FINITE GENERATION OF K-GROUPS OF RINGS OF INTEGERS IN NUMBER FIELDS

TAKUMI MURAYAMA

ABSTRACT. I present Quillen's proof of the fact that the K-groups K_iA where A is the ring of integers in a number field F are finitely generated. This talk was given in the Algebraic K-theory seminar at Michigan during the Winter semester of 2015.

CONTENTS

INTRODUCTION

We follow [Qui73a]. The main result we will show in this talk is the following

Theorem 1. *If A is the ring of algebraic integers in a number field F (finite over $\mathbf{Q}$) then K_iA is a finitely generated group for all $i \geq 0$.*

To show this, we use the definition of K_iA as

$$K_iA \coloneqq \pi_{i+1}(N(Q\mathscr{P}), 0)$$

where $Q-$ denotes Quillen's q-construction [Qui73b], $N-$ denotes the nerve of a category, and $\mathscr{P}$ is the category of finitely-generated projective A-modules.

We pause to note the arithmetic significance of this result. Lichtenbaum in [Lic73] conjectured that higher K-groups K_iA can give information about special values of the Dedekind zeta functions $\zeta_F(s)$. In particular, the ranks of K_iA computed by Borel in [Bor74] give multiplicities of trivial zeros of $\zeta_F(s)$. We unfortunately don't have much time to go deeply into this, but see [Kah05; Wei05] for surveys on the subject.

1. BUILDINGS

To prove Theorem 1, we introduce the notion of a building.

Definition. The *building* $\boxed{V}$ of an n-dimensional vector space V over a field F is the nerve associated to the poset of nontrivial proper subspaces of V, i.e., p-simplices are chains $0 \subsetneq W_0 \subsetneq \cdots \subsetneq W_p \subsetneq V$ of subspaces W_i of V. If $n \leq 1$, then $\boxed{V} = \emptyset$; if $n = 2$, then $\boxed{V}$ is the projective space $\mathbf{P}(V)$ of lines in V as a discrete space.

A fundamental theorem about buildings is the following

Theorem 2 (Solomon-Tits [Sol69])**.** *If $n \geq 2$, then $\boxed{V}$ has the homotopy type of a wedge of $(n-2)$-spheres.*

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To prove this, we first state a corollary of the following theorem by Quillen:

Theorem A*. *Let $f\colon \mathscr{C} \to \mathscr{C}'$ be a functor, and let $f/Y := \{(X,u) \mid u\colon f(X) \to Y\}$. Then, if $N(f/Y)$ is contractible for every object $Y \in \mathscr{C}'$, then f induces a homotopy equivalence of nerves.*

Lemma 3. Let $g\colon K \to K'$ is a simplicial map of simplicial complexes. If for every closed subsimplex $\sigma \subset K'$, its inverse image $g^{-1}(\sigma)$ is contractible, then g is a homotopy equivalence.

Proof. Let $\operatorname{Simpl} K$ and $\operatorname{Simpl} K'$ denote the posets of subsimplices of K, K' ordered by inclusion, and let h be the map on posets such that g is homeomorphic to Nh. Then, h/σ is the poset of subsimplices of $g^{-1}(\sigma)$, and the claim follows by Theorem A*. ☐

Now to prove the Solomon-Tits theorem, we introduce a new simplicial complex. We call $\underline{\underline{\boxed{V}}}$ the simplicial complex with p-simplices being chains $0 \subseteq W_0 \subsetneq \cdots \subsetneq W_p \subsetneq V$, i.e., it is defined in the same way as $\boxed{V}$ but W_0 can be 0. Note $\operatorname{Cone}\boxed{V} \simeq \underline{\underline{\boxed{V}}}$.

Proof of Theorem 2 (by induction). If $n = 2$, then the claim is trivial since $\boxed{V}$ is discrete hence is trivially a wedge of 0-spheres.

Now suppose $n \geq 3$. Fix a line $L \subseteq V$, and let $\mathscr{H}$ be the set of hyperplanes H such that $V = H \oplus L$. Now let Y be the full subcomplex of $\boxed{V}$ obtained by removing the vertices $\mathscr{H}$.

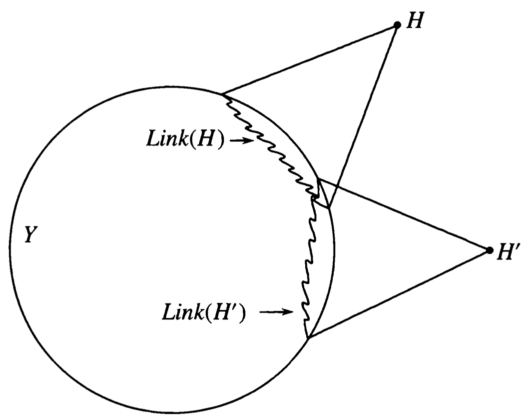
Claim. Y is contractible.

Proof of Claim. If $V \to V/L$ is the projection, we get an induced simplicial map $q\colon Y \to \underline{\underline{\boxed{V/L}}}$; the latter is contractible since it has a minimal element 0. Now by the Lemma 3, it suffices to show for every subsimplex $\sigma = (W_0/L \subsetneq \cdots \subsetneq W_p/L)$ of $\underline{\underline{\boxed{V/L}}}$, its inverse image $q^{-1}(\sigma)$ in Y is contractible. Now if $U \in q^{-1}(\sigma)$, then $q(U) = W_i/L$ for some i, so $U + L = W_i$. We can then visualize the simplices in Y and $\underline{\underline{\boxed{V/L}}}$ as follows:

$$
\begin{array}{ccccccccc}
Y & & U_0 + L & \subsetneq & U_1 + L & \subsetneq & \cdots & \subsetneq & U_p + L \\
\Big\downarrow{\scriptstyle q} & & \cup\!| & & \cup\!| & & & & \cup\!| \\
 & & U_0 & \subsetneq & U_1 & \subsetneq & \cdots & \subsetneq & U_p \\
\underline{\underline{\boxed{V/L}}} & & \sigma = W_0/L & \subsetneq & W_1/L & \subsetneq & \cdots & \subsetneq & W_p/L
\end{array}
$$

"Pushing up" then defines a deformation retraction $q^{-1}(\sigma) \simeq \Delta^p$. ☐

We then have the following schematic picture of $\boxed{V}$ from [Qui10, p. 483]:



where for each $H \in \mathscr{H}$, $Link(H)$ is the subcomplex of $\boxed{V}$ formed by simplices σ such that $H \notin \sigma$ but $\sigma \cup \{H\}$ is a simplex. Note $Link(H) \subset Y$, and that $\boxed{V}$ is the union of Y with the cones over these links, glued along the $Link(H)$ as H varies in $\mathscr{H}$. Thus,

$$\boxed{V} \simeq \boxed{V}/Y \simeq \bigvee_{H \in \mathscr{H}} \sum Link(H).$$

Now $Link(H) = \boxed{H}$ for any $H \in \mathscr{H}$, so the theorem follows by induction since $\dim H = n-1$. □

Now we define another poset $J(V)$ which will be useful later because it is simpler to analyze.

Let $J(V)$ be the set of subspaces $W_0 \subseteq W_1$ of V such that $\dim(W_1/W_0) < n$, ordered by $(W_0, W_1) \leq (W_0', W_1')$ if $W_0' \leq W_0$ and $W_1 \leq W_1'$. For $n = 1$, $J(V)$ consists of $(0,0)$ and (V,V), which are incomparable, hence $NJ(V) = S^0$.

Proposition. If $n \geq 2$, there is a $\mathrm{GL}(V)$-equivariant homotopy equivalence

$$\overline{\underline{\boxed{V}}} \longrightarrow N(J(V))$$

where $\overline{\underline{\boxed{V}}}$ is the simplicial complex with p-simplices being chains $0 \subseteq W_0 \subsetneq \cdots \subsetneq W_p \subseteq V$ such that $\dim(W_p/W_0) < n$, which is a subsimplicial complex of the complex $\textcircled{V}$ formed without this restriction.

Proof. Define a map $g\colon \mathrm{Simpl}\overline{\underline{\boxed{V}}} \to J(V)$ by

$$g(W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_p) = (W_0, W_p).$$

g is a $\mathrm{GL}(V)$-equivariant functor. Now $N\left(\mathrm{Simpl}\overline{\underline{\boxed{V}}}\right)$ is a barycentric subdivision of $\overline{\underline{\boxed{V}}}$, so it suffices to show Ng is a homotopy equivalence. By Theorem A*, it suffices to show for each $(U_0, U_1) \in J(V)$, the category $g/(U_0, U_1)$ is contractible. But the objects of $g/(U_0, U_1)$ are simplices $W_0 \subsetneq W_1 \subsetneq \cdots W_p$ such that $U_0 \subsetneq W_0$ and $W_p \subsetneq U_1$ ordered by inclusion. Now this is isomorphic to $\mathrm{Simpl}\,\textcircled{V}$, which is contractible since it has an initial object. □

Noting that $\overline{\underline{\boxed{V}}} \simeq \sum \boxed{V}$, we have the following

Corollary 4. Suppose $n \geq 1$. The reduced homology $\tilde{H}_i(NJ(V))$ vanishes for $i \neq n-1$, and is a free $\mathbf{Z}$-module for $i = n-1$.

Definition. The $\mathbf{Z}$-module $\tilde{H}_{n-1}(NJ(V)) \cong \tilde{H}_{n-2}(\boxed{V})$ with the natural $\mathrm{GL}(V)$ action is called the *Steinberg module* of V, and is denoted $\mathrm{st}(V)$.

2. A long exact sequence

The main homological input in this theorem is a long exact sequence, which we will prove in this section.

Let A be a Dedekind ring with field of fractions F. For each $n \geq 0$, let Q_n be the full subcategory of $Q\mathscr{P}(A)$ formed by projective modules of rank $\leq n$. Then, Q_0 is the trivial category with one object and no morphisms, $Q_n \subset Q_{n+1}$, and $Q = \bigcup_n Q_n$.

Theorem 5. *Let $n \geq 1$. The inclusion $w\colon Q_{n-1} \to Q_n$ induces a long exact sequence*

$$\cdots \longrightarrow H_i(NQ_{n-1}) \longrightarrow H_i(NQ_n) \longrightarrow \coprod_\alpha H_{i-n}(\mathrm{GL}(P_\alpha), \mathrm{st}(V_\alpha)) \longrightarrow H_{i-1}(NQ_{n-1}) \longrightarrow \cdots$$

where P_α represent isomorphism classes of projective modules of rank n, and $V_\alpha = P_\alpha \otimes_A F$.

To prove this theorem, we first recall the following fact about computing homology of nerves:

Proposition 6 ([GZ67, App. II, 3.3])**.** If $F\colon \mathscr{C} \to \mathsf{Ab}$ is a functor, then $H_n(N\mathscr{C}, F)$ can be computed by $\varinjlim_n^{\mathscr{C}} F$, the nth derived functor of $\varinjlim^{\mathscr{C}}\colon \mathsf{Ab}^{\mathscr{C}} \to \mathsf{Ab}$.

Example. If L is the constant functor $\mathbf{Z}$, then $\varinjlim_n^{\mathscr{C}} \mathbf{Z}$ gives integral homology $H_n(N\mathscr{C})$.

Proof of Theorem 5. The main ingredient in this proof is a Grothendieck spectral sequence obtained from the following commutative diagram of functors:

$$\begin{array}{ccc} \mathsf{Ab}^{Q_{n-1}} & \xrightarrow{\;w^*\;} & \mathsf{Ab}^{Q_n} \\ & {\scriptstyle \varinjlim^{Q_{n-1}}} \searrow \quad \swarrow {\scriptstyle \varinjlim^{Q_n}} & \\ & \mathsf{Ab} & \end{array}$$

where w^* is defined (on objects) by

$$(w^*f)(P) = \varinjlim_{(P',u)\in w/P} f(P') = \varinjlim_{w(P')\to P} f(P')$$

for any functor $f\colon Q_{n-1} \to \mathsf{Ab}$. Note this "looks like" the pullback functor for sheaves, and indeed the functor w_* defined by precomposition is a right adjoint to this functor that preserves epimorphisms since it does so componentwise. Thus, w^* preserves projectives and so we have the following Grothendieck spectral sequence:

$$E^2_{pq} = \varinjlim_p^{Q_n}((L_q w^*)(f)) \Rightarrow \varinjlim_{p+q}^{Q_{n-1}}(f)$$

for any functor $f\colon Q_{n-1} \to \mathsf{Ab}$. We can simplify this further by looking at the following commutative diagram:

$$\begin{array}{ccc} \mathsf{Ab}^{Q_{n-1}} & \xrightarrow{\;w^*\;} & \mathsf{Ab}^{Q_n} \\ {\scriptstyle i_{P*}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{eval}_P} \\ \mathsf{Ab}^{(w/P)} & \xrightarrow[\varinjlim^{(w/P)}]{} & \mathsf{Ab} \end{array}$$

for each $P \in Q_n$, where i_P is the projection $w/P \to Q_{n-1}$ defined by $(P', u) \mapsto P'$. The two vertical functors are exact, hence we have that

$$(L_q w^*)(f) \cong \Big(P \mapsto \varinjlim_q^{w/P} f \circ i_P\Big)$$

and the spectral sequence becomes

$$E^2_{pq} = \varinjlim_p^{Q_n}\Big(P \mapsto \varinjlim_q^{(w/P)} f \circ i_P\Big) \Rightarrow \varinjlim_{p+q}^{Q_{n-1}}(f).$$

In particular, for $f = \mathbf{Z}$, by Proposition 6 this becomes the spectral sequence

$$E^2_{pq} = \varinjlim_p^{Q_n}(P \mapsto H_q(N(w/P))) \Rightarrow H_{p+q}(NQ_{n-1}). \tag{1}$$

Now to use this spectral sequence, we would like to know what $H_q(N(w/P))$ looks like. Recall $w/P = \{(P', u) \mid u\colon P' \to P\}$. We can write down the following bijection, noting that the morphisms $u\colon P' \to P$ in Q_n are of the form on the left by the q-construction:

$$\{P' \twoheadleftarrow P_1 \rightarrowtail P\} \leftrightarrow \left\{\begin{array}{c}\text{pairs } (P_0, P_1) \text{ of submodules } P_0 \subseteq P_1 \text{ of } P \\ \text{such that } P' \xrightarrow{\sim} P_1/P_0 \text{ is an isomorphism}\end{array}\right\}$$

so w/P is equivalent to the poset J of pairs of submodules $P_0 \subseteq P_1$ of P such that $\mathrm{rk}(P_1, P_0) < n$, with the ordering $(P_0, P_1) \le (P_0', P_1')$ if $P_0' \subseteq P_0$ and $P_1 \subseteq P_1'$.

Now if $\operatorname{rk} P < n$, then J has a maximal element $(0, P)$, so $N(w/P)$ is contractible. If instead $\operatorname{rk} P = n$, then the map $P' \mapsto P' \otimes_A F \subset V = P \otimes_A F$ induces an equivalence $J \simeq J(V)$, hence Corollary 4 applies. Thus, if $n = 1$, then

$$\begin{aligned} H_q(N(w/P)) &= 0 \qquad\qquad \text{if } q > 0 \\ H_0(N(w/P)) &= \begin{cases} \mathbf{Z} & \text{if } P = 0 \\ \mathbf{Z} \oplus \mathbf{Z} & \text{if } \operatorname{rk} P = 1 \end{cases} \end{aligned}$$

and if $n \geq 2$,

$$\begin{aligned} H_0(N(w/P)) &= \mathbf{Z} \\ H_q(N(w/P)) &= 0 \qquad\qquad \text{if } q \neq 0, n-1 \\ H_{n-1}(N(w/P)) &= \begin{cases} 0 & \text{if } \operatorname{rk} P < n \\ \operatorname{st}(V) & \text{if } \operatorname{rk} P = n \end{cases} \end{aligned} \tag{2}$$

We first analyze the case when $n \geq 2$. The E^2_{pq} terms in the spectral sequence (1) are given by

$$E^2_{pq} = \varinjlim\nolimits_p^{Q_n}(P \mapsto H_q(N(w/P))) = \begin{cases} H_p(NQ_n) & \text{if } q = 0 \\ 0 & \text{if } q \neq 0, n-1 \end{cases}$$

It remains to analyze what happens when $q = n - 1$. To do so, consider the full subcategory Q' of Q_n consisting of rank n projectives. Since functor $P \mapsto H_q(N(w/P))$ is the zero functor for objects not in Q', we have an isomorphism of functors

$$\varinjlim\nolimits_p^{Q_n}(P \mapsto H_q(N(w/P))) \cong \varinjlim\nolimits_p^{Q'}(P \mapsto H_q(N(w/P)))$$

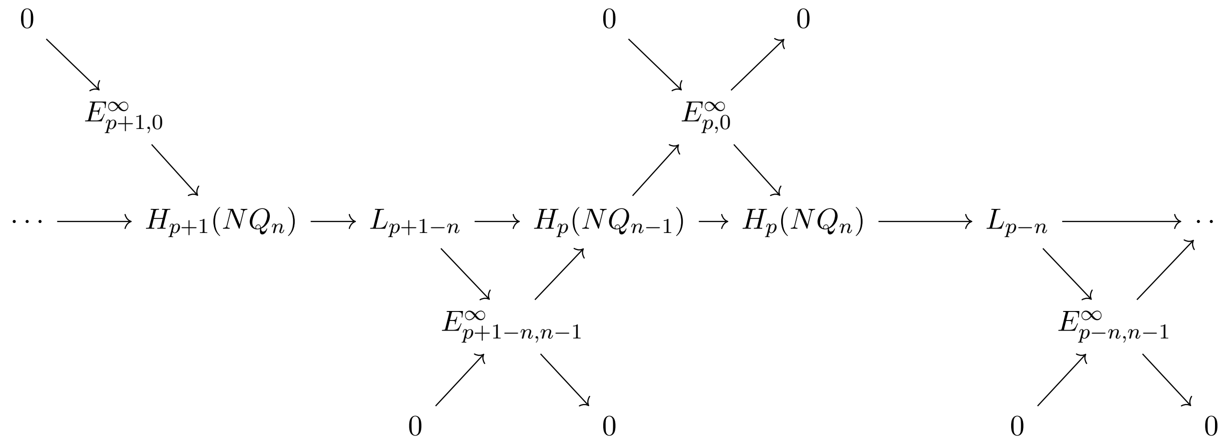
where in the latter, the functor is restricted to $P \in Q'$ (to be precise, it is necessary to understand the details of the calculation of left-derived functors $\varinjlim_p$ from [GZ67, App. II, 3.2]). Q' is equivalent to the groupoid of rank n projectives and their isomorphisms, which is in turn equivalent to the full skeletal subcategory with one object P_α from each isomorphism class, and this is the category for the groupoid $Q'' = \coprod_\alpha \operatorname{GL}(P_\alpha)$. On this category, the functor $P_\alpha \mapsto H_{n-1}(w/P)$ maps P_α to the $\operatorname{GL}(P_\alpha)$-modules $\operatorname{st}(V_\alpha)$ where $V_\alpha = P_\alpha \otimes_A F$ by (2). Thus,

$$E^2_{p,n-1} = \varinjlim\nolimits_p^{Q_n}(P \mapsto H_{n-1}(N(w/P))) = \coprod_\alpha H_p(\operatorname{GL}(P_\alpha), \operatorname{st}(V_\alpha)) =: L_p$$

and the E^2 page of our spectral sequence looks like

$$\begin{array}{c|ccccc} E^2 \quad q & & & & & \\ n-1 & L_0 & L_1 & \cdots & L_p & \cdots \\ \vdots & & 0 & & & \\ 0 & H_0(NQ_n) & H_1(NQ_n) & \cdots & H_p(NQ_n) & \cdots \\ \hline & 0 & 1 & \cdots & p & \to p \end{array}$$

which is the same until the E^n page, where it converges to E^∞. The desired long exact sequence is obtained by splicing together the maps from the E^∞ page:



Finally, for $n = 1$, the spectral sequence (1) degenerates to an isomorphism

$$\varinjlim\nolimits_p^{Q_1}(P \mapsto H_0(w/P)) \cong H_p(Q_0) = \begin{cases} \mathbf{Z} & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}$$

We also have the short exact sequence

$$0 \longrightarrow (P \mapsto \tilde{H}_0(w/P)) \longrightarrow (P \mapsto H_0(w/P)) \longrightarrow \mathbf{Z} \longrightarrow 0$$

in Ab^{Q_1} and just like the case $n \geq 2$, we have the isomorphism

$$\varinjlim\nolimits_p^{Q_1}(P \mapsto H_0(N(w/P))) = \coprod_\alpha H_p(\mathrm{GL}(P_\alpha), \mathrm{st}(V_\alpha)) =: L_p$$

The desired long exact sequence is exactly that obtained by the long exact sequence on homology for $\varinjlim_p^{Q_1}$:

$$\cdots \longrightarrow H_p(NQ_0) \longrightarrow H_p(NQ_1) \longrightarrow L_{p-1} \longrightarrow H_{p-1}(NQ_0) \longrightarrow \cdots \qquad \square$$

3. Proof of Theorem 1

3.1. **Reduction to a group homology calculation.** We are now ready to prove the main theorem. Grayson in [Gra82] noted that Quillen's argument boils down to the following statement:

Proposition 7. Let A be a Dedekind domain with fraction field F. Then, K_iA is finitely generated for all $i \geq 0$ if $\operatorname{Pic} A$ is finite and $H_i(\mathrm{GL}(P), \mathrm{st}(V))$ is finitely generated for all $P \in \mathscr{P}$ and $V = P \otimes_A F$.

Proof. By Theorem 5, there is a long exact sequence

$$\cdots \longrightarrow H_i(NQ_{n-1}) \longrightarrow H_i(NQ_n) \longrightarrow \coprod_\alpha H_{i-n}(\mathrm{GL}(P_\alpha), \mathrm{st}(V_\alpha)) \longrightarrow H_{i-1}(NQ_{n-1}) \longrightarrow \cdots$$

Note for each n there are only finitely many isomorphism classes of projective A-modules of rank n by the finiteness of $\operatorname{Pic} A$ and the classification of finitely generated modules over a Dedekind domain [DF04, Ch. 6, Thm. 22], so the groups $\coprod_\alpha H_{i-n}(\mathrm{GL}(P_\alpha), \mathrm{st}(V_\alpha))$ are finitely generated.

We first claim $H_i(NQ_n)$ is finitely generated for all i, n. First, $H_0(NQ_0) = \mathbf{Z}$ and $H_i(NQ_0) = 0$ for all $i > 0$ by the fact that Q_0 is the trivial category. Using the long exact sequence above, the claim follows by induction.

Now $H_i(NQ\mathscr{P}(A))$ is finitely generated for each i, since fixing i and letting $n \gg i+2$, the long exact sequence above gives

$$\coprod_\alpha \underset{\substack{\| \\ 0}}{H_{i-n+1}(\mathrm{GL}(P_\alpha), \mathrm{st}(V_\alpha))} \longrightarrow H_i(NQ_{n-1}) \longrightarrow H_i(NQ_n) \longrightarrow \coprod_\alpha \underset{\substack{\| \\ 0}}{H_{i-n}(\mathrm{GL}(P_\alpha), \mathrm{st}(V_\alpha))}$$

hence the homology groups stabilize.

Finally, $NQ\mathscr{P}(A)$ is an H-space with addition induced by $\oplus\colon \mathscr{P}(A) \times \mathscr{P}(A) \to \mathscr{P}(A)$, hence $NQ\mathscr{P}(A)$ is simple [Hat02, Ex. 4A.3]. Thus, by (a modification of) the generalized Hurewicz theorem, the finite generation of homology groups implies finite generation of homotopy groups [Hat04, Thm. 1.7], so $K_i A = \pi_{i+1}(NQ\mathscr{P}(A))$ is finitely generated for all $i \geq 0$. □

We note that Grayson in [Gra82] went on to show that the hypotheses for Proposition 7 hold for coordinate rings of smooth affine curves over finite fields.

Now let A be the ring of integers in a number field F. Since $\operatorname{Pic} A$ is finitely generated in this setting [Sam70, §4.3, Thm. 2], it suffices to show that

Theorem 8. *If $P \in \mathscr{P}(A)$ and $V = P \otimes_A F$, then $H_i(\mathrm{GL}(P), \mathrm{st}(V))$ is finitely generated for all i.*

Note that $\mathrm{GL}(P)$ is an arithmetic subgroup of $\mathrm{GL}(V \otimes_{\mathbf{Q}} \mathbf{R})$, so the main technical tool we use here is the *Borel-Serre compactification*, which is used to compute the (co)homology of arithmetic groups.

3.2. **The Borel-Serre compactification.** We give a short description of the compactification and its applications to computing (co)homology of arithmetic groups in our particular case; see [Bor75; Bor06] for surveys on the cohomology of arithmetic groups, and [Gor05; Sap03] for surveys on the Borel-Serre (and other compactifications) in the reductive case. Note that most treatments of the Borel-Serre compactification only deal with the semi-simple case.

Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over $\mathbf{Q}$; we write $G = \mathbf{G}(\mathbf{R})$. Fix an arithmetic subgroup $\Gamma \subset \mathbf{G}(F)$ where F is a number field, that is, a subgroup a subgroup that is commensurable with $\mathbf{G}(A)$ where A is the ring of integers in F. We make the assumption that Γ is torsion-free. One motivation for the Borel-Serre compactification is the following

Question. What is a good finite classifying space $B\Gamma$ for an arithmetic group Γ, from which we can deduce facts about the group cohomology of Γ?

We outline the construction of such a classifying space in the sequel.

Let $K \subset G$ be a fixed maximal compact subgroup, and let A_G be the (topologically) connected identity component of the group of real points of the greatest $\mathbf{Q}$-split torus $\mathbf{A_G}$ in the center of $\mathbf{G}$. Define $X \coloneqq G/KA_G$. Then, Borel and Serre proved the following

Theorem 9 ([BS73])**.** *There exists an enlargement $\overline{X}$ of X satisfying the following properties:*

1. *$\overline{X}$ is contractible* [BS73, Lem. 8.6.4];
2. *$\partial\overline{X}$ has the homotopy type of the Tits building T associated to $\mathbf{G}(\mathbf{Q})$, and has dimension ℓ, the $\mathbf{Q}$-rank of G/RG, where RG denotes the radical of G* [BS73, Thm. 8.4.1];
3. *Γ acts freely on $\overline{X}$, hence $\overline{X}/\Gamma$ is a space $B\Gamma$ that is compact* [BS73, Thm. 9.3, n° 9.5].

We give a brief outline of the construction.

Construction. First, for any $\mathbf{G}$ as above, define ${}^0\mathbf{G}$ as follows:

$${}^0\mathbf{G} \coloneqq \bigcap_\chi \ker(\chi^2),$$

where the intersection is taken over all rationally defined characters $\chi\colon \mathbf{G} \to \mathbf{G}_m$ [BS73, n° 1.1]. Then, the group G decomposes as $G = {}^0G A_G$, where ${}^0G = {}^0\mathbf{G}(\mathbf{R})$ [BS73, Prop. 9.2].

Let $\mathbf{P}$ be a rational parabolic subgroup of $\mathbf{G}$, and let $P = \mathbf{P}(\mathbf{R})$ be its group of real points. Then, P has a *Langlands decomposition*

$$P = U_P A_P {}^0 L_P,$$

where U_P is the unipotent radical of P, L_P is the Levi quotient of P, and the decomposition follows as above. Now define $e(P) = X/A_P$. $\overline{X}$ is obtained set-theoretically by glueing each space $e(P)$ to X, and the topology is defined in [BS73, §7]. The closures of the $e(P)$ form a locally finite cover of the boundary $\partial\overline{X}$, whose nerve is the Tits building [BS73, Thm. 8.4.1]. The nerve of a locally finite cover of a space is homotopy equivalent to the space itself by [BS73, Thm. 8.2.2], so we have (b). □

3.3. **Proof of Theorem 8.** We are now ready to prove Theorem 8. Recall that A is the ring of integers in a number field F, and P is a finitely-generated projective module over A.

Let $\mathbf{G}$ be the general linear group, such that $\mathbf{G}(F) = \mathrm{GL}(V)$, where $V = P \otimes_A F$. By [BS73, Thm. 8.2.2], $\partial\overline{X}$ has the homotopy type of the Tits building T associated to $\mathrm{GL}(V)$. In our case, there is a natural isomorphism $\boxed{V} \to T$ where the simplex $W_0 \subsetneq \cdots \subsetneq W_p$ of $\boxed{V}$ corresponds to the reverse chain of stabilizers of each space in $\mathrm{GL}(V)$. Its homology is then described by Theorem 2.

Now we can compute the cohomology (with compact support) of $\overline{X}$:

Proposition 10 ([BS73, Thm. 8.6.5])**.** The groups $H^i_c(\overline{X})$ are 0 for $i \neq d - \ell$, where $d = \dim X$. The group $H^{d-\ell}_c(\overline{X})$ is free abelian, and is isomorphic to $\mathrm{st}(V)$.

Proof. If $\ell = 0$, $\overline{X} = X$, hence is a orientable manifold, and Poincaré duality gives that

$$H^i_c(\overline{X}) \cong H_{d-i}(\overline{X})$$

and the result follows by contractibility of $\overline{X}$.

For $\ell \geq 1$, since $\overline{X}$ is contractible, the long exact sequence of the pair gives isomorphisms $\widetilde{H}_j(\partial\overline{X}) \cong H_{j+1}(\overline{X}, \partial\overline{X})$. Then, Poincaré duality for manifolds with boundary [Hat02, Thm. 3.35] gives isomorphisms

$$H^i_c(\overline{X}) \cong H_{d-i}(\overline{X}, \partial\overline{X}) \cong \widetilde{H}_{d-i-1}(\partial\overline{X})$$

and the result follows by Theorem 2 and Theorem 9. □

Now consider the arithmetic subgroup $\mathrm{GL}(P)$ of $\mathrm{GL}(V) = \mathbf{G}(F)$. It has a finite index torsion-free subgroup Γ by Minkowski's theorem (see [Sou07, Thm. 8] for a proof). Then, $\overline{X}/\Gamma$ is a classifying space $B\Gamma$, hence we can calculate the cohomology of Γ by calculating the cohomology of this space. We then have the following

Theorem 11 (Duality, [BS73, Thm. 11.4.2])**.** *There is an isomorphism*

$$H^{d-\ell-i}(\Gamma, \mathbf{Z}) \cong H_i(\Gamma, \mathrm{st}(V))$$

for each i.

Proof. General facts about classifying spaces in, say, [BE73, n° 6.3] imply the isomorphisms

$$H^i(\Gamma, \mathbf{Z}[\Gamma]) \cong H^i(\overline{X}/\Gamma, \mathbf{Z}[\Gamma]) \cong H^i_c(\overline{X}, \mathbf{Z})$$

hence $H^i(\Gamma, \mathbf{Z}[\Gamma]) = 0$ for all $i \neq d - \ell$, and $H^{d-\ell}(\Gamma, \mathbf{Z}[\Gamma]) \cong \mathrm{st}(V)$ is free abelian by Proposition 10. Γ is thus a duality group in the sense of [BE73, Thm. 4.5], and the isomorphism follows. □

By [BS73, n° 11.1], $H^{d-\ell-i}(\Gamma, \mathbf{Z})$ is finitely generated for each i, hence by the Duality theorem above, $H_i(\Gamma, \mathrm{st}(V))$ is as well. Finally, the homology spectral sequence

$$H_p(\mathrm{GL}(P)/\Gamma, H_q(\Gamma, \mathrm{st}(V)) \Rightarrow H_{p+q}(\mathrm{GL}(V), \mathrm{st}(V))$$

implies that since $\mathrm{GL}(V)/\Gamma$ is finite, the groups $H_i(\Gamma, \mathrm{st}(V))$ are finitely generated. □

We finally note that while it is beyond the scope of this talk, we know the ranks of the K-groups $K_i A$:

Theorem 12 ([Bor74]). *The ranks are given by the formula*

$$\operatorname{rk} K_i A = \begin{cases} r_1 + r_2 & \text{if } n \equiv 1 \pmod 4 \\ r_2 & \text{if } n \equiv 3 \pmod 4 \\ 0 & \text{otherwise} \end{cases}$$

where r_1, r_2 are the numbers of real and complex places in A, respectively.

On the other hand, the torsion parts of $K_i A$ were not known until recently; see [Wei05].

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