THE PROPERTY OF BEING JAPANESE (AKA N-2) IS NOT LOCAL

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The goal of this note is to show the following result, answering an old question on MathOverflow.[1]

Proposition 1. *There exists a noetherian domain T such that for every prime ideal $\mathfrak{p}$ of T, the localization $T_\mathfrak{p}$ is Japanese, but T itself is not Japanese. In fact, one can choose T so that its local rings are essentially of finite type over appropriate fields.*

Recall that a noetherian domain R with fraction field K is *Japanese* or *N-2* if for every finite field extension L/K, the integral closure of R in L is a module-finite R-algebra. A noetherian ring A, not necessarily a domain, is *Nagata* if for every prime ideal $\mathfrak{p}$ of A, the quotient ring $A/\mathfrak{p}$ is Japanese.

Example 2. A one-dimensional noetherian domain is Japanese if and only if it is Nagata.

We will use the following meta theorem of Melvin Hochster. We simplify the setting of his result for convenience.

Theorem 3 [Hoc73, Prop. 1 and 2]**.** *Let $\mathcal{P}$ be a property of noetherian local rings. Let k be an algebraically closed field, and let $(R, \mathfrak{m})$ be a local ring essentially of finite type over k such that*

- *(i) R is domain (hence geometrically integral);*
- *(ii) $R/\mathfrak{m} = k$; and*
- *(iii) for every field extension $L \supseteq k$, the ring $(L \otimes_k R)_{\mathfrak{m}(L\otimes_k R)}$ fails to satisfy $\mathcal{P}$.*

Moreover, suppose every field extension $L \supseteq k$ satisfies $\mathcal{P}$. For all $n \in \mathbf{N}$, let R_n be a copy of R with maximal ideal $\mathfrak{m}_n = \mathfrak{m}$. Let $R' \coloneqq \bigotimes_{n\in\mathbf{N}} R_n$, where the infinite tensor product is taken over k. Then, each $\mathfrak{m}_n R'$ is a prime ideal of R'. Moreover, if $S = R' \smallsetminus (\bigcup_n \mathfrak{m}_n R')$, then the ring

$$T \coloneqq S^{-1}R'$$

is a noetherian domain whose locus of primes that satisfy $\mathcal{P}$ is not open in $\operatorname{Spec}(T)$*. Moreover, the map $n \mapsto \mathfrak{m}_n T$ induces a one-to-one correspondence between $\mathbf{N}$ and the maximal ideals of T, and*

$$T_{\mathfrak{m}_n T} \cong (L_n \otimes_k R_n)_{\mathfrak{m}_n(L_n\otimes_k R_n)},$$

where L_n is the fraction field of the domain $\bigotimes_{m\neq n} R_m$. Thus, each local ring of T is essentially of finite type over some appropriate field extension of k. In particular, all local rings of T are excellent.

Remark 4. Note that implicit in Theorem 3 is the assertion that for each $n \in \mathbf{N}$, $\mathfrak{m}_n(L_n \otimes_k R_n)$ is a prime ideal of $L_n \otimes_k R_n$ (Hochster calls this property *absolutely prime* in his paper). This follows from the fact that since $R_n/\mathfrak{m}_n$ is isomorphic to the base field k, then for any field extension L of k

$$L \otimes_k R_n/\mathfrak{m}_n \cong L.$$

Thus, $\mathfrak{m}_n(L_n \otimes_k R_n)$ is in fact a maximal ideal of $L_n \otimes_k R_n$. Moreover, as a consequence of Hochster's construction, it follows that the dimension of each local ring $T_{\mathfrak{m}_n T}$ equals the dimension of R. Indeed, the extension $R_n \hookrightarrow L_n \otimes_k R_n$ is faithfully flat, and induces a faithfully flat local extension $R_n = (R_n)_{\mathfrak{m}_n} \hookrightarrow (L_n \otimes_k R_n)_{\mathfrak{m}_n(L_n\otimes_k R_n)}$. But the closed fiber of this local extension is a field (the maximal ideal of R_n expands to the maximal ideal of $(L_n \otimes_k R_n)_{\mathfrak{m}_n(L_n\otimes_k R_n)}$), hence zero dimensional. Therefore

$$\dim(R) = \dim(R_n) = \dim\big((L_n \otimes_k R_n)_{\mathfrak{m}_n(L_n\otimes_k R_n)}\big) = \dim(T_{m_n T}),$$

where the first equality follows because R_n is just a copy of R, the second equality follows by faithful flatness of the local extension and [Stacks, Tag 00ON], and the third equality follows by the isomorphism in Hochster's result. In other words, each maximal ideal of T has the same height, that is, T is equicodimensional and $\dim(T) = \dim(R)$.

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[1] See Jaakko's question at `https://mathoverflow.net/q/40935`.

We can now prove Proposition 1.

Proof of Proposition 1. Let $\mathcal{P}$ be the property of a noetherian local ring being regular. Let k be an algebraically closed field and $(R, \mathfrak{m})$ be a local domain essentially of finite type over k of dimension 1 such that R is not normal (equivalently, not regular) and $R/\mathfrak{m} = k$. Note that for every field extension L of k, the ring $(L \otimes_k R)_{\mathfrak{m}(L\otimes_k R)}$ is not regular because R is not regular and regularity descends under faithfully flat maps [Stacks, Tag 07NG]. Then R satisfies the three conditions of Theorem 3 for the property $\mathcal{P}$ of being regular. Let T be the ring of Theorem 3 constructed using infinitely many copies of R. Then all the local rings of T are excellent, but the regular locus of $\operatorname{Spec}(T)$ is not open. Thus, T is not an excellent ring.

Since every local ring of T is excellent, it follows that T is a G-ring (i.e., all formal fibers of the local rings of T are geometrically regular). Moreover, since the property of being universally catenary can be checked locally at prime ideals [Stacks, Tag 0AUN], it follows that T is also universally catenary. Finally, T is an equicodimensional domain of dimension 1 by our choice of R and Remark 4.

Since the formal fibers of the local rings of T are geometrically reduced, by the Zariski–Nagata theorem we find that each local ring of T is Nagata [EGAIV_2, Thm. 7.6.4], hence equivalently, Japanese by Example 2. But T cannot be Japanese by the following lemma since it has non-open regular locus. □

Lemma 5. *Let T be a 1-dimensional noetherian domain. If T is Japanese, then for any finite type T-algebra B, the regular locus of $\operatorname{Spec}(B)$ is open.*

Proof. Let K be the fraction field of T. By [Mat80, Thm. 73], it suffices to show that for any $\mathfrak{p} \in \operatorname{Spec}(T)$ and for any finite field extension K' of the residue field $\kappa(\mathfrak{p})$ at $\mathfrak{p}$, there exists a finite T-algebra T' such that $T/\mathfrak{p} \hookrightarrow T' \hookrightarrow K'$, the regular locus of T' contains a non-empty open set and K' is the fraction field of T'. Since T is a one-dimensional domain, $\mathfrak{p}$ is either a maximal ideal or the zero ideal. If $\mathfrak{p}$ is maximal, then one can just take T' to equal K'. If $\mathfrak{p} = (0)$, then $\kappa(\mathfrak{p})$ is the fraction field K of T. By hypothesis T is a Japanese ring. Taking T' to be the integral closure of T in K', it follows that T' is a 1-dimensional normal domain, hence regular (hence has a non-empty regular locus). Moreover, T' is module-finite over T by the Japanese property. □

Our original motivation was the following related question in prime characteristic that we still do not know the answer to.

Question 6. Let A be a noetherian domain of prime characteristic $p > 0$. Suppose for each prime ideal $\mathfrak{p}$ of A, the (absolute) Frobenius map of $A_{\mathfrak{p}}$ is finite. Is the Frobenius map of A finite?

References

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