POINCARÉ HOMOLOGY SPHERES AND EXOTIC SPHERES FROM LINKS OF HYPERSURFACE SINGULARITIES

TAKUMI MURAYAMA

Abstract. Singularities arise naturally in algebraic geometry when trying to classify algebraic varieties. However, they are also a rich source of examples in other fields, particularly in topology. We will explain how complex hypersurface singularities give rise to knots and links, some of which are manifolds that are homologically spheres but not topologically (Poincaré homology spheres), or manifolds that are homeomorphic to spheres but not diffeomorphic to them (exotic spheres). The talk should be accessible to anyone with some basic knowledge of differential and algebraic topology, although even that is not strictly necessary.

We list some references:

(1) For a general overview on exotic spheres and the technical background necessary for some proofs, see [Ran12], who also lists many useful references.
(2) For plane curve singularities, see [BK12].
(3) The speaker first learned this material from attending a course on the topology of algebraic singularities [Ném14].
(4) For an introduction on singularities in higher dimensions, see [Kau87, Ch. XIX].
(5) For a technical reference on singularities in higher dimensions, see [Mil68].

1. Motivation

Our goal for today is to see why singularities are interesting. In geometry, we are mostly interested in smooth objects, such as smooth manifolds or smooth varieties. However, there are two ways in which singularities naturally arise in algebraic geometry:

(1) In moduli theory, to obtain a compact moduli space, you want to include "limits" of spaces, which inevitably become singular.
(2) In the minimal model program, performing various surgery operations leads to spaces with *terminal* singularities.

One might ask why one should care about singularities for reasons *external* to algebraic geometry, and indeed, we have the following:

(3) Complex algebraic singularities give rise to interesting smooth manifolds. For example,
   (a) Some curve singularities in $\mathbf{C}^2$ give rise to *algebraic knots* K^1,
   (b) A surface singularity in $\mathbf{C}^3$ gives rise to the *Poincaré homology sphere* K^3, and
   (c) A fourfold singularity in $\mathbf{C}^5$ gives rise to an *exotic sphere* K^7,

   which satisfy the following properties:

$$\begin{aligned} K^1 &\underset{\text{diffeo}}{\cong} S^1 && \text{but} && (K^1 \subset S^3) \underset{\text{isotopy}}{\ncong} (S^1 \subset S^3),\\ H_*(K^3,\mathbf{Z}) &\underset{\text{groups}}{\cong} H_*(S^3,\mathbf{Z}) && \text{but} && K^3 \underset{\text{homeo}}{\ncong} S^3,\\ K^7 &\underset{\text{homeo}}{\cong} S^7 && \text{but} && K^7 \underset{\text{diffeo}}{\ncong} S^7. \end{aligned}$$

We will spend today constructing these examples. We will also explain some of the ideas and constructions used in showing the claims in both columns, although we cannot give full proofs.

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2. Hypersurface singularities and their associated links

All of our examples today will be *hypersurface singularities*, i.e., singularities defined by one equation in complex affine space. Before we begin, we recall what a singularity is in the first place.

Definition 2.1. Let $f \in \mathbf{C}[z_0, z_2, \ldots, z_n]$, which defines a function

$$f\colon \mathbf{C}^{n+1} \longrightarrow \mathbf{C}.$$

Then, the *vanishing locus* of f is

$$V(f) \coloneqq f^{-1}(0) \subset \mathbf{C}^{n+1}.$$

The vanishing locus $V(f)$ is *singular* at $P \in V(f)$ if the (complex) gradient

$$\nabla f \coloneqq \left[\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n}\right]$$

vanishes at P, and is *smooth* at P if $\nabla f(P) \neq 0$.

Remark 2.2. This condition on the gradient is known as the *Jacobian criteron*. In higher codimension, the condition is that the matrix $(\partial f_i/\partial z_j)$ has full rank. In differential topology, the fact that $V(f)$ is a smooth manifold is sometimes known as the *preimage theorem* [GP10, p. 21].

We will be interested in the following special case:

Setup 2.3. Suppose $V(f)$ has an *isolated singularity* at $\vec{0}$, i.e., suppose $\vec{0}$ is a singular point of $V(f)$ and $V(f)$ is smooth in a punctured neighborhood of $\vec{0}$.

Given such an isolated singularity, we can associate a smooth manifold to it that can be used to study the singularity. This was first done for curve singularities by Brauner in 1928 [BK12, p. 223].

Definition 2.4. Given an isolated singularity $V(f)$, the *link* of f is the intersection

$$K(f) \coloneqq V(f) \cap S_\varepsilon^{2n+1},$$

where S_ε^{2n+1} is the $(2n+1)$-dimensional sphere of radius $\varepsilon > 0$ centered at $\vec{0} \in \mathbf{C}^{n+1}$.

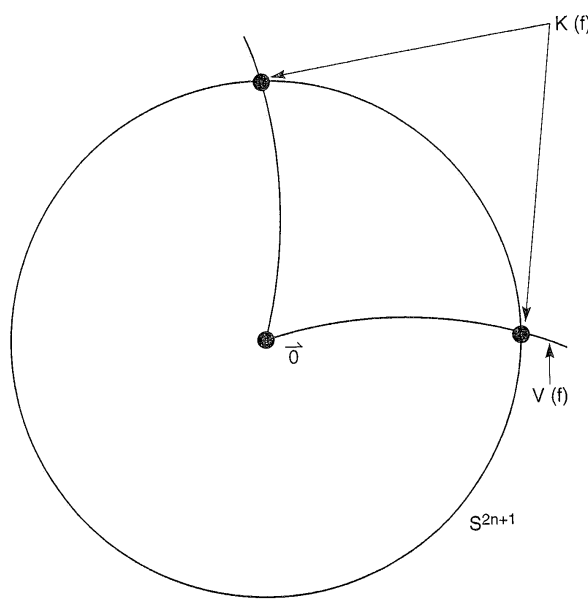

Figure 1. The link of an isolated singularity (from [Kau96, Fig. 14.1]).

We implicitly used the following corollary of Sard and Ehresmann's theorems in our definition:

Fact 2.5 (see [Mil68, Cor. 2.9])**.** If $V(f)$ is an isolated singularity, then the diffeomorphism type of $K(f)$ does not depend on ε for $0 < \varepsilon \ll 1$, and is a $(2n-1)$-dimensional smooth manifold.

3. Knots from plane curve singularities

We now start studying a 1-dimensional example.

Example 3.1. Let $f = x^3 + y^5$. Then,

$$\nabla f = \left[3x^2,\ 5y^4\right]$$

vanishes only at the origin, and so we have an isolated singularity. Let us compute what this looks like (up to diffeomorphism). First,

$$S^3 = \partial B^3 \underset{\text{diffeo}}{\cong} \partial \begin{Bmatrix} |x| \leq 1 \\ |y| \leq 1 \end{Bmatrix} = \begin{Bmatrix} |x| = 1 \\ |y| \leq 1 \end{Bmatrix} \cup \begin{Bmatrix} |x| \leq 1 \\ |y| = 1 \end{Bmatrix},$$

an we will replace S^3 with the boundary of a polydisc for simplicity. Since

$$|f| \geq \left| |x|^3 - |y|^5 \right| > 0$$

on the interior of the first component of this decomposition, we only need to consider the second component. Parametrizing y by $e^{i\theta}$, we obtain that $x = \zeta_3 e^{i5\theta/3}$, where ζ_3 is a cube root of -1. Thus, $K(f)$ is the $(3,5)$-torus knot, and in particular, it is diffeomorphic to S^1.

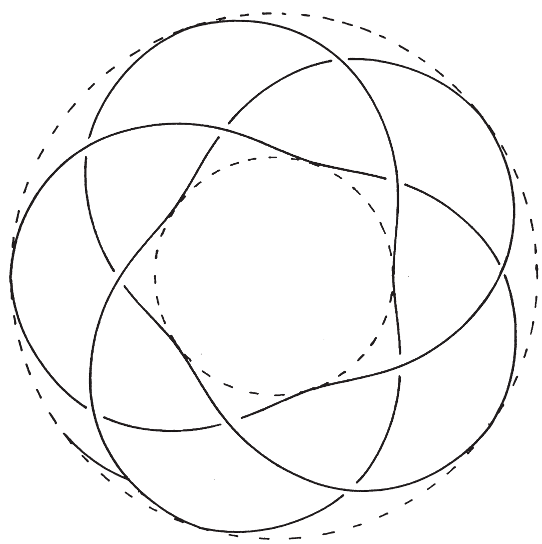

Figure 2. The torus knot $(3,5)$ (from [BK12, p. 432]).

A similar analysis shows the following:

Fact 3.2. If $\gcd(p,q) = 1$, then $K(x^p + y^q)$ is the torus knot (p,q), and all right-handed torus knots can be obtained in this way.

More complicated polynomials give more complicated knots, but these are well-understood.

Remark 3.3. If $\gcd(p,q) \neq 1$, then $K(x^p + y^q)$ is a disjoint union of torus knots that are linked together in S^3. Using *Newton diagrams* and the theory of *Puiseux pairs,* one can show that more complicated equations give rise to *iterated torus knots;* see [BK12, §8.3].

4. The Alexander polynomial

To finish Example 3.1, we want to show that the $(3,5)$-torus knot is not isotopic to the unknot $S^1 \subset \mathbf{R}^3$. The invariant we will use to distinguish these is the *Alexander polynomial*, which has a purely knot-theoretic definition in terms of skein relations when $n = 1$ [Lic97, Ch. 6].

We will not define the Alexander polynomial, but we list the properties that we will need:

Theorem 4.1 (see [Lic97, Ch. 6]; [Mil68, §§8,10])**.** *There is a Laurent polynomial $\Delta_f(t)$ associated to the link $K(f)$ of an isolated singularity, such that*

(i) *If $n = 1$ and if $K(f)$ is the unknot, then $\Delta_f(t) = 1$;*
(ii) *$\Delta_f(t)$ is invariant under isotopy;*
(iii) *$\Delta_f(1) = \pm 1$ if and only $K(f)$ is a homology sphere.*

Definition 4.2. The Laurent polynomial in Theorem 4.1 is (the higher-dimensional version of) the *Alexander polynomial*.

Remark 4.3. We give a brief description of how to construct the Alexander polynomial. First, the mapping $f\colon \mathbf{C}^{n+1} \to \mathbf{C}$ gives rise to a map

$$\begin{aligned} \phi\colon S^{2n+1}_\varepsilon \smallsetminus K(f) &\longrightarrow S^1 \\ z &\longmapsto \frac{f(z)}{|f(z)|} \end{aligned}$$

This is a smooth fiber bundle [Mil68, Thm. 4.8] with fiber F, called the *Milnor fiber*; see Figure 4 for a visualization. The monodromy action on S^1 induces an action on $H_n(F, \mathbf{Z})$, and the characteristic polynomial of this action is the Alexander polynomial.

Lemma 4.4 [Mil68, Thm. 9.1]**.** *Let*

$$f = z_0^{a_0} + z_1^{a_n} + \cdots + z_n^{a_n}.$$

In this case,

$$\Delta_f(t) = \prod_{0<i_k<a_k} \left(t - \xi_0^{i_0}\xi_1^{i_1}\cdots\xi_n^{i_n}\right) \quad \textit{where} \quad \xi_k = e^{2\pi i/a_k}.$$

All of our examples will be of this form; the $V(f)$ are called *Brieskorn varieties*.

Example 4.5. We return to the $(3,5)$-torus knot from Example 3.1. The Alexander polynomial is

$$\Delta(t) = \frac{(t^{15}-1)(t-1)}{(t^5-1)(t^3-1)},$$

and $\Delta(1) = 1$. Thus, $K(x^3 + y^5)$ is homeomorphic to S^1, but is not the unknot.

5. Higher-dimensional examples

We are now ready to discuss the promised higher-dimensional examples: the Poincaré homology sphere, and some exotic spheres.

Example 5.1 (The Poincaré homology sphere)**.** Consider

$$K^3 = K(x^3 + y^5 + z^2).$$

The Alexander polynomial is

$$\Delta(t) = \frac{(t^{30}-1)(t^5-1)(t^3-1)(t^2-1)}{(t^{15}-1)(t^{10}-1)(t^6-1)(t-1)},$$

and $\Delta(1) = 1$. Thus, K^3 is a homology sphere. To show it is not homeomorphic to S^3, we have:

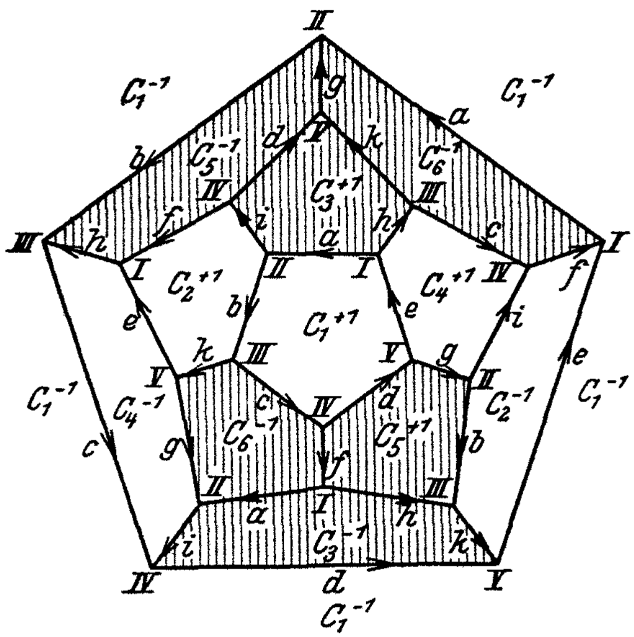

FIGURE 3. The Poincaré homology sphere as a dodecahedral space (from [TS31, Fig. 11]).

Proposition. *K^3 is homeomorphic to $S^3/\hat{G}$, where $\hat{G}$ is the inverse image of the dodecahedral group via the projection $\pi\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$. In particular, $\pi_1(K^3) \simeq \hat{G}$.*

Idea of Proof. Consider the action of $SU(2)$ on $\mathbf{C}[x, y]$, given by

$$f(x, y) \cdot \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} = f(ax + by, -\bar{b}x + \bar{a}y).$$

The invariants of $\hat{G}$ under this action are

$$R^{\hat{G}} \simeq \frac{\mathbf{C}[x, y, z]}{(x^3 + y^5 + z^2)}.$$

Taking spectra and intersecting with S^3, we obtain that K^3 is homeomorphic to $S^3/\hat{G}$. □

The group $\hat{G}$ is called the *binary dodecahedral group*. For more details, see [Mil68, Ex. 9.8], [Kau87, Ex. 19.6], and [Mil75]. One can also compute the fundamental group of K^3 algebraically via resolution of singularities for surfaces. For this argument, see [Mum61], who showed that π_1 is trivial if and only if $V(f)$ is smooth; see also [Hir63].

Example 5.2 (Some exotic spheres)**.** Consider

$$K^{2n-1} = K(z_0^3 + z_1^5 + z_2^2 + z_3^2 + \cdots + z_n^2)$$

for $n \geq 4$. The Alexander polynomial can be computed as before to obtain that $\Delta(1) = 1$, i.e., K^{2n-1} are homology spheres. Since they are simply connected by [Mil68, Thm. 5.2], Whitehead's theorem implies K^{2n-1} are homotopy-equivalent to S^{2n-1}.* Furthermore, by Smale and Stalling's proof of the Poincaré conjecture in dimensions ≥ 5, this implies K^{2n-1} is homeomorphic to S^{2n-1}.

The case when $n = 4$ gives Milnor's original exotic sphere, which he constructed using S^3-bundles over S^4. Similarly, the case $n = 5$ gives Kervaire's exotic 9-sphere. Since it is a bit hard to show these are not diffeomorphic to spheres, we will consider a higher-dimensional example.

*The non-trivial homology class in $H_{2n-1}(K^{2n-1}, \mathbf{Z})$ corresponds to a degree one map $S^{2n-1} \to K^{2n-1}$, which induces isomorphisms on homology, hence is a homotopy equivalence.

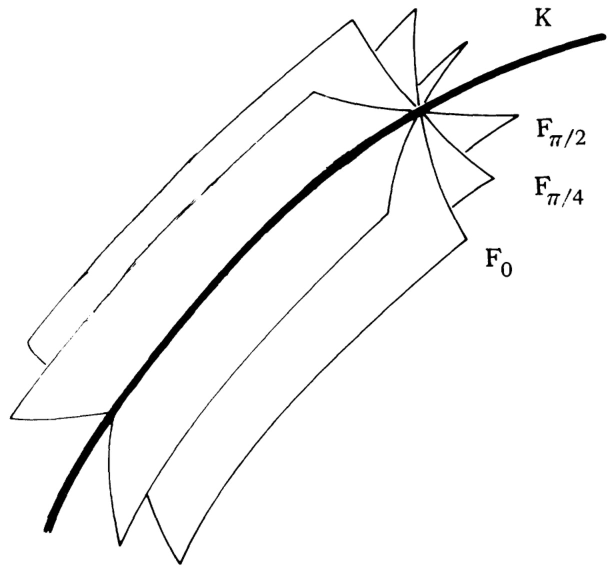

FIGURE 4. The Milnor fiber (from [Mil68, Fig. 2]).

We will consider when $n = 6$, so that K^{11} is an 11-manifold. Then, K^{11} is the boundary of a 12-manifold F (the *Milnor fiber* from Remark 4.3), whose non-trivial homology lives in $H_6(F, \mathbf{Z})$ [Mil68, Lem. 6.4]. The Hirzebruch signature formula [MS74, Thm. 19.4] says

$$\sigma(X) = \frac{1}{3^3 \cdot 5 \cdot 7}(62p_3 - 13p_2p_1 + 2p_1^3) \in \mathbf{Z},$$

where the left-hand side is the *signature* of the manifold X,† and the cohomology classes $p_i(X) \in H^{4i}(X, \mathbf{Z})$ are the *Pontrjagin classes*. The Mayer–Vietoris sequence says that $H_4(X, \mathbf{Z}) = H_4(F, \mathbf{Z}) = 0$ and $H_8(X, \mathbf{Z}) = H_8(F, \mathbf{Z}) = 0$. Thus, we have $p_1 = p_2 = 0$, hence

$$\sigma(X) = \frac{62p_3}{3^3 \cdot 5 \cdot 7} \in \mathbf{Z}.$$

Note that $p_3 \in H^{12}(X, \mathbf{Z}) \simeq \mathbf{Z}$, and so we have the divisibility relations $3^3 \cdot 5 \cdot 7 \mid p_3$ and $62 \mid \sigma(X)$. But for this example, one can show $\sigma(X) = \sigma(F) = -8$ (see, e.g., [Hir66, p. 19]), a contradiction.

6. Other applications

We conclude with some other examples of using topological methods to study algebraic varieties.

(1) The Abhyankar–Moh theorem says that if $\mathbf{C} \hookrightarrow \mathbf{C}^2$ is a closed embedding defined by polynomials $p(t), q(t) \in \mathbf{C}[t]$, then either $\deg p \mid \deg q$ or $\deg q \mid \deg p$. This theorem can be proved by studying the "knot at infinity" of this embedding; see [Rud82], or the speaker's talk in this seminar from last year [Mur16]. The knot at infinity is obtained by intersecting the image of $\mathbf{C}$ with a sphere $S_R^3 \subset \mathbf{C}^2$ where $R \gg 0$.

(2) The so-called Zariski cancellation problem asks if there exist non-isomorphic complex varieties X and Y such that $X \times \mathbf{C}$ and $Y \times \mathbf{C}$ *are* isomorphic. One positive answer to this question was given by Danielewski, who showed using "fundamental groups at infinity" that such a pair exists. The speaker has written about this on Mathematics Stack Exchange; see [Mur15] and the references therein.

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†The signature of a smooth manifold of even dimension is the signature of the intersection form on the middle homology group.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109-1043, USA
E-mail address: takumim@umich.edu
URL: http://www-personal.umich.edu/~takumim/