ASCENT OF FINITENESS OF FLAT DIMENSION

TAKUMI MURAYAMA

Abstract. The main focus of this talk is to prove ascent of finiteness of flat dimension through local homomorphisms. Descent is known classically, e.g., it is in Cartan–Eilenberg's book on homological algebra, but the corresponding ascent property is surprising because of the need to use derived categories. We present a short proof of ascent due to Dwyer–Greenlees–Iyengar, and discuss the main ingredient of the proof, which is the classification due to Hopkins and Neeman of thick subcategories of the category of perfect complexes over a commutative noetherian ring R. At the end, we discuss other directions these ideas of Hopkins and Neeman lead, in particular Balmer's result that says one can reconstruct a scheme from its derived category.

The main sources for this talk are [Iye06] and [DGI06]. For basic results on unbounded complexes of modules at a (hopefully) accessible level, and for more thorough proofs for many of the preliminary results we do not prove here, we point the reader to [BN93], and a previous note by the author [Mur15].

1. Introduction: Homological algebra in commutative algebra

Let $(R, \mathfrak{m}, k)$ be a (noetherian) local ring. Recall that we say R is *regular* if $\mathfrak{m}$ is generated by a regular sequence, which is equivalent to saying $\mathfrak{m}$ is generated by $\dim R$ elements.

Regular local rings are important in algebraic geometry because of their role as local rings of algebraic varieties, and they had been studied thoroughly by the 1950s, as can be seen in, e.g., Zariski–Samuel's *Commutative Algebra* [ZS60], Vol. II. The issue at the time, as pointed out by [BH93, p. 86], however, was that there were two major gaps in the theory at the time:

(1) Is the localization of a regular local ring regular?
(2) Are regular local rings UFD's?

Nowadays, a graduate student would know these facts are true from a homological algebra course, as a consequence of the following

Theorem 1 (Auslander–Buchsbaum–Serre [BH93, Thm. 2.2.7])**.** *R is regular if and only if the residue field k has finite projective dimension over R.*

This theorem was a breakthrough for commutative algebra, because it showed that *homological* algebra, something that arose out of topology, could be made useful to prove statements purely about the theory of commutative rings.

Now we want to share the main idea from [DGI06]. We can turn this statement into one about the derived category of R as follows:

- Since finiteness of projective dimension $\operatorname{pd}_R(k)$ is detected by $\operatorname{Ext}^\bullet_R(k, M)$, regularity can be detected on the level of *derived categories;*
- This finiteness can be turned into the structural statement that k is quasi-isomorphic to a *perfect complex,* that is, a finite complex consisting of finitely generated projective modules.

It is even true that in Theorem 1, there is no need to choose k in particular, and the Theorem can be reformulated as follows:

Theorem 1*. *R is regular if and only if every homologically finite complex M is quasi-isomorphic to a perfect complex.*

The main goal of [DGI06] is to reformulate questions about commutative rings to ones about derived categories, and solve them in that setting. We will mainly be interested in following result:

Theorem 2 ([FI03, Thm. IV])**.** *Let $(R, \mathfrak{m}, k)$ and $(Q, \mathfrak{q}, h)$ be local rings, and let $\psi\colon Q \to R$ be a local homomorphism, i.e., a ring homomorphism such that $\psi(\mathfrak{q}) \subseteq \mathfrak{m}$. Then, for any finitely generated R-module M, if the flat dimension of M over Q is finite, then the flat dimension of R over Q is finite.*

Date: February 9, 2016.

We want to point out how this theorem is interesting. Classically [CE56], it is known that qualitative information about homological dimension *descends* through local homomorphisms, i.e., the following statement is known: if $\operatorname{fd}_Q R$ and $\operatorname{fd}_R M$ are both finite, then so is $\operatorname{fd}_Q M$ (essentially, you can take the tensor product of flat resolutions $F_\bullet \to R \to 0$ over Q and $F'_\bullet \to M \to 0$). On the other hand, this theorem is much more surprising: it says that information about homological dimension *ascends,* and that we can choose *any* M to do our check.

It is also interesting to note that this gives a characterization of regular rings of characteristic p, which extends results of [Kun69]. Denote ${}^{\varphi^n}F$ for an R-module F to be F with the R-module structure on F induced by restriction of scalars along φ^n.

Theorem 3 ([FI03, Thm. V])**.** *Let R be a local ring of characteristic p, and let $\varphi\colon R \to R$ be the Frobenius endomorphism of R. Then, the following are equivalent:*

1. *R is regular;*
2. *φ^n is flat for each integer $n \geq 0$;*
3. *there exists a positive integer n and a nonzero finitely generated R-module F such that both* $\operatorname{fd}_R F$ *and* $\operatorname{fd}_R({}^{\varphi^n}F)$ *are finite.*

To prove this theorem, it turns out that this theorem is a consequence of general facts about thick subcategories of the derived category of a noetherian ring R. So we begin with some preliminary results about the derived category of modules over a noetherian ring R. Local cohomology serves a key role in the proof what is the main ingredient in the proof of the theorem.

2. Preliminaries on the derived category

We start with some preliminaries about the derived category of chain complexes over a ring R. We follow [BN93], whose ideas are primarily topological in origin.

2.1. **Resolutions.** Recall that the *derived category* $\mathbf{D}(R)$ of R is constructed by inverting quasi-isomorphisms in the homotopy category $\mathbf{K}(R)$ of chain complexes over R. We give a brief overview of how to construct resolutions of chain complexes.

First, recall that any single module M has a projective resolution:

$$\cdots \longrightarrow P_q \longrightarrow P_{q-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

Moreover, even if we replace M by a bounded below chain complex $M_\bullet$, we can combine all resolutions of the M_p at once to get a double complex

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P_{p\bullet} & \longrightarrow & P_{p-1,\bullet} & \longrightarrow & \cdots & \longrightarrow & P_{1\bullet} & \longrightarrow & P_{0\bullet} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & M_p & \longrightarrow & M_{p-1} & \longrightarrow & \cdots & \longrightarrow & M_1 & \longrightarrow & M_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & & & 0 & & 0 & &
\end{array}$$

such that the induced map on the totalization $\operatorname{Tot}^{\oplus}(P_{\bullet\bullet}) \to M_\bullet$ is a quasi-isomorphism; this is called a *Cartan–Eilenberg resolution* [Wei94, §5.7].

It was an open question until [AJS00; Ser03] how to construct analogous unbounded resolutions for *arbitrary* Grothendieck abelian categories, but luckily in our setting, we have

Theorem 4 ([BN93])**.** *Every chain complex over R is quasi-isomorphic to an object in $\mathbf{K}(R)_{\mathsf{proj}}$, the subcategory generated by chain complexes of* projective *modules. Moreover, the composition $\mathbf{K}(R)_{\mathsf{proj}} \hookrightarrow \mathbf{K}(R) \to \mathbf{D}(R)$ is in fact an equivalence.*

Proof Sketch. The idea is to take brutal truncations $\tau_{\geq n} M_\bullet$ of our chain complex, and use the Cartan–Eilenberg resolution there. Then, we can combine them into a larger chain complex by using a homotopy colimit. See [Mur15, n° 1.1] for details. The second statement is by general results in [BK90, §1]. ☐

Dualizing we also get that

Theorem 4* ([BN93]). *Every chain complex over R is quasi-isomorphic to an object in $\mathbf{K}(R)_{\mathsf{inj}}$, the subcategory generated by chain complexes of* injective *modules. Moreover, the composition $\mathbf{K}(R)_{\mathsf{inj}} \hookrightarrow \mathbf{K}(R) \to \mathbf{D}(R)$ is in fact an equivalence.*

Using both equivalences, we can construct derived functors in the unbounded setting:

Theorem 5. *There is a right derived functor $\mathbf{R}\mathrm{Hom}\colon \mathbf{D}(R)^{\mathsf{op}} \times \mathbf{D}(R) \to \mathbf{D}(R)$ and a left derived functor $\otimes^{\mathbf{L}}\colon \mathbf{D}(R) \times \mathbf{D}(R) \to \mathbf{D}(R)$ satisfying all reasonable good properties.*

Proof. Both $\mathbf{R}\mathrm{Hom}$ and $\otimes^{\mathbf{L}}$ can be defined easily on $\mathbf{K}(R)$ as the totalizations of the Hom and tensor product bicomplexes [Wei94, 2.7.1, 2.7.4]. But now using Propositions 4 and 4*, we can define $\mathbf{R}\mathrm{Hom}$ as the functor $\mathbf{K}(R)^{\mathsf{op}}_{\mathsf{proj}} \times \mathbf{K}(R)_{\mathsf{inj}} \to \mathbf{D}(R)$ and $\otimes^{\mathbf{L}}$ as the functor $\mathbf{K}(R)_{\mathsf{proj}} \times \mathbf{K}(R)_{\mathsf{proj}} \to \mathbf{D}(R)$ by restricting the functors on $\mathbf{K}(R)$. □

3. The flat dimension theorem

Our main goal was a result on flat dimension, so we give a generalized definition of flat dimension that works for complexes here:

Definition 6. The *flat dimension* of a chain complex $M_\bullet$ over R is defined to be the integer

$$\mathrm{fd}_R M_\bullet \coloneqq \inf \left\{ n \in \mathbf{Z} \;\middle|\; \begin{array}{c} \text{there is a flat complex } F_\bullet \text{ with} \\ F_\bullet \simeq M_\bullet, \text{ and } F_i = 0 \text{ for } i \geq n+1 \end{array} \right\}.$$

With this definition, we can state the version of Theorem 2 we will prove:

Theorem 2* ([DGI06, Thm. 5.5]). *Let $(R, \mathfrak{m}, k)$ and $(Q, \mathfrak{q}, h)$ be local rings, and let $\psi\colon Q \to R$ be a local homomorphism, i.e., a ring homomorphism such that $\psi(\mathfrak{q}) \subseteq \mathfrak{m}$. Then, for any perfect complex $M_\bullet$ of R-modules, if $\mathrm{fd}_Q M_\bullet < \infty$, then $\mathrm{fd}_Q R < \infty$.*

Recall that a *perfect complex* is a finite complex consisting of finitely generated projective modules, which we will denote by $\mathbf{D}^b(R)_{\mathsf{proj}}$.

The proof in [DGI06] breaks down into a couple of easy steps. Note this is in contrast to the proof in [FI03], which seems to be a much more complicated commutative algebra proof.

Step 1. $\mathrm{fd}_Q R < \infty$ if and only if $\sup(h \otimes^{\mathbf{L}}_Q R) \coloneqq \sup\{n \mid H_n(h \otimes^{\mathbf{L}}_Q R) \neq 0\} < \infty$.

Proof. $\Rightarrow$ is clear. For the converse, let $P_\bullet \to R \to 0$ be a minimal free resolution over Q; then, $\sup(h \otimes^{\mathbf{L}}_Q R) < \infty$ implies $h \otimes P_\bullet = (h \otimes P)_\bullet$ is eventually zero, and so $P_\bullet$ was eventually zero, i.e., $\mathrm{fd}_Q R \leq \mathrm{pd}_Q R < \infty$. □

Step 2. The full triangulated subcategory $\{X_\bullet \in \mathbf{D}(R) \mid \sup(h \otimes^{\mathbf{L}}_Q X_\bullet) < \infty\}$ of $\mathbf{D}(R)$ is *thick,* that is, it is closed under taking direct summands.

Proof. This is a straightforward check: for the direct summand condition, we see that taking a direct summand should *decrease* the number $\sup(h \otimes^{\mathbf{L}}_Q X_\bullet)$, and that if two objects have finite $\sup(h \otimes^{\mathbf{L}}_Q X_\bullet)$, then their mapping cone would also have finite $\sup(h \otimes^{\mathbf{L}}_Q X_\bullet)$ by using the long exact sequence on homology. □

Step 3. Let $K_\bullet$ be the Koszul complex on a finite set of elements in R, and let $Y_\bullet$ be a complex of R-modules such that $H_n(Y_\bullet)$ is finitely generated for each n. Then, if $\sup(Y_\bullet \otimes^{\mathbf{L}}_Q K_\bullet)$ is finite, then $\sup(Y_\bullet)$ is finite.

Proof. We induce on the number of elements used to define the Koszul complex. For one element, since $K_\bullet$ is the mapping cone of the morphism $R \xrightarrow{x} R$, we have the short exact sequence of complexes

$$0 \longrightarrow Y_\bullet \longrightarrow Y_\bullet \otimes^{\mathbf{L}}_Q K_\bullet \longrightarrow Y[-1] \longrightarrow 0.$$

The long exact sequence on homology and Nakayama's lemma imply that when $\sup(Y_\bullet \otimes^{\mathbf{L}}_Q K_\bullet)$ is finite, so is $\sup(Y_\bullet)$, as claimed. For multiple elements used to define the Koszul complex, it suffices to notice that the corresponding Koszul complex is an iterated mapping cone. □

Now if we take for granted the following theorem, we can prove our result on flat dimension. First, denote

$$\mathrm{Supp}_R(M_\bullet) \coloneqq \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_\bullet \otimes^{\mathbf{L}} \kappa(\mathfrak{p}) \neq 0\},$$

where $\kappa(\mathfrak{p})$ is the residue field $\mathrm{Frac}(R/\mathfrak{p}) \cong R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ at $\mathfrak{p}$.

Theorem 7 ([Nee92a]). *If $M_\bullet$ and $N_\bullet$ are perfect complexes such that $\operatorname{Supp}_R M_\bullet \subseteq \operatorname{Supp}_R N_\bullet$, then $M_\bullet$ is in the thick subcategory of $\mathbf{D}^b(R)_{\mathsf{proj}}$ generated by $N_\bullet$.*

Step 4. Theorem 2, assuming Theorem 7.

Proof. Let $K_\bullet$ be the Koszul complex on a finite set of generators for $\mathfrak{m}$. Then, since $\mathfrak{m}$ is the unique closed point of $\operatorname{Spec} R$, and $\operatorname{Supp}_R M_\bullet$ is a closed subset of $\operatorname{Spec} R$, we see that Theorem 7 implies that $K_\bullet$ is in the thick subcategory generated by $N_\bullet$. We then have the chain of implications:

$$\begin{aligned}
\operatorname{fd}_Q M_\bullet < \infty &\overset{(1)}{\Longrightarrow} \sup(h \otimes_Q^{\mathbf{L}} M_\bullet) < \infty \\
&\Longrightarrow \sup((h \otimes_Q^{\mathbf{L}} R) \otimes_R^{\mathbf{L}} M_\bullet) < \infty \\
&\overset{(2)}{\Longrightarrow} \sup((h \otimes_Q^{\mathbf{L}} R) \otimes_R^{\mathbf{L}} K_\bullet) < \infty \\
&\overset{(3)}{\Longrightarrow} \sup(h \otimes_Q^{\mathbf{L}} R) < \infty \\
&\overset{(1)}{\Longrightarrow} \operatorname{fd}_Q R < \infty,
\end{aligned}$$

where the implications are marked with their corresponding steps above. The second implication is just by associativity of the tensor product, and the fourth is by applying (3) to $Y_\bullet = h \otimes_Q^{\mathbf{L}} R$. □

We therefore see that once we have a handle of what the thick subcategories of $\mathbf{D}^b(R)_{\mathsf{proj}}$ look like, we would get Theorem 2* with minimal effort.

4. Thick subcategories of $\mathbf{D}^b(R)_{\mathsf{proj}}$ and their classification

We now want to give an idea for how the statement in Theorem 7 is proved. Recall that a subcategory of a triangulated category $\mathcal{D}$ is *thick* if it is closed under direct summands.

Recall that the *derived category of perfect complexes* is the full subcategory of $\mathbf{D}(R)$ consisting of complexes quasi-isomorphic to bounded complexes of finitely generated projective modules, and is denoted by $\mathbf{D}^b(R)_{\mathsf{proj}}$.

The main statement about thick subcategories of $\mathbf{D}^b(R)_{\mathsf{proj}}$ is actually stronger than Theorem 7:

Theorem 8. *There is an inclusion-preserving bijection of sets*

$$\left\{\begin{matrix}\textit{Thick subcategories}\\ \textit{of } \mathbf{D}^b(R)_{\mathsf{proj}}\end{matrix}\right\} \underset{g}{\overset{f}{\rightleftarrows}} \left\{\begin{matrix}\textit{Specialization closed}\\ \textit{subsets of } \operatorname{Spec}(R)\end{matrix}\right\}$$

where

$$f(\mathcal{L}) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \exists X_\bullet \in \mathcal{L} \textit{ with } X_\bullet \otimes^{\mathbf{L}} \kappa(\mathfrak{p}) \neq 0\},$$
$$g(P) = \{X_\bullet \mid \operatorname{Supp}(X_\bullet) \subset P\}.$$

The main outline of a proof of this Theorem is as follows:

(1) First, we classify localizing subcategories of $\mathbf{D}(R)$, that is, thick subcategories that are also closed under direct sum:

Theorem 9. *There is an inclusion-preserving bijection of sets*

$$\left\{\begin{matrix}\textit{Localizing subcategories}\\ \textit{of } \mathbf{D}(R)\end{matrix}\right\} \underset{g}{\overset{f}{\rightleftarrows}} \left\{\begin{matrix}\textit{Subsets}\\ \textit{of } \operatorname{Spec}(R)\end{matrix}\right\}$$

where

$$f(\mathcal{L}) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \exists X_\bullet \in \mathcal{L} \textit{ with } X_\bullet \otimes^{\mathbf{L}} \kappa(\mathfrak{p}) \neq 0\},$$
$$g(P) = \textit{the smallest localizing category containing } \kappa(\mathfrak{p}), \textit{ for all } \mathfrak{p} \in P.$$

A word about the proof of this: the idea is to characterize when an object of $\mathbf{D}(R)$ is in a localizing subcategory by writing it as a complex consisting of injectives, each of which is a direct sum of copies of $I_\mathfrak{p}$, the injective hulls of $R/\mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Spec}(R)$, and then using the derived version of the local cohomology $\mathbf{R}\Gamma_{Z/Z'}(-)$ functor.

(2) Since we are only interested in Theorem 7 here, we will sketch that. If $\operatorname{Supp}_R M_\bullet \subset \operatorname{Supp}_R N_\bullet$, then $M_\bullet$ is in the localizing subcategory of $\mathbf{D}(R)$ generated by $N_\bullet$. But since $M_\bullet$ and $N_\bullet$ are both in $\mathbf{D}^b(R)_{\mathsf{proj}}$, they are compact objects in $\mathbf{D}(R)$, and so a result of Neeman [Nee92b, Lem. 2.2] implies $M_\bullet$ is in fact in the thick subcategory of $\mathbf{D}^b(R)_{\mathsf{proj}}$ generated by $N_\bullet$.

Finally, we want to mention an interesting consequence of Theorem 8.

First, out of the category $\mathbf{D}^b(R)_{\mathsf{proj}}$, we can construct some sort of spectral space (in the sense of Hochster [Hoc69]) $\operatorname{Spc}(\mathbf{D}^b(R)_{\mathsf{proj}})$, where the construction is basically taking the spectrum of the Grothendieck ring of $\mathbf{D}^b(R)_{\mathsf{proj}}$. Then, Theorem 8 can show

Theorem 10 ([Bal05])**.** *There is a homeomorphism* $\varphi\colon \operatorname{Spec}(R) \xrightarrow{\sim} \operatorname{Spc}(\mathbf{D}^b(R)_{\mathsf{proj}})$*, with*

$$\varphi(\mathfrak{p}) = \{X_\bullet \in \mathbf{D}^b(R)_{\mathsf{proj}} \mid \mathfrak{p} \notin \operatorname{Supp}(X_\bullet)\}.$$

In fact, this can be turned into an isomorphism of locally ringed spaces; see [Bal05, Thm. 6.3]. The idea is to translate conditions for a subset to be closed in the statement of Theorem 8.

We end with some notes about how much this result can be generalized. In [Bal05], the main result is that this theorem holds for X a quasi-compact quasi-separated scheme, and it is known [Bal05, Rem. 5.7] that it has no possibility of holding for a scheme that is not quasi-compact and quasi-separated (since the construction of Spc always gives a quasi-compact and quasi-separated space). On the other hand, if we do not start with the derived category of a scheme, and instead start with an arbitrary (tensor) triangulated category, it is unknown how the original triangulated category and the ring R such that $\operatorname{Spec} R$ is homeomorphic to the Spc construction are related; see [Bal10b] for a survey, and [Bal10a] for recent commentary and examples of applications of this construction Spc in other contexts.

References

[AJS00] L. Alonso Tarrío, A. Jeremías López, and M. J. Souto Salorio. "Localization in categories of complexes and unbounded resolutions." *Canad. J. Math.* 52.2 (2000), pp. 225–247.

[Bal05] P. Balmer. "The spectrum of prime ideals in tensor triangulated categories." *J. Reine Angew. Math.* 588 (2005), pp. 149–168.

[Bal10a] P. Balmer. "Spectra, spectra, spectra—tensor triangular spectra versus Zariski spectra of endomorphism rings." *Algebr. Geom. Topol.* 10.3 (2010), pp. 1521–1563.

[Bal10b] P. Balmer. "Tensor triangular geometry." In: *Proceedings of the International Congress of Mathematicians (Hyderabad, 2010).* Vol. II. New Delhi: Hindustan Book Agency, 2010, pp. 85–112.

[BH93] W. Bruns and J. Herzog. *Cohen-Macaulay rings.* Cambridge Studies in Advanced Mathematics 39. Cambridge University Press, Cambridge, 1993.

[BK90] A. I. Bondal and M. M. Kapranov. "Representable functors, Serre functors, and mutations." *Math. USSR Izv.* 35.3 (1990), pp. 519–541.

[BN93] M. Bökstedt and A. Neeman. "Homotopy limits in triangulated categories." *Compositio Math.* 86.2 (1993), pp. 209–234.

[CE56] H. Cartan and S. Eilenberg. *Homological algebra.* Princeton, N. J.: Princeton University Press, 1956.

[DGI06] W. Dwyer, J. P. C. Greenlees, and S. Iyengar. "Finiteness in derived categories of local rings." *Comment. Math. Helv.* 81.2 (2006), pp. 383–432.

[FI03] H.-B. Foxby and S. Iyengar. "Depth and amplitude for unbounded complexes." In: *Commutative algebra (Grenoble/Lyon, 2001).* Contemp. Math. 331. Providence, RI: Amer. Math. Soc., 2003, pp. 119–137.

[Hoc69] M. Hochster. "Prime ideal structure in commutative rings." *Trans. Amer. Math. Soc.* 142 (1969), pp. 43–60.

[Iye06] S. B. Iyengar. "Thick subcategories of perfect complexes over a commutative ring" (2006), pp. 495–499. In: "Mini-workshop: Thick Subcategories—Classifications and Applications." Abstracts from the mini-workshop held February 19–25, 2006, organized by R.-O. Buchweitz, H. Krause and S. Schwede. *Oberwolfach Rep.* 3.1 (2006), pp. 461–509.

[Kun69] E. Kunz. "Characterizations of regular local rings for characteristic p." *Amer. J. Math.* 91 (1969), pp. 772–784.

[Mur15] T. Murayama. "The classification of thick subcategories and Balmer's reconstruction theorem." Unpublished notes, available at `http://www-personal.umich.edu/~takumim/NeemanBalmer.pdf`. Apr. 2015.

[Nee92a] A. Neeman. "The chromatic tower for $D(R)$." With an appendix by Marcel Bökstedt. *Topology* 31.3 (1992), pp. 519–532.

[Nee92b] A. Neeman. "The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel." *Ann. Sci. École Norm. Sup. (4)* 25.5 (1992), pp. 547–566.

[Ser03] C. Serpé. "Resolution of unbounded complexes in Grothendieck categories." *J. Pure Appl. Algebra* 177.1 (2003), pp. 103–112.

[Wei94] C. A. Weibel. *An introduction to homological algebra.* Cambridge Studies in Advanced Mathematics 38. Cambridge: Cambridge University Press, 1994.

[ZS60] O. Zariski and P. Samuel. *Commutative algebra. Vol. II.* The University Series in Higher Mathematics. Princeton, N. J.-Toronto-London-New York: D. Van Nostrand Co., Inc., 1960.