F-MODULES AND FINITENESS OF ASSOCIATED PRIMES

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Abstract. We present Lyubeznik's theory of F-modules, which are then used to show finiteness of associated primes of local cohomology modules for regular rings in positive characteristic.

Contents

1. Introduction

We are concerned with the following:

Conjecture 1.1 [Lyu93, Rem. 3.7]**.** *Let R be a regular ring, and consider an ideal $\mathfrak{a} \subseteq R$. Then,*

$$\# \operatorname{Ass}_R\bigl(H^i_{\mathfrak{a}}(R)\bigr) < \infty.$$

Recall that $\mathfrak{p} \in \operatorname{Ass}_R(M)$ if there exists $m \in M$ such that $\mathfrak{p} = \operatorname{ann}(m)$. One geometric interpretation is that the "scheme-theoretic support" has finitely many components, and so local cohomology "looks like" it is of finite type.

We quickly review what is known about this question:

(1) $\operatorname{char} R = p > 0$ [HS93, Cor. 2.3].
(2) $\operatorname{char} k = 0$ when R finite type over k, or R local [Lyu93, Rem. 2.9, Cor. 3.6].
(3) R unramified regular local in mixed characteristic [Lyu00, Thm. 1].
(4) R smooth over $\mathbf{Z}$ [BBLSZ14, Thm. 1.2].

In the opposite direction, the regularity condition cannot be weakened substantially:

(5) There are counterexamples to Conjecture 1.1 when R has mild singularities (rational singularities for $\operatorname{char} R = 0$ or F-regular singularities for $\operatorname{char} R = p > 0$) [SS04, Thm. 1.1].

Our goal is to show the following:

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Theorem 1.2 [Lyu97, Cor. 2.14]. *Let R be a regular ring with $\operatorname{char} R = p > 0$. Let T be any functor of the form*

$$T = T_1 \circ T_2 \circ \cdots \circ T_t,$$

where for each j, there exists a locally closed subscheme $Y_j = Y_j'' \smallsetminus Y_j' \subseteq \operatorname{Spec}(R)$ where Y_j'', Y_j' are closed, such that $T_j = H^i_{Y_j}(-)$ or T_j is the kernel, image, or cokernel of any arrow appearing in the long exact sequence

$$0 \longrightarrow H^0_{Y_j'}(-) \longrightarrow H^0_{Y_j''}(-) \longrightarrow H^0_{Y_j}(-) \longrightarrow H^1_{Y_j'}(-) \longrightarrow \cdots.$$

Then, $\#\operatorname{Ass}_R\big(T(R)\big) < \infty$.

1.1. **The approach.** We explain the general approach taken to show finiteness of associated primes.

Theorem 1.3 [Mat89, Thm. 6.5(*i*)]. *If R is noetherian and $M \in \mathsf{Mod}_R$ is finitely generated, then*

$$\#\operatorname{Ass}_R(M) < \infty.$$

The usual proof goes through finding a composition series and arguing about the factors that show up. Instead, we give a proof that will generalize to local cohomology.

We need the following:

Definition 1.4. Let $M \in \mathsf{Mod}_R$ and let $\mathfrak{a} = (f_1, f_2, \ldots, f_r) \subseteq R$ be an ideal. The $\mathfrak{a}$-*torsion submodule* of M is

$$\Gamma_{\mathfrak{a}}(M) \coloneqq \big\{m \in M \bigm| \exists n \text{ such that } \mathfrak{a}^n m = 0\big\} = \ker\Big(M \longrightarrow \bigoplus_{i=1}^r M_{f_i}\Big).$$

Lemma 1.5. *Let R be a noetherian ring, and let $M \in \mathsf{Mod}_R$. If $\mathfrak{p}$ is maximal in $\operatorname{Ass}_R(M)$, then*

$$\operatorname{Ass}_R\big(\Gamma_{\mathfrak{p}}(M)\big) = \{\mathfrak{p}\}.$$

Proof. To show "$\subseteq$" it suffices to show that for every $\mathfrak{q} \in \operatorname{Ass}_R\big(\Gamma_{\mathfrak{p}}(M)\big)$, we have $\mathfrak{p} \subseteq \mathfrak{q}$. This is because $\mathfrak{q}$ is an associated prime of M as well, and by maximality of $\mathfrak{p}$. Let $m \in \Gamma_{\mathfrak{p}}(M)$ such that $\operatorname{ann}(m) = \mathfrak{q}$. Then, $m \in \Gamma_{\mathfrak{p}}(M)$ implies that for every $f \in \mathfrak{p}$, there exists some n such that $f^n \in \operatorname{ann}(m)$. Since $\mathfrak{q}$ is prime, we see that $f \in \mathfrak{q}$, hence $\mathfrak{p} \subseteq \mathfrak{q}$. For "$\supseteq$" it suffices to note that $\Gamma_{\mathfrak{p}}(M) \neq 0$ by definition, hence $\operatorname{Ass}_R\big(\Gamma_{\mathfrak{p}}(M)\big) \neq \emptyset$. ☐

We can now show Theorem 1.3.

Proof of Theorem 1.3. We construct a chain

$$0 \subsetneq M_1 \subseteq M_2 \subseteq \cdots \subseteq M$$

such that $\#\operatorname{Ass}_R(M_i/M_{i-1}) = 1$ as follows. By Lemma 1.5, we can let $M_1 = \Gamma_{\mathfrak{p}_1}(M)$ for $\mathfrak{p}_1 \in \operatorname{Ass}_R(M)$ maximal. For $i > 0$, we let $\mathfrak{p}_i \in \operatorname{Ass}_R(M/M_{i-1})$ be maximal, and set M_i to be the preimage of $\Gamma_{\mathfrak{p}_i}(M/M_{i-1}) \subseteq M/M_{i-1}$ in M. We then have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{i-1} & \longrightarrow & M_i & \longrightarrow & \Gamma_{\mathfrak{p}_i}(M/M_{i-1}) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_{i-1} & \longrightarrow & M & \longrightarrow & M/M_{i-1} & \longrightarrow & 0 \end{array}$$

hence $\operatorname{Ass}_R(M_i/M_{i-1}) = \{\mathfrak{p}_i\}$. Since M is finitely generated, it is noetherian, hence the ascending chain must stabilize at some point r. By looking at the short exact sequences

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow M_i/M_{i-1} \longrightarrow 0$$

for each i, we therefore have that

$$\begin{aligned}\operatorname{Ass}_R(M) = \operatorname{Ass}_R(M_r) &\subseteq \operatorname{Ass}_R(M_{r-1}) \cup \operatorname{Ass}_R(M_r/M_{r-1}) \subseteq \cdots \\ &\subseteq \bigcup_{i=1}^{r} \operatorname{Ass}_R(M_i/M_{i-1}) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r\},\end{aligned}$$

which is finite. □

1.2. **What we need.** To carry out this strategy for local cohomology modules (Theorem 1.2), we need a category $\mathscr{A}$ such that

(I) The category $\mathscr{A}$ is abelian.
(II) There is an abelian subcategory $\mathscr{C}$ of $\mathscr{A}$ such that every object of $\mathscr{C}$ satisfies ACC in $\mathscr{C}$.
(III) $R \in \mathscr{C}$.
(IV) If $M \in \mathscr{C}$, then $M_f \in \mathscr{C}$ for every $f \in R$.

Note that (II–IV) imply that if $M \in \mathscr{C}$, then $H^i_{\mathfrak{a}}(M) \in \mathscr{C}$. In particular, $H^i_{\mathfrak{a}}(R) \in \mathscr{C}$.

We are using the notation "$\in$" a bit loosely here: in actuality, the category $\mathscr{C}$ will consist of R-modules with some extra information. In our proof of Theorem 1.3, we had $\mathscr{A} = \mathsf{Mod}_R$ and $\mathscr{C} = \mathsf{Mod}_F^{\mathrm{fg}}$, although this category does not satisfy (IV).

Lyubeznik in [Lyu93] used that in characteristic zero, one can set $\mathscr{C} = \mathsf{D}\text{-}\mathsf{Mod}_R^{\mathrm{hol}} \subseteq \mathsf{D}\text{-}\mathsf{Mod}_R = \mathscr{A}$. Our main goal is to construct an analogous category in positive characteristic.

From now on, all rings will be of characteristic $p > 0$.

2. Preliminaries

Let R be a ring. Recall that the *Frobenius morphism* is the ring homomorphism

$$\begin{aligned} F\colon R &\longrightarrow R \\ r &\longmapsto r^p \end{aligned}$$

We state some preliminary facts about the Frobenius morphism.

2.1. **Kunz's theorem.** We will use the following result repeatedly.

Theorem 2.1 [Kun69, Cor. 2.7]**.** *Let R be a noetherian ring. Then, R is regular if and only if $F\colon R \to R$ is flat and R is reduced.*

2.2. **The Frobenius functor of Peskine–Szpiro.** We also need to define the Frobenius functor of Peskine–Szpiro.

Definition 2.2 (cf. [PS73, Def. I.1.2])**.** Let R be a ring. The *Frobenius functor* is the extension of scalars functor $\mathsf{F}_R\colon \mathsf{Mod}_R \to \mathsf{Mod}_R$, which is right-exact, and is exact if R is regular. We denote F_R by F when it (hopefully) cannot cause any confusion.

Remark 2.3. We define the functor more explicitly. Let R be a ring. If M is an R-module, we denote by $M^{(1)}$ the (R,R)-bimodule given by the usual R-module structure on the left, and the right R-module structure given by $m \cdot r = r^p m$ for all $r \in R$. One can define the Frobenius functor as the functor

$$\begin{aligned} \mathsf{F}\colon \mathsf{Mod}_R &\longrightarrow \mathsf{Mod}_R \\ M &\longmapsto R^{(1)} \otimes_R M \\ \bigl(M \xrightarrow{h} N\bigr) &\longmapsto \bigl(R^{(1)} \otimes_R M \xrightarrow{\mathrm{id}\otimes_R h} R^{(1)} \otimes_R N\bigr) \end{aligned}$$

where the R-module structure on $R^{(1)} \otimes_R M$ is given by the left R-module structure on $R^{(1)}$.

We will need the following facts:

Lemma 2.4 (cf. [PS73, Prop. I.1.4] and [Lyu97, Rem. 1.0(i)])**.** *Let R be a ring, and let $T \subseteq R$ be a multiplicative set. Then, the diagram*

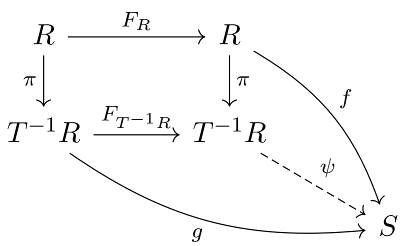
$$\begin{array}{ccc} R & \xrightarrow{F_R} & R \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ T^{-1}R & \xrightarrow{F_{T^{-1}R}} & T^{-1}R \end{array} \tag{1}$$

is cocartesian, and there is a natural isomorphism

$$\beta\colon {}_R\big(\mathsf{F}_{T^{-1}R}(-)\big) \xrightarrow{\sim} \mathsf{F}_R({}_R-)$$

of functors.

Proof. Let S be a ring fitting into the commutative diagram below:



We want to show there is a unique ring homomorphism ψ making the diagram commute. First, note that if $t \in T$, then the image $f(t)$ is invertible since $g(t)$ is invertible and

$$f(t) \cdot f(t^{p-1}) \cdot g(t)^{-1} = f(\varphi_R(t)) \cdot g(t)^{-1} = g(t) \cdot g(t)^{-1} = 1$$

by the commutativity of the diagram. There is therefore a unique ring homomorphism ψ making the diagram commute by the universal property of localization [Stacks, Tag 00CP].

The final assertion follows from cohomology and base change [Stacks, Tag 02KG]. Explicitly, in the notation of Remark 2.3, we have natural isomorphisms

$$(T^{-1}R)^{(1)} \otimes_{T^{-1}R} M \xrightarrow{\sim} \big(R^{(1)} \otimes_R T^{-1}R\big) \otimes_{T^{-1}R} M \xrightarrow{\sim} R^{(1)} \otimes_R M$$

as R-modules. □

Lemma 2.5 [Lyu97, Rem. 1.0(b)]**.** *Let R be a regular ring. Then, the Frobenius functor commutes with finite intersections of submodules, i.e., if $\{N_i\}_{i \in I}$ is a finite set of submodules of an R-module M, then*

$$\mathsf{F}\Big(\bigcap_{i \in I} N_i\Big) = \bigcap_{i \in I} \mathsf{F}(N_i) \subseteq \mathsf{F}(M).$$

Proof. There exists an exact sequence

$$0 \longrightarrow \bigcap_{i \in I} N_i \longrightarrow M \xrightarrow{m \mapsto \bigoplus_i \overline{m}_i} \bigoplus_{i \in I} M/N_i,$$

where $\overline{m}_i \in M/N_i$ is the image of $m \in M$ under the natural projection map $M \to M/N_i$. By the fact that F is flat, F is exact, and so we have an exact sequence

$$0 \longrightarrow \mathsf{F}\Big(\bigcap_{i \in I} N_i\Big) \longrightarrow \mathsf{F}(M) \xrightarrow{m \mapsto \bigoplus_i \overline{m}_i} \bigoplus_{i \in I} \mathsf{F}(M/N_i),$$

where $\overline{m}_i \in \mathsf{F}(M/N_i)$ is the image of $m \in \mathsf{F}(M)$ under the natural projection map $\mathsf{F}(M) \to \mathsf{F}(M/N_i)$. Since this sequence is exact, it follows that F commutes with finite intersections. □

3. F-modules

The Frobenius functor is important since for regular rings, $\mathsf{F}(H^i_{\mathfrak{a}}(R)) \cong H^i_{\mathfrak{a}}(R)$ [HS93, Lem. 1.8]. We will prove this eventually, but for now this suggests the following candidate for $\mathscr{A}$:

Definition 3.1 (cf. [Lyu97, Def. 1.1])**.** Let R be a ring. An *F-module* over R is a pair (M, θ_M), where M is an R-module and $\theta_M \colon M \to \mathsf{F}(M)$ is an R-module isomorphism, which we call the *structure morphism* of M. A *morphism* $f\colon (M,\theta_M) \to (N,\theta_N)$ of F-modules is an R-module homomorphism $f\colon M \to N$ for which the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & N \\ {\scriptstyle\theta_M}\downarrow & & \downarrow{\scriptstyle\theta_N} \\ \mathsf{F}(M) & \xrightarrow{\mathsf{F}(f)} & \mathsf{F}(N) \end{array}$$

commutes in Mod_R. The *category of F-modules* over R is denoted $\mathsf{F}\text{-}\mathsf{Mod}_R$. There is a natural "forgetful" functor

$$\begin{aligned} \mathsf{Forget}\colon \mathsf{F}\text{-}\mathsf{Mod}_R &\longrightarrow \mathsf{Mod}_R \\ (M,\theta_M) &\longmapsto M \end{aligned}$$

If (M,θ_M) is an F-module, we will often abuse terminology by simply referring to M as an F-module over R. In this case, we are simply asserting that M is already equipped with a structure morphism $\theta_M \colon M \xrightarrow{\sim} \mathsf{F}(M)$.

Example 3.2 [Lyu97, Ex. 1.2(*a*)]**.** Any R-module isomorphism $\theta_R\colon R \to \mathsf{F}(R)$ makes R into an F-module. In particular, there is a canonical F-module structure on R defined by the R-module isomorphism

$$\begin{aligned} R &\xrightarrow{\sim} R^{(1)} \otimes_R R =: \mathsf{F}(R) \\ r &\longmapsto r \otimes 1 \end{aligned}$$

3.1. *F*-modules form an abelian category.

We first show some elementary properties of the category $\mathsf{F}\text{-}\mathsf{Mod}_R$ for an arbitrary ring R. See [Stacks, Tag 09SE] for definitions of preadditive and additive categories.

Proposition 3.3. *Let R be a ring. The category $\mathsf{F}\text{-}\mathsf{Mod}_R$ of F-modules over R is additive and has cokernels.*

Proof. The forgetful functor $\mathsf{F}\text{-}\mathsf{Mod}_R \to \mathsf{Mod}_R$ is faithful, and the subset of morphisms of F-modules $M \to N$ is a subgroup of $\operatorname{Hom}_R(M,N)$, hence $\mathsf{F}\text{-}\mathsf{Mod}_R$ is preadditive. Moreover, the zero module 0 is trivially an F-module, and if $(M,\theta_M),(N,\theta_N)$ are two F-module, then $(M \oplus N, \theta_M \oplus \theta_N)$ is an F-module. We therefore see that $\mathsf{F}\text{-}\mathsf{Mod}_R$ is additive.

Now let $f\colon M \to N$ be a morphism of F-modules. Consider the exact sequence

$$M \xrightarrow{f} N \longrightarrow \operatorname{coker}(f) \longrightarrow 0.$$

Applying F, we have the commutative diagram

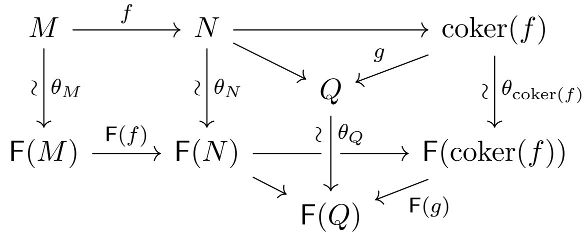
$$\begin{array}{ccccccc} M & \xrightarrow{\ f\ } & N & \longrightarrow & \operatorname{coker}(f) & \longrightarrow & 0 \\ \wr\downarrow{\scriptstyle\theta_M} & & \wr\downarrow{\scriptstyle\theta_N} & & \downarrow{\scriptstyle\theta_{\operatorname{coker}(f)}} & & \\ \mathsf{F}(M) & \xrightarrow{\mathsf{F}(f)} & \mathsf{F}(N) & \longrightarrow & \mathsf{F}\big(\operatorname{coker}(f)\big) & \longrightarrow & 0 \end{array}$$

in Mod_R, where the bottom row is exact since φ_R^* is right-exact. Since θ_M and θ_N are isomorphisms by definition, we see that $\theta_{\operatorname{coker}(f)}$ is also an isomorphism, hence $\operatorname{coker}(f)$ is an F-module.

Finally, we show that $\operatorname{coker}(f)$ is the cokernel for f in $\mathsf{F}\text{-}\mathsf{Mod}_R$. Suppose we have a sequence

$$M \xrightarrow{f} N \longrightarrow Q$$

of morphisms of F-modules whose composition is zero. We then have the diagram



in Mod_R, where g is uniquely constructed from the universal property of $\operatorname{coker}(f)$, and the bottom layer is the top layer with F applied. The universal property of $\operatorname{coker}(f)$ applied to

$$M \xrightarrow{f} N \longrightarrow Q \xrightarrow[\sim]{\theta_Q} \mathsf{F}(Q)$$

implies that $\theta_Q \circ g = \mathsf{F}(g) \circ \theta_{\operatorname{coker}(f)}$, hence $g \colon \operatorname{coker}(f) \to Q$ is a morphism of F-modules. □

On the other hand, for $\mathsf{F}\text{-}\mathsf{Mod}_R$ to be abelian, we need a stronger assumption on R. See [Stacks, Tag 00ZX] for the definition of an abelian category.

Proposition 3.4. *Let R be a regular ring. The category $\mathsf{F}\text{-}\mathsf{Mod}_R$ of F-modules over R is abelian, and the forgetful functor $\mathsf{Forget} \colon \mathsf{F}\text{-}\mathsf{Mod}_R \to \mathsf{Mod}_R$ reflects exactness.*

Proof. Let $f \colon M \to N$ be a morphism of F-modules. By Proposition 3.3, to show that $\mathsf{F}\text{-}\mathsf{Mod}_R$ is abelian, it remains to show that $\ker(f)$ exists, and that the natural map $\operatorname{coim}(f) \to \operatorname{im}(f)$ is an isomorphism. Consider the exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow M \xrightarrow{f} N.$$

Applying F, we have the commutative diagram

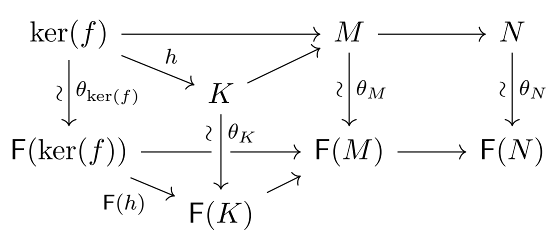
$$\begin{array}{ccccccc}
0 & \longrightarrow & \ker(f) & \longrightarrow & M & \xrightarrow{f} & N \\
 & & \Big\downarrow{\scriptstyle \theta_{\ker(f)}} & & \wr\Big\downarrow{\scriptstyle \theta_M} & & \wr\Big\downarrow{\scriptstyle \theta_N} \\
0 & \longrightarrow & \mathsf{F}(\ker(f)) & \longrightarrow & \mathsf{F}(M) & \xrightarrow{\mathsf{F}(f)} & \mathsf{F}(N)
\end{array}$$

in Mod_R, where the bottom row is exact since F is exact by the flatness of Frobenius. Since θ_M and θ_N are isomorphisms by definition, we see that $\theta_{\ker(f)}$ is also an isomorphism, hence $\ker(f)$ is an F-module.

We now show that $\ker(f)$ is a kernel for f in $\mathsf{F}\text{-}\mathsf{Mod}_R$. Suppose we have a sequence

$$K \longrightarrow M \xrightarrow{f} N$$

of morphisms of F-modules whose composition is zero. We then have the diagram

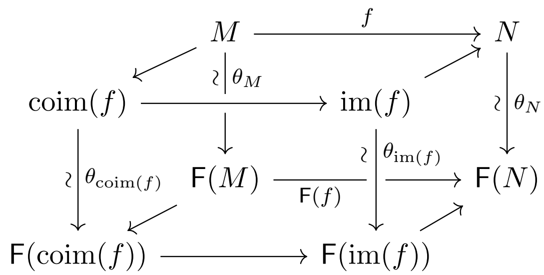


in Mod_R, where h is uniquely constructed from the universal property of $\ker(f)$, and the bottom layer is the top layer with F applied. The universal property of $\ker(f)$ applied to

$$K \xrightarrow[\sim]{\theta_K} \mathsf{F}(K) \longrightarrow \mathsf{F}(M) \xrightarrow{\mathsf{F}(f)} \mathsf{F}(N)$$

implies that $\theta_K \circ h = \mathsf{F}(h) \circ \theta_{\ker(f)}$, hence $h\colon K \to \ker(f)$ is a morphism of F-modules.

To show that the natural map $\operatorname{coim}(f) \to \operatorname{im}(f)$ is an isomorphism, consider the diagram



where the bottom layer is the top layer with F applied. The diagram commutes since all maps are constructed using universal properties of kernels and cokernels. Since the category Mod_R is abelian, the morphism $\operatorname{coim}(f) \to \operatorname{im}(f)$ is an isomorphism, hence $\mathsf{F}(\operatorname{coim}(f)) \to \mathsf{F}(\operatorname{im}(f))$ is an isomorphism as well. Thus, $\operatorname{coim}(f) \to \operatorname{im}(f)$ is an isomorphism in $\mathsf{F}\text{-}\mathsf{Mod}_R$.

Finally, since the underlying modules for kernels and cokernels are the kernels and cokernels from Mod_R, we see that the forgetful functor Forget reflects exactness. □

We now can set $\mathscr{A} = \mathsf{F}\text{-}\mathsf{Mod}_R$ for goal (I).

3.2. **Local cohomology as an F-module.** We disregard (II) for a moment, and first explain why local cohomology gives an example of an F-module. The easiest way to do so is to define restriction and extension of scalars for R-modules.

Definition 3.5 [Lyu97, Def.-Prop. 1.3(*a*)]**.** Let R and S be rings, and let $\pi\colon R \to S$ be a ring homomorphism. Since the diagram

$$\begin{array}{ccc} R & \xrightarrow{F_R} & R \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ S & \xrightarrow{F_S} & S \end{array}$$

commutes, there is a natural isomorphism

$$\alpha\colon S \otimes_R \mathsf{F}_R(-) \xrightarrow{\sim} \mathsf{F}_S(S \otimes_R -)$$

of functors $\mathsf{Mod}_R \to \mathsf{Mod}_S$. The *extension of scalars functor*

$$\pi_*\colon \mathsf{F}\text{-}\mathsf{Mod}_R \longrightarrow \mathsf{F}\text{-}\mathsf{Mod}_S$$

is defined as follows. First, we define $\pi_*(M, \theta_M)$ to be the S-module $\pi_* M \coloneqq S \otimes_R M$ equipped with the structure morphism

$$\pi_* M \xrightarrow[\sim]{\pi_*\theta_M} \pi_*\mathsf{F}_R(M) \xrightarrow[\sim]{\alpha(M)} \mathsf{F}_S(\pi_* M).$$

Second, if $f\colon M \to N$ is a morphism of F-modules over R, then we define $\pi_* f \coloneqq S \otimes_R f$, which defines a morphism since the diagram

$$\begin{array}{ccccc} \pi_* M & \xrightarrow[\sim]{\pi_*\theta_M} & \pi_*\mathsf{F}_R(M) & \xrightarrow[\sim]{\alpha(M)} & \mathsf{F}_S(\pi_* M) \\ \downarrow{\scriptstyle \pi_* f} & & \downarrow{\scriptstyle \pi_*\mathsf{F}_R(f)} & & \downarrow{\scriptstyle \mathsf{F}_S(\pi_* f)} \\ \pi_* N & \xrightarrow{\pi_*\theta_N} & \pi_*\mathsf{F}_S(N) & \xrightarrow[\sim]{\alpha(N)} & \mathsf{F}_S(\pi_* N) \end{array}$$

commutes. We then have the commutative diagram

$$\begin{array}{ccc} \mathsf{F\text{-}Mod}_R & \xrightarrow{\pi_*} & \mathsf{F\text{-}Mod}_S \\ {\scriptstyle \mathsf{Forget}}\downarrow & & \downarrow{\scriptstyle \mathsf{Forget}} \\ \mathsf{Mod}_R & \xrightarrow{\pi_*} & \mathsf{Mod}_S \end{array}$$

of functors.

Definition 3.6 [Lyu97, Def.-Prop. 1.3(b)]**.** Let R be a ring, and let $T \subseteq R$ be a multiplicative set. The *restriction of scalars functor*

$$\pi^* \colon \mathsf{F\text{-}Mod}_{T^{-1}R} \longrightarrow \mathsf{F\text{-}Mod}_R$$

is defined as follows. First, we define $\pi^*(M, \theta_M)$ to be the R-module $\pi^* M \coloneqq {}_R M$ equipped with the structure morphism

$$\pi^* M \xrightarrow[\sim]{\pi^*\theta_M} \pi^* \mathsf{F}_{T^{-1}R}(M) \xrightarrow[\sim]{\beta(M)} \mathsf{F}_R(\pi^* M)$$

where $\beta(M)$ is the isomorphism from Lemma 2.4. Second, if $f \colon M \to N$ is a morphism of F-modules over $T^{-1}R$, then we define $\pi^* f \coloneqq {}_S f$, which defines a morphism since the diagram

$$\begin{array}{ccccc} \pi^* M & \xrightarrow[\sim]{\pi^*\theta_M} & \pi^* \mathsf{F}_{T^{-1}R}(M) & \xrightarrow[\sim]{\beta(M)} & \mathsf{F}_R(\pi^* M) \\ \downarrow{\scriptstyle \pi^* f} & & \downarrow{\scriptstyle \pi^*\mathsf{F}_{T^{-1}R}(f)} & & \downarrow{\scriptstyle \mathsf{F}_R(\pi^* f)} \\ \pi^* N & \xrightarrow[\sim]{\pi^*\theta_N} & \pi^* \mathsf{F}_{T^{-1}R}(N) & \xrightarrow[\sim]{\beta(N)} & \mathsf{F}_R(\pi^* N) \end{array}$$

commutes. We then have the commutative diagram

$$\begin{array}{ccc} \mathsf{F\text{-}Mod}_{T^{-1}R} & \xrightarrow{\pi^*} & \mathsf{F\text{-}Mod}_R \\ {\scriptstyle \mathsf{Forget}}\downarrow & & \downarrow{\scriptstyle \mathsf{Forget}} \\ \mathsf{Mod}_{T^{-1}R} & \xrightarrow{\pi^*} & \mathsf{Mod}_R \end{array}$$

of functors.

Proposition 3.7 [Lyu97, Def.-Prop. 1.3(b)]**.** *Let R be a ring, and let $T \subseteq R$ be a multiplicative set. There is an adjunction $\pi^* \vdash \pi_*$ of functors such that for every F-module M over R, the morphism*

$$M \longrightarrow \pi^* \pi_* M$$

induced by the unit of the adjunction is the natural localization map $\ell \colon M \to T^{-1}M$.

Proof. By the universal property of localization [Stacks, Tag 07K0], there is a natural bijection

$$\operatorname{Hom}_{T^{-1}R}(\pi_* M, N) \xrightarrow{\sim} \operatorname{Hom}_R(M, \pi^* N)$$

where M is an R-module and N is a $T^{-1}R$-module. The R-module homomorphism $M \to \pi^*\pi_* M$ induced by the unit of the adjunction is the natural localization map $\ell \colon M \to T^{-1}M$.

It therefore suffices to show that if M is an F-module over R and N is an F-module over $T^{-1}R$, then under this bijection, morphisms of F-modules over $T^{-1}R$ correspond to morphisms of

F-modules over R. Consider the following diagram in Mod_R:

$$\begin{array}{ccccc} M & \xrightarrow{\ell} & \pi^*T^{-1}M & \xrightarrow{\pi^* f} & \pi^*N \\ & & \wr\downarrow \pi^*\theta_{T^{-1}M} & & \wr\downarrow \pi^*\theta_N \\ \wr\downarrow \theta_M & & \pi^*\mathsf{F}_{T^{-1}R}(T^{-1}M) & \xrightarrow{\mathsf{F}_{T^{-1}R}(\pi^* f)} & \pi^*\mathsf{F}_{T^{-1}R}(N) \\ & & \wr\downarrow \beta(T^{-1}M) & & \wr\downarrow \beta(N) \\ \mathsf{F}(M) & \xrightarrow{\mathsf{F}_R(\ell)} & \mathsf{F}_R(\pi^*T^{-1}M) & \xrightarrow{\pi^*\mathsf{F}_R(f)} & \mathsf{F}_R(\pi^*N) \end{array}$$

Since the bottom right square automatically commutes by Lemma 2.4, we see that the left square commutes if and only if the top right square commutes as desired. □

We can now give our first non-trivial example of an F-module.

Proposition 3.8 [Lyu97, pp. 80–81]**.** *Let R be a regular ring, and let M be an F-module over R. Consider an ideal $\mathfrak{a} \subseteq R$. Then, the local cohomology modules $H^i_{\mathfrak{a}}(M)$ have natural F-module structures, and form a δ-functor*

$$H^i_{\mathfrak{a}} \colon \mathsf{F}\text{-}\mathsf{Mod}_R \longrightarrow \mathsf{F}\text{-}\mathsf{Mod}_R.$$

Proof. Consider the Čech complex for a choice of generators $f_1, f_2, \ldots, f_r$ of $\mathfrak{a}$ together with its image under F:

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & M & \xrightarrow{d^0} & \bigoplus_i M_{f_i} & \xrightarrow{d^1} & \bigoplus_{i_1<i_2} M_{f_{i_1}f_{i_2}} & \xrightarrow{d^2} & \cdots & \xrightarrow{d^{r-1}} & M_{f_1 f_2 \cdots f_r} & \longrightarrow & 0 \\ & & \wr\downarrow \theta_M & & \wr\downarrow \oplus\theta_{M_{f_i}} & & \wr\downarrow \oplus\theta_{M_{f_{i_1}f_{i_2}}} & & & & \wr\downarrow \oplus\theta_{M_{f_1 f_2\cdots f_r}} & & \\ 0 & \longrightarrow & \mathsf{F}(M) & \xrightarrow{d^0} & \bigoplus_i \mathsf{F}(M_{f_i}) & \xrightarrow{d^1} & \bigoplus_{i_1<i_2} \mathsf{F}(M_{f_{i_1}f_{i_2}}) & \xrightarrow{d^2} & \cdots & \xrightarrow{d^{r-1}} & \mathsf{F}(M_{f_1 f_2 \cdots f_r}) & \longrightarrow & 0 \end{array}$$

The bottom row is exact by the flatness of F, and the F-module structures on each localization of M are induced by extending scalars via the localization map, and restricting scalars back to R. The diagram commutes by Proposition 3.7 and the definition of the differentials d^j. Since the category $\mathsf{F}\text{-}\mathsf{Mod}_R$ is abelian by Proposition 3.4, we see that $H^i_{\mathfrak{a}}(M)$ has a natural F-module structure for each i. Finally, by performing the proof of [BS13, Lem. 5.1.9] in the category $\mathsf{F}\text{-}\mathsf{Mod}_R$, the functor $H^i_{\mathfrak{a}}$ indeed defines a δ-functor. □

Remark 3.9. The F-module structure seems to depend on the choice of generators $f_1, f_2, \ldots, f_r$. One can show that local cohomology modules can be given F-module structures using injective resolutions [Lyu97, Ex. 1.2(b)], and that this definition will match the description in Proposition 3.8 by [Lyu97, Prop. 1.8].

4. F-finite F-modules

We now discuss (II). The idea is that the subcategory $\mathscr{C}$ should consist of F-modules that "come from" finitely generated R-modules in a precise sense, which we describe first.

4.1. Generating morphisms.

Definition 4.1 [Lyu97, Def. 1.9]**.** Let R be a ring. Let M be an F-module over R. A *generating morphism* for M is an R-module homomorphism $\vartheta\colon M_0 \to \mathsf{F}(M_0)$ where M_0 is an R-module, such that M is the direct limit of the top row of the commutative diagram

$$\begin{array}{ccccccc}
M_0 & \xrightarrow{\vartheta} & \mathsf{F}(M_0) & \xrightarrow{\mathsf{F}(\vartheta)} & \mathsf{F}^2(M_0) & \xrightarrow{\mathsf{F}^2(\vartheta)} & \cdots \\
\downarrow{\scriptstyle \vartheta} & & \downarrow{\scriptstyle \mathsf{F}(\vartheta)} & & \downarrow{\scriptstyle \mathsf{F}^2(\vartheta)} & & \\
\mathsf{F}(M_0) & \xrightarrow{\mathsf{F}(\vartheta)} & \mathsf{F}^2(M_0) & \xrightarrow{\mathsf{F}^2(\vartheta)} & \mathsf{F}^3(M_0) & \xrightarrow{\mathsf{F}^3(\vartheta)} & \cdots
\end{array}$$

and such that the structure morphism $\theta\colon M \overset{\sim}{\to} \mathsf{F}(M)$ is induced by the vertical arrows in this diagram.

Example 4.2 [Lyu97, Rem. 1.10(a)]**.** The structure morphism of an F-module M is automatically a generating morphism for M, and a generating morphism ϑ for an F-module is the structure morphism if and only if ϑ is an isomorphism.

We now describe how generating morphisms interact with localization.

Proposition 4.3 [Lyu97, Rem. 1.10(c)]**.** *Let R be a regular ring. Let M be an F-module with a generating morphism $\vartheta\colon M_0 \to \mathsf{F}(M_0)$. Let $f \in R$, and denote by $\pi\colon R \to R_f$ the localization homomorphism. Then, $\vartheta \circ f^{p-1}\colon M_0 \to \mathsf{F}(M_0)$ is a generating morphism for $\pi^*\pi_*M$.*

Proof. Denote $\pi^*\pi_*M = (M_f, \theta_{M_f})$, and denote by $\ell\colon M \to M_f$ the natural localization map. Consider the diagram

$$\begin{array}{ccccccc}
M_0 & \xrightarrow{\vartheta\circ f^{p-1}} & \mathsf{F}(M_0) & \xrightarrow{\mathsf{F}(\vartheta\circ f^{p-1})} & \mathsf{F}^2(M_0) & \xrightarrow{\mathsf{F}^2(\vartheta\circ f^{p-1})} & \cdots \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \mathsf{F}(\gamma)} & & \downarrow{\scriptstyle \mathsf{F}^2(\gamma)} & & \\
M & \xrightarrow{\theta_M\circ f^{p-1}} & \mathsf{F}(M) & \xrightarrow{\mathsf{F}(\theta_M\circ f^{p-1})} & \mathsf{F}^2(M) & \xrightarrow{\mathsf{F}^2(\theta_M\circ f^{p-1})} & \cdots \\
\downarrow{\scriptstyle f^{-1}\circ\ell} & & \downarrow{\scriptstyle \mathsf{F}(f^{-1}\circ\ell)} & & \downarrow{\scriptstyle \mathsf{F}^2(f^{-1}\circ\ell)} & & \\
M_f & \xrightarrow{\theta_{M_f}} & \mathsf{F}(M_f) & \xrightarrow{\mathsf{F}(\theta_{M_f})} & \mathsf{F}^2(M_f) & \xrightarrow{\mathsf{F}^2(\theta_{M_f})} & \cdots
\end{array}$$

where the vertical morphisms in the top row are those induced by ϑ. The top half of the diagram commutes by definition of a generating morphism; the bottom half of the diagram commutes since

$$\mathsf{F}^i(f^{-1} \circ \ell) = f^{-p^i} \circ \mathsf{F}(\ell), \qquad \mathsf{F}^i(\theta_M \circ f^{p-1}) = \mathsf{F}^i(\theta_M) \circ f^{p^{i+1}-p^i}$$

and ℓ is a morphism of F-modules.

Let N_0 and N be the F-modules generated by the first and second rows, respectively, and let $g\colon N_0 \to N$ be the corresponding morphism of F-modules. This morphism is an isomorphism since the direct limit of the $\mathsf{F}^i(\gamma)$ is an isomorphism by the assumption that ϑ is a generating morphism for M.

Now consider the morphism $h\colon N \to M_f$ induced by the second and third rows. We claim that g is an isomorphism. First, the morphism h is injective since every element in

$$\ker\bigl(\mathsf{F}^i(f^{-1} \circ \ell)\bigr) = \ker\bigl(f^{-p^i} \circ \mathsf{F}^i(\ell)\bigr) = \ker\bigl(\mathsf{F}^i(\ell)\bigr) = \mathsf{F}^i \ker(\ell)$$

is annihilated by some power of f, and is therefore eventually goes to zero in the directed system of the middle row. Second, the morphism h is surjective since every morphism in the bottom row is an isomorphism, so we have a map

$$\bigl(\theta_{M_f}^{-1} \circ (\mathsf{F}(\theta_{M_f}))^{-1} \circ \cdots \circ (\mathsf{F}^{i-1}(\theta_{M_f}))^{-1}\bigr) \circ \mathsf{F}^i(f^{-1} \circ \ell)\colon \mathsf{F}^i M \longrightarrow M_f$$

whose image is the R-submodule of M_f consists of all elements of the form m/f^{p^i} where $m \in M$, and every element of M_f may be written in such a form for some i. ☐

4.2. **Definition of $\boldsymbol{F}$-finite $\boldsymbol{F}$-modules.** We now define the category $\mathscr{C}$ we wanted to find in (II).

Definition 4.4 [Lyu97, Def. 2.1]**.** Let R be a ring, and let M be an F-module. We say M is *F-finite* if M has a generating morphism $\vartheta\colon M_0 \to \mathsf{F}(M_0)$, where M_0 is a finitely generated R-module.

If ϑ is injective, then we say M_0 is a *root* of M and we call ϑ a *root morphism*.

The full subcategory of F-finite F-modules is denoted $\mathsf{F}\text{-}\mathsf{Mod}_R^{\mathrm{fin}}$.

Example 4.5 [Lyu97, Ex. 2.2(a)]**.** The ring R with the F-module structure from Example 3.2 is F-finite, since R is finitely generated over itself.

Example 4.6 (cf. [Lyu97, Prop. 2.9(b)])**.** Let R be a regular ring, and let M be an F-finite F-module over R with generating morphism $\vartheta\colon M_0 \to \mathsf{F}(M_0)$. Let $f \in R$, and let $\pi\colon R \to R_f$ be the natural localization homomorphism. Then, $M_f \coloneqq \pi^*\pi_*M$ is F-finite with generating morphism $\vartheta \circ f^{p-1}\colon M_0 \to \mathsf{F}(M_0)$ by Proposition 4.3. Once we show that $\mathsf{F}\text{-}\mathsf{Mod}_R^{\mathrm{fin}}$ is an abelian subcategory of $\mathsf{F}\text{-}\mathsf{Mod}_R$, this implies $H^i_{\mathfrak{a}}(M)$ and more generally $T(M)$ is F-finite for all $M \in \mathsf{F}\text{-}\mathsf{Mod}_R^{\mathrm{fin}}$, where $T(-)$ is a functor as in Theorem 1.2.

This shows (III) and (IV). It therefore remains to show (II).

4.3. **$\boldsymbol{F}$-finite $\boldsymbol{F}$-modules form a Serre subcategory.** We start with the following:

Proposition 4.7 [Lyu97, Prop. 2.3]**.** *Every F-finite F-module over a regular ring R has a root.*

Proof. Let $M \in \mathsf{F}\text{-}\mathsf{Mod}_R^{\mathrm{fin}}$, and let $\vartheta\colon M_0 \to \mathsf{F}(M_0)$ be a generating morphism. Denote

$$\vartheta_i \coloneqq \mathsf{F}^{i-1}(\vartheta) \circ \cdots \circ \mathsf{F}(\vartheta) \circ \vartheta.$$

Since M_0 is finitely generated over a noetherian ring, the ascending chain

$$\ker(\vartheta_1) \subseteq \ker(\vartheta_2) \subseteq \cdots \subseteq M_0$$

of submodules of M_0 eventually stabilizes with common value $\ker(\vartheta_i)$. The generating morphism ϑ then induces an injection

$$f\colon \operatorname{im}(\vartheta_i) \longrightarrow \operatorname{im}\bigl(\mathsf{F}(\vartheta_i)\bigr) \simeq \mathsf{F}\bigl(\operatorname{im}(\vartheta_i)\bigr),$$

which we claim is a root for M. The isomorphism here is by the fact that F is flat.

The commutative diagram

$$\begin{array}{ccccccc}
M_0 & \xrightarrow{\ \vartheta\ } & \mathsf{F}(M_0) & \xrightarrow{\ \mathsf{F}(\vartheta)\ } & \mathsf{F}^2(M_0) & \xrightarrow{\ \mathsf{F}^2(\vartheta)\ } & \cdots \\
\downarrow{\scriptstyle \vartheta_i} & & \downarrow{\scriptstyle \mathsf{F}(\vartheta_i)} & & \downarrow{\scriptstyle \mathsf{F}^2(\vartheta_i)} & & \\
\operatorname{im}(\vartheta_i) & \xrightarrow{\ f\ } & \mathsf{F}(\operatorname{im}(\vartheta_i)) & \xrightarrow{\ \mathsf{F}(f)\ } & \mathsf{F}^2(\operatorname{im}(\vartheta_i)) & \xrightarrow{\ \mathsf{F}^2(f)\ } & \cdots
\end{array}$$

induces a morphism on the corresponding F-finite modules that each row generates. The morphism is surjective since every vertical arrow is surjective, and is injective since the kernel of every vertical arrow eventually goes to zero in the direct limit of the top row. $\square$

Now recall that a *Serre subcategory* $\mathscr{C}$ of an abelian category $\mathscr{A}$ is a strictly full subcategory containing 0 such that $\mathscr{C}$ is closed under the formation of subobjects, quotient objects, and extensions [Stacks, Tag 02MN]. The subcategory $\mathsf{F}\text{-}\mathsf{Mod}_R^{\mathrm{fin}}$ of F-finite F-modules contains 0 and is strictly full, so it suffices to show that $\mathsf{F}\text{-}\mathsf{Mod}_R^{\mathrm{fin}}$ is closed under formation of subobjects, quotient objects, and extensions. Note that Serre subcategories are automatically abelian, and that the inclusion functor is exact [Stacks, Tag 02MP].

Proposition 4.8 [Lyu97, Prop. 2.5(b)]**.** *Let R be a regular ring, and let (M, θ) be an F-module. If M is F-finite with root $M_0 \subseteq M$ and N is an F-submodule of M, then N is F-finite and $N_0 \coloneqq N \cap M_0$ is a root of N. In particular, $\mathsf{F}\text{-}\mathsf{Mod}_R^{\mathrm{fin}}$ is closed under the formation of subobjects.*

Proof. We claim that $N \cap \mathsf{F}^r(M_0) = \mathsf{F}^r(N_0)$. This follows from definition of N_0 for $r = 0$. For $r > 0$, we have

$$\mathsf{F}^r(N_0) = \mathsf{F}\big(\mathsf{F}^{r-1}(N_0)\big) = \mathsf{F}(N \cap \mathsf{F}^{r-1}(M_0)) = \mathsf{F}(N) \cap \mathsf{F}^r(M_0) = N \cap \mathsf{F}^r(M_0)$$

by inductive hypothesis and Lemma 2.5, where the last equality is by the fact that θ maps N isomorphically onto $\mathsf{F}(N)$.

Now the fact that M_0 is a root implies

$$M = \bigcup_r \mathsf{F}^r(M_0),$$

hence

$$N = \bigcup_r \big(N \cap \mathsf{F}^r(M_0)\big) = \bigcup_r \mathsf{F}^r(N_0).$$

Thus, the direct limits of the rows in the commutative diagram

$$\begin{array}{ccccccc} N_0 & \xrightarrow{\vartheta'} & \mathsf{F}(N_0) & \xrightarrow{\mathsf{F}(\vartheta')} & \mathsf{F}^2(N_0) & \xrightarrow{\mathsf{F}^2(\vartheta')} & \cdots \\ \downarrow{\scriptstyle \vartheta'} & & \downarrow{\scriptstyle \mathsf{F}(\vartheta')} & & \downarrow{\scriptstyle \mathsf{F}^2(\vartheta')} & & \\ \mathsf{F}(N_0) & \xrightarrow{\mathsf{F}(\vartheta')} & \mathsf{F}^2(N_0) & \xrightarrow{\mathsf{F}^2(\vartheta')} & \mathsf{F}^3(N_0) & \xrightarrow{\mathsf{F}^3(\vartheta')} & \cdots \end{array}$$

are isomorphic to N, and the structure isomorphism of N is the direct limit of the vertical arrows of this diagram. □

We also take care of another property we need now:

Corollary 4.9 [Lyu97, Cor. 2.6, Prop. 2.7]**.** *Let R be a regular ring, and let M be an F-finite F-module with root $M_0 \subseteq M$.*

(*a*) *There is a one-to-one correspondence*

$$\begin{array}{ccc} \Big\{F\text{-submodules } N \subseteq M\Big\} & \xleftrightarrow{\text{1-1}} & \Big\{R\text{-submodules } N_0 \subseteq M_0 \;\Big|\; N_0 = M_0 \cap \mathsf{F}(N_0)\Big\} \\ N & \longmapsto & M_0 \cap N \\ \bigcup_r \mathsf{F}^r(N_0) & \longleftarrow\!\shortmid & N_0 \end{array}$$

(*b*) *Every object in $\mathsf{F\text{-}Mod}_R^{\mathrm{fin}}$ satisfies ACC in the category of F-modules.*

Proof. For (a), the proof of Proposition 4.8 shows that $N = \bigcup_r \mathsf{F}^r(M_0 \cap N)$, and so it suffices to consider the opposite composition. Let $N_0 \subseteq M_0$ be an R-submodule such that $N_0 = M_0 \cap \mathsf{F}(N_0)$. By Lemma 2.5, we have

$$\mathsf{F}^r(N_0) = \mathsf{F}^r(M_0) \cap \mathsf{F}^{r+1}(N_0)$$

for all $r \geq 0$. Let N be the F-finite F-module with root morphism $\vartheta'\colon N_0 \to \mathsf{F}(N_0)$ induced by the root morphism $\vartheta\colon M_0 \to \mathsf{F}(M_0)$. Let $i\colon N_0 \hookrightarrow M_0$ be the inclusion. Then, the diagram

$$\begin{array}{ccccccc} N_0 & \xrightarrow{\vartheta'} & \mathsf{F}(N_0) & \xrightarrow{\mathsf{F}(\vartheta')} & \mathsf{F}^2(N_0) & \xrightarrow{\mathsf{F}^2(\vartheta')} & \cdots \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \mathsf{F}(i)} & & \downarrow{\scriptstyle \mathsf{F}^2(i)} & & \\ M_0 & \xrightarrow{\vartheta} & \mathsf{F}(M_0) & \xrightarrow{\mathsf{F}(\vartheta)} & \mathsf{F}^2(M_0) & \xrightarrow{\mathsf{F}^2(\vartheta)} & \cdots \end{array}$$

commutes. Since every vertical arrow is injective, it makes N a sub-F-module of M. Since

$$N = \bigcup_r \mathsf{F}^r(N_0) \qquad \text{and} \qquad \mathsf{F}^r(N_0) = \mathsf{F}^r(M_0) \cap \mathsf{F}^{r+1}(N_0),$$

we have that for all $r > 0$,

$$M_0 \cap \mathsf{F}^r(N_0) = M_0 \cap \mathsf{F}^{r-1}(M_0) \cap \mathsf{F}^r(N_0) = M_0 \cap \mathsf{F}^{r-1}(N_0) = N_0,$$

hence $N \cap M_0 = N_0$ as desired.

For (b), an ascending chain

$$N_1 \subseteq N_2 \subseteq \cdots$$

of F-submodules of M induces an ascending chain

$$N_1 \cap M_0 \subseteq N_2 \cap M_0 \subseteq \cdots$$

of R-submodules of M_0, which must eventually stabilize since M_0 is finitely generated over a noetherian ring. By (a), the chain $N_1 \subseteq N_2 \subseteq \cdots$ must then also stabilize. □

We now show the other two properties that a Serre subcategory must satisfy.

Theorem 4.10 [Lyu97, Thm. 2.8]**.** *Let R be a regular ring. The subcategory $\mathsf{F}\text{-}\mathsf{Mod}_R^{\mathrm{fin}} \subseteq \mathsf{F}\text{-}\mathsf{Mod}_R$ of the category of F-modules consisting F-finite F-modules is a Serre subcategory. In particular, $\mathsf{F}\text{-}\mathsf{Mod}_R^{\mathrm{fin}}$ is an abelian subcategory of $\mathsf{F}\text{-}\mathsf{Mod}_R$.*

Proof. By Proposition 4.8, it remains to show that $\mathsf{F}\text{-}\mathsf{Mod}_R^{\mathrm{fin}}$ is closed under formation of quotients and extensions. Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence in $\mathsf{F}\text{-}\mathsf{Mod}_R$.

We first show that $\mathsf{F}\text{-}\mathsf{Mod}_R^{\mathrm{fin}}$ is closed under formation of quotients, i.e., that if M is F-finite, then M'' is F-finite. Let (M_0, ϑ) be a root for M, and set $M_0' = M_0 \cap M'$ and $M_0'' = M_0/M_0' \subseteq M''$. We then have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_0' & \longrightarrow & M_0 & \longrightarrow & M_0'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \vartheta'} & & \downarrow{\scriptstyle \vartheta} & & \downarrow{\scriptstyle \vartheta''} & & \\
0 & \longrightarrow & \mathsf{F}(M_0') & \longrightarrow & \mathsf{F}(M_0) & \longrightarrow & \mathsf{F}(M_0'') & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathsf{F}(\vartheta')} & & \downarrow{\scriptstyle \mathsf{F}(\vartheta)} & & \downarrow{\scriptstyle \mathsf{F}(\vartheta)''} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where the morphism ϑ'' is induced by the universal property of cokernels. Since the direct limits along the left and middle columns give the F-modules M' and M, respectively, we see that (M_0'', ϑ'') is a generating morphism for M''. Since M_0' and M_0 are finitely generated, the module M_0'' is also finitely generated, hence M'' is F-finite.

We now show that $\mathsf{F}\text{-}\mathsf{Mod}_R^{\mathrm{fin}}$ is closed under extensions, i.e., that if M' and M'' are F-finite, then M is F-finite. Let (M_0', ϑ') be a root of M' and (M_0'', ϑ'') a root of M''. Let $x_1, x_2, \ldots, x_s$ be elements of M whose images in M'' generate M_0'' as an R-module. Let N be the R-submodule of M generated by M' and $x_1, x_2, \ldots, x_s$. We have the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \longrightarrow & N & \longrightarrow & M_0'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \vartheta''} & & \\
0 & \longrightarrow & \mathsf{F}(M') & \longrightarrow & \mathsf{F}(N) & \longrightarrow & \mathsf{F}(M_0'') & \longrightarrow & 0
\end{array}$$

and we claim that $\theta\colon M \to \mathsf{F}(M)$ restricts to $N \to \mathsf{F}(N)$ in the way shown above. We know that the submodule $M' \subseteq N$ has image in $\mathsf{F}(N)$, hence it suffices to show that each x_i has image in $\mathsf{F}(N)$. But the images of the x_i in M_0'' map into $\mathsf{F}(M_0'')$ under ϑ'', hence $\theta(x_i) \in \mathsf{F}(N)$ as well.

Now let N_i be the R-submodule of M generated by $\mathsf{F}^i(M_0')$ and $x_1, x_2, \ldots, x_s$. Then,

$$N = \bigcup_i N_i.$$

Combining these two facts, we see that $x_1, \ldots, x_s \in \theta^{-1}(\mathsf{F}(N_r))$ for large enough r, and so

$$\theta(N_r) \subseteq \mathsf{F}(N_r)$$

for some r. We therefore have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_0' & \longrightarrow & N_r & \longrightarrow & M_0'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \vartheta'} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \vartheta''} & & \\ 0 & \longrightarrow & \mathsf{F}(M_0') & \longrightarrow & \mathsf{F}(N_r) & \longrightarrow & \mathsf{F}(M_0'') & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathsf{F}(\vartheta')} & & \downarrow{\scriptstyle \mathsf{F}(\theta)} & & \downarrow{\scriptstyle \mathsf{F}(\vartheta'')} & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

Since the left and right columns have direct limits isomorphic to M' and M'', respectively, we see that the middle column has direct limit isomorphic to M. Thus, M is F-finite. □

This completes the proof of Theorem 1.2, since $T(R)$ carries the structure of an F-finite F-module by Example 4.6, and then mimicking the proof of Theorem 1.3.

Proof of Theorem 1.2. We show more generally that if M is an F-finite F-module, then it has finitely many associated primes. We construct a chain

$$0 \subsetneq M_1 \subseteq M_2 \subseteq \cdots \subseteq M$$

such that $\#\operatorname{Ass}_R(M_i/M_{i-1}) = 1$ as follows. By Lemma 1.5, we can let $M_1 = \Gamma_{\mathfrak{p}_1}(M)$ for $\mathfrak{p}_1 \in \operatorname{Ass}_R(M)$ maximal; note it is an F-finite F-submodule of M by Example 4.6, since it is the kernel of the F-module morphism

$$M \longrightarrow \bigoplus_i M_{f_i}$$

where f_i is a choice of generators for $\mathfrak{p}_1$. For $i > 0$, we let $\mathfrak{p}_i \in \operatorname{Ass}_R(M/M_{i-1})$ be maximal, and set M_i to be the preimage of $\Gamma_{\mathfrak{p}_i}(M/M_{i-1}) \subseteq M/M_{i-1}$ in M. We then have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{i-1} & \longrightarrow & M_i & \longrightarrow & \Gamma_{\mathfrak{p}_i}(M/M_{i-1}) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_{i-1} & \longrightarrow & M & \longrightarrow & M/M_{i-1} & \longrightarrow & 0 \end{array}$$

hence $\operatorname{Ass}_R(M_i/M_{i-1}) = \{\mathfrak{p}_i\}$, and moreover, M_i is an F-finite F-module since it is the extension of F-finite F-modules. Since M is F-finite, the ascending chain must stabilize at some point r. By looking at the short exact sequences

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow M_i/M_{i-1} \longrightarrow 0$$

for each i, we therefore have that

$$\begin{aligned} \operatorname{Ass}_R(M) = \operatorname{Ass}_R(M_r) &\subseteq \operatorname{Ass}_R(M_{r-1}) \cup \operatorname{Ass}_R(M_r/M_{r-1}) \subseteq \cdots \\ &\subseteq \bigcup_{i=1}^{r} \operatorname{Ass}_R(M_i/M_{i-1}) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r\}, \end{aligned}$$

which is finite. □

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