away from isolated points. Then, the localization $R_\mathfrak{m}$ is weakly F-regular for every maximal ideal $\mathfrak{m} \subseteq R$ by [HH90, Cor. 4.15], and to show that R is strongly F-regular, it suffices to show that 0 is tightly closed in

$$E_\mathfrak{m} \coloneqq E_{R_\mathfrak{m}}(R/\mathfrak{m})$$

for every maximal ideal $\mathfrak{m} \subseteq R$ [Has10, Lem. 3.6]. Since $R_\mathfrak{m}$ is weakly R-regular, every submodule of a finitely generated module is tightly closed [HH90, Prop. 8.7], hence the finitistic tight closure $0^{*\mathrm{fg}}_{E_\mathfrak{m}}$ as defined in [HH90, Def. 8.19] is zero. Finally, since $0^{*\mathrm{fg}}_{E_\mathfrak{m}} = 0^*_{E_\mathfrak{m}}$ under the assumptions on R [LS01, Thm. 8.8], we see that 0 is tightly closed in $E_\mathfrak{m}$, hence R is strongly F-regular.

Weakly F-regular + $\mathbf{N}$-graded ⇒ split F-regular. We adapt the proof of [LS99, Cor. 4.4]. Let R be the $\mathbf{N}$-graded ring with irrelevant ideal $\mathfrak{m}$; note that by assumption in [LS99, §3], the ring R is finitely generated over a field $R_0 = k$ of characteristic $p > 0$. The localization $R_\mathfrak{m}$ of R is weakly F-regular by [HH90, Cor. 4.15]. Now let L be the perfect closure of k, and let $\mathfrak{m}'$ be the expansion of $\mathfrak{m}$ in $R \otimes_k L$; since R is graded, $\mathfrak{m}'$ is the irrelevant ideal in $R \otimes_k L$. The ring homomorphism

$$R_\mathfrak{m} \longrightarrow R_\mathfrak{m} \otimes_k L \cong (R \otimes_k L)_{\mathfrak{m}'}$$

is purely inseparable and $\mathfrak{m}$ expands to $\mathfrak{m}'$, hence $(R \otimes_k L)_{\mathfrak{m}'}$ is weakly F-regular by [HH94, Thm. 6.17(b)]. By the proof of [LS99, Cor. 4.3], $R \otimes_k L$ is strongly F-regular. Finally, R is a direct summand of $R \otimes_k L$ as an R-module, hence R is strongly F-regular as well [HH94, Thm. 5.5(e)].

F-rational + F-finite ⇒ strongly F-rational. The hypotheses of [Vél95, Thm. 1.12] are satisfied when the ring is F-finite since an F-finite ring is excellent and is isomorphic to a quotient of a regular ring of finite Krull dimension by [Gab04, Rem. 13.6]. □

Remark 9. The condition that R is the image of a Cohen–Macaulay ring is not too restrictive in practice. For instance, it suffices for R to have a dualizing complex [Kaw02, Cor. 1.4], which in turn is implied by F-finiteness [Gab04, Rem. 13.6].

Remark 10. In the implication *Weakly F-regular + Gorenstein away from isolated points + Cohen–Macaulay ⇒ strongly F-regular,* MacCrimmon [Mac96, Thm. 3.3.2] showed that for F-finite rings, the Gorenstein condition can be weakened to being $\mathbf{Q}$-Gorenstein away from isolated points. The implication *weakly F-regular + F-finite ⇒ split F-regular* is a famous open problem, and is known in dimensions at most three by [Wil95, §4]. See also [Abe02] for other situations in which this implication is known and for a proof of MacCrimmon's theorem.

Remark 11. The stated cases for the implication "F-rational ⇒ F-injective" follow by reducing to the local case, which is proved in [QS17, Thm. 3.7]. Thus, the implication "F-rational ⇒ F-injective" holds under different hypotheses by using [AHH93, Thm. 5.21], which shows that F-rationality localizes under various assumptions. In particular, by [AHH93, Thm. 5.21(b)], it suffices to assume that R has a weak test element and that $R/\mathfrak{p}$ is of acceptable type (in the sense of [AHH93, p. 87]) for every minimal prime ideal $\mathfrak{p} \subseteq R$.

Acknowledgments. I would like to thank Rankeya Datta for pointing out the implication "weakly F-regular + $\mathbf{N}$-graded ⇒ split F-regular," and for finding a correct reference for the implication "F-rational + local ⇒ F-injective."

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