USING GROTHENDIECK GROUPS TO DEFINE AND RELATE THE HILBERT AND CHERN POLYNOMIALS AXIOMATICALLY

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Abstract. The Hilbert polynomial can be defined axiomatically as a group homomorphism on the Grothendieck group $K(X)$ of a projective variety X, satisfying certain properties. The Chern polynomial can be similarly defined. We therefore define these rather abstract notions to try and find a nice description of this relationship.

Introduction

Let $X = \mathbb{P}^r$ be a complex projective variety over an algebraically closed field k with coordinate ring $S = k[x_0, \ldots, x_r]$. Recall that the *Hilbert function* of a finitely generated graded S-module M is defined as

$$h_M^{\text{func}}(t) \coloneqq \dim_k M_t,$$

and that we have the following

Theorem (Hilbert). *There is a polynomial $h_M(t)$ such that*

$$h_M(t) = h_M^{func}(t) \quad \text{for } t \gg 0.$$

We call $h_M(t)$ the *Hilbert polynomial* of M. The Hilbert polynomial satisfies a nice property: given an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of graded S-modules, where each homomorphism is *graded*, i.e., it respects the grading on the modules M, M', M'', then

$$h_M(t) = h_{M'}(t) + h_{M''}(t), \tag{1}$$

by the fact that the sequence above is exact at each degree t, and by the rank-nullity theorem. The Hilbert polynomial therefore said to be "additive on exact sequences"; this is an extremely useful fact that enables us to compute the Hilbert polynomial in many cases. For example, we computed the following for homework:

Example (HW8, #3). Let $f(w, x, y, z)$ and $g(w, x, y, z)$ be relatively prime nonzero homogeneous polynomials of degree a and b in $S \coloneqq k[w, x, y, z]$. Then, $\{f, g\}$ forms a

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regular sequence, and so by [Eis95, Thm. 17.6], the Koszul sequence

$$0 \longrightarrow S(-a-b) \xrightarrow{\begin{pmatrix} g \\ -f \end{pmatrix}} S(-a) \oplus S(-b) \xrightarrow{(f\ g)} S \longrightarrow S/(f,g) \longrightarrow 0$$

is a graded free resolution for $S/(f,g)$. By the additivity of the Hilbert polynomial,

$$\begin{aligned} h^{\text{poly}}_{S/(f,g)}(t) &= h^{\text{poly}}_S(t) - (h^{\text{poly}}_S(t-a) + h^{\text{poly}}_S(t-b)) + h^{\text{poly}}_S(t-a-b) \\ &= \binom{t+3}{3} - \binom{t+3-a}{3} - \binom{t+3-b}{3} + \binom{t+3-a-b}{3}. \end{aligned}$$

This additivity property, however, is not unique to Hilbert polynomials. By a general construction in [FAC, n$^{\text{o}}$ 57] and [Ser55, n$^{\text{o}}$ 4], it is often[1] possible to associate a vector bundle $\mathscr{E}$ to M. From a topological point of view (in which case we assume $k = \mathbb{C}$), to such a bundle we can associate *Chern classes* $c_{\mathscr{E}}(i) \in H^{2i}(X, \mathbb{Z}) = \mathbb{Z}$ that are useful as an invariant of $\mathscr{E}$. We can encode these Chern classes into a polynomial we call the *Chern polynomial*

$$c_{\mathscr{E}} = \sum_{i=0}^{q} c_{\mathscr{E}}(i) x^i,$$

and an important formula used in computing Chern classes is the *Whitney multiplication formula*, which says

$$c_{\mathscr{E}\oplus\mathscr{E}'} = \sum_{i=0}^{q+q'} c_{\mathscr{E}\oplus\mathscr{E}'}(i) x^i = \left(\sum_{i=0}^{q} c_{\mathscr{E}}(i) x^i\right)\left(\sum_{i=0}^{q'} c_{\mathscr{E}'}(i) x^i\right) = c_{\mathscr{E}} c_{\mathscr{E}'}, \tag{2}$$

which is found in, for example, [Hir95, Thm. 4.4.3] or [MS74, 14.7]. The Chern polynomial therefore transforms direct sums into products of polynomials.

In this manner, Hilbert polynomials and Chern polynomials both behave "nicely" with respect to exact sequences, although the former is *additive* with respect to exact sequences, while the latter are *multiplicative*. It is goal in this note to define both *axiomatically* in the hope that this elucidates their relationship.

We have striven to make most ideas as simple as possible, hopefully so that those who have taken a first semester course in commutative algebra following [AM69] or [Eis95] and in algebraic geometry following [Sha13] or [Har77, I], for example, will ideally be able to follow the arguments made. The exception is §2, which we regrettably were not able to make more elementary; we have instead tried to use [FAC] as our main source for cohomological results, since it does not require schemes.

1. Additive Functions and the Grothendieck Group

Let $\mathscr{C}$ be an abelian category. The Hilbert and Chern polynomials are specific examples of the following general construction, following [SGA5, VIII 1]:

[1] In general, you only get a coherent sheaf.

Definition. A function λ on the objects of $\mathscr{C}$ with values in an abelian monoid G is *additive* if for all short exact sequences

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

in $\mathscr{C}$, where M', M'', M are objects in $\mathscr{C}$, we have

$$\lambda(M) = \lambda(M') + \lambda(M'').$$

Example 1. We explain here why exactly the Hilbert and Chern polynomials are examples of this construction. Let S and X as in the Introduction, where we recall $X = \mathbb{P}^r$.

Let $\mathsf{gr}\text{-}S$ be the category of finitely generated graded S-modules. Then, the function $M \mapsto h_M(t) \in \mathbb{Q}[t]$ is additive by (1).

Now let $\mathsf{vect}\text{-}X$ be the category of vector bundles[2] over X. Then, the function $\mathscr{E} \mapsto c_{\mathscr{E}}(t) \in \mathbb{Z}[t]/(t^{r+1})$ is multiplicative by (2). Note that it is therefore additive if we consider $\mathbb{Z}[t]/(t^{r+1})$ as an abelian monoid under multiplication.

Now let $F(\mathscr{C})$ denote the free abelian group generated by isomorphism classes of objects in $\mathscr{C}$. Let $Q(\mathscr{C})$ be the subgroup of $F(\mathscr{C})$ generated by formal sums of the form $[M'] - [M] + [M'']$ for each short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

in $\mathscr{C}$, where $[M]$ is the isomorphism class of M. We then have the following

Definition. The quotient group $F(\mathscr{C})/Q(\mathscr{C})$ is called the *Grothendieck group* of $\mathscr{C}$, and is denoted $K(\mathscr{C})$. If M is an object of $\mathscr{C}$, then denote $\gamma(M)$ to be the image of $[M]$ in $K(\mathscr{C})$.

Now if the abelian monoid G is actually a group, the Grothendieck group $K(\mathscr{C})$ is characterized by the following

Universal Property. *For each additive function λ on $\mathscr{C}$, with values in an abelian group G, there exits a unique homomorphism $\lambda_0\colon K(\mathscr{C}) \to G$ such that $\lambda(M) = \lambda_0(\gamma(M))$ for all M.*

Proof. This follows from the universal property for quotient groups, which gives us a unique group homomorphism λ_0 fitting into the diagram

$$\begin{array}{ccc} F(\mathscr{C}) & \xrightarrow{\quad\lambda\quad} & G \\ & \searrow \qquad \nearrow_{\lambda_0} & \\ & K(\mathscr{C}) & \end{array}$$

□

The study of additive functions on a category $\mathscr{C}$ with values in an abelian group G, then, can be reduced to the study of group homomorphisms $K(\mathscr{C}) \to G$. We compute some elementary examples of Grothendieck groups:

[2]Technically we only defined additive functions on abelian categories. However, the same definition goes through for non-abelian categories, as long as there is a good notion of an exact sequence.

Example 2 (Modules over a Noetherian ring, cf. [AM69, Exc. 7.26])**.** Let A be a Noetherian ring, and let $K(\mathsf{mod}\text{-}A)$ be the Grothendieck group of $\mathsf{mod}\text{-}A$, the category of finitely generated A-modules. By *dévissage* [AM69, Exc. 7.18], for each finitely generated A-module M, there exists a chain of submodules

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M,$$

such that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for each i, where $\mathfrak{p}_i \subset A$ are prime ideals. We then have a short exact sequence

$$0 \longrightarrow M_{r-1} \longrightarrow M \longrightarrow M/M_{r-1} \longrightarrow 0,$$

so

$$\gamma(M) = \gamma(M_{r-1}) + \gamma(M/M_{r-1}) = \gamma(M_{r-1}) + \gamma(A/\mathfrak{p}_r)$$

in $K(\mathsf{mod}\text{-}A)$. By induction,

$$\gamma(M) = \sum_{i=1}^{r} \gamma(A/\mathfrak{p}_i),$$

and so $K(\mathsf{mod}\text{-}A)$ is generated by $\gamma(A/\mathfrak{p})$ for each prime ideal $\mathfrak{p} \subset A$.

In particular, if A is a PID and M is a finitely-generated A-module, then by the structure theorem for finitely generated modules over a PID,

$$M \cong A^{\oplus r} \oplus A/(f_1) \oplus A/(f_2) \oplus \cdots \oplus A/(f_n)$$

where $f_i \mid f_{i+1}$ for all i. Since we have the short exact sequence

$$0 \longrightarrow A \xrightarrow{\cdot f_i} A \longrightarrow A/(f_i) \longrightarrow 0,$$

we see $\gamma(A/(f_i)) = \gamma(A) - \gamma(A) = 0$. Thus, $\gamma(M) = \gamma(A^r)$, and since γ is additive by the fact that the short exact sequence

$$0 \longrightarrow M \longrightarrow M \oplus M' \longrightarrow M' \longrightarrow 0$$

gives the relation $\gamma(M \oplus M') = \gamma(M) + \gamma(M')$, we have that γ is additive and moreover $\gamma(M) = r \cdot \gamma(A)$. Thus, $K(\mathsf{mod}\text{-}A)$ is the abelian group generated by $\gamma(A)$, and mapping $\gamma(A)$ to 1 gives an isomorphism $K(\mathsf{mod}\text{-}A) \cong \mathbb{Z}$ since the order of $\gamma(A)$ is infinite.

Example 3 (Graded modules over a Noetherian graded ring)**.** Let S be a Noetherian graded ring, and let $K(\mathsf{gr}\text{-}S)$ be the Grothendieck group of the category of finitely generated graded S-modules. As in the previous example, we can apply dévissage [Har77, I, Prop. 7.4]: for each finitely generated graded S-module M, there exists a chain of graded submodules

$$0 = M^0 \subsetneq M^1 \subsetneq \cdots \subsetneq M^r = M,$$

such that $M^i/M^{i-1} \cong (S/\mathfrak{p}_i)(l_i)$ for each i, where $\mathfrak{p}_i \subset S$ are homogeneous prime ideals, and $l_i \in \mathbb{Z}$ denotes a degree shift $M(l)_t = M_{t+l}$. Just like in the previous example,

$$\gamma(M) = \sum_{i=1}^{r} \gamma((S/\mathfrak{p}_i)(l_i)),$$

and so $K(\mathsf{gr}\text{-}S)$ is generated by $\gamma((S/\mathfrak{p})(l))$ for each homogeneous prime ideal $\mathfrak{p} \subset S$ and grade shifts $l \in \mathbb{Z}$.

Also in analogy with the previous example, it is possible, with more work, to show that if $S = k[x, y]$, then $K(\mathsf{gr}\text{-}S) \cong \mathbb{Z} \oplus \mathbb{Z}$. See [Har77, II, Exc. 6.11].

Remark. We end this section with a short historical note. Grothendieck groups were first introduced in Grothendieck's proof of his version of the Riemann-Roch theorem [SGA6, 0 App.],[BS58]. Since then, they have enjoyed applications in other fields such as algebraic topology [Ati67], culminating in Atiyah-Singer's proof of their index theorem [AS68]. For the interested reader, there is a recent book by Weibel [Wei13] on the subject.

2. An Axiomatic Approach to the Hilbert Polynomial

In this section, we provide an axiomatic definition of the Hilbert polynomial, following [FAC, §6],[EGAIII$_1$, 2.5],[Har77, III, Exc. 5.1–5.4]. Note in particular that this means we will be using the theory of coherent sheaves and their cohomology.

Let X be a projective variety, and $\mathsf{coh}\text{-}X$ the category of coherent sheaves on X. Let $\mathscr{F}$ be a coherent sheaf on X. We define the *Euler-Poincaré characteristic* of X with values in $\mathscr{F}$ to be the alternating sum

$$\chi(\mathscr{F}) \coloneqq \sum_{q=0}^{\infty} (-1)^q \dim_k H^q(X, \mathscr{F}),$$

where $H^q(X, \mathscr{F})$ is the qth cohomology module of $\mathscr{F}$, which is finite dimensional over k and 0 for all $q > \dim X$ [FAC, n° 66, Thm. 1].

Remark. By the general theory of the correspondence between algebraic and analytic geometry [GAGA, Thm. 1], this definition can be used to recover the topological Euler-Poincaré characteristic by using the constant sheaf $\widetilde{\mathbb{Z}}$ and [God73, Thm. 5.10.1].

The Euler-Poincaré characteristic is the prototypical example of an additive function.

Proposition 1. *Let*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

be a short exact sequence of coherent sheaves on X. Then,

$$\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}''),$$

i.e., $\chi(-)$ is an additive function on $\mathsf{coh}\text{-}X$ with values in $\mathbb{Z}$.

Proof. We reduce to the case $X = \mathbb{P}^r$ by noting that $H^q(X, \mathscr{F}) = H^q(\mathbb{P}^r, i_*\mathscr{F})$ for each q,[3] where i_* denotes pushforward by the inclusion $i\colon X \to \mathbb{P}^r$. Then, the long exact sequence on sheaf cohomology [FAC, n° 47, Cor. 2]

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathscr{F}') \longrightarrow H^0(X, \mathscr{F}) \longrightarrow H^0(X, \mathscr{F}'') \longrightarrow H^1(X, \mathscr{F}') \longrightarrow \cdots \\ \cdots \longrightarrow H^r(X, \mathscr{F}') \longrightarrow H^r(X, \mathscr{F}) \longrightarrow H^r(X, \mathscr{F}'') \longrightarrow 0 \end{aligned}$$

[3] This is either obvious if we use Čech cohomology, or is true because the pushforward of a flasque resolution is flasque; see [Har77, Lem. 2.10].

terminates by Serre vanishing [FAC, n° 66, Thm. 1] and additivity of vector space dimension on long exact sequences gives us the desired result. □

We are now ready to state and prove our main result on the Hilbert polynomial.

Theorem 1. *There exists a unique additive function* $h_-\colon \mathsf{coh}\text{-}X \to \mathbb{Q}[t]$ *such that* $\chi(\mathscr{F}(n)) = h_{\mathscr{F}}(n)$ *for all* $n \in \mathbb{Z}$*. Moreover, if* $\mathscr{F}$ *is the associated coherent sheaf to some finitely generated graded* S*-module* M*, then* $h_M(n) = h_{\mathscr{F}}(n)$*.*

Proof. Uniqueness is immediate since any polynomial in $\mathbb{Q}[t]$ is determined by a finite number of values, hence the requirement $\chi(\mathscr{F}(n)) = h_{\mathscr{F}}(n)$ ensures $h_{\mathscr{F}}(n)$ is unique.

For existence, we apply the same trick as before and reduce to the case $X = \mathbb{P}^r$. By [FAC, n° 60, Thm. 2], there then exists some finitely generated graded S-module M (not necessarily unique) such that the associated sheaf $\widetilde{M}$ of M is equal to $\mathscr{F}$. By the Hilbert syzygy theorem [Eis95, Thm. 1.13], there then exists a resolution

$$0 \longrightarrow L_q \longrightarrow L_{q-1} \longrightarrow \cdots \longrightarrow L_1 \longrightarrow M \longrightarrow 0,$$

where each L_i is free. Applying $\widetilde{(-)}$ gives the resolution

$$0 \longrightarrow \widetilde{L}_q \longrightarrow \widetilde{L}_{q-1} \longrightarrow \cdots \longrightarrow \widetilde{L}_1 \longrightarrow \widetilde{M} \longrightarrow 0$$

by the exactness of $\widetilde{(-)}$ [FAC, n° 58, Prop. 3]. By additivity on exact sequences, and then by decomposing each free sheaf $\widetilde{L}_i$ into direct summands $\mathcal{O}(n_j)$, it therefore suffices to show $\chi(\mathcal{O}(n))$ is a polynomial for every n. But this follows since $\dim H^0(X, \mathcal{O}(n)) = \binom{n+r}{r}$ and $H^q(X, \mathcal{O}(n)) = 0$ for all other q except $H^{-n-1}(X, \mathcal{O}(n)) = k$; note these follow by the fact that we can compute cohomology using modules and free resolutions of modules [FAC, n° 64, Cor.], and by calculations in [FAC, n° 62, Prop. 2]. The calculation $\dim H^0(X, \mathcal{O}(n)) = \binom{n+r}{r}$ is exactly the same as when we showed $h_S^{\text{func}}(n) = \binom{n+r}{r}$. Note in particular that the statements above show that for $n \gg 0$, $\chi(\mathcal{O}(n)) = \dim H^0(X, \mathcal{O}(n))$.

Finally, the last statement showing our new definition of Hilbert polynomial matches the old one follows by the same unicity argument as given above: $h_{S(n)}(t)$ and $h_{\mathcal{O}(n)}(t)$ must match for high enough n by the same argument as in the previous paragraph, so $h_M(t)$ and $h_{\mathscr{F}}(t)$ must as well; since these two polynomials match on infinitely many points, they must be equal. □

3. An Axiomatic Approach to the Chern Polynomial

We now return to the Chern polynomial. While before with the Hilbert polynomial the reduction to projective space was trivial, since it is a bit more involved for the Chern polynomial, we restrict to the case when $X = \mathbb{P}^r$. In this case, we claim we can characterize the Chern polynomial relatively simply:

Theorem 2 (cf. [Har77, A.3])**.** *The Chern polynomial* $c_{\mathscr{E}}(t)$ *is characterized uniquely as the additive function on* $\mathsf{vect}\text{-}X$ *with values in* $\mathbb{Z}[t]/(t^{r+1})$ *(under multiplication) such that* $c_{\mathcal{O}(-a)}(t) = 1 - at$*.*

Note that by Serre's equivalence [FAC, n° 59, Props. 7,8, n° 65, Prop. 6], we can reduce[4] the question to asking about graded modules over $S = k[x_0, \dots, x_r]$, that is

Theorem 2* (cf. [Eis95, Exc. 19.18])**.** *The Chern polynomial $c_M(t)$ is characterized uniquely as the additive function on* gr-X *with values in $\mathbb{Z}[t]/(t^{r+1})$ (under multiplication) such that $c_{S(-a)}(t) = 1 - at$.*

Proof. We use the Hilbert syzygy theorem [Eis95, Thm. 1.13] again to get a resolution

$$0 \longrightarrow L_q \longrightarrow L_{q-1} \longrightarrow \cdots \longrightarrow L_1 \longrightarrow M \longrightarrow 0,$$

where each $L_i = \bigoplus_j S(-a_{ij})$ is free. Then, by using the multiplicativity of the Chern polynomial and induction on length q of the free resolution (this is the same argument as for additivity for the Hilbert polynomial), we get that

$$c_M(t) = \frac{\prod_{i \text{ even},j}(1 - a_{ij}t)}{\prod_{i \text{ odd},j}(1 - a_{ij}t)} \bmod t^{r+1},$$

which is a polynomial since we can formally write $\left(\prod_{i \text{ odd},j}(1 - a_{ij}t)\right)^{-1}$ as a formal power series, which terminates since we are working mod t^{r+1}.

We need to check whether this definition is well-defined. But this follows since we can use the unique *minimal* free resolution of M to define its Chern polynomial, and then every other free resolution can be obtained by direct summing the minimal resolution with a trivial complex [Eis95, Thm. 20.2]; since the Chern polynomial is defined to be additive, this extra trivial direct summand does not affect the Chern polynomial.

Finally, uniqueness also follows by the uniqueness of the minimal free resolution above. □

4. A Relationship between the Hilbert and Chern Polynomials

We end this note with a surprising relationship between the Hilbert and Chern polynomials due to [Cha04], in the special case of $X = \mathbb{P}^r$.

In order to describe the relationship between the two, we have to introduce even more notation. Suppose the Chern polynomial can be written

$$c_M(t) = (1 - \alpha_1 t)(1 - \alpha_2 t) \cdots (1 - \alpha_r t)$$

in $\mathbb{Z}[t]/(t^{r+1})$; note this is possibe for $X = \mathbb{P}^r$ by the previous section. Then, the *Chern character* of M is a power series defined by

$$\operatorname{ch}(M) = e^{\alpha_1 t} + e^{\alpha_2 t} + \cdots + e^{\alpha_r t}.$$

For any power series $s(t)$ in t, define

$$\Phi(s(t)) := \text{coefficient of } t^r \text{ in the Taylor expansion of } s(t) \left(\frac{t}{1 - e^{-t}}\right)^{r+1}.$$

We also need to use the following (difficult) theorem:

[4]Calling this a "reduction" is a bit unfair—in fact, there are many graded modules M that don't give rise to vector bundles. Instead, $\widetilde{M}$ would just be a coherent sheaf.

Theorem 4 (Hirzebruch-Riemann-Roch, [Hir95, Lem. 1.7.1])**.** *Let $X = \mathbb{P}^r$. Then, for any vector bundle $\mathscr{E}$ on X,*

$$\Phi(\mathrm{ch}(\mathscr{E})) = \chi(\mathscr{E}).$$

We can now describe a relationship between the Hilbert and Chern polynomials:

Corollary 1. *Let $X, \mathscr{E}$ as before. Let $C_{\mathscr{E}}(x)$ be the Chern polynomial of $\mathscr{E}$. Then, the Hilbert polynomial of $\mathscr{E}$ is*

$$h_{\mathscr{E}}(t) = \Phi(e^{tx}\,\mathrm{ch}(\mathscr{E})).$$

Proof. Let M be a module such that $\widetilde{M} = \mathscr{E}$. As in the proof of Theorem 1, $\chi(\mathscr{E}(n)) = \dim M_n$ for $n \gg 0$. On the other hand, $\Phi(\mathrm{ch}(\mathscr{E}(n))) = \chi(\mathscr{E}(n))$ by the Hirzebruch-Riemann-Roch theorem. The Chern character satisfies $\mathrm{ch}(\mathscr{E} \otimes \mathscr{E}') \cong \mathrm{ch}(\mathscr{E})\,\mathrm{ch}(\mathscr{E}')$ by the splitting principle (see, e.g., [Har77, A.3]), hence

$$\mathrm{ch}(\mathscr{E}(n)) = \mathrm{ch}(\mathscr{E} \otimes \mathcal{O}(n)) = e^{nx}\,\mathrm{ch}(\mathscr{E}).$$

Then, $\Phi(\mathrm{ch}(\mathscr{E}(n)))$ becomes a polynomial in t which matches $h_{\mathscr{E}}(t)$ for large enough t, hence equals the Hilbert polynomial. ☐

We end here with some notes on this result. [Cha04] has shown a more combinatorial description of the corollary above to the Hirzebruch-Riemann-Roch theorem. The nice thing about their proof is that since in our case $X = \mathbb{P}^r$ with coordinate ring S, the Chern polynomial is easily described as an element of $\mathbb{Z}[t]/(t^{r+1})$. In general, however, the Chern polynomial has coefficients in $H^{2i}(X, \mathbb{Z})$ (topologically) or the Chow ring $A(X)$, which makes the description much harder. In fact, the author is not aware if such a description is possible for projective varieties other than $\mathbb{P}^r$.

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