CONTINUOUS VALUATIONS AND THE ADIC SPECTRUM

TAKUMI MURAYAMA

Abstract. Following [Hub93, §3], we introduce the spectrum of continuous valuations $\operatorname{Cont}(A)$ for a Huber ring A and the adic spectrum $\operatorname{Spa}(A, A^+)$ for a Huber pair (A, A^+). We also draw heavily from [Con14; Wed12]. These notes are from the arithmetic geometry learning seminar on adic spaces held at the University of Michigan during the Winter 2017 semester, organized by Bhargav Bhatt. See [Dat17; Ste17] for other notes from the seminar.

Contents

1. Introduction

Last time, we defined the following space of valuations on a (commutative, unital) ring A:

Definition 1.1. Let A be a ring. The *valuation spectrum* of A is

$$\operatorname{Spv}(A) \coloneqq \left\{ \begin{matrix} \text{valuations} \\ v\colon A \to \Gamma \cup \{0\} \end{matrix} \right\} \Bigg/ \begin{tikzcd} & \Gamma_v \cup \{0\} \arrow[dd, "\wr"', "\text{ordered}"] \\ A \arrow[ur, "v"] \arrow[dr, "w"'] & \\ & \Gamma_w \cup \{0\} \end{tikzcd}$$

where $\Gamma_v = \langle \operatorname{im}(v) \smallsetminus 0 \rangle \subseteq \Gamma$ is the *value group* of v. The topology on $\operatorname{Spv}(A)$ is generated by open sets of the form

$$R\left(\frac{f}{g}\right) \coloneqq \bigl\{ v \in \operatorname{Spv}(A) \bigm| v(f) \le v(g) \ne 0 \bigr\} \qquad f, g \in A.$$

We spent a long time discussing topological rings, but $\operatorname{Spv}(A)$ is not able to detect this topology. For Huber rings, our goal today is the following:

Goal 1.2. For A a Huber ring, define spectral subspaces

$$\operatorname{Spa}(A, A^+) \subseteq \operatorname{Cont}(A) \subseteq \operatorname{Spv}(A).$$

The *spectrum of continuous valuations* $\operatorname{Cont}(A)$ will respect the topology of A, and the *adic spectrum* $\operatorname{Spa}(A, A^+)$ will keep track of a subring of "integral elements."

Date: February 16, 2017. Compiled February 16, 2017.

These adic spectra will accomplish some of our motivational goals in this seminar:

- They give an algebro-geometric notion of "punctured tubular neighborhoods;"
- Affinoid perfectoid spaces [Sch12] will be of the form $\mathrm{Spa}(A, A^+)$ for A a perfectoid algebra;
- If A is Tate, then $\mathrm{Spa}(A, A^\circ)$ satisfies nice comparison results connecting Huber's theory to Tate's theory of rigid analytic spaces ([Hub93, §4], to be discussed next time).

A helpful way to organize our work will be the following diagram of functors. The functor $A \mapsto \mathrm{Cont}(A)$ factors Spv:

$$\begin{tikzcd} \mathsf{Ring}^{\mathsf{op}} \arrow[r, "\text{discrete}"] \arrow[drr, bend right, "\mathrm{Spv}"'] & \mathsf{HubRing}^{\mathsf{op}}_{\mathsf{ad}} \arrow[r, "\mathrm{Cont}"] \arrow[dr, dashed, "\exists?"] & \mathsf{Top} \arrow[d, phantom, "\cup"] \\ & & \mathsf{SpecSp} \end{tikzcd} \tag{1}$$

We have seen that $\mathrm{Spv}(A)$ is spectral [Hub93, Prop. 2.6(i)], hence we have the factorization of the functor $\mathrm{Spv}\colon \mathsf{Ring}^{\mathsf{op}} \to \mathsf{Top}$ through SpecSp, the category of spectral spaces with spectral maps. To show the factorization of Cont through SpecSp exists, we will use the construction $\mathrm{Spv}(A, I)$ from last time [Hub93, §2]. There is a similar story for *Huber pairs* (A, A^+):

$$\begin{tikzcd} \mathsf{Ring}^{\mathsf{op}} \arrow[r, "\text{discrete}"] \arrow[drr, bend right, "\mathrm{Spv}"'] & \mathsf{HubRing}^{\mathsf{op}}_{\mathsf{ad}} \arrow[r, "A \mapsto (A, \overline{\mathbf{Z}\cdot 1 + A^{\circ\circ}})"] \arrow[dr, "\mathrm{Cont}"'] & \mathsf{HubPair}^{\mathsf{op}}_{\mathsf{ad}} \arrow[d, "\mathrm{Spa}"] \\ & & \mathsf{SpecSp} \end{tikzcd}$$

The subscripts ad denote that morphisms in the corresponding categories are restricted to adic homomorphisms. In particular, each new space we introduce is more general than the last.

2. The spectrum of continuous valuations

From now on, let A be a Huber ring.

2.1. Definitions. Note that the following definition works for an arbitrary topological ring A, although we will only discuss it in the Huber case.

Definition 2.1. A valuation $v \in \mathrm{Spv}(A)$ is *continuous* if, equivalently,

- $\{f \in A \mid v(f) < \gamma\}$ is open for every $\gamma \in \Gamma_v$;
- $v\colon A \to \Gamma_v \cup \{0\}$ is continuous, where Γ_v is given the order topology; or
- The topology on A is finer than the valuation topology induced by v.

The *continuous valuation spectrum* is

$$\mathrm{Cont}(A) \coloneqq \{\text{continuous valuations}\} \subseteq \mathrm{Spv}(A),$$

which we equip with the subspace topology induced by $\mathrm{Spv}(A)$.

All valuation spectra are continuous valuation spectra, in the following sense:

Example 2.2. If A is a ring with the discrete topology, then $\mathrm{Cont}(A) = \mathrm{Spv}(A)$.

Example 2.3. Let $v \in \mathrm{Spv}(A)$ with $\Gamma_v = 1$. Then, v is continuous if and only if $\mathrm{supp}(v)$ is open.

Example 2.4. Consider $k(\!(y)\!)[\![x]\!]$ with the x-adic topology. We can visualize some points of the valuation spectrum $\mathrm{Spv}\bigl(k(\!(y)\!)[\![x]\!]\bigr)$ as in Figure 1. Since $(0) \subseteq k(\!(y)\!)[\![x]\!]$ is not open, we see that η is not continuous by Example 2.3. Every other valuation depicted in Figure 1 is continuous, where we note the xy-adic valuation w has $\Gamma_w = w(x)^{\mathbf{Z}} \times w(y)^{\mathbf{Z}}$ with the lexicographic ordering $w(x) < w(y)$.

The argument in [Con14, Ex. 6.2.1] shows that the x-adic and xy-adic topologies coincide, and so $\mathrm{Cont}\bigl(k(\!(y)\!)[\![x]\!]\bigr)$ is the same subset of $\mathrm{Spv}\bigl(k(\!(y)\!)[\![x]\!]\bigr)$ in either topology. For the y-adic topology, however, only the point $\overline{w}$ is continuous (see [Con14, Ex. 8.2.2] for a proof that w is not continuous).

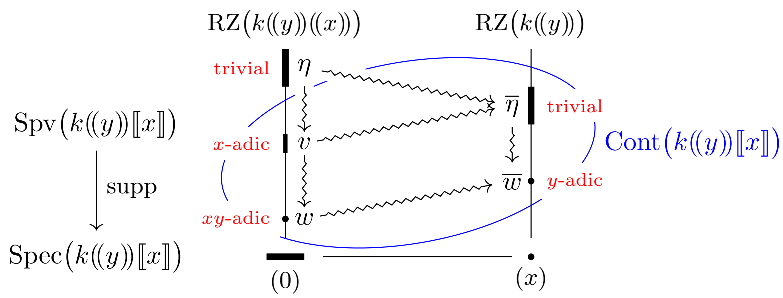

FIGURE 1. A picture of $\mathrm{Cont}\big(k((y))[\![x]\!]\big)$.

Just as for valuation spectra, continuous valuation spectra define a functor. A continuous map $f\colon A \to B$ of Huber rings induces a map

$$\begin{aligned}\mathrm{Cont}(f)\colon\ \mathrm{Cont}(B) &\longrightarrow \mathrm{Cont}(A)\\ v &\longmapsto v\circ f\end{aligned}$$

since the pullback of a continuous valuation is continuous. This map $\mathrm{Cont}(f)$ is continuous since it is the restriction of the continuous map $\mathrm{Spv}(B) \to \mathrm{Spv}(A)$ induced by f. Thus, we have a functor

$$\mathrm{Cont}\colon \mathsf{HubRing}^{\mathsf{op}} \longrightarrow \mathsf{Top}$$

which factors Spv as in (1).

2.2. **Spectrality.** We now want to show that $\mathrm{Cont}(A)$ is spectral, and the factorization of functors through SpecSp in (1) exists. The idea is to realize $\mathrm{Cont}(A)$ as a closed subspace of the space $\mathrm{Spv}(A, I)$, which we showed was spectral last time.

Definition 2.5. Let $v \in \mathrm{Spv}(v)$. The *characteristic subgroup* $c\Gamma_v$ of v is

$$c\Gamma_v \coloneqq \text{convex subgroup of } \Gamma \text{ generated by } \big\{v(a) \bigm| v(a) \geq 1\big\}, \tag{2}$$

We say an element $\gamma \in \Gamma \cup \{0\}$ is *cofinal* in a subgroup $H \subseteq \Gamma$ if for every $h \in H$, there exists $n \in \mathbf{N}$ such that $\gamma^n < h$.

Proposition 2.6 [Hub93, Prop. 2.6]**.** *The space*

$$\mathrm{Spv}(A, I) = \left\{ v \in \mathrm{Spv}(A) \;\middle|\; \begin{array}{l} \bullet\ \Gamma_v = c\Gamma_v,\ \textit{or} \\ \bullet\ v(a)\ \textit{is cofinal in}\ \Gamma_v\ \textit{for all}\ v \in I \end{array} \right\} \tag{3}$$

is spectral with a quasi-compact basis of constructible sets

$$R\left(\frac{T}{s}\right) = \big\{v \in \mathrm{Spv}(A, I) \bigm| v(f_i) \leq v(s) \neq 0 \textit{ for all } i\big\}$$
$$\emptyset \neq T = \{f_1, \ldots, f_n\} \subset A,\ \ s \in A,\ \ I \subseteq \sqrt{T \cdot A}$$

called rational domains*. This basis is stable under finite intersections.*

Here, we are using [Hub93, Lem. 2.5] to identify the description on the right-hand side of (3) with the usual definition for $\mathrm{Spv}(A, I)$. To use Proposition 2.6 to show that $\mathrm{Cont}(A)$ is spectral, we first need to find a suitable ideal I for this construction. Recall that the *topologically nilpotent elements* of A are

$$A^{\circ\circ} \coloneqq \big\{a \in A \bigm| a^n \to 0 \text{ as } n \to \infty\big\}.$$

The following suggests what we could do:

Lemma 2.7. *Let $T = \{f_1, \ldots, f_n\} \subset A$ be nonempty. Then, $T \cdot A$ is open in A if and only if $A^{\circ\circ} \cdot A \subseteq \sqrt{T \cdot A}$.*

Proof. Let I be an ideal of definition for A. Then, $T \cdot A$ is open if and only if $I^n \subseteq T \cdot A$ for some $n \geq 0$. By properties of radical ideals, this holds if and only if $A^{\circ\circ} \cdot A \subseteq \sqrt{T \cdot A}$, since $\sqrt{A^{\circ\circ} \cdot A} = \sqrt{I \cdot A}$ for any ideal of definition I: "$\supseteq$" holds since any element of I is topologically nilpotent, and "$\subseteq$" holds since $A^{\circ\circ} \subseteq \sqrt{I \cdot A}$ [Con14, Rem. 8.4.3]. See Figure 2. □

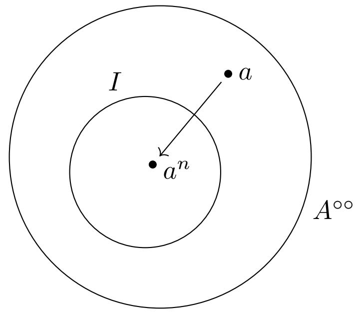

FIGURE 2. $\sqrt{A^{\circ\circ} \cdot A} = \sqrt{I \cdot A}$ for any ideal of definition I.

Thus, choosing $I = A^{\circ\circ} \cdot A$ makes $\operatorname{Spv}(A, I)$ detect the topology of A, and seems like a good candidate for $\operatorname{Cont}(A)$. This guess is *almost* correct; we have to restrict further to a particular subset of $\operatorname{Spv}(A, A^{\circ\circ} \cdot A)$ to ensure that topologically nilpotent elements are nilpotent with respect to continuous valuations.

Theorem 2.8 [Hub93, Thm. 3.1]**.** *We have*

$$\operatorname{Cont}(A) = \bigl\{v \in \operatorname{Spv}(A, A^{\circ\circ} \cdot A) \bigm| v(a) < 1 \text{ for all } a \in A^{\circ\circ}\bigr\} \tag{4}$$

in $\operatorname{Spv}(A)$.

Theorem 2.8 will be the key result necessary to achieve Goal 1.2 for $\operatorname{Cont}(A)$:

Corollary 2.9 [Hub93, Cor. 3.2]**.** $\operatorname{Cont}(A)$ *is a closed subset of* $\operatorname{Spv}(A, A^{\circ\circ} \cdot A)$*, hence is spectral and closed under specialization.*

Proof of Corollary 2.9, following [Wed12, Cor. 7.12]. The set

$$\operatorname{Spv}(A, A^{\circ\circ} \cdot A) \smallsetminus \operatorname{Cont}(A) = \bigcup_{a \in A^{\circ\circ}} \operatorname{Spv}(A, A^{\circ\circ} \cdot A)\bigl(\tfrac{1}{a}\bigr)$$

is open since each set on the right-hand side is open. Thus, $\operatorname{Cont}(A)$ is closed in $\operatorname{Spv}(A, A^{\circ\circ} \cdot A)$, hence spectral by [Hub93, Rem. 2.1(*iv*)]. □

Proof of Theorem 2.8. "$\subseteq$". Let $w \in \operatorname{Cont}(A)$ and $a \in A^{\circ\circ}$. For $n \gg 0$, we have

$$w(a^n) = w(a)^n < \gamma$$

by continuity for any $\gamma \in \Gamma_w$. Thus, $w(a) < 1$ by choosing $\gamma = 1$, and $w(a)$ is cofinal in Γ_w, so $w \in \operatorname{Spv}(A, A^{\circ\circ} \cdot A)$ by (3).

"$\supseteq$". Let v as on the right-hand side of (4).

Step 1. $v(a)$ is cofinal in Γ_v for all $a \in A^{\circ\circ}$.

If $\Gamma_v \neq c\Gamma_v$, then we are done by (3).

If $\Gamma_v = c\Gamma_v$, then let $\gamma \in \Gamma_v$ be given. If $\gamma \geq 1$, then we are done since $v(a) < 1$ by hypothesis. Otherwise, suppose $\gamma < 1$. Then, by the definition of the characteristic subgroup (2), there exist $t, t' \in A$ such that $v(t) \neq 0$, and

$$v(t)^{-1} \leq \frac{v(t')}{v(t)} \leq \gamma < 1.$$

Now choose $n \in \mathbf{N}$ such that $ta^n \in A^{\circ\circ}$. Then, $v(ta^n) < 1$, hence $v(a)^n < \gamma$. We can visualize this situation as in Figure 3.

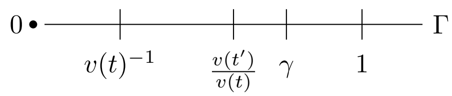

FIGURE 3. A visualization of Theorem 2.8, Step 1.

Step 2. $v \in \mathrm{Cont}(A)$.

Let $S = \{t_1, \ldots, t_r\}$ be a set of generators for an ideal of definition I of A. Set $\delta = \max\{v(t_i)\}$; we have $\delta < 1$ since $t_i \in A^{\circ\circ}$ for all i. Since each $t_i \in A^{\circ\circ}$, by Step 1 there exists $n \in \mathbf{N}$ such that $\delta^n < \gamma$. Thus, $v(S^n \cdot I) = v(I^{n+1}) < \gamma$ and so $I^{n+1} \subseteq \{f \in A \mid v(f) < \gamma\}$. □

What is left is to show that the spectrality of $\mathrm{Cont}(A)$ gives a factorization of the functor Cont through SpecSp as in (1).

Proposition 2.10 [Hub93, Prop. 3.8(*iv*)]**.** *The functor* $\mathrm{Spa}\colon \mathsf{HubRing}^{\mathsf{op}} \to \mathsf{Top}$ *maps adic homomorphisms to spectral maps.*

Proof. It suffices to check rational domains pull back to rational domains, since the constructible topology is generated by finite boolean combinations of rational domains. We see that

$$g^{-1}\left(R\left(\frac{T}{s}\right)\right) = R\left(\frac{f(T)}{f(s)}\right),$$

where the adic condition ensures the set on the right is indeed rational. □

2.3. **Analytic points.** We now come to a notion that is a bit unmotivated at first glance, but will come up again when we discuss the relationship between adic spaces and other flavors of non-Archimedean geometry using formal schemes and rigid-analytic spaces in [Hub94].

Definition 2.11. We say $v \in \mathrm{Cont}(A)$ is *analytic* if the support $\mathrm{supp}(v)$ is not open in A. We put

$$\begin{aligned} \mathrm{Cont}(A)_a &\coloneqq \{v \in \mathrm{Cont}(A) \mid v \text{ is analytic}\} \\ \mathrm{Cont}(A)_{na} &\coloneqq \mathrm{Cont}(A) \smallsetminus \mathrm{Cont}(A)_a \end{aligned}$$

Example 2.12. If A has the discrete topology, then every point is not analytic.

We give an alternative characterization for analyticity, which is related to our original goal of finding an algebro-geometric definition for a punctured tubular neighborhood:

Proposition 2.13 [Con14, Prop. 8.3.2]**.** *Let $T \subset A^{\circ\circ}$ be a finite set such that $A^{\circ\circ} \cdot A \subseteq \sqrt{T \cdot A}$. Then, $v \in \mathrm{Cont}(A)$ is analytic if and only if $v(t) \neq 0$ for some $t \in T$.*

Proof. $\mathrm{supp}(v) \subseteq A$ is open if and only if $(T \cdot A)^n \subseteq \mathrm{supp}(v)$ for some $n \gg 0$. But $\mathrm{supp}(v)$ is prime, hence radical, so this is equivalent to having $T \subseteq \mathrm{supp}(v)$, i.e., $v(T) = 0$. □

The following statement justifies why we will not study analytic points in too much detail, since perfectoid algebras are Tate.

Corollary 2.14 [Con14, Cor. 8.3.3]**.** *If A is Tate, then* $\mathrm{Cont}(A) = \mathrm{Cont}(A)_a$.

Proof. This follows from Proposition 2.13 since the topologically nilpotent unit u satisfies $u^n \in T \cdot A$, and $v(u)$ cannot be zero. □

We now illustrate analyticity with an example:

Example 2.15. Let $A = k(\!(y)\!)[\![x]\!]$ with the x-adic topology as in Example 2.4. Then,

$$(x) = \{f \in A \mid v_x(f) < 1\}$$

is open in A, and so the analytic points are those lying over (0) that are also in $\mathrm{Cont}(A)$. Alternatively, an ideal of definition for A is given by (x), and so the analytic points are those such that $v(x) \neq 0$, using Proposition 2.13. See Figure 4. This suggests that the analytic points of $\mathrm{Cont}(A)$ look like a punctured tubular neighborhood.

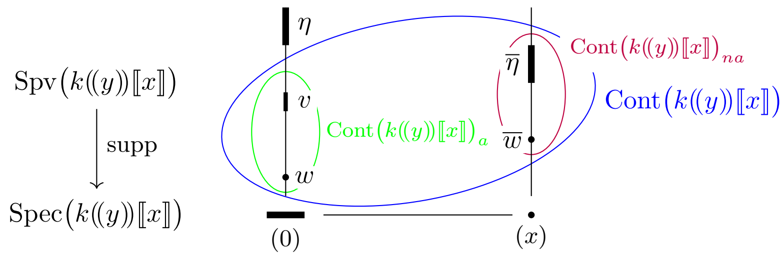

FIGURE 4. (Non-)analytic points in $\mathrm{Cont}\big(k((y))[[x]]\big)$.

Proposition 2.16 [Con14, Prop. 8.3.8]**.** *As subsets of* $\mathrm{Cont}(A)$*, the analytic points* $\mathrm{Cont}(A)_a$ *form an open set, and the non-analytic points* $\mathrm{Cont}(A)_{na}$ *form a closed set.*

Proof. Let $T \subset A^{\circ\circ}$ be a finite subset of $A^{\circ\circ}$ such that $A^{\circ\circ} \cdot A \subseteq \sqrt{T \cdot A}$. Then, Proposition 2.13 implies

$$\mathrm{Cont}(A)_a = \big\{v \in \mathrm{Cont}(A) \;\big|\; v(t) \neq 0 \text{ for some } t \in T\big\} = \bigcup_{t \in T} R\left(\frac{T}{t}\right),$$

which is open. ☐

Remark 2.17. One can show that $A\left(\frac{T}{t}\right)$ is a Tate ring, whose adic spectrum can be identified with $R\left(\frac{T}{t}\right)$ [Con14, Rem. 8.3.9]. This suggests another way to think about analytic points: x is analytic if and only if there is an open neighborhood of x that is the adic spectrum of a Tate ring [Hub94, Rem. 3.1]. Spaces where all points are analytic are the well-behaved spaces in Huber's theory, reminiscent of "good" k-analytic spaces in Berkovich's theory.

Lemma 2.18. *There are no horizontal specializations in* $\mathrm{Cont}(A)_a$*. In particular, if A is Tate, then there are no horizontal specializations.*

Proof. A horizontal specialization $v|_H$ satisfies

$$\mathrm{supp}(v|_H) = \bigcup_{\gamma \in \Gamma_v \smallsetminus H} \big\{a \in A \;\big|\; v(a) < \gamma\big\},$$

which is open. Thus, $v|_H$ is not analytic. The last statement follows from Corollary 2.14. ☐

This next result suggests that restricting to analytic points takes out the trivially valued points:

Lemma 2.19. *For every* $v \in \mathrm{Cont}(A)_a$*,* $\mathrm{rk}\,\Gamma_v \geq 1$*, and* $\mathrm{rk}\,\Gamma_v = 1$ *if and only if v is a maximal point of* $\mathrm{Cont}(A)_a$*, i.e., a point with no generizations.*

Proof Sketch. The first statement follows since any continuous valuation such that $\Gamma_v = \{1\}$ must have open support Example 2.3. Now if $\mathrm{rk}\,\Gamma_v = 1$, then only way a generization could occur is if it were vertical by Lemma 2.18. Let w be such a generization. One can show that v and w both induce the same topology, hence (since they are of rank 1) must coincide [Con14, Prop. 9.1.5]. ☐

Proposition 2.20 [Hub93, Prop. 3.8]**.** *Let* $f\colon A \to B$ *be continuous, and let* $g\colon \mathrm{Cont}(B) \to \mathrm{Cont}(A)$ *be the map induced by f. Then,*

(*i*) *g preserves non-analytic points;*
(*ii*) *If f is adic, then g preserves analytic points;*

(*iii*) *If B is complete and g preserves analytic points, then f is adic;*

We won't prove (*iii*); see [Hub93, Prop. 3.8(*iii*)].

Proof of (*i*) *and* (*ii*). Consider the composition

$$A \xrightarrow{f} B \xrightarrow{v} \Gamma_v \cup \{0\},$$

and consider the preimage of zero $\operatorname{supp}(v)$.

For (*i*), if $\operatorname{supp}(v)$ is open, then $f^{-1}(\operatorname{supp}(v)) = \operatorname{supp}(v \circ f)$ is open by continuity.

For (*ii*), let $I \subseteq A_0$ be an ideal of definition for A. Then, if $\operatorname{supp}(v) \subseteq B$ is not open, it does not contain the ideal of definition $f(I)B_0$, and so $\operatorname{supp}(v \circ f) \subseteq B$ cannot contain $I \subseteq A_0$. □

3. The adic spectrum

To give better comparison results with rigid-analytic geometry [Hub93, §4] and for applications [Hub94; Sch12], we need to restrict to even smaller pro-constructible subspaces of $\operatorname{Spv}(A)$.

3.1. **Definitions and the "adic Nullstellensatz".** To not lose "too much information" when we pass to a smaller pro-constructible set, we will restrict to the case when these subspaces are dense. The following Nullstellensatz-type result motivates our definition for which pro-constructible sets should be permissible.

Lemma 3.1 ("adic Nullstellensatz"[1] [Hub93, Lem. 3.3])**.**

(*i*) *There is a inclusion-reversing bijection*

$$\mathscr{G}_A := \left\{\begin{matrix}\textit{open, integrally closed}\\ \textit{subrings of } A\end{matrix}\right\} \underset{\tau}{\overset{\sigma}{\rightleftarrows}} \left\{\begin{matrix}\textit{pro-constructible subsets}\\ \textit{of } \operatorname{Cont}(A) \textit{ that are}\\ \textit{intersections of sets}\\ \{v \in \operatorname{Cont}(A) \mid v(a) \le 1\}\end{matrix}\right\} =: \mathscr{F}_A$$

$$G \longmapsto \{v \in \operatorname{Cont}(A) \mid v(g) \le 1 \textit{ for all } g \in G\}$$

$$\{a \in A \mid v(a) \le 1 \textit{ for all } v \in F\} \longleftarrow F$$

(*ii*) *If $G \in \mathscr{G}_A$ satisfies $G \subseteq A^\circ$, then $\sigma(G)$ is dense in* $\operatorname{Cont}(A)$.
(*iii*) *The converse of* (*ii*) *holds if A is a Tate ring that has a noetherian ring of definition.*

Example 3.2. The subring A° of power-bounded elements is open (it is the union of all rings of definition by [Hub93, Cor. 1.3(*iii*)]) and integrally closed, so $\sigma(A^\circ)$ is dense in $\operatorname{Cont}(A)$ by (*ii*).

This description of pro-constructible sets in $\operatorname{Cont}(A)$ motivates the following:

Definition 3.3.

(*i*) A subring $A^+ \subseteq A$ that is open, integrally closed, and contained in A° is called a *ring of integral elements* of A.
(*ii*) A *Huber pair*[2] is a pair (A, A^+) where A is a Huber ring and A^+ is a ring of integral elements of A. A *morphism* of Huber pairs $(A, A^+) \to (B, B^+)$ is a ring homomorphism $f\colon A \to B$ such that $f(A^+) \subseteq B^+$, and $(A, A^+) \to (B, B^+)$ is *continuous* or *adic* if f is.

[1] This name is inspired by the discussion in [Con14, §10.3].
[2] These are called *affinoid rings* in [Hub93; Wed12, §7.3]. Affinoid algebras are something different in [Sch12, Def. 2.6], so we use Conrad's terminology instead [Con14, Def. 10.3.3].

(*iii*) For a Huber pair (A, A^+), the *adic spectrum* is
$$\mathrm{Spa}(A, A^+) \coloneqq \sigma(A^+) = \{v \in \mathrm{Cont}(A) \mid v(a) \le 1 \text{ for all } a \in A^+\} \subseteq \mathrm{Cont}(A),$$
where the topology is the subspace topology induced from $\mathrm{Cont}(A)$. If $f\colon (A, A^+) \to (B, B^+)$ is continuous, we get a continuous map
$$\mathrm{Spa}(f)\colon\ \mathrm{Spa}(B, B^+) \longrightarrow \mathrm{Spa}(A, A^+)$$
via restriction from $\mathrm{Cont}(f)$. We therefore obtain a functor
$$\mathrm{Spa}\colon \mathsf{HubPair}^{\mathsf{op}} \longrightarrow \mathsf{Top},$$
where $\mathsf{HubPair}$ is the category of Huber pairs with continuous morphisms.

Remark 3.4. By Lemma 3.1(*i*), if $f \in A$ is such that $v(f) \le 1$ for all $v \in \mathrm{Spa}(A, A^+)$, then $f \in A^+$. This justifies the idea that $\mathrm{Spa}(A, A^+)$ keeps track of a ring of integral elements A^+.

Remark 3.5. In [Hub94, §1], Huber constructs a presheaf on $\mathrm{Spa}(A, A^+)$ for any Huber pair (A, A^+). This is what is necessary to make the statement in Remark 2.17 make sense.

We will see in §3.2 that the functor Spa factors through SpecSp.

Example 3.6. Let A be a Huber ring. Let B be the integral closure of $\mathbf{Z} \cdot 1 + A^{\circ\circ}$ in A. This is the smallest ring of integral elements of A, since any other open subring B' contains a power of $A^{\circ\circ}$, and if B' is integrally closed, then it contains $A^{\circ\circ}$. Moreover, $\mathrm{Cont}(A) = \mathrm{Spa}(A, B)$, since $v(a) \le 1$ for all $a \in B$.

Remark 3.7. One may think the only example we need to consider is when $A^+ = A^\circ$. We give two reasons why we need the flexibility of changing A^+ from [Sch12, p. 254]:

(1) Points $v \in \mathrm{Spa}(A, A^+)$ give rise to pairs (L, L^+) where L is some non-Archimedean extension of $K = \mathrm{Frac}(A/\operatorname{supp}(v))$, where $L^+ \subset L^\circ$ is an open valuation subring [Sch12, Prop. 2.27]. If $\mathrm{rk}(\Gamma_v) \neq 1$, then $L^+ \neq L^\circ$.
(2) The condition $R^+ = R^\circ$ is not necessarily preserved under passage to a rational domain.

We only show (*i*) and (*ii*) in Lemma 3.1; for (*iii*), see [Hub93, Lems. 3.3(*iii*), 3.4].

Proof of Lemma 3.1(i). We first note $\sigma(G)$ is pro-constructible since every set of the form
$$\{v \in \mathrm{Cont}(A) \mid v(a) \le 1\}$$
is constructible Proposition 2.6. The fact that $\sigma \circ \tau = \mathrm{id}$ follows by definition, and so the hard part is showing that $\tau \circ \sigma = \mathrm{id}$.

Let $G \in \mathscr{G}_A$. Then, by definition, we have $G \subseteq \tau(\sigma(G))$. Suppose, for the sake of contradiction, that there exists $a \in \tau(\sigma(G)) \smallsetminus G$. We will show that $v(a) > 1$ for some valuation v on A. The idea will be to construct a valuation on A, and then to use horizontal specialization to ensure it is continuous. See Figure 5 for a geometric representation of the steps involved.

Consider the inclusion of rings
$$G[a^{-1}] \subseteq A_a.$$
Step 1. There exists $s \in \mathrm{Spv}\big(G[a^{-1}]\big)$ such that $s(a) > 1$ and $s(g) \le 1$ for all $g \in G$.

Note $a^{-1} \notin G^\times$; otherwise, a is integral over G, hence in G. Thus, there exists $\mathfrak{p} \in \mathrm{Spec}\big(G[a^{-1}]\big)$ containing a^{-1}, and a minimal prime $\mathfrak{q}$ contained in $\mathfrak{p}$. Now consider a valuation ring
$$R \subseteq \mathrm{Frac}\big(G[a^{-1}]/\mathfrak{q}\big)$$
dominating the local ring $\big(G[a^{-1}]/\mathfrak{q}\big)_{\mathfrak{p}/\mathfrak{q}}$. The valuation ring R corresponds to $s \in \mathrm{Spv}\big(G[a^{-1}]\big)$.

- $s(g) \le 1$ for all $g \in G$, since $\big(G[a^{-1}]/\mathfrak{q}\big)_{\mathfrak{p}/\mathfrak{q}} \subseteq R$.
- $s(x) < 1$ for all $x \in \mathfrak{p}$, since R dominates the local ring $\big(G[a^{-1}]/\mathfrak{q}\big)_{\mathfrak{p}/\mathfrak{q}}$. Thus, $s(a^{-1}) < 1$.

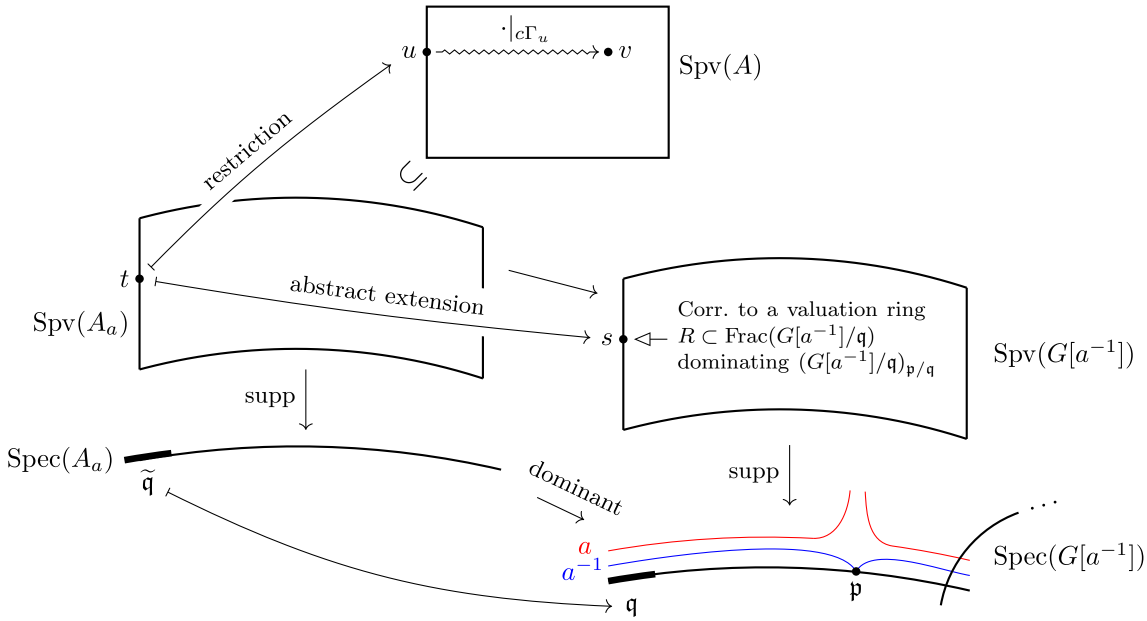

FIGURE 5. A visualization of the proof of Lemma 3.1.

Step 2. There exists $u \in \mathrm{Spv}(A)$ such that $u(a) > 1$ and $u(g) \leq 1$ for all $g \in G$.

We first claim s extends to a valuation $t \in \mathrm{Spv}(A_a)$. But we have an inclusion

$$G[a^{-1}]_{\mathfrak{q}} \hookrightarrow (A_a)_{\mathfrak{q}},$$

so the latter is nonzero, and contains a prime $\widetilde{\mathfrak{q}}$ whose contraction is contained in $\mathfrak{q}$ hence equals $\mathfrak{q}$ by minimality. We then get an extension of fields

$$\mathrm{Frac}\big(G[a^{-1}]/\mathfrak{q}\big) \hookrightarrow \mathrm{Frac}\big(A_a/\widetilde{\mathfrak{q}}\big),$$

hence s abstractly extends to some valuation $t \in \mathrm{Spv}(A_a)$ by Zorn's lemma. Finally, the restriction $u = t|_A \in \mathrm{Spv}(A)$ satisfies $u(a) > 1$ and $u(g) \leq 1$ for all $g \in G$.

Step 3. There exists $v \in \mathrm{Cont}(A)$ such that $v(a) > 1$ and $v(g) \leq 1$ for all $g \in G$.

Let $v = u|_{c\Gamma_u} \in \mathrm{Spv}(A)$ be the horizontal specialization of u along $c\Gamma_u$; this satisfies $v(a) > 1$ and $v(g) \leq 1$ for all $g \in G$ by definition since these hold for u, and so it suffices to show $v \in \mathrm{Cont}(A)$. By Theorem 2.8, it suffices to show

- $v(x) < 1$ for all $x \in A^{\circ\circ}$;
- $v \in \mathrm{Spv}(A, A^{\circ\circ} \cdot A)$.

The latter holds by (3) since $v = u|_{c\Gamma_u}$ satisfies $\Gamma_v = c\Gamma_v = c\Gamma_u$. For the latter, let $x \in A^{\circ\circ}$. Then, G is open, so there exists $n \in \mathbf{N}$ with $x^n a \in G$. Thus, $v(x^n a) \leq 1$, so $v(x)^n \leq v(a^{-1}) < 1$, hence $v(x) < 1$.

Finally, $v(a) > 1$ implies $a \notin \tau(\sigma(G))$, contradicting our assumption that $a \in \tau(\sigma(G))$. $\square$

Proof of Lemma 3.1(ii). We show something a bit stronger: Every point $v \in \mathrm{Cont}(A)$ is a vertical specialization of a point in $\sigma(G)$. Let $v \in \mathrm{Cont}(A)$.

If v is not analytic, i.e., $\mathrm{supp}(v)$ is open, then the trivial valuation v/Γ_v is in $\sigma(G) \subseteq \mathrm{Cont}(A)$ by Example 2.3.

Suppose v is analytic, i.e., $\mathrm{supp}(v)$ is not open. Then, $\mathrm{supp}(v)$ cannot contain $A^{\circ\circ}$, and so there exists $a \in A^{\circ\circ}$ such that $v(a) > 0$. Let H be the largest convex subgroup of Γ_v with $v(a) \notin H$. We claim that $w \coloneqq v/H \in \sigma(G)$. Note w is continuous since it is the composition

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[dr, "w"'] & \Gamma_v \cup \{0\} \arrow[d, two heads] \\ & \Gamma_v/H \cup \{0\} \end{tikzcd}$$

and the vertical quotient map is continuous. Now let $g \in G$; we have to show that $w(g) \leq 1$. Assume $w(g) > 1$. Since Γ_w has rank 1 and $w(a) \neq 0$, there exists $n \in \mathbf{N}$ with $w(g^n a) > 1$. On the other hand, since $a \in A^{\circ\circ}$ and $g \in A^{\circ}$, we have $g^n a \in A^{\circ\circ}$ hence $w(g^n a) < 1$ by continuity of w, which is a contradiction. $\square$

Remark 3.8. This proof shows that any non-trivial vertical generization of a continuous valuation remains continuous [Con14, Thm. 8.2.1], and that any $v \in \mathrm{Cont}(A)_a$ has a vertical generization $w \in \mathrm{Cont}(A)_a$ with $\mathrm{rk}(\Gamma_w) = 1$ [Con14, Prop. 9.1.5].

3.2. **Spectrality.** We saw in [Hub93, Prop. 2.6] that $\mathrm{Spv}(A, I)$ is spectral, and rational domains form a basis; we want an analogous result for $\mathrm{Spa}(A, A^+)$. We first define rational domains:

Definition 3.9. Let (A, A^+) be a Huber pair. A *rational domain* in $\mathrm{Spa}(A, A^+)$ is a set

$$R\left(\frac{T}{s}\right) \coloneqq \left\{v \in \mathrm{Spa}(A, A^+) \;\middle|\; v(t) \leq v(s) \neq 0 \text{ for all } t \in T\right\}$$

where $s \in A$ and $T \subset A$ is a finite nonempty subset such that $T \cdot A$ is open in A.

We can now state what Huber calls his "first main theorem," which is an immediate consequence of our work so far.

Theorem 3.10 [Hub93, Thm. 3.5]**.** *Let* $X = \mathrm{Spa}(A, A^+)$.

- (i) *X is a spectral space.*
- (ii) *Rational domains form a quasi-compact basis of X that is closed under finite intersection, and every rational domain is constructible in X.*

Proof of Theorem 3.10(i). For (i), we note any pro-constructible subset of a spectral space is spectral [Hub93, Rem. 2.1(iv)]. But $\mathrm{Spa}(A, A^+)$ is a pro-constructible subset of $\mathrm{Cont}(A)$ by Lemma 3.1(i).

To prove (ii), we recall that rational domains of the form

$$\left\{v \in \mathrm{Spv}(A, A^{\circ\circ} \cdot A) \;\middle|\; v(t) \leq v(s) \neq 0 \text{ for all } t \in T\right\} \subseteq \mathrm{Spv}(A, A^{\circ\circ} \cdot A)$$

for $s \in A$ and $T \subset A$ a finite subset such that $A^{\circ\circ} \cdot A \subseteq \sqrt{T \cdot A}$ form a basis for $\mathrm{Spv}(A, A^{\circ\circ} \cdot A)$ by Proposition 2.6. We showed in Lemma 2.7 that this condition on T is equivalent to the condition in Definition 3.9. Constructible sets remain constructible after restriction to a pro-constructible subspace [Hub93, Rem. 2.1(iv)], so rational domains are constructible. Finally, constructible opens are quasi-compact [Hub93, Rem. 2.1(i)]. $\square$

3.3. **Nonemptiness criteria.** Recall that for a ring B, $\mathrm{Spec}(B) = \emptyset$ if and only if $B = 0$. We have a similar statement for $\mathrm{Spa}(A, A^+)$:

Proposition 3.11 . *Let (A, A^+) be a Huber pair. Then,*

- (i) $\mathrm{Spa}(A, A^+) = \emptyset$ *if and only if* $A/\overline{\{0\}} = 0$.
- (ii) $\mathrm{Spa}(A, A^+)_a = \emptyset$ *if and only if the topology of* $A/\overline{\{0\}}$ *is discrete.*

Proof. We first note that the map

$$\begin{aligned} \mathrm{Spa}\big(A/\overline{\{0\}}, A^+/\overline{\{0\}}\big) &\longrightarrow \mathrm{Spa}(A, A^+) \\ \big(A/\overline{\{0\}} \xrightarrow{v} \Gamma_v \cup \{0\}\big) &\longmapsto \big(A \twoheadrightarrow A/\overline{\{0\}} \xrightarrow{v} \Gamma_v \cup \{0\}\big) \end{aligned}$$

is a bijection, since $v(\overline{\{0\}}) = 0$ for any $v \in \mathrm{Cont}(A)$ by continuity.

$\Leftarrow$. For (*i*), note that the zero ring has no valuations. For (*ii*), a discrete ring has no analytic points by Example 2.12.

$\Rightarrow$. We first show (*i*), assuming (*ii*). If $\mathrm{Spa}(A, A^+) = \emptyset$, then $\mathrm{Spa}(A, A^+)_a = \emptyset$, and so $A/\overline{\{0\}}$ is discrete. If it were not zero, then the trivial valuation at a residue field of $A/\overline{\{0\}}$ would be in $\mathrm{Spa}(A, A^+)$, contradicting that $\mathrm{Spa}(A, A^+) = \emptyset$. We now show (*ii*) in three steps.

Step 1 [Hub93, Lem. 3.7]. Let B be an open subring of A. Let $f\colon \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ be the morphism of schemes induced by the inclusion $B \subseteq A$. Let

$$T = \big\{\mathfrak{p} \in \mathrm{Spec}(B) \bigm| \mathfrak{p} \text{ is open}\big\} \subseteq \mathrm{Spec}(B)$$

be the locus of primes that support non-analytic valuations. Then,

$$f^{-1}(T) = \big\{\mathfrak{p} \in \mathrm{Spec}(A) \bigm| \mathfrak{p} \text{ is open}\big\} \subseteq \mathrm{Spec}(A)$$

and the restriction $\mathrm{Spec}(A) \smallsetminus f^{-1}(T) \to \mathrm{Spec}(B) \smallsetminus T$ of f is an isomorphism.

Let $\mathfrak{p} \in \mathrm{Spec}(B) \smallsetminus T$, and let $s \in B^{\circ\circ}$ such that $s \notin B$. For every $a \in A$ there exists $n \in \mathbf{N}$ with $s^n a \in B$ since B is open in A. Then, the ring homomorphism $B_s \to A_s$ is an isomorphism. The description of $f^{-1}(T)$ follows from Proposition 2.20(*i*).

Step 2. Let B be a ring of definition for A, with ideal of definition I. Let $\mathfrak{p} \subseteq \mathfrak{q}$ be two prime ideals in B. If $I \subseteq \mathfrak{q}$, then $I \subseteq \mathfrak{p}$, that is, we have a diagram

$$\begin{tikzcd} \mathfrak{p} \arrow[dr, hook] & \\ & \mathfrak{q} \\ I \arrow[ur, hook] \arrow[uu, hook, dashed] & \end{tikzcd}$$

Geometrically, $V(I)$ contains every irreducible component of $\mathrm{Spec}(B)$ that it touches.[3]

Suppose $I \not\subseteq \mathfrak{p}$. Let u be a valuation of B with $\mathfrak{p} = \mathrm{supp}(u)$ such that the valuation ring for u dominates the local ring $(B/\mathfrak{p})_{\mathfrak{q}/\mathfrak{p}}$. Let $r\colon \mathrm{Spv}(B) \to \mathrm{Spv}(B, I)$ be the retraction from [Hub93, Prop. 2.6(*iii*)]. Then, $r(u)$ is a continuous valuation of B with $I \not\subseteq \mathrm{supp}(r(u))$ by Theorem 2.8 and [Hub93, Prop. 2.6(*iv*)], and so $\mathrm{supp}(r(u))$ is not open. By Step 1, there then exists $v \in \mathrm{Cont}(A)$ with $r(u) = v|_B$, and by Lemma 3.1(*ii*), there exists a vertical generization $w \in \mathrm{Spa}(A, A^+)$ of v, which is analytic by Step 1, a contradiction.

Step 3. The topology of $A/\overline{\{0\}}$ is discrete.

Consider the localization

$$\varphi\colon B \longrightarrow (1 + I)^{-1}B =: C.$$

Then, $\varphi(I) \cdot C \subseteq \mathfrak{R}(C)$, where $\mathfrak{R}$ denotes the Jacobson radical [AM69, Exc. 3.2]. Thus, every maximal ideal in C contains I, and Step 2 implies that $\varphi(I) \cdot C$ is contained in every *prime* ideal of C, i.e., $\varphi(I) \cdot C \subseteq \mathfrak{N}(C)$, the nilradical of C. Since I is finitely generated, there exists $n \in \mathbf{N}$ with

$$\varphi(I^n) \cdot C = \{0\}.$$

By definition of the localization, there exists $i \in I$ with $(1 + i)I^n = \{0\}$ in B. Thus, $I^n \subset I^{n+1}$, so $I^n = I^{n+1}$ and $I^n = I^k$ for every $k \geq n$ by multiplying by appropriate powers of I on both sides. Thus, the topology of $A/\overline{\{0\}}$ is discrete. $\square$

[3]We owe this geometric interpretation to [Con14, Prop. 11.6.1], who also says "Huber employ[s] a fluent command of valuation theory (using vertical generization and horizontal specialization i[n] clever ways)" to prove Step 2.

3.4. **Invariance under completion.** We now come to our last result, which really gives credence to the interpretation of $\operatorname{Spa}(A, A^+)$ as a "punctured tubular neighborhood." Let (A, A^+) be a Huber pair. Then, $(\widehat{A}, \widehat{A^+})$ is also a Huber pair (after possibly taking the integral closure of $\widehat{A^+}$; see [Con14, Rem. 11.5.2]).

Proposition 3.12 [Hub93, Prop. 3.9]**.** *The canonical map*

$$g\colon \operatorname{Spa}(\widehat{A}, \widehat{A^+}) \longrightarrow \operatorname{Spa}(A, A^+)$$

is a homeomorphism identifying rational domains.

We start with two preparatory Lemmas.

Lemma 3.13 [Hub93, Lem. 3.11]**.** *Let X be a quasi-compact subset of* $\operatorname{Spa}(A, A^+)$*, and let $s \in A$ such that $v(s) \neq 0$ for all $x \in X$. Then, there exists a neighborhood U of* 0 *in A such that $v(u) < v(s)$ for all $v \in X, u \in U$.*

Proof. Let $T \subset A^{\circ\circ}$ be finite such that $T \cdot A^{\circ\circ}$ is open. For each $n \in \mathbf{N}$, put

$$X_n = R\biggl(\frac{T^n}{s}\biggr) \subseteq \operatorname{Spa}(A, A^+).$$

Each X_n is open, and $X \subseteq \bigcup_{n\in\mathbf{N}} X_n$. By quasi-compactness, $X \subseteq X_m$ for some $m \in \mathbf{N}$. The set

$$U = T^m \cdot A^{\circ\circ}$$

is then an open neighborhood of 0 in A, and $v(u) < v(s)$ for all $v \in X, u \in U$. ☐

The next Lemma says that rational domains in $\operatorname{Spa}(\widehat{A}, \widehat{A^+})$ are insensitive to small perturbations in defining parameters. This is the trickiest part of the proof; see [Con14, §11.5].

Lemma 3.14 [Hub93, Lem. 3.10]**.** *Suppose A is complete, and let $s, t_1, \ldots, t_n \in A$ such that the ideal $I = (t_1, \ldots, t_n)A$ is open in A. Then, there is a neighborhood $U \subseteq A$ of* 0 *such that*

$$R\biggl(\frac{t_1, \ldots, t_n}{s}\biggr) = R\biggl(\frac{t'_1, \ldots, t'_n}{s'}\biggr)$$

for all $s' \in s + U$ and $t'_i \in t_i + U$ such that $I' = (t'_1, \ldots, t'_n)A$ is open in A.

Proof. Let B be a ring of definition of A. Let $r_1, \ldots, r_m \in B \cap I$ such that $J \coloneqq (r_1, \ldots, r_m)B$ is open in B. By [Bou98, Ch. III, §2, n° 8, Cor. 2 to Thm. 1], there exists a neighborhood $V \subseteq B$ of 0 such that $J = (r'_1, \ldots, r'_m)B$ for any $r'_i \in r_i + V$. There therefore exists a neighborhood U' of 0 in A such that $(t'_1, \ldots, t'_n)A$ is open in A where $t'_i \in t_i + U$.

Now let $t_0 \coloneqq s$. For each $i \in \{0, \ldots, n\}$, let

$$R_i = R\biggl(\frac{t_0, \ldots, t_n}{t_i}\biggr).$$

Then, R_i is quasi-compact by Theorem 3.10(*ii*), and $v(t_i) \neq 0$ for every $v \in R_i$. By applying Lemma 3.13 to each R_i separately, and then taking the intersection of the resulting open sets, there exists a neighborhood U'' of 0 in A such that $v(u) < v(t_i)$ for every $u \in U'', i \in \{0, \ldots, n\}, v \in R_i$.

Claim. The open set $U = U' \cap U'' \cap A^{\circ\circ}$ works.

Step 1. $R_0 \subseteq R\bigl(\frac{t'_1, \ldots, t'_n}{t'_0}\bigr)$.

Let $v \in R_0$ be given. Since $t'_i - t_i \in U''$ for $i = 0, \ldots, n$, we have

$$v(t'_i - t_i) < v(t_0)$$

for $i = 0, \ldots, n$. This implies for every $i = 1, \ldots, n$,

$$v(t'_i) = v\bigl(t_i + (t'_i - t_i)\bigr) \leq \max\bigl\{v(t_i), v(t'_i - t_i)\bigr\} \leq v(t_0) = v\bigl(t_0 + (t'_0 - t_0)\bigr) = v(t'_0).$$

Thus, $v \in R\left(\frac{t'_1,\ldots,t'_n}{t'_0}\right)$.

Step 2. $R_0 \supseteq R\left(\frac{t'_1,\ldots,t'_n}{t'_0}\right)$.

Suppose $v \notin R_0$. First suppose $v(t_i) = 0$ for all i. Then, $\operatorname{supp}(v) \supseteq I$, hence $\operatorname{supp}(v)$ is open. Thus, $t'_0 - t_0 \in \operatorname{supp}(v)$ (since $t'_0 - t_0 \in A^{\circ\circ}$) which implies $t'_0 \in \operatorname{supp}(v)$. Thus, $v \notin R\left(\frac{t'_1,\ldots,t'_n}{t'_0}\right)$.

Otherwise, suppose $v(t_i) \neq 0$ for some i. Let j such that

$$v(t_j) = \max\bigl\{v(t_0), \ldots, v(t_n)\bigr\}.$$

We have $v(t_0) < v(t_j)$, for otherwise $v \in R_0$. Since $t'_i - t_i \in U''$ for every i and $v \in R_j$, we have $v(t'_i - t_i) < v(t_j)$ for all i. Then,

$$v(t'_0) = v\bigl(t_0 + (t'_0 - t_0)\bigr) \leq \max\bigl\{v(t_0), v(t'_0 - t_0)\bigr\} < v(t_j) = v\bigl(t_j + (t'_j - t_j)\bigr) = v(t'_j),$$

hence $v \notin R\left(\frac{t'_1,\ldots,t'_n}{t'_0}\right)$. □

We can now show Proposition 3.12.

Proof of Proposition 3.12. Since continuous valuations extend continuously when taking completions in a unique way, the map g is a bijection. Since we already know rational domains pull back Proposition 2.10, it suffices to show that if $U \subseteq \operatorname{Spa}(\widehat{A}, \widehat{A^+})$ is a rational domain, then $g(U)$ is a rational domain in $\operatorname{Spa}(A, A^+)$.

Let $i\colon A \to \widehat{A}$ be the natural map. By Lemma 3.14, since $i(A)$ is dense in $\widehat{A}$, there exist $s \in A$ and $T \subseteq A$ such that

$$U = R\left(\frac{i(T)}{i(S)}\right).$$

Since U is quasi-compact by Theorem 3.10(*ii*), and since $v\bigl(i(s)\bigr) \neq 0$ for every $v \in G$, there exists a neighborhood G of 0 in A such that $v\bigl(i(g)\bigr) \leq v\bigl(i(s)\bigr)$ for all $v \in U$ and $g \in G$ by Lemma 3.13. Finally, let D be a finite subset of G such that $D \cdot A$ is open. Then,

$$g(U) = R\left(\frac{T \cup D}{s}\right). \qquad \square$$

References

[AM69] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra.* Addison-Wesley Ser. Math. Reading, MA: Addison-Wesley, 1969. MR: 0242802.

[Bou98] N. Bourbaki. *Elements of Mathematics. Commutative algebra. Chapters 1–7.* Translated from the French, Reprint of the 1989 English translation. Berlin: Springer-Verlag, 1998. MR: 1727221.

[Con14] B. Conrad. *Number theory learning seminar.* 2014–2015. URL: http://math.stanford.edu/~conrad/Perfseminar.

[Dat17] R. Datta. *Huber rings.* 2017. URL: http://www-personal.umich.edu/~rankeya/Huber%20Rings.pdf.

[Hub93] R. Huber. "Continuous valuations." *Math. Z.* 212.3 (1993), pp. 455–477. DOI: 10.1007/BF02571668. MR: 1207303.

[Hub94] R. Huber. "A generalization of formal schemes and rigid analytic varieties." *Math. Z.* 217.4 (1994), pp. 513–551. DOI: 10.1007/BF02571959. MR: 1306024.

[Sch12] P. Scholze. "Perfectoid spaces." *Publ. Math. Inst. Hautes Études Sci.* 116 (2012), pp. 245–313. DOI: 10.1007/s10240-012-0042-x. MR: 3090258.

[Ste17] M. Stevenson. *Arithmetic geometry learning seminar: Adic spaces.* 2017. URL: http://www-personal.umich.edu/~stevmatt/adic_spaces.pdf.

[Wed12] T. Wedhorn. *Adic spaces.* June 19, 2012. URL: http://www2.math.uni-paderborn.de/fileadmin/Mathematik/People/wedhorn/Lehre/AdicSpaces.pdf.

E-mail address: takumim@umich.edu
URL: http://www-personal.umich.edu/~takumim/