APPLICATIONS OF LOCAL COHOMOLOGY

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Abstract. Local cohomology was discovered in the 1960s as a tool to study sheaves and their cohomology in algebraic geometry, but have since seen wide use in commutative algebra. An example of their use is to answer the question: how many elements are necessary to generate a given ideal, up to radical?

For example, consider two planes in 4-space meeting at a point. The vanishing ideal $I = (x, y) \cap (u, v) \subseteq k[x, y, u, v]$ can be generated up to radical by $xu, yv, xv + yu$. Krull's Hauptidealsatz implies that one element is not enough, but local cohomology is used to show two elements also do not work.

The main sources for this talk are [Hun07] and [Eis05, App. 1]. For a more "homological" introduction, see [Wei94, §4.6]. The speaker would like to point out that the algebro-geometric literature on the topic takes a different approach via sheaf cohomology; see [Har67; Har77].

1. Introduction: Complete intersections

We start off with our favorite counterexample from Algebraic Geometry I.

Example 1 (Twisted Cubic [Har77, Exc. I.2.17])**.** Let C be the twisted cubic curve, defined by the equations

$$V(x^2 - yw, xz - y^2, xy - zw) \subseteq \mathbf{P}^3.$$

In algebraic geometry, this is usually the first example of a variety that is not a *complete intersection,* that is, its vanishing ideal $I(C)$ cannot be generated by r elements, where r is the codimension of C. You can see this by looking at the degree 2 piece of $I(C)$, which is three-dimensional. However, the vanishing ideal of $I(C)$ is generated up to radical by

$$x^2 - yw,\ z(xy - zw) + y(xz - y^2),$$

since

$$\begin{aligned}(xz - y^2)^2 &= x^2z^2 - 2xy^2z + y^4 &&= z^2(x^2 - yw) - y(z(xy - zw) + y(xz - y^2))\\ (xy - zw)^2 &= x^2y^2 - 2xyzw + z^2w^2 &&= y^2(x^2 - yw) - w(z(xy - zw) + y(xz - y^2))\end{aligned}$$

We can then naïvely ask: can we generate $I(C)$, up to radical, by one element? This is impossible by Krull's Hauptidealsatz, since then C would have dimension 2, not 1. We say that C is a *set-theoretic complete intersection,* that is, there are two hypersurfaces in $\mathbf{P}^3$ that cut out C set-theoretically.

Even in one dimension higher, it's already hard to answer such questions.

Example 2 (Hartshorne's Example [Har70, Exc. III.5.12; Har77, Exc. III.4.9])**.** Let $Y \subseteq \mathbf{A}^2$ be the union of two planes meeting at a point (you can also think of this example as two skew lines in $\mathbf{P}^3$). Its vanishing ideal is

$$I(Y) = (x, y) \cap (u, v) = (xu, xv, yu, yv) \subseteq k[x, y, u, v],$$

which is generated up to radical by the elements $xu, yv, xv + yu$, since

$$(xv)^2 = xv(xv + yu) - (xu)(yv) \in (xu, yv, xv + yu).$$

We can ask again: can we generate $I(Y)$, up to radical, by less than three elements? We know it cannot be generated by one element by Krull's Hauptidealsatz again, since then Y would have dimension 3. But we can't rule out two elements by dimension arguments alone!

We want to answer the question: *is Y a set-theoretic complete intersection?* To do so, we want to construct some sort of obstruction for $I(Y)$ to be generated by two elements, up to radical. To a ring R and an ideal J, we will construct a sequence of modules $H^i_J(R)$ such that

Properties 3.

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(1) $H^i_J(R) = H^i_{\sqrt{J}}(R)$, and
(2) if J is generated by k elements, then $H^i_J(R) = 0$ for all $i > k$.

These modules are what we will call *local cohomology modules*.

2. Local cohomology as a derived functor

Just like sheaf cohomology, there are multiple ways to define local cohomology. We will describe the derived functor construction first.

Let R be a noetherian ring, $I \subseteq R$ an ideal, and M an R-module. We define the *I-power torsion module of M* to be

$$H^0_I(M) := \{m \in M \mid I^d m = 0 \text{ for some } d \in \mathbf{N}\} = \varinjlim \operatorname{Hom}(R/I^d, M).$$

Note that $\operatorname{Hom}(R/I^d, -)$ is left exact and $\varinjlim$ is exact, and so H^0_I is a left exact functor. Like any left exact functor, we can define its right derived functors, by choosing an injective resolution

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots,$$

applying the functor $H^0_I(-)$ term by term to get a new complex $H^0_I(E^\bullet)$, and then computing cohomology to get modules $H^i_I(M)$. We automatically get some really nice properties!

Properties 4.
(1) If I and J have the same radical, they define the same functor $H^i_I(-)$ (Property (1) from before).
(2) If $0 \to N \to M \to L \to 0$ is a short exact sequence of R-modules, there is a long exact sequence

$$0 \longrightarrow H^0_I(N) \longrightarrow H^0_I(M) \longrightarrow H^0_I(L) \longrightarrow H^1_I(N) \longrightarrow H^1_I(M) \longrightarrow \cdots.$$

(3) Every element of $H^i_I(M)$ is killed by some power of I.
(4) Since $\varinjlim$ is exact, we can compute local cohomology as

$$H^i_I(M) \cong \varinjlim \operatorname{Ext}^i(R/I^d, M).$$

Before we get too carried away with technical details, let's compute some examples.

Example 5. Let $p \in \mathbf{Z}$ be a prime number. We will compute $H^i_I(\mathbf{Z})$, where $I = (p)$. Since $\mathbf{Z}$ is a PID, the injective modules are exactly the divisible modules. We have the following injective resolution of $\mathbf{Z}$:

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Q} \longrightarrow \mathbf{Q}/\mathbf{Z} \longrightarrow 0.$$

Now since H^0_I simply computes p^d-torsion for all d, the only non-vanishing term after applying $H^0_I(-)$ is $H^0_I(\mathbf{Q}/\mathbf{Z})$, and so we have

$$H^1_I(\mathbf{Z}) = H^0_I(\mathbf{Q}/\mathbf{Z}) \cong \mathbf{Z}[p^{-1}]/\mathbf{Z},$$

where the last isomorphism is by unique factorization.

Similarly, we can compute the local cohomology of $R = k[x]$, where k is a field:

Exercise 6 ([Hun07, Ex. 2.12])**.** Let $R = k[x]$ be a polynomial ring in one variable over a field k. Our goal is to completely describe $H^i_I(M)$ for $I = (x)$ and M an arbitrary finitely generated module.

(1) Using the structure theorem for finitely generated modules over a PID, reduce to the case of computing $H^i_I(R/(g))$ for $g \in R$.

Proof. The structure theorem says

$$M = \bigoplus_{j=1}^{s} R/(g_j)$$

for some elements $g_j \in R$, possibly equal to zero. We conclude by the fact that computing local cohomology commutes with direct sums. □

(2) First compute the case when $g = 0$ by considering the short exact sequence

$$0 \longrightarrow R \longrightarrow K \longrightarrow K/R \longrightarrow 0,$$

where $K = k(x)$ is the fraction field of $k[x]$.

Proof. As was the case for $H^i_I(\mathbf{Z})$, the only nonzero term after applying $H^0_I(-)$ is $H^0_I(K/R)$, and so

$$H^1_I(R) = H^0_I(K/R) \cong k[x,x^{-1}]/k[x],$$

using unique factorization as before. Note this has a k-basis $\langle x^{-n} \mid n \geq 1\rangle$ such that $x \cdot x^{-1} = 0$. □

(3) Now compute the case when $g \neq 0$ by considering the short exact sequence

$$0 \longrightarrow R \xrightarrow{g} R \longrightarrow R/(g) \longrightarrow 0.$$

Hint: Writing $g = x^n h$ for $(h, x) = 1$, note that h acts as a unit on $H^1_I(R)$ since there exist $a, b \in R$ such that $ah = 1 - bx$, and $1 - bx$ acts as a unit on this module.

Proof. Using (b), we have that $R^i_I(R)$ is nonzero if and only if $i = 1$, and so the long exact sequence on cohomology is

$$0 \longrightarrow H^0_I(R/(g)) \longrightarrow H^1_I(R) \xrightarrow{g} H^1_I(R) \longrightarrow H^1_I(R/(g)) \longrightarrow 0.$$

Now consider the hint; $1 - bx$ indeed acts as a unit on $H^1_I(R)$ since it does on each basis element:

$$\frac{1}{x^n}(1-bx)(1+bx+b^2x^2+\cdots+b^{n-1}x^{n-1}) = \frac{1}{x^n}(1-b^nx^n) = \frac{1}{x^n}.$$

Now, $H^0_I(R/(g))$ is the kernel by multiplication by x^n, and $H^1_I(R/(g))$ is the cokernel by multiplication by x^n. The set of elements in $H^1_I(R)$ annihilated by x^n is generated by $\frac{1}{x^n}$, hence $H^1_I(R/(g)) \cong R/(x^n)$. Finally, $H^1_I(R) = k[x,x^{-1}]/k[x]$ is divisible by R, hence multiplication by x^n is surjective, and so $H^1(R/(g)) = 0$. □

3. Computing local cohomology

One disadvantage of working with derived functors is that computing injective resolutions in practice is rather difficult. We already saw how the long exact sequence associated to a short exact sequence can be very useful, since we can break apart local cohomology modules into pieces we can understand. We first give another example where a similar breaking apart procedure is useful.

3.1. The Mayer–Vietoris sequence.

In algebraic topology, we learn about the Mayer–Vietoris sequence, which allows us to break apart a topological space into smaller pieces whose (co)homology we hopefully understand. We have a similar result for local cohomology:

Theorem 7. *Let I and J be ideals in a noetherian ring R, and let M be a finitely generated module M. Then, there is a long exact sequence in local cohomology*

$$0 \longrightarrow H^0_{I+J}(M) \longrightarrow H^0_I(M) \oplus H^0_J(M) \longrightarrow H^0_{I\cap J}(M) \longrightarrow H^1_{I+J}(M) \longrightarrow H^1_J(M) \oplus H^1_J(M) \longrightarrow \cdots.$$

Proof. We will use the identification $H^i_I(M) \cong \varinjlim \operatorname{Ext}^i(R/I^d, M)$ from before. Take the short exact sequences

$$0 \longrightarrow R/(I^n \cap J^n) \longrightarrow R/I^n \oplus R/J^n \longrightarrow R/(I^n + J^n) \longrightarrow 0$$

for each n, and consider their corresponding long exact sequences on Ext modules. Then, since the system $\{I^n \cap J^n\}$ is cofinal with $\{(I+J)^n\}$, and since the system $\{I^n + J^n\}$ is cofinal with $\{(I+J)^n\}$, we can take direct limits over these long exact sequences to get the desired long exact sequence. □

3.2. Čech cohomology.

Just like with sheaf cohomology in algebraic geometry, in nice cases we have a very concrete way of computing local cohomology. This is called *Čech cohomology*.

Let R be a noetherian ring, and let I be an ideal in R generated by elements $(x_1, \ldots, x_t)$. Write $[t] = \{1, \ldots, t\}$ for the set of integers from 1 to t, and for any subset $J \subset [t]$ let $x_J = \prod_{j \in J} x_j$. Denote $M[x_J^{-1}]$ to be the localization of M by inverting x_J.

Theorem 8. *For any R-module M, the local cohomology module $H^i_I(M)$ is the i-th cohomology of the complex*

$$C(x_1, \ldots, x_t; M) := \left\{ 0 \longrightarrow M \xrightarrow{d} \bigoplus_{i=1}^{t} M[x_i^{-1}] \xrightarrow{d} \cdots \xrightarrow{d} \bigoplus_{|J|=s} M[x_J^{-1}] \xrightarrow{d} \cdots \xrightarrow{d} \bigoplus M[x^{-1}_{\{1,\ldots,t\}}] \longrightarrow 0 \right\}$$

where the differential takes an element

$$m_J \in M[x_J^{-1}] \subseteq \bigoplus_{|J|=s} M[x_J^{-1}]$$

to the element

$$d(m_J) = \sum_{k \notin J} (-1)^{o_J(k)} m_{J \cup \{k\}},$$

where $o_J(k)$ denotes the number of elements of J less than k, and $m_{J\cup\{k\}}$ denotes the image of m_J in the further localization $M[x_{J\cup\{k\}}^{-1}] = M[x_J^{-1}][x_k^{-1}]$.

We omit the proof, and instead point the reader to [Eis05, Thm. A1.3] or [Hun07, Prop. 2.13].

We point out that this theorem has the following nice corollary, which is Property (2) from before:

Corollary 9. *If $I = (x_1, \ldots, x_t)$, then $H_I^i(M) = 0$ for all $i > t$.*

Proof. The length of the Čech complex $C(x_1, \ldots, x_t; M)$ is t. □

We have therefore found a nice functor that satisfies the properties we wanted!

4. Hartshorne's example

We now turn back to Example 2.

Exercise 10 (Hartshorne's Example [Har70, Exc. III.5.12; Har77, Exc. III.4.9])**.** Consider the ideal

$$I(Y) = (x, y) \cap (u, v) = (xu, xv, yu, yv) \subseteq k[x, y, u, v].$$

We want to show that $I(Y)$ cannot be generated, up to radical, by two elements.

(1) Using our two main properties, show that it suffices to show $H^3_{I(Y)}(R) \neq 0$.

Proof. Suppose J is an ideal generated by two elements such that $\sqrt{J} = I(Y)$. Then, by Corollary 9 of the identification of local cohomology with Čech cohomology, we have $H_J^3 = H^3_{I(Y)} = 0$, using our other main property that local cohomology does not change under taking the radical of J. □

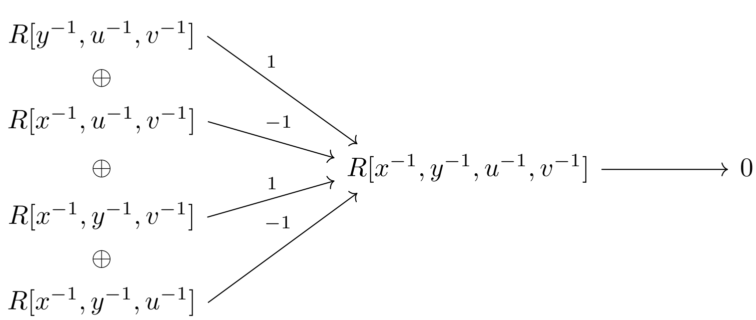
(2) Use the Mayer–Vietoris sequence on $I(Y)$ and the vanishing property to show $H^3_{I(Y)}(R) \cong H^4_{\mathfrak{m}}(R)$, where $\mathfrak{m} = (x, y) + (u, v)$.

Proof. The Mayer–Vietoris sequence gives

$$H^3_{(x,y)}(R) \oplus H^3_{(u,v)}(R) \longrightarrow H^3_{I(Y)}(R) \longrightarrow H^4_{\mathfrak{m}}(R) \longrightarrow H^4_{(x,y)}(R) \oplus H^4_{(u,v)}(R).$$

The two edge modules vanish since (x, y) and (u, v) are generated by two elements each, and by applying Corollary 9. □

(3) Compute $H^4_{\mathfrak{m}}(R)$ using Čech cohomology. *Hint:* The relevant part of the Čech complex is



Proof. The hard part is writing down the complex. Using the above representation of the complex, we obtain

$$H^3_{I(Y)}(R) \cong H^4_{\mathfrak{m}}(R) \cong k\langle x_1^a x_2^b x_3^c x_4^d \mid a, b, c, d < 0\rangle \neq 0.$$ □

5. Back to Curves

In Example 1, we saw that it was fairly easy to construct examples of non-complete intersections by considering monomial curves in $\mathbf{P}^3$. The twisted cubic ended up being a set-theoretic complete intersection. However, the generalization of this fact is not known:

Open Question ([Har77, Exc. I.2.17(d)]). *Can all irreducible curves in* $\mathbf{P}^3$ *be defined set-theoretically by two equations?*

This question is surprisingly difficult: while local cohomology gives us an *obstruction* for a variety to be a set-theoretic complete intersection, if the relevant local cohomology module vanishes, we get no information. Even for the following simple example, we have no idea:

Example 11. Consider the smooth rational quartic curve C in $\mathbf{P}^3$ defined as the image of the map

$$\mathbf{P}^1 \longrightarrow \mathbf{P}^3$$
$$[s:t] \longmapsto [s^4 : s^3t : st^3 : t^4]$$

It is known that $H^i_{I(C)}(M) = 0$ for all $i > 2$ and all modules M [Har70, Ch. III], so local cohomology does not provide an obstruction to C being a set-theoretic complete intersection. On the other hand, it *is* known that if we work over a field of characteristic $p > 0$, we *do* get a set-theoretic complete intersection [Har79].

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