Perverse Sheaves

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The goal of this class is to introduce perverse sheaves, and how to work with it; plus some applications.

Background

For more background, see Kleiman's paper entitled "The development/history of intersection homology theory". On manifolds, the idea is that you can intersect cycles via Poincaré duality—we want to be able to do this on singular spces, not just manifolds. Deligne figured out how to compute intersection homology via sheaf cohomology, and does not use anything about cycles—only pullbacks and truncations of complexes of sheaves. In any derived category you can do this—even in characteristic p. The basic summary is that we define an abelian subcategory that lives inside the derived category of constructible sheaves, which we call the category of perverse sheaves. We want to get to what is called the decomposition theorem.

Outline of Course

1. Derived categories, t-structures
2. Six Functors
3. Perverse sheaves—definition, some properties
4. Statement of decomposition theorem—"yoga of weights"
5. Application 1: Beilinson, et al., "there are enough perverse sheaves", they generate the derived category of constructible sheaves
6. Application 2: Radon transforms. Use to understand monodromy of hyperplane sections.
7. Some geometric ideas to prove the decomposition theorem.

If you want to understand everything in the course you need a lot of background. We will assume Hartshorne-level algebraic geometry. We also need constructible sheaves—look at *Sheaves in Topology*. Problem sets will be given, but not collected; will be on the webpage. There are more references than BBD; they will be online.

1.1 Classical Results in Complex Algebraic Geometry

All sheaves I consider will be sheaves of $\mathbf{Q}$-vector spaces in the analytic topology.

Theorem 1.1 (Poincaré Duality)**.** *If X is a projective smooth variety of dimension d, then $H^i(X, \mathbf{Q}) \cong H^{2d-i}(X, \mathbf{Q})^\vee$, given by the cup product. More precisely, the cup product pairing*

$$H^i(X, \mathbf{Q}) \otimes H^{2d-i}(X, \mathbf{Q}) \longrightarrow H^{2d}(X, \mathbf{Q}) \cong \mathbf{Q}$$

is a perfect pairing. There is a dual version for homology.

This is one of the results we will extend to the singular setting. There is another:

Theorem 1.2 (Deligne)**.** *Let $f\colon X \to Y$ be a smooth projective morphism of varieties over $\mathbf{C}$. Then, the Leray spectral sequence degenerates:*

$$E_2^{p,q}\colon H^p(Y, \mathbf{R}^q f_* \underline{\mathbf{Q}}) \Rightarrow H^{p+q}(X, \underline{\mathbf{Q}})$$

(You can think of this sheaf $\mathbf{R}^q f_* \underline{\mathbf{Q}}$ as a local system on Y.) Moreover, each $\mathbf{R}^q f_* \underline{\mathbf{Q}}$ is semisimple, as in the action of the fundamental group on the automorphism group of fibres is semisimple.

Why is this useful?

Corollary 1.3 (Invariant cycle theorem)**.** *Fix $y \in Y$. Then, the restriction map $H^n(X, \mathbf{Q}) \to H^n(X_y, \mathbf{Q})$ lands inside of the invariants of the fundamental group $\pi_1(Y, y)$, and this map is surjective.*

Proof. $H^n(X_y, \mathbf{Q})^{\pi_1(Y,y)} = H^0(Y, \mathbf{R}^q f_* \underline{\mathbf{Q}})$; use first statement in theorem. □

These are some examples of useful statements about complex projective and smooth things; we want to move to the singular setting.

Example 1.4. Let $X = \mathbf{P}^1 \vee \mathbf{P}^1$, the union of two lines in $\mathbf{P}^2$, homeomorphic to $S^2 \vee S^2$. The claim is that Poincaré duality fails in this setting (but actually works for intersection homology!).

$$H_0(X, \mathbf{Q}) = \mathbf{Q}$$

because it is connected.

$$H_1(X, \mathbf{Q}) = 0$$

by van Kampen.

$$H_2(X, \mathbf{Q}) = \mathbf{Q} \oplus \mathbf{Q}$$

by Mayer–Vietoris. The dimensions are off, hence Poincaré duality fails.

Example 1.5. We want an example where the Leray spectral sequence does not degenerate. Let Y be the nodal cubic, and let X be its normalization, $\mathbf{P}^1$. Let $f\colon X \to Y$ be the resolution. The Leray spectral sequence still degenerates since $\mathbf{R}^q f_* \underline{\mathbf{Q}} = 0$. Maybe the blowup of a surface gives a counterexample, says Mircea.

Remark 1.6. If Y is a normal singular variety such that $H^i(Y, \mathbf{Q})$ is not "pure", then the Leray spectral sequence cannot degenerate for a resolution. The pullback must be injective if it were to degenerate, but a pure thing cannot live in an impure thing. This is where we should be able to find a counterexample.

Example 1.7. Leray degeneration fails in the holomorphic setting. Let $X = (\mathbf{C}^2 \setminus \{0\})/q^{\mathbf{Z}}$, where $q \in \mathbf{C}$ and $|q| < 1$. Let $Y = \mathbf{P}^1 = (\mathbf{C}^2 \setminus \{0\})/\mathbf{C}^*$, and let $f\colon X \to Y$ be the canonical map. It is easy to find the fibres: they are genus 1 curves, since they are all just $\mathbf{C}^*/q^{\mathbf{Z}}$, so f is indeed "smooth and proper" in this setting. They form a non-algebraic family of elliptic curves over $\mathbf{P}^1$. It is non-algebraic for then Leray must degenerate. Then, we must have $h^1(X) = h^1(Y) + \dim(H^0(Y, \mathbf{R}^1 f_* \underline{\mathbf{Q}})$. But Y is simply connected, hence $\mathbf{R}^1 f_* \underline{\mathbf{Q}} = \underline{\mathbf{Q}} \oplus \underline{\mathbf{Q}}$, so $h^1(X) = 2$. But, $X = (\mathbf{C}^2 \setminus \{0\})/q^{\mathbf{Z}}$, and so $\pi_1(X) = q^{\mathbf{Z}}$ since $\mathbf{C}^2 \setminus \{0\}$ is simply connected. By Hurewicz, this means $h^1(X) = 1$, which is a contradiction.

So Deligne's theorem is really a non-trivial statement in algebraic geometry.

Deligne's proof uses Hard Lefschetz—it's only intuitive once we know about weights.

1.2 Perverse sheaves

1.2.1 Goals

We will only work in the setting of algebraic geometry—even though there are analytic analogues.

Let X be a variety over $\mathbf{C}$ (or any field that is algebraically closed).

1. First construct $\mathrm{Perv}(X) \subseteq \mathsf{D}^b(X, \mathbf{Q})$. It is an abelian category that is noetherian and artinian, e.g., each object has finite length. It is also the heart of a t-structure, which implies we get cohomology functors, that is ${}^pH^i\colon \mathsf{D}^b(X, \mathbf{Q}) \to \mathrm{Perv}(X)$, which behaves cohomologically. Note perverse sheaves are not sheaves, but actually complexes.

2. The simple objects occurring in decompositions of these objects arise in a simple fashion. First, if $U \hookrightarrow Z \hookrightarrow X$ where $U \to Z$ is an open immersion and $Z \to X$ is a closed immersion, and U is smooth, then a local system L on U defines a complex $\underline{\mathrm{IC}}_Z(L) \in \mathrm{Perv}(X)$, such that if L is simple, that is the fundamental group acts simply, then $\underline{\mathrm{IC}}_Z(L)$ is simple in $\mathrm{Perv}(X)$. In fact, all simple objects arise in this fashion. This gives the theory a combinatorial flavor, by stratifying X and thinking about how local systems on U can extend to the boundary.
3. Poincaré duality (Goresky–Macpherson). In the theory of constructible sheaves, you have a constant sheaf that you use to get other cohomology. In this case, we also have the intersection cohmology complex $\underline{\mathrm{IC}}_X = \underline{\mathrm{IC}}_X(\underline{\mathbf{Q}})$ for $Z = X$, $U = X^{\mathrm{sm}}$, and $L = \underline{\mathbf{Q}}$. We can therefore obtain
$$\mathrm{IH}_*(X) = \mathbf{H}^{-*}(X, \underline{\mathrm{IC}}_X).$$
Theorem 1.8. $\mathrm{IH}_i(X) = (\mathrm{IH}^{-i}(X))^\vee$ *via "intersection product," for* all X.

   In fact, Goresky–Macpherson were able to prove this for more general topological spaces than just varieties.
4. Decomposition theorem, a complete generalization of Deligne's theorem:

   Theorem 1.9 (BBD(G))**.** *For any proper map $f\colon X \to Y$ of varieties, we have a decomposition*
$$Rf_*(\underline{\mathrm{IC}}_X) \cong \bigoplus_i \underline{\mathrm{IC}}_{Z_i}(L_i)[n_i],$$
   where $Z_i \subseteq Y$ are closed, L_i are irreducible (or simple) local systems on an open subset of Z_i, and the $n_i \in \mathbf{Z}$.

   What this is saying is that
$$Rf_*(\mathrm{IC}_X) \cong \bigoplus_i {}^p\mathbf{H}^i(Rf_*\underline{\mathrm{IC}}_X)[-i],$$
   cf. the usual case: $Rf_*\underline{\mathbf{Q}} = \bigoplus \mathbf{H}^i(Rf_*\mathbf{Q})[-i]$. This implies the perverse Leray spectral sequence degenerates.

There was a second proof by Saito using mixed Hodge structures, as well as a third proof using Hodge theory.

1.3 Intersection homology via singular chains

Goresky–Macpherson had two papers—"Intersection homology I & II."

Let X be a proper algebraic variety over $\mathbf{C}$ (the properness isn't essential, but it makes technicalities simpler) of complex dimension n. The homology we construct will depend on choices at first, but ends up being independent of said choices. Fix a stratification $X = X_n \supset X_{n-1} \supset X_{n-2} \supset \cdots \supset X_0$ by closed subsets, satisfying the properties that $\dim(X_i) \leq i$, and that $X_i - X_{i-1}$ is smooth of dimension i. Let $C_i(X)$ be the singular i-chains on X with $\mathbf{Q}$-coefficients.

Note the general definition is more complicated, and needs extra conditions to work with topological pseudo-manifolds; Goresky–Macpherson also use real dimensions. Our motivating example is $\mathbf{P}^1 \vee \mathbf{P}^1$.

We look at a certain subset of all singular chains that work well with our stratification.

Definition 1.10. $\xi \in C_i(X)$ is *allowable* if for all $c > 0$, we have $\dim(|\xi| \cap X_{n-c}) \leq i - 2c + c - 1 = i - c - 1$. $|\xi|$ is the support (i.e., closure of the image) of the chain, that is the subset of X you hit with the singular chain. $i - 2c$ is the expected dimension if everything were generic; we allow a bit of extra information with the $c - 1$ summand; this summand is called the "perversity".

Note that this case is called "middle perversity"; there are other definitions in other cases.

There is an issue in that dimensions are not well-defined for singular simplices, but you can either take analytic chains, or take direct limits over triangulations and using actual simplices.

Definition 1.11. Let $\mathrm{IC}_i(X) = \{\xi \in C_i(X) \mid \xi \text{ is allowable in } C_i(X), \text{ and } d(\xi) \text{ is allowable in } C_{i-1}(X)\}$. This gives a subcomplex $\mathrm{IC}_\bullet(X) \subseteq C_\bullet(X)$. Set $\mathrm{IH}_i(X) = H_i(\mathrm{IC}_\bullet(X))$.

Remark 1.12.

1. $\mathrm{IH}_*(X)$ only depends on $|X|$. Goresky–Macpherson used sheaf-theoretic formalism to prove this.
2. $\mathrm{IH}_*(X)$ is *not* homotopy invariant.
3. To get precise definitions, we need to use subanalytic chains.
4. $\dim(\emptyset) = -\infty$.

Example 1.13. Let X be smooth, $X = X_n$, $X_i = \emptyset$ if $i < n$. Then, every chain $\xi \in C_i(X)$ is allowable.

Example 1.14. Let $X = \mathbf{P}^1 \vee \mathbf{P}^1$, and let $x \in X$ be the node. Let a be a loop on one copy A of $\mathbf{P}^1$, and α a point in A; similarly for b, B, β. Let $X = X_1 \supset X_0 = \{x\}$. Recall

$$H_i(X) = \begin{cases} \mathbf{Q} & i = 0 \\ 0 & i = 1 \\ \mathbf{Q} \cdot A \oplus \mathbf{Q} \cdot B & i = 2 \end{cases}$$

We claim Poincaré duality is fixed in this case.

First, $\xi \in C_i(X)$ is allowable if and only if $\dim(|\xi| \cap \{x\}) \leq i - 2$. So,

$$\begin{aligned} \mathrm{IC}_0(X) &= \{\xi \in C_0(X) \mid |\xi| \not\ni x\} = C_0(X \setminus \{x\}) \\ \mathrm{IC}_1(X) &= \{\xi \in C_1(X) \mid |\xi| \not\ni x\} = C_1(X \setminus \{x\}) \\ \mathrm{IC}_2(X) &= \{\xi \in C_2(X) \mid |d\xi| \not\ni x\} = C_1(X \setminus \{x\}) \end{aligned}$$

and so

$$\mathrm{IH}_i(X) = \begin{cases} H_0(X \setminus \{x\}) \cong \mathbf{Q} \cdot \alpha \oplus \mathbf{Q} \cdot \beta & i = 0 \\ 0 & i = 1 \\ H_2(X) = \mathbf{Q} \cdot A \oplus \mathbf{Q} \cdot B & i = 2 \end{cases}$$

For $i = 1$, it is zero since any loops not going through x can be shrunk to a point. So, in this case we get Poincaré duality!

Remark 1.15. If X is a proper variety of dimension n with an isolated singularity at $\{x\}$, then by the same reasoning implies that

$$\mathrm{IH}_i(X) = \begin{cases} H_i(X \setminus \{x\}) & i < n \\ \mathrm{im}(H_n(X \setminus \{x\}) \to H_n(X)) & i = n \\ H_i(X) & i > n \end{cases}$$

Recommended to prove this on your own.

Next time we will talk about derived categories.

2 September 10, 2015

Last time, we wanted an example of the Leray spectral sequence not degenerating.

Example 2.1. Choose X to be a smooth projective, rational surface, and $E \hookrightarrow X$ an elliptic curve on X, such that there exists a contraction $\pi \colon X \to Y$ that contracts E. In this case, you can check that the Leray spectral sequence does not degenerate for π.

In this first part of this course, we want to understand the category of perverse sheaves; in order to do so, we need to build some background.

2.1 Recollections on homological algebra

We remind ourselves of our goal: for a topological space/scheme X, we can associate the category $\mathsf{Ab}(X)$ of abelian sheaves on X (in this course, we will use abelian sheaves, not just coherent ones). Then, we can embed this category in the category $\mathsf{Ch}(X)$ of chain complexes over X. Both categories are abelian. These map to the homotopy category $\mathsf{K}(X)$, which is a triangulated category, and has the same objects but identifies homotopy equivalences. Inverting quasi-isomorphisms gives the derived category $\mathsf{D}(X)$, which is also triangulated.

Perverse sheaves $\mathsf{Perv}(X)$ will be a nice abelian subcategory of $\mathsf{D}(X)$.

We will first review things about additive and abelian categories.

2.1.1 Additive categories

Everything I wrote down so far has been an additive category.

Definition 2.2. A category $\mathscr{C}$ is *additive* if

1. All Hom-sets in $\mathscr{C}$ are endowed with an abelian group structure that is compatible with composition (pre-additive category).
2. Finite coproducts exist.

Example 2.3.

Ab, the category of abelian groups.

Mod_R, the category of R-modules, where R is a ring.

$\mathsf{Ab}(X)$, the category of abelian sheaves on a topological space, the most interesting example for us.

In the category of abelian groups, there is a subcategory of free abelian groups, which is still additive.

If $\mathscr{I}$ is any category, and $\mathscr{C}$ is an additive category, then $\mathsf{Fun}(\mathscr{I}, \mathscr{C})$ is additive (ex. $\operatorname{Rep}(G)$, the category of representations of a group G).

Exercises 2.4.

1. Being additive is a property.
2. If $\mathscr{C}$ is an additive category, there exists a zero object "0" which is simultaneously an initial and final object. In particular, the category of rings is not abelian.

Definition 2.5. Let $\mathscr{C}$ be an additive category, and $f\colon X \to Y$ a map in $\mathscr{C}$, with zero object 0. Then, a *kernel* for f is a map $g\colon Z \to X$ satisfying the following property:

$$\operatorname{Hom}(T, Z) \xrightarrow{\sim} \{\alpha \in \operatorname{Hom}(T, X) \mid f\alpha = 0\}$$

By abstract nonsense, this is unique, as long as it exists. There is a dual notion called the *cokernel*.

Warnings 2.6.

1. Kernels/cokernels might not exist. E.g., consider $R = k[x_1, \ldots, x_n, x_{n+1}, \ldots]$, and let $\mathscr{C}$ be the category of finitely generated R-modules. Since R is not noetherian, the kernel of a map does not have to be finitely generated. For example, $R \twoheadrightarrow k$ is a map in $\mathscr{C}$, but has no kernel, since if any finitely generated module was the kernel, then you can enlarge it to a larger module that is still finitely generated.
2. Kernels/cokernels might not look "correct." E.g., consider $R = \mathbf{Z}$, and $\mathscr{C}$ the category of free abelian groups. Then, $\mathbf{Z} \xrightarrow{2} \mathbf{Z}$ has a trivial cokernel, since if a module collapses $\mathbf{Z}$, then it must collapse the source $\mathbf{Z}$, since $\mathbf{Z}$ is torsion-free.
3. Vanishing of kernel and cokernel do not imply an isomorphism; see the example above. The kernel is zero since it is zero in the larger category.

All these pathologies go away for abelian categories.

2.1.2 Abelian categories

Definition 2.7. $\mathscr{C}$ is *abelian* if

1. $\mathscr{C}$ is additive,
2. kernels and cokernels exists,
3. each monic is a kernel of its cokernel. This is confusing so we write out what it means: If $f\colon X \to Y$ is a monomorphism, then $X \cong \ker(Y \to \operatorname{cok}(f))$, which is induced by the natural map.
4. each epi is a cokernel of its kernel.

Examples 2.8. Ab; Mod_R, R a ring; $\mathsf{Ab}(X)$, X a topological space; $\mathsf{PAb}(X)$, presheaves of abelian groups on X.

Another example: if $Z \hookrightarrow X$ is a closed subspace, then the category $\mathsf{Ab}_Z(X)$ of abelian sheaves such that $F|_{X\setminus Z} = 0$ is abelian.

$\mathsf{Mod}_R^{\mathrm{fp}}$ is abelian if and only if R is coherent, in particular if it is noetherian.

If $\mathscr{A}$ is any abelian category, and $\mathscr{I}$ is another category, then $\mathsf{Fun}(\mathscr{I}, \mathscr{A})$ is also abelian. Presheaves are a special case of this. Just compute everything pointwise; the axioms follow from the axioms on $\mathscr{A}$.

Remark 2.9. There exists an "obvious" notion of an exact sequence in an abelian category, that is,

$$X \xrightarrow{a} Y \xrightarrow{b} Z$$

is exact at Y if and only if $ba = 0$, and $\ker(b) \xleftarrow{\sim} \mathrm{im}(a) =: \ker(Y \to \mathrm{cok}(a))$; note that this latter object can be defined with the dual objects and it would not affect the definition. The map is the natural map.

Warning 2.10. This notion of exactness depends on the ambient abelian category. E.g., take $X = \mathbf{P}^1$ and the Euler sequence

$$\mathcal{O}^{\oplus 2}(-1) \xrightarrow{\binom{x}{y}} \mathcal{O} \longrightarrow 0$$

is exact in $\mathsf{Ab}(X)$, but not in $\mathsf{PAb}(X)$.

Exercises 2.11.

1. Let $\mathscr{A}$ be an abelian category. We can ask if it admits infinite direct sums. This is equivalent to asking if it admits all colimits. $\Leftarrow$ is obvious. (This fails in other categories: consider the category of rings).
2. Find an abelian category $\mathscr{A}$ where taking infinite direct sums is not exact.

Grothendieck figured out how to get around this by defining a better notion.

Definition 2.12. An abelian category $\mathscr{A}$ is *Grothendieck* if

1. $\mathscr{A}$ admits infinite direct sums;
2. filtered colimits are exact, i.e., if $X_i \to Y_i \to Z_i$ are exact sequences indexed by a filtered poset $I \ni i$, then $\operatorname{colim} X_i \to \operatorname{colim} Y_i \to \operatorname{colim} Z_i$ is exact.
3. there exists a generator $X \in \mathscr{A}$, that is for all $Y \in \mathscr{A}$, there exists an epimorphism $X^{\oplus I} \twoheadrightarrow Y$.

Warning 2.13. This notion is not self-dual.

Examples 2.14. Mod_R is Grothendieck abelian (colimits come from those of abelian groups, and R is a generator).

(Gabber) $\mathsf{QCoh}(X)$ is Grothendieck abelian. This is a bit difficult—you need to actually do something to find a generator.

Now I would like to say why we care about Grothendieck abelian categories the most, and why we see them all the time.

Lemma 2.15. *If X is a topological space, then $\mathsf{Ab}(X)$ is Grothendieck abelian.*

Proof. 1. Infinite direct sums exist: Just sheafify the direct sum presheaf.

2. Filtered colimits are exact: Let I be a filtered poset, and $X_i \to Y_i \to Z_i$ are exact for all $i \in I$. We want the colimit to be exact. You can just compute this on stalks: $(X_i)_x \to (Y_i)_x \to (Z_i)_x$ is exact for all $x \in X$. Then use that the stalk of a colimit is the colimit of the stalks (since direct limits commute with each other). The statement for abelian groups implies our claim.

3. Generators exist: Given $j\colon U \to X$ an open immersion, then we can define $j_!\colon \mathsf{Ab}(U) \to \mathsf{Ab}(X)$ which "extends by zero":

$$j_! F = \text{sheafification of} \left(V \mapsto \begin{cases} F(V) & \text{if } V \subseteq U \\ 0 & \text{otherwise} \end{cases} \right)$$

in which case

$$(j_! F)_x = \begin{cases} F_x & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

In this case, check that $\mathrm{Hom}(j_! \underline{\mathbf{Z}}, G) \cong G(U)$. Set

$$F = \bigoplus_{\substack{j\colon U \hookrightarrow X \\ \text{open immersions}}} j_! \underline{\mathbf{Z}}$$

For all $G \in \mathsf{Ab}(X)$,

$$\mathrm{Hom}(F, G) = \prod_{j\colon U \hookrightarrow X} \mathrm{Hom}(j_! \underline{\mathbf{Z}}, G) = \prod_{j\colon U \hookrightarrow X} G(U)$$

Check that then, the canonical map $\bigoplus_{f \in \mathrm{Hom}(F,G)} F \to G$ is an epimorphism. Thus, F is a generator. $\square$

This shows that the generator is not nice at all. This is why Gabber's result is nontrivial—there is no analogue for $j_!$ in the context of quasi-coherent sheaves.

Theorem 2.16 (Grothendieck)**.** *If $\mathscr{A}$ is a Grothendieck abelian category, then $\mathscr{A}$ has "enough" injectives (and in particular the formalism of derived categories make sense). More precisely, given $X \in \mathscr{A}$, there exists a monomorphism $X \hookrightarrow I(X)$ such that*

1. *$I(X)$ is injective, that is,* $\mathrm{Hom}(-, I)$ *is exact;*
2. *$X \mapsto (X \hookrightarrow I(X))$ is functorial in X.*

Upshot 2.17. Functorial injective resolutions exist!

If you go through Weibel's book, for example, he struggles with the fact that resolutions are not functorial. In our cases, this will not be a problem.

Corollary 2.18. *Suppose $F\colon \mathscr{A} \to \mathscr{B}$ is a left-exact functor between abelian categories, and $\mathscr{A}$ is in fact Grothendieck abelian. Then, derived functors $\mathbf{R}^i F\colon A \to B$ exist.*

Proof. Fix $X \in \mathscr{A}$, and set $I^{-1} = X$. Choose $X \overset{d^{-1}}{\hookrightarrow} I(X) = I^0$ as in the theorem. Set $Q_0 = I^0/\operatorname{im}(d^{-1})$. Choose $Q_0 \hookrightarrow I(Q_0) = I^1$. Continuing in this way, we get the following picture:

$$X \xrightarrow{d^{-1}} I^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} X^2 \cdots$$

and

$$(\mathbf{R}^i F)(X) = H^i(F(I^\bullet)) = \frac{\ker(d\colon F(I^i) \to F(I^{i+1}))}{\operatorname{im}(F(I^{i-1}) \to F(I^i))}$$ □

2.1.3 Chain complexes

Fix an abelian category $\mathscr{A}$.

Definition 2.19.

1. A *chain complex* $K^\bullet$ over $\mathscr{A}$ is given by

$$\cdots \longrightarrow K^{i-1} \xrightarrow{d} K^i \xrightarrow{d} K^{i+1} \longrightarrow \cdots$$

   such that $d^2 = 0$.
2. $\mathsf{Ch}(A)$ forms a category with all chain complexes.

I will always use cohomological indexing; technically these are cochain complexes but I would like to avoid saying cochain so often.

Exercise 2.20.

1. $\mathsf{Ch}(\mathscr{A})$ is an abelian category.
2. If $\mathscr{A}$ is Grothendieck abelian, then there exists a functor $\mathscr{A} \to \mathsf{Ch}(\mathscr{A})$ such that $X \mapsto I^\bullet$ is an injective resolution of X (as in the corollary).

As we said, we want to pass to the derived category. In order to do so, I have to remind you how certain operations work with chain complexes.

Operations 2.21.

0. Shifts: If $K^\bullet \in \mathsf{Ch}(\mathscr{A})$, and $n \in \mathbf{Z}$, then define $(K^\bullet[n])^i = K^{n+i}$, and $d_{K^\bullet[n]} = (-1)^n d_{K^\bullet}$.
1. "Stupid" (or "brutal") truncation. If $K^\bullet \in \mathsf{Ch}(\mathscr{A})$, $n \in \mathbf{Z}$, then

$$\sigma^{\le n} K^\bullet = (\cdots \longrightarrow K^{n-2} \longrightarrow K^{n-1} \longrightarrow K^n \longrightarrow 0 \longrightarrow 0)$$
$$\sigma^{\ge n} K^\bullet = (0 \longrightarrow 0 \longrightarrow K^n \longrightarrow K^{n+1} \longrightarrow K^{n+2} \longrightarrow \cdots),$$

   which give two functors $\sigma^{\le n}, \sigma^{\ge n}\colon \mathsf{Ch}(\mathscr{A}) \to \mathsf{Ch}(\mathscr{A})$.

2. Canonical truncation. If $K^\bullet \in \mathsf{Ch}(\mathscr{A})$, you can do something better than just killing terms: we can make sure homology is preserved.

$$\begin{aligned}
\tau^{\leq n} K^\bullet &= (\cdots \longrightarrow K^{n-2} \longrightarrow K^{n-1} \longrightarrow \ker(K^n \longrightarrow K^{n+1}) \longrightarrow 0) \\
\tau^{\geq n} K^\bullet &= (0 \longrightarrow \operatorname{cok}(K^{n-1} \longrightarrow K^n) \longrightarrow K^{n+1} \longrightarrow K^{n+2} \longrightarrow \cdots)
\end{aligned}$$

which give two functors $\tau^{\leq n}, \tau^{\geq n} \colon \mathsf{Ch}(\mathscr{A}) \to \mathsf{Ch}(\mathscr{A})$.

3. Homology: $n \in \mathbf{Z}$, $K^\bullet \in \mathsf{Ch}(\mathscr{A})$, then we can define homology in the usual way:

Definition 2.22. $H^n(K^\bullet) = \ker(K^n \xrightarrow{d} K^{n+1}) / \operatorname{im}(K^{n-1} \xrightarrow{d} K^n)$, or equivalently, $H^n(K^\bullet) = \tau^{\geq n}\tau^{\leq n}(K^\bullet)[n] \cong \tau^{\leq n}\tau^{\geq n}(K^\bullet)[n]$ (identify $\mathscr{A}$ with $K^\bullet \in \mathsf{Ch}(\mathscr{A})$ such that $K^i = 0$ for all $i \neq 0$).

The reason we call them "stupid" is that they do not pass to the derived category.

The point of explaining it this way is that for any functors like this, we can get a notion of homology. To get to the derived category, you have to invert certain maps in the category of chain complexes.

Definition 2.23.
1. A map $f \colon K \to L$ is a *quasi-isomorphism* (or *qis*) if $H^n(f)$ is an isomorphism for all n.
2. $\mathsf{D}(\mathscr{A}) = \mathsf{Ch}(\mathscr{A})[(\text{qis})^{-1}]$

We will give another formula to compute maps.

Examples 2.24. $K^\bullet = (\mathbf{Z} \xrightarrow{2} \mathbf{Z})$ in degrees $-1, 0$. This maps to $(0 \to \mathbf{Z}/2) = L^\bullet$. This is a quasi-isomorphism, but is not an isomorphism in any reasonable sense.

Exercise 2.25. The objects $\mathbf{Z}/2[0] \oplus \mathbf{Z}/2[1]$ and $(\mathbf{Z}/2 \xrightarrow{2} \mathbf{Z}/4)$ are isomorphic in $\mathsf{D}(\mathsf{Ab})$, but *not* in the category $\mathsf{D}(\mathsf{Mod}_{\mathbf{Z}/4})$.

Next time we will talk about the homotopy category.

3 September 15, 2015

Problem set #1 is up. See Milicic's derived categories notes for (lots of) details.

Recall $\mathscr{A}$ abelian categories, and in particular Grothendieck abelian categories for which derived categories are easily defined. Also recall $\mathsf{Ch}(\mathscr{A})$ the category of chain complexes, from which we defined the derived category $\mathsf{D}(\mathscr{A}) \coloneqq \mathsf{Ch}(\mathscr{A})[(\mathsf{qis})^{-1}]$, in which objects are identified with their resolutions.

Our goal is to make $\mathsf{D}(\mathscr{A})$ more explicit via the triangulated structure. It is slightly easier to construct the triangulated structure for the case of the homotopy category, which lives between chain complexes and the derived category.

3.1 Homotopy category

In topology, there is a notion of homotopy between maps of spaces; there is an analogue in homological algebra. This is an example of a larger phenomenon where algebraic topology gives notions in homological algebra.

Fix an abelian category $\mathscr{A}$.

Definition 3.1. Given $f, g \colon K \to L$ in $\mathsf{Ch}(\mathscr{A})$, we say *f is homotopic to g*, denoted $f \sim g$, if there exist maps $h^i \colon K^i \to L^{i-1}$ satisfying the following: $f^n - g^n = d_L^{n-1} h^n + h^{n+1} d_K^n$ (or simply: $f - g = dh + hd$).

Note these h^i are not maps of chain complexes!

$$\begin{tikzcd}
K^{n-1} \arrow[r] \arrow[d, shift right, "f^{n-1}"'] \arrow[d, shift left, "g^{n-1}"] & K^n \arrow[r] \arrow[dl, "h^n"'] \arrow[d, shift right, "f^n"'] \arrow[d, shift left, "g^n"] & K^{n+1} \arrow[r] \arrow[dl, "h^{n+1}"'] \arrow[d, shift right] \arrow[d, shift left] & \cdots \\
L^{n-1} \arrow[r, "d^{n-1}"] & L^n \arrow[r, "d^n"] & L^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Why do we like this?

Lemma 3.2. *If $f, g\colon K \to L$ are homotopic, then $H^n(f) = H^n(g)$ as maps $H^n(K) \to H^n(L)$.*

Proof. Choose homotopies $h^i\colon K^i \to L^{i-1}$ such that $f - g = dh + hd$ (note that there is a choice involved; the set of homotopies actually creates an algebraic structure of some sort, however). Choose an element $\alpha \in \ker(d\colon K^n \to K^{n+1})$; we know $f(\alpha) - g(\alpha) = dh(\alpha) + hd(\alpha) = dh(\alpha)$. So $\overline{f(\alpha)} - \overline{g(\alpha)} = 0$ in the homology group $\ker(L^n \xrightarrow{d} L^{n+1}) / \operatorname{im}(L^{n-1} \xrightarrow{d} L^n)$. $\square$

Proving things about homotopies is straightforward: just write everything out.

Exercise 3.3. Check "$f \sim g$" gives an equivalence relation on the set of all possible maps $\operatorname{Hom}_{\mathsf{Ch}(\mathscr{A})}(K, L)$, which is compatible with composition and addition, i.e., if $f \sim g$ then $h \circ f \sim h \circ g$ and $f + h \sim g + h$. Transitivity involves adding homotopies.

Definition 3.4. $\mathsf{K}(\mathscr{A}) = \mathsf{Ch}(\mathscr{A})$/homotopy equivalences, i.e., objects in $\mathsf{K}(\mathscr{A})$ are the objects in $\mathsf{Ch}(\mathscr{A})$, and $\operatorname{Hom}_{\mathsf{K}(\mathscr{A})}(K, L) = \operatorname{Hom}_{\mathsf{Ch}(\mathscr{A})}(K, L)/\sim$.

Observations 3.5.

1. $\mathsf{K}(\mathscr{A})$ is additive (using exercise).
2. There is a canonical factorization

$$\begin{tikzcd} \mathsf{Ch}(\mathscr{A}) \arrow[rr, "H^n(-)"] \arrow[dr, "\text{canonical}"'] & & \mathscr{A} \\ & \mathsf{K}(\mathscr{A}) \arrow[ur, "H^n(-)"'] & \end{tikzcd}$$

Example 3.6.

$$K = (\mathbf{Z} \xrightarrow{2} \mathbf{Z}), \quad L = (\mathbf{Z} \oplus \mathbf{Z} \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)} \mathbf{Z} \oplus \mathbf{Z})$$

Claim 3.7. *$K \cong L$ in $\mathsf{K}(\mathscr{A})$.*

We want to give two maps between K and L whose compositions in either direction are homotopic to the identity.

$$\begin{tikzcd} \mathbf{Z} \arrow[r, "2"] \arrow[d, shift left, "i_2"] & \mathbf{Z} \arrow[d, shift left, "i_2"] \\ \mathbf{Z} \oplus \mathbf{Z} \arrow[r, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)"] \arrow[u, bend left, "\mathrm{pr}_2"] & \mathbf{Z} \oplus \mathbf{Z} \arrow[u, bend left, "\mathrm{pr}_2"] \end{tikzcd}$$

Get $i\colon K \to L$ and $p\colon L \to K$. $pi = \mathrm{id}$ in $\mathsf{Ch}(\mathscr{A})$. ip is the map

$$\begin{tikzcd} \mathbf{Z} \oplus \mathbf{Z} \arrow[r, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)"] \arrow[d, "(a,b) \mapsto (0,b)"'] & \mathbf{Z} \oplus \mathbf{Z} \arrow[d, "(a,b) \mapsto (0,b)"] \\ \mathbf{Z} \oplus \mathbf{Z} \arrow[r, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)"] & \mathbf{Z} \oplus \mathbf{Z} \end{tikzcd}$$

Choose $h\colon \mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z} \oplus \mathbf{Z}$ where $(a, b) \mapsto (a, 0)$. We want to show: $(\mathrm{id} - ip) = dh + hd$ (check). Thus, $ip = \mathrm{id}$ in $\mathsf{K}(\mathscr{A})$, and p, i are isomorphisms in $\mathsf{K}(\mathscr{A})$.

One reason this works that both complexes resolve $\mathbf{Z}/2$; the first factor doesn't matter since the map is just the identity on this summand.

Remark 3.8. Both K, L are projective resolutions in $\mathbf{Z}/2$ in $\mathsf{Ch}(\mathsf{Ab})$. More generally, we have the following:

Lemma 3.9. *Let R be a ring and $M \in \mathsf{Mod}_R$, where $P^\bullet \to M$ and $Q^\bullet \to M$ are both projective resolutions of M. In that case, they are homotopy equivalent, i.e., $P^\bullet \cong Q^\bullet$ in $\mathsf{K}(\mathsf{Mod}_R)$.*

Warning 3.10. $M \not\cong P^\bullet$ in $\mathsf{K}(\mathsf{Mod}_R)$ unless M is projective.

This is why the homotopy category is a step toward the derived category. We will first show $\mathsf{K}(\mathscr{A})$ is a triangulated category, which will induce a triangulated structure on $\mathsf{D}(\mathscr{A})$.

3.2 More constructions on chain complexes

Note the homotopy category of chain complexes is not abelian, so we need different tools than in that case.

3.2.1 Cones

Recall that if $f\colon X \to Y$ is a map of topological spaces, then

$$\mathrm{Cone}(f) = \left(\frac{X \times [0,1]}{X \times \{1\}}\right) \amalg Y \Big/ (x,0) \sim f(x)$$

Categorically,

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[d, hook] & & Y \arrow[d] \\ \mathrm{Cone}(X) = \left(\frac{X\times[0,1]}{X\times\{1\}}\right) \arrow[rr] & & \mathrm{Cone}(f) \end{tikzcd}$$

is a pushout, i.e.,

1. For all Z, $\mathrm{Map}(\mathrm{Cone}(f), Z) \cong \{(g,h) \mid g\colon Y \to Z,\ h\colon gf \sim \text{constant map}\}$. This is why cones are like cokernels. The "fattening" of X allows us to keep track of the null-homotopy.
2. If f is a nice inclusion, then $\mathrm{Cone}(f) \simeq Y/X$ is a homotopy-equivalence, i.e., they are an analogue for quotients.

Example 3.11.

1. $\mathrm{Cone}(\mathrm{id}) = \mathrm{Cone}(X)$.
2. $\mathrm{Cone}(X \to *) = \Sigma X =$ suspension of X. For example, let $X = S^1$. Then, $\mathrm{Cone}(S^1 \to *) = \Sigma S^1 = S^2$. This will give the shift functor on our category (some people use Σ as the notation for it!).

Definition 3.12. Let $\mathscr{A}$ be an abelian category, and $f\colon K \to L$ in $\mathsf{Ch}(\mathscr{A})$. Then, $\mathrm{Cone}(f) \in \mathsf{Ch}(\mathscr{A})$ is defined by

$$\mathrm{Cone}(f)^i = K^{i+1} \oplus L^i,\ d(x,y) = (-dx, d(y) + f(x)) = \begin{pmatrix} d_{K[1]} & 0 \\ f[1] & d_L \end{pmatrix}$$

where there are no signs because of our conventions about shifting.

Observations 3.13.

1. $d^2 = 0$: $d(d(x,y)) = d(-dx, dy + f(x)) = (d(-dx), d(dy + f(x)) + f(-dx)) = (0, df(x) - fd(x)) = 0$, so the cone is a chain complex.
2. Inclusion, projection give $L \overset{i_f}{\hookrightarrow} \mathrm{Cone}(f)$ and $\mathrm{Cone}(f) \overset{\pi_f}{\twoheadrightarrow} K[1]$, and a short exact sequence

$$0 \longrightarrow L \longrightarrow \mathrm{Cone}(f) \longrightarrow K[1] \longrightarrow 0$$

in $\mathsf{Ch}(\mathscr{A})$. The topological analogue of the inclusion is the vertical map in the picture; also, both squares below are pushouts

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d] & * \arrow[d] \\ \mathrm{Cone}(X) \arrow[r] & \mathrm{Cone}(f) \arrow[r] & \Sigma X \end{tikzcd}$$

and this diagram is the topological analogue of the sequence above.

Example 3.14. Say $f\colon M \to N$ in $\mathscr{A}$ (viewed in $\mathsf{Ch}(\mathscr{A})$). In this case,

$$\mathrm{Cone}(f) = \left(M \xrightarrow{f} N\right).$$

Then, $H^0(\mathrm{Cone}(f)) = \mathrm{cok}(f)$, $H^{-1}(\mathrm{Cone}(f)) = \ker(f)$. If f is injective, then the cone is a cokernel; this is the analogue of saying that if f is a nice inclusion of spaces, then the cone is a quotient. It is also true that if $\mathrm{Cone}(f) \to \mathrm{cok}(f)$ is a quasi-isomorphism if $\ker(f) = 0$.

Exercises 3.15.

1. Calculate $\mathrm{Hom}_{\mathsf{Ch}(\mathscr{A})}(\mathrm{Cone}(f), -)$.
2. Given $f\colon K \to L$ with L *acyclic* (no homology), then $H^i(\mathrm{Cone}(f)) \cong H^{i+1}(K)$.
3. If $K^\bullet \xrightarrow{f} L^\bullet$ is termwise split injective (in that K^i is a direct summand of L^i but this splitting does not have to glue to a chain complex splitting), then $\mathrm{Cone}(f) \cong L/K$ in $\mathsf{K}(\mathscr{A})$. This is the analogue of a "nice inclusion" in homological algebra.

The moral of this is that the cone is a generalization of the cokernel if we have nice inclusions. The notion of cylinders makes this precise.

3.2.2 Cylinders

Recall that if $f\colon X \to Y$ is a map of topological spaces, then we can define something called the mapping cylinder of f: $M_f = (X \times [0,1]) \amalg Y/(x,0) \sim f(x)$. You get

$$X \xhookrightarrow[i]{\text{nice inclusion}} M_f \xrightarrow{\quad\pi\quad} Y$$

where $x \mapsto (x,1)$, and $Y \to Y$ is the identity, while $f \circ p\colon X \times [0,1] \to Y$. i is a nice inclusion (a cofibration) and π is a homotopy equivalence. The inverse map is given by sending Y to the second slot. Thus if we work in the homotopy category, then we can always replace Y by M_f to assume $X \to Y$ is a cofibration.

Definition 3.16. Given $f\colon K \to L$ in $\mathsf{Ch}(\mathscr{A})$, we can define the mapping cylinder as

$$\mathrm{Cyl}(f)^i = K^i \oplus K[1]^i \oplus L^i,\ d = \begin{pmatrix} d_K & -1 & 0 \\ 0 & d_{K[1]} & 0 \\ 0 & f[1] & d_L \end{pmatrix}$$

Check that $d^2 = 0$, and so the cylinder is a chain complex.

Lemma 3.17. *Given $f\colon K \to L$ in $\mathsf{Ch}(\mathscr{A})$, we get a commutative diagram*

$$\begin{tikzcd}
 & 0 \arrow[r] & L \arrow[r, "i_f"] \arrow[d, "\alpha_f"] & \mathrm{Cone}(f) \arrow[r, "\pi_f"] \arrow[d, equal] & K[1] \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[r, "\delta_f"] \arrow[d, equal] & \mathrm{Cyl}(f) \arrow[r, "\gamma_f"] \arrow[d, "\beta_f"] & \mathrm{Cone}(f) \arrow[r] & 0 & \\
 & K \arrow[r, "f"] & L & & &
\end{tikzcd}$$

with all rows exact, and all vertical maps are isomorphisms in $\mathsf{K}(\mathscr{A})$.

Proof. i_f, π_f are as before. δ_f is inclusion into the first factor, and γ_f is projection to the second two factors. α_f is inclusion in third factor, and β_f is the only "interesting" map

$$\beta_f = (f, 0, \mathrm{id})\colon K^i \oplus K[1]^i \oplus L^i \longrightarrow L^i$$

Checking all rows are exact is obvious: in each degree you have the decomposition of the piece in the middle to the pieces on the left and right. We need to check that the vertical maps are isomorphisms in the homotopy category. $\alpha_f \circ \beta_f = \mathrm{id}_L$ in $\mathsf{Ch}(\mathscr{A})$, and so we only have to check the other direction, i.e.,

Claim 3.18. $\alpha_f\beta_f \sim \mathrm{id}$.

Proof of Claim. $(\alpha_f\beta_f)(x,y,z) = \alpha_f(f(x)+z) = (0,0,f(x)+z)$. We need to show this is homotopy equivalent to doing nothing, so somehow y isn't doing anything. The homotopy is: $h^i\colon \mathrm{Cyl}(f)^i \to \mathrm{Cyl}(f)^{i-1}$, where $(x,y,z) \mapsto (0,x,0)$. Check: $\alpha_f\beta_f - \mathrm{id} = dh + hd$. ∎

Thus, α_f, β_f are mutually inverse isomorphisms in $\mathsf{K}(\mathscr{A})$. ∎

This is sort of the point: δ_f injects into the cylinder; α_f is "fattening up" L to make sure that the map δ_f is actually an injection without changing the homotopy type of L.

Corollary 3.19. *Given $f\colon K \to L$ in $\mathsf{Ch}(\mathscr{A})$, you get the long exact sequence*

$$\begin{tikzcd} \cdots \arrow[r] & H^i(K) \arrow[r, "f_*"] & H^i(L) \arrow[r, "i_{f,*}"] & H^i(\mathrm{Cone}(f)) \arrow[dll, out=0, in=180, "\pi_{f,*}"'] & \\ & H^{i+1}(K) \arrow[r] & H^{i+1}(L) \arrow[r] & H^{i+1}(\mathrm{Cone}(f)) \arrow[r] & \cdots \end{tikzcd}$$

Proof. Take long exact sequence for the first or second row. Check that it gives the same result as the "snake" map induced by π_f. ☐

Right now we do not even know that there is a functor $\mathsf{K}(\mathscr{A}) \to \mathsf{D}(\mathscr{A})$. One of the steps involved is to show the homotopy category is a triangulated category.

Lemma 3.20 (What is the cone of a cone?)**.** *Given $f\colon K \to L$ in $\mathsf{Ch}(\mathscr{A})$, the natural map* $\mathrm{Cone}(L \xrightarrow{i_f} \mathrm{Cone}(f)) \to K[1]$ *is a homotopy equivalence.*

Let me point something out: the exact sequence above shows the homologies are nicely related. You would expect the statement in the lemma to follow in that the homologies are indeed the same; the statement though is stronger in that it says there is a homotopy equivalence, not just an isomorphism of homologies.

If you like, this is the triangulated category version of the fact that $(B/A)/B \cong 1/A$, which is what this lemma is saying since cones are like quotients. Most axioms for triangulated categories can be thought of as analogues of facts about fractions!

Proof. $\mathrm{Cone}(f)^i = K[1]^i \oplus L^i$, so $\mathrm{Cone}(i_f)^i = L^{i+1} \oplus K^{i+1} \oplus L^i$, where

$$d(x,(y,z)) = (-dx, i_f(x) + d(y,z)) = (-dx, (0,x) + (-dy, f(y)+dz)) = (-dx, -dy, x + f(y) + dz)$$

The projection onto the second factor gives $\mathrm{Cone}(i_f) \xrightarrow{\mathrm{can}} K[1]$ in $\mathsf{Ch}(\mathscr{A})$. The claim is that this is a homotopy equivalence.

We first start by writing down an inverse. The map $K^{i+1} \to L^{i+1} \oplus K^{i+1} \oplus L^i$ is the map $y \mapsto (-f(y), y, 0)$. We claim this gives a map of chain complexes $K[1] \xrightarrow{j_f} \mathrm{Cone}(i_f)$. Check: $\mathrm{can} \circ j_f = \mathrm{id}$. $j_f \circ \mathrm{can} \simeq \mathrm{id}$, where $(a,b,c) \mapsto (-c,0,0)$ gives the homotopy. ☐

This will be useful later when showing one of the axioms of triangulated categories.

Next time we will talk about triangulated categories.

4 September 17, 2015

Next week, Mircea will be filling in for Bhargav.

Let us quickly recall from last time: so far we've constructed $\mathsf{K}(\mathscr{A})$, from which we will construct $\mathsf{D}(\mathscr{A})$. If $f\colon K \to L$ in $\mathsf{Ch}(\mathscr{A})$, we have defined $\mathrm{Cone}(f)$ and $\mathrm{Cyl}(f)$, which are also chain complexes, that are related to f in a nice ways:

1. There is a short exact sequence $0 \to K \to \mathrm{Cyl}(f) \to \mathrm{Cone}(f) \to 0$ in $\mathsf{Ch}(\mathscr{A})$.
2. $\mathrm{Cyl}(f) \cong L$ in $\mathsf{K}(\mathscr{A})$.
3. $\mathrm{Cone}(L \xrightarrow{i_f} \mathrm{Cone}(f)) \xrightarrow{\sim} K[1]$ in $\mathsf{K}(\mathscr{A})$.

The first two properties are saying "$\mathrm{Cone}(f) \cong L/K$", while the last is saying "$(L/K)/L \cong 1/K$". Today we will talk about why this category is a triangulated category.

Definition 4.1. A *triangle* in $\mathsf{K}(\mathscr{A})$ is a sequence

$$K \longrightarrow L \longrightarrow M \longrightarrow K[1]$$

of maps. This triangle is an *exact triangle* (exact Δ) if it is isomorphic to a triangle of the form

$$K \xrightarrow{f} L \xrightarrow{i_f} \mathrm{Cone}(f) \xrightarrow{\mathrm{can}} K[1]$$

in $\mathsf{K}(\mathscr{A})$; by isomorphism we mean an isomorphism of sequences in the homotopy category.

The axioms for a triangulated category say we have a category with shifts and triangles satisfying some nice properties.

Lemma 4.2 (TR1)**.** *If $K \in \mathsf{K}(\mathscr{A})$, the triangle*

$$K \xrightarrow{\mathrm{id}} K \longrightarrow 0 \longrightarrow K[1]$$

is an exact triangle.

Already it is important to be in the homotopy category; Cone(id) cannot possibly be 0 in the category of chain complexes.

Proof. We know

$$K \xrightarrow{\mathrm{id}} K \longrightarrow \mathrm{Cone(id)} \longrightarrow K[1]$$

is exact, and so it suffices to show that $\mathrm{Cone(id)} \cong 0$ in $\mathsf{K}(\mathscr{A})$, since commutativity follows by the fact that 0 is the zero object. $\mathrm{Cone(id)} = K^{i+1} \oplus K^i$, $d(x, y) = (-dx, x + dy)$. Use $h^i : K^{i+1} \oplus K^i \to K^i \oplus K^{i-1}$ defined by $(x, y) \mapsto (y, 0)$; check that $dh + hd = \mathrm{id}_{\mathrm{Cone(id)}}$ and so $\mathrm{Cone(id)} \cong 0$ in $\mathsf{K}(\mathscr{A})$.

What we are doing is we have a diagram

$$\begin{tikzcd}
K \arrow[r, "\mathrm{id}"] \arrow[d, equal] & K \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, shift left, "0"] & K[1] \arrow[d, equal] \\
K \arrow[r, "\mathrm{id}"] & K \arrow[r] & \mathrm{Cone(id)} \arrow[r] \arrow[u, shift left, "0"] & K[1]
\end{tikzcd}$$

and the homotopy h^i makes the compositions in either direction in the third column equal to the identity in the homotopy category. ☐

We recall that cones are like cokernels, but we want all maps to have cokernels, so we have

Lemma 4.3 (TR2)**.** *Any $f : K \to L$ in $\mathsf{K}(\mathscr{A})$ fits into an exact triangle*

$$K \xrightarrow{f} L \longrightarrow M \longrightarrow K[1].$$

In general, this is non-trivial, but in our case it's stupid.

Proof. Lift f to some $\tilde{f}$ in $\mathsf{Ch}(\mathscr{A})$, and apply cones. ☐

The notion of cones is not functorial in the homotopy category, which is why we are emphasizing the fact that there is a choice being made. This is why cones are not a perfect analogue for cokernels: the latter is always a functor. But nevertheless we can do it.

There is a nice discussion on MathOverflow about why Cones are not functorial and how we might be able to fix it—David Speyer had an attempt which didn't work.

Lemma 4.4 (TR3)**.** *If*

$$K \xrightarrow{a} L \xrightarrow{b} M \xrightarrow{c} K[1]$$

is an exact triangle, so are

$$L \xrightarrow{b} M \xrightarrow{c} K[1] \xrightarrow{-a[1]} L[1]$$
$$M[-1] \xrightarrow{-c[-1]} K \xrightarrow{a} L \xrightarrow{b} (M[-1])[1] \cong M$$

Proof sketch. We can assume we are working with the standard example

$$K \xrightarrow{f} L \xrightarrow{i_f} \mathrm{Cone}(f) \xrightarrow{\mathrm{can}} K[1]$$

We want

$$L \xrightarrow{i_f} \mathrm{Cone}(f) \xrightarrow{\mathrm{can}} K[1] \xrightarrow{-f[1]} L[1]$$

to be exact (and the dual statement). But last time, we saw that $\mathrm{Cone}(L \xrightarrow{i_f} \mathrm{Cone}(f)) \cong K[1]$, so on the level of objects it looks promising. You can now check this gives an isomorphism in $\mathsf{K}(\mathscr{A})$ to the standard exact triangle

$$L \xrightarrow{i_f} \mathrm{Cone}(f) \xrightarrow{\mathrm{can}} \mathrm{Cone}(i_f) \xrightarrow{\mathrm{can}} L[1]$$

The point is that the sign convention is there to make the signs work out. $\square$

This will be a very useful device to get long exact sequences. The reason why we call it rotation is because you can draw exact triangles like so:

$$\begin{tikzcd} & K \arrow[dr, "a"] & \\ M \arrow[ur, "+1", "c"'] & & L \arrow[ll, "b"] \end{tikzcd}$$

and then TR3 is literally rotating this triangle.

We recall that cones are not functorial. But nevertheless cones are "weakly functorial", and this is what TR4 says.

Lemma 4.5 (TR4)**.** *Given a commutative diagram in* $\mathsf{K}(\mathscr{A})$

$$\begin{tikzcd} K \arrow[r, "u"] \arrow[d, "f"] & L \arrow[r, "v"] \arrow[d, "g"] & M \arrow[r, "w"] & K[1] \\ K' \arrow[r, "u'"] & L' \arrow[r, "v'"] & M' \arrow[r, "w'"] & K'[1] \end{tikzcd}$$

with exact rows, we can extend it to a commutative diagram

$$\begin{tikzcd} K \arrow[r, "u"] \arrow[d, "f"] & L \arrow[r, "v"] \arrow[d, "g"] & M \arrow[r, "w"] \arrow[d, "h"] & K[1] \arrow[d, "f[1]"] \\ K' \arrow[r, "u'"] & L' \arrow[r, "v'"] & M' \arrow[r, "w'"] & K'[1] \end{tikzcd}$$

So there is some category $\mathsf{Map}(\mathsf{K}(\mathscr{A}))$, where morphisms are commutative squares; there is no functor $\mathsf{Map}(\mathsf{K}(\mathscr{A})) \to \mathsf{K}(\mathscr{A})$ for cones. But the Lemma shows that we can do this non-uniquely. The proof will show that the reason why this occurs is because we are only working in the homotopy category.

If we had actual maps of chain complexes, we can apply Cones in that category, but we cannot do that (even though it's in common sources, it's wrong!).

Proof. May assume both rows are standard exact triangles:

$$\begin{tikzcd} K \arrow[r, "u"] \arrow[d, "f"] & L \arrow[r] \arrow[d, "g"] & \mathrm{Cone}(u) \arrow[r] & K[1] \\ K' \arrow[r, "u'"] & L' \arrow[r] & \mathrm{Cone}(u') \arrow[r] & K'[1] \end{tikzcd}$$

which only commutes in $\mathsf{K}(\mathscr{A})$. The problem with doing the obvious construction is that this first square only commutes up to homotopy; this homotopy will come up in our construction for h.

Choose representatives $\tilde{u}, \tilde{u'}, \tilde{f}, \tilde{g}$ of u, u', f, g in $\mathsf{Ch}(\mathscr{A})$. What we know is that $\tilde{u'}\tilde{f} - \tilde{g}\tilde{u} = ds^i + s^{i+1}d$ for some homotopy $s^i\colon K^i \to L'^{\,i-1}$. Now we want to find $\tilde{h}$ that works. Set

$$\begin{array}{ccc} \mathrm{Cone}(\tilde{u}) & \xrightarrow{\quad\tilde{h}\quad} & \mathrm{Cone}(\tilde{u'}) \\ \shortparallel & & \shortparallel \\ K^{i+1} \oplus L^i & & K'^{\,i+1} \oplus L'^{\,i} \\ & & \\ (x, y) & \longmapsto & (\tilde{f}(x), \tilde{g}(y) + s^{i+1}(x)) \end{array}$$

Now check

1. $\tilde{h}$ is a map of chain complexes (use s^i).
2. All relevant diagrams commute.

$$\begin{tikzcd} K \arrow[r, "u"] \arrow[d, "\tilde{f}"] & L \arrow[r] \arrow[d, "\tilde{g}"] & \mathrm{Cone}(u) \arrow[r] \arrow[d, "\tilde{h}"] & K[1] \arrow[d, "\tilde{f}[1]"] \\ K' \arrow[r, "u'"] & L' \arrow[r] & \mathrm{Cone}(u') \arrow[r] & K'[1] \end{tikzcd}$$

and the two right squares commute in $\mathsf{Ch}(\mathscr{A})$; the left square only commutes in $\mathsf{K}(\mathscr{A})$. □

The reason why cones were not functorial is because we had to choose the homotopy that makes the left square commutative, and they were needed to define the map $\tilde{h}$.

Lemma 4.6 (TR5; see Gelfand–Manin)**.** *A triangulated version of the fact that in* Ab*, if there is a sequence* $A \subset B \subset C$ *of subgroups, then* $C/B \cong (C/A)/(B/A)$.

All of what I am doing is Verdier's thesis. I have no idea how he (or his adviser) was able to come up with this. It is remarkable that somehow these axioms give a nice theory.

Definition 4.7. An additive category $\mathscr{C}$ is *triangulated* if we are given

1. A shift functor $\mathscr{C} \to \mathscr{C}$, $M \mapsto M[1]$
2. A class of sequences

   $$K \longrightarrow L \longrightarrow M \longrightarrow K[1]$$

   called exact triangles satisfying the conclusions of TR1–TR5 (just replace $\mathsf{K}(\mathscr{A})$ with $\mathscr{C}$ in the statements for the Lemmas).

Theorem 4.8. *If* $\mathscr{A}$ *is an abelian category, then* $\mathsf{K}(\mathscr{A})$ *is a triangulated category.*

I hope it's clear that all over the place we used the fact that we are in the homotopy category. $\mathsf{Ch}(\mathscr{A})$ is not triangulated.

The only algebraic triangulated categories are $\mathsf{K}(\mathscr{A})$ and $\mathsf{D}(\mathscr{A})$, and there are some topological ones like the category of spectra. We can always create more like in the process for $\mathsf{D}(\mathscr{A})$. There is also something called the "stable module category" which is useful in finite group theory; there the suspension is the identity. For spectra, it is suspension.

We should really be doing ∞-categories! There the triangulatedness of the category would just be a property. In this setup I do not know if there are any categories with more than one triangulated structure.

Definition 4.9.

1. A functor $\mathscr{C} \xrightarrow{F} \mathscr{D}$ between triangulated categories is *exact* if
   (a) $F \circ [1]_{\mathscr{C}} \cong [1]_{\mathscr{D}} \circ F$, using given isomorphisms;
   (b) F preserves exact triangles.
2. If $\mathscr{C}$ is a triangulated category, and $\mathscr{A}$ is an abelian category, then a functor $H\colon \mathscr{C} \to \mathscr{A}$ is *cohomological* if for all exact triangles $K \xrightarrow{a} L \xrightarrow{b} M \xrightarrow{c} K[1]$ in $\mathscr{C}$, you get a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H(K) \arrow[r, "H(a)"] & H(L) \arrow[r, "H(b)"] & H(M) \arrow[dll, "H(c)"'] \\ & H(K[1]) \arrow[r, "H(-a[1])"] & H(L[1]) \arrow[r, "H(-b[1])"] & H(M[1]) \arrow[r] & \cdots \end{tikzcd}$$

Example 4.10. Letting $\mathscr{C} = \mathsf{K}(\mathscr{A})$, the functor $H^0 \colon \mathscr{C} \to \mathscr{A}$ is cohomological. When we do perverse sheaves, we will have another category $\mathscr{A}$ and a functor H^0 which we will call "perverse cohomology."

Lemma 4.11. *If $\mathscr{C}$ is a triangulated category, and $X \in \mathscr{C}$, then the functor* $\operatorname{Hom}_{\mathscr{C}}(X, -) \colon \mathscr{C} \to \mathscr{A}$ *is cohomological.*

Proof. Fix an exact triangle $K \xrightarrow{a} L \xrightarrow{b} M \to K[1]$ in $\mathscr{C}$. Since I can always just rotate this triangle, it suffices to show exactness in one spot, i.e.,

$$\operatorname{Hom}_{\mathscr{C}}(X, K) \xrightarrow{a_*} \operatorname{Hom}_{\mathscr{C}}(X, L) \xrightarrow{b_*} \operatorname{Hom}_{\mathscr{C}}(X, M)$$

is exact in the middle (by rotation invariance TR3). This is already a complex because the composition of two adjacent maps in a triangle is 0 (which follows from the axioms).

Choose $\alpha \in \ker(b_*)$. Then, I get a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[r] \arrow[d, "0"] & X[1] \arrow[r, "\mathrm{id}"] & X[1] \\ L \arrow[r, "b"] & M \arrow[r, "c"] & K[1] \arrow[r, "-a[1]"] & L[1] \end{tikzcd}$$

where both rows are exact by TR1. This digram commutes, which is what it means for α to be in the kernel of b_*. Now TR4 says that there exists a $\beta \colon X[1] \to K[1]$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[r] \arrow[d, "0"] & X[1] \arrow[r, "\mathrm{id}"] \arrow[d, "\beta"] & X[1] \arrow[d, "\alpha[1]"] \\ L \arrow[r, "b"] & M \arrow[r, "c"] & K[1] \arrow[r, "-a[1]"] & L[1] \end{tikzcd}$$

Then, $-\beta[-1] \in \operatorname{Hom}_{\mathscr{C}}(X, K)$ lifts α under a_*. ∎

All we used was the axioms; we did not need anything about the homotopy category!

In the problem set, there are exercises about triangulated categories. You can show that monomorphisms in a triangulated category are extremely rigid: they always split. This is why they are very different from abelian categories.

Localizations of categories

Fact 4.12. Given a category $\mathscr{C}$ and S a set of maps in $\mathscr{C}$, then there is always a localization $\mathscr{C}[S^{-1}]$ exists, that is there is a functor $\mathscr{C} \xrightarrow{q} \mathscr{C}[S^{-1}]$ which is universal for $q(s)$ being isomorphisms for all $s \in S$.

The construction is just formal: it is some sort of zig-zagging like what we do for localization of rings. In a special setting, these zig-zagging calculations become more manageable.

Definition 4.13. A set S of maps in $\mathscr{C}$ is *localizing* if

L1: $\mathrm{id}_M \in S$ for all $M \in \mathscr{C}$.

L2: S is closed under composition.

L3$_a$: Given

$$\begin{tikzcd} & K \arrow[d, "s"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

in $\mathscr{C}$ with $s \in S$, there exists an extension

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, "t"] & K \arrow[d, "s"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

with $t \in S$. (We want to say S is closed under fibre product, but this is a bit ridiculous since we would need to assume they exist).

L3$_b$: is the dual notion of L3$_a$.
L4: Given $f, g\colon M \to N$ in $\mathscr{C}$, the following are equivalent:
  (a) $fs = gs$ for some $s \in S$;
  (b) $tf = tg$ for some $t \in S$.

The claim will be that quasi-isomorphisms in $\mathsf{K}(\mathscr{A})$ satisfy this. But first, we prove

Theorem/Construction 4.14. *Say $\mathscr{C}$ is a category, and S is a localizing set of maps. Then construct $\mathscr{C}[S^{-1}]$ as follows:*

1. $\mathrm{Obj}(\mathscr{C}[S^{-1}]) = \mathrm{Obj}(\mathscr{C})$*;*
2. *A fraction in $\mathscr{C}$ is a diagram $X \xleftarrow{s} X' \xrightarrow{f} Y$ with $s \in S$. Then,*

$$\mathrm{Hom}_{\mathscr{C}[S^{-1}]}(X, Y) = \frac{\{\textit{fractions } X \xleftarrow{s} X' \xrightarrow{f} Y\}}{\sim}$$

where

$$\left(X \xleftarrow{s} X' \xrightarrow{f} Y\right) \sim \left(X \xleftarrow{t} X'' \xrightarrow{g} Y\right)$$

if there exists a commutative diagram

$$\begin{tikzcd}
& X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & X''' \arrow[l, "u"'] \arrow[r, "h"] \arrow[u] \arrow[d] & Y \\
& X'' \arrow[ul, "t"] \arrow[ur, "g"'] &
\end{tikzcd}$$

with $u \in S$ (note the vertical maps don't have to be in S). Equivalently:

$$\mathrm{Hom}_{\mathscr{C}[S^{-1}]}(X, Y) = \operatorname*{colim}_{\substack{X' \xrightarrow{s} X \\ s \in S}} \mathrm{Hom}_{\mathscr{C}}(X', Y)$$

Exercise 4.15. Our axioms ensure the colimit above is filtered.

3. *Given two fractions*

$$F \coloneqq \left(X \xleftarrow{s} X' \xrightarrow{f} Y\right)$$
$$G \coloneqq \left(Y \xleftarrow{t} Y' \xrightarrow{g} Z\right)$$

define $G \circ F$ as follows: choose a picture

$$\begin{tikzcd}
F \arrow[r, "h"] \arrow[d, "u"] & Y' \arrow[d, "t"] \\
X' \arrow[r, "f"] & Y
\end{tikzcd}$$

with $u \in S$. Then, the diagram

$$X \xleftarrow{s} X' \xrightarrow{f} Y \xleftarrow{t} Y' \xrightarrow{g} Z$$

is equivalent to

$$\begin{tikzcd}
X \arrow[d, equal] & X' \arrow[l, "s"'] & F \arrow[l, "u"'] \arrow[r, "h"] \arrow[d, equal] & Y' \arrow[r, "g"] & Z \arrow[d, equal] \\
X & & F \arrow[ll, "us"'] \arrow[rr, "gh"] & & Z
\end{tikzcd}$$

Define $G \circ F$ to be this bottom row.

Theorem 4.16 (Gabriel–Zisman)**.** *This construction works, i.e., what we defined is a category satisfying the universal property desired. Moreover,*

1. *If $\mathscr{C}$ is additive, $\mathscr{C}[S^{-1}]$ is additive.*
2. *If $\mathscr{C}$ is abelian, $\mathscr{C}[S^{-1}]$ is abelian.*

Some exercises in the problem set ask you to compute examples; another asks you to prove this theorem.

Theorem 4.17 (Verdier)**.** *Let $\mathscr{C}$ be a triangulated category, and S a localizing set of maps. Assume*

1. *$f \in S$ if and only if $f[1] \in S$;*
2. *Given a commutative diagram*

$$\begin{tikzcd}
K \arrow[r] \arrow[d, "f"] & L \arrow[r] \arrow[d, "g"] & M \arrow[r] & K[1] \\
K' \arrow[r] & L' \arrow[r] & M' \arrow[r] & K'[1]
\end{tikzcd}$$

with exact rows such that $f, g \in S$, then there exists $h\colon M \to M'$ in S making the diagram commute. Then, $\mathscr{C}[S^{-1}]$ has a natural triangulated structure, and $\mathscr{C} \to \mathscr{C}[S^{-1}]$ is exact.

For us, this second axiom is just the snake lemma.

I don't have time for what comes next: that quasi-isomorphisms satisfy our axioms, which will show $\mathsf{D}(\mathscr{A})$ is triangulated.

5 September 22, 2015

Last time: if S is a class of morphisms in a category $\mathscr{C}$, which is a localizing class, then $\mathscr{C}[S^{-1}]$ has an explicit description.

Example 5.1. If $\mathscr{C}$ additive, then $\mathscr{C}[S^{-1}]$ is additive.

Localizing condition tells you that you have "common denominators", i.e., for a diagram

$$\begin{tikzcd}
& W \arrow[dl, dashed, "s'"'] \arrow[dr, dashed, "t'"] & \\
U \arrow[d, "s"'] \arrow[drr, "f"] & & V \arrow[dll, "t"'] \arrow[d, "g"] \\
X & & Y
\end{tikzcd}$$

with $s, t \in S$, then there exists W filling in the diagram such that $s' \in S$, making the diagram commute, i.e., $ss' = tt' \in S$. Note $(s, f) \sim (ss', fs')$ and $(t, g) \sim (tt', gt')$ and so $(s, f) + (t, g) := (ss', fs' + gt')$ will give an additive structure.

Theorem 5.2 (Verdier)**.** *Under mild conditions, if $\mathscr{C}$ is triangulated, then $\mathscr{C}[S^{-1}]$ is triangulated such that a triangle in $\mathscr{C}[S^{-1}]$ is exact if and only if it is isomorphic to the image of an exact triangle in $\mathscr{C}$. In particular, $\mathscr{C} \to \mathscr{C}[S^{-1}]$ is exact.*

Remark 5.3. The class of quasi-isomorphisms in $\mathsf{Ch}(\mathscr{A})$ where $\mathscr{A}$ is an abelian category is not a localizing class.

Example 5.4. Let $X^\bullet$ be an exact complex, and

$$\begin{tikzcd}
W^\bullet \arrow[r, dashed, "\psi"] \arrow[d, dashed, "g?"', "\mathsf{qis}"] & X^\bullet \arrow[d, "0"] \\
Z^\bullet \arrow[r, "\varphi"] & Y^\bullet
\end{tikzcd}$$

where
$$Y^\bullet = [0 \longrightarrow M \xrightarrow{\text{id}} M \longrightarrow 0]$$
If there are such ψ, g, then $\psi \circ g = 0$, and so g must factor through the kernel of φ. Hence, $\operatorname{im}(g) \subseteq \ker(\varphi)$. Now take
$$Z^\bullet = [0 \longrightarrow N \xrightarrow{\text{incl}} M \longrightarrow 0]$$
with φ the inclusion, such that $N \subsetneq M$. φ being injective would imply $g = 0$, but since g is supposed to be a quasi-isomorphism, this forces $Z^\bullet$ to be exact, which is a contradiction since $H^1(Z^\bullet) \neq 0$.

Theorem 5.5. *The class of quasi-isomorphisms in $\mathsf{K}(\mathscr{A})$ for $\mathscr{A}$ an abelian category is a localizing class which satisfies the conditions in Verdier's theorem.*

Remark 5.6. Given an exact triangle in $\mathsf{K}(\mathscr{A})$
$$X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow X^\bullet[1]$$
then we have a long exact sequence in cohomology
$$H^i(X^\bullet) \longrightarrow H^i(Y^\bullet) \longrightarrow H^i(Z^\bullet) \longrightarrow H^{i+1}(X^\bullet) \longrightarrow \cdots$$
This is the case since it holds for the triangle
$$X^\bullet \xrightarrow{u} Y^\bullet \longrightarrow \operatorname{Cone}(u) \longrightarrow X^\bullet[1]$$
In particular, given an exact triangle in $\mathsf{K}(\mathscr{A})$,
$$X^\bullet \xrightarrow{f} Y^\bullet \longrightarrow Z^\bullet \longrightarrow X^\bullet[1]$$
f is a quasi-isomorphism if and only if $Z^\bullet$ is acyclic ($H^i(Z^\bullet) = 0$ for all i).

Proof of Theorem. Let S be the class of quasi-isomorphisms in $\mathsf{K}(\mathscr{A})$. It is trivial that $\mathrm{id}_{X^\bullet} \in S$. If $u, v \in S$, it is also true that $vu \in S$ (if defined). Now for Verdier's theorem, we need that

1. $f \in S$ if and only if $f[1] \in S$ (which is trivial);
2. Given a commutative diagram between two exact triangles
$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, dashed, "h"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$
with exact rows $f, g \in S$, there exists $h\colon Z \to Z'$ making squares commute such that $h \in S$. We know that there exists h making the diagram commute by TR4; any such $h \in S$ by using the long exact sequence and the five lemma.

Now we show the "interesting" conditions for a localizing set.

First consider a diagram
$$\begin{tikzcd} W^\bullet \arrow[r, dashed] \arrow[d, dashed, "S\ni"'] & Z^\bullet \arrow[d, "s\in S"] \\ X^\bullet \arrow[r, "f"] & Y^\bullet \end{tikzcd}$$
s being a quasi-isomorphism implies $\operatorname{Cone}(s)$ is acyclic. So, in the diagram below, the map t is a quasi-isomorphism:
$$\begin{tikzcd} \operatorname{Cone}(i_s f)[-1] \arrow[r, dashed, "g"] \arrow[d, "t"] & Z^\bullet \arrow[d, "s"] \\ X^\bullet \arrow[r, "f"] \arrow[d, "i_s f"] & Y^\bullet \arrow[d, "i_s"] \\ \operatorname{Cone}(s) \arrow[r, equal] & \operatorname{Cone}(s) \end{tikzcd}$$

by using the fact that both vertical maps are part of an exact triangle. Therefore, it is enough to show there exists a $g\colon \mathrm{Cone}(i_s f)[-1] \to Z^\bullet$ such that the top square commutes up to homotopy, i.e., $ft \sim sg$. But this follows from TR4, by first rotating the two triangles formed by the vertical maps. For the dual property, simply replace $\mathscr{A}$ by $\mathscr{A}^0$.

Finally, we need to check that if

$$\begin{tikzcd} W^\bullet \arrow[r, dashed, "t"] & X^\bullet \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y^\bullet \arrow[r, "s", "\mathsf{qis}"'] & Z^\bullet \end{tikzcd}$$

such that $su \sim sv$, then there exists a quasi-isomorphism $t\colon W^\bullet \to X^\bullet$ such that $ut \sim vt$. So, let $w = u - v$; then $sw \sim 0$ via a homotopy $h\colon X^n \to Z^{n-1}$. Consider the diagram

$$\begin{tikzcd} \mathrm{Cone}(\alpha)[-1] \arrow[r, "\beta"] & X^\bullet \arrow[r, dashed, "\alpha"] \arrow[d, equal] & \mathrm{Cone}(s)[-1] \arrow[d, "q"] \\ & X^\bullet \arrow[r, "w"] \arrow[d] & Y^\bullet \arrow[d, "s"', "\mathsf{qis}"] \\ & 0 \arrow[r] & Z^\bullet \end{tikzcd}$$

s being a quasi-isomorphism implies $\mathrm{Cone}(s)[-1]$ is acyclic. Then we

Claim. *There exists $\alpha\colon X^\bullet \to \mathrm{Cone}(s)[-1]$ such that $q\alpha \sim w$.*

But this follows from TR4. Now the fact that $\mathrm{Cone}(s)[-1]$ is acyclic implies β is a quasi-isomorphism, and so $w\beta \sim (q\alpha)\beta = q(\alpha\beta) \sim 0$. For the converse, replace $\mathscr{A}$ by $\mathscr{A}^0$. $\square$

Upshot 5.7. $\mathsf{D}(\mathscr{A}) := \mathsf{K}(\mathscr{A})[\mathsf{qis}^{-1}]$ is a triangulated category such that $\mathsf{K}(\mathscr{A}) \to \mathsf{D}(\mathscr{A})$ is an exact functor.

Consider also $\mathscr{D} := \mathsf{Ch}(\mathscr{A})[\mathsf{qis}^{-1}]$. The universal property of localization implies the functor $\mathsf{Ch}(\mathscr{A}) \to \mathsf{K}(\mathscr{A})$ induces a functor $\mathscr{D} \to \mathsf{D}(\mathscr{A})$.

Proposition 5.8. *This is an equivalence of categories.*

Proof. We need to show that if $f, g\colon X^\bullet \to Y^\bullet$ are homotopic, then they induce the same map in $\mathscr{D}$. Once we know this, we get a functor $\mathsf{K}(\mathscr{A}) \to \mathscr{D}$, and using the universal property of localization we get a functor $\mathsf{D}(\mathscr{A}) \to \mathscr{D}$. It is easy to show that the two functors are mutual inverses.

Recall that we have

$$\begin{tikzcd} & Y^\bullet \arrow[d, "\alpha_f"] & \\ X^\bullet \arrow[r, "j_f"] \arrow[dr, "f"'] & \mathrm{Cyl}(f) \arrow[r] \arrow[d, "\beta_f"] & \mathrm{Cone}(f) \\ & Y^\bullet & \end{tikzcd}$$

where the left triangle commutes on the nose, $\beta_f \circ \alpha_f = \mathrm{id}_{Y^\bullet}$, and $\alpha_f \circ \beta_f \sim \mathrm{id}_{\mathrm{Cyl}(f)}$.

Claim. $\alpha_f \circ f = j_f$ *in* $\mathscr{D}$.

This follows from $\beta_f \circ j_f = f$, and the fact that the relations $\beta_f \circ \alpha_f = \mathrm{id}_{Y^\bullet}$ and $\alpha_f \circ \beta_f \sim \mathrm{id}_{\mathrm{Cyl}(f)}$ imply that α_f, β_f are quasi-isomorphisms that are inverse to each other in $\mathscr{D}$. This gives the diagram

$$\begin{tikzcd} & Y^\bullet \arrow[d, "\alpha_f"] \\ X^\bullet \arrow[ur, "f"] \arrow[r, "j_f"] \arrow[d, equal] & \mathrm{Cyl}(f) \arrow[d, dashed, "\eta"] \\ X^\bullet \arrow[r, "j_g"] \arrow[dr, "g"'] & \mathrm{Cyl}(g) \arrow[d, "\beta_g"] \\ & Y^\bullet \end{tikzcd}$$

where both triangles are commutative on the nose. We now note that if we have $\eta\colon \mathrm{Cyl}(f) \to \mathrm{Cyl}(g)$ such that $\beta_g \circ \eta \circ \alpha_f = \mathrm{id}_Y$ in $\mathscr{D}$ and $\eta \circ j_f = j_g$, then we would be done. Let $h\colon X^n \to Y^{n-1}$ gives the homotopy between f, g. Then,

$$\begin{tikzcd} \mathrm{Cyl}(f)^n = X^n \oplus X^{n+1} \oplus Y^n \arrow[d, "\eta"] \\ \mathrm{Cyl}(g)^n = X^n \oplus X^{n+1} \oplus Y^n \end{tikzcd}$$

can be defined by $\eta(x_1, x_2, y) = (x_1, x_2, y + h(x_2))$, which is a morphism of complexes by definition of h. Finally, note $\beta_g \circ \eta \circ \alpha_f = \mathrm{id}_{Y^\bullet}$ and $\eta \circ j_f = j_g$ clearly hold, even on the level of complexes. $\square$

Lemma 5.9. *If we have a short exact sequence*

$$0 \longrightarrow X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow 0$$

in $\mathsf{Ch}(\mathscr{A})$*, then there exists* $Z^\bullet \to X^\bullet[1]$*, such that*

$$X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow X^\bullet[1]$$

is exact in $\mathsf{D}(\mathscr{A})$*.*

Example 5.10. Given any $X^\bullet$, there is a short exact sequence

$$0 \longrightarrow \tau_{\leq a} X^\bullet \longrightarrow X^\bullet \longrightarrow \tau_{\geq a+1} X^\bullet \longrightarrow 0$$

which will later be a condition we need to define t-structures.

This is somehow related to the three operations.

Proof. It is enough to construct a commutative diagram

$$\begin{tikzcd} X^\bullet \arrow[r, "f"] & Y^\bullet \arrow[r, "g"] & Z^\bullet \\ X^\bullet \arrow[u, equal] \arrow[r, "j_f"] & \mathrm{Cyl}(f) \arrow[u, "\beta_f"] \arrow[r] & \mathrm{Cone}(f) \arrow[u, dashed, "\psi"] \end{tikzcd}$$

such that the vertical arrows are quasi-isomorphisms. The first square is OK, so we need to find ψ. Let

$$\psi\colon X^{n+1} \oplus Y^n \longrightarrow Z^n,\ \psi(x, y) = g(y).$$

To check commutativity, look at the diagram above in each degree n:

$$\begin{tikzcd} Y^n \arrow[r, "g"] & Z^n \\ X^n \oplus X^{n+1} \oplus Y^n \arrow[u, "\beta_f"] \arrow[r] & X^{n+1} \oplus Y^n \arrow[u, dashed, "\psi"] \end{tikzcd}$$

in which case

$$\begin{tikzcd} y \arrow[r, mapsto] & g(y) \\ (x_1, x_2, y) \arrow[u, mapsto] \arrow[r, mapsto] & (x_2, y) \arrow[u, mapsto] \end{tikzcd}$$

Now by definition, ψ is an epimorphism since g is; $\ker(\psi) = \mathrm{Cone}(\mathrm{id}_X : X^\bullet \to X^\bullet)$ is acyclic. Thus, the long exact sequence in cohomology implies ψ is a quasi-isomorphism. This gives the dashed map in the diagram below

$$\begin{tikzcd} X^\bullet \arrow[r, "f"] & Y^\bullet \arrow[r, "g"] & Z^\bullet \arrow[r, dashed] & X^\bullet[1] \\ X^\bullet \arrow[u, equal] \arrow[r, "j_f"] & \mathrm{Cyl}(f) \arrow[u, "\beta_f"] \arrow[r] & \mathrm{Cone}(f) \arrow[u, "\psi"] \arrow[r] & X^\bullet[1] \arrow[u, equal] \end{tikzcd}$$

in the *derived* category. $\square$

Definition 5.11. Define the following full subcategories of $\mathsf{D}(\mathscr{A})$:

$$\begin{aligned}
\mathsf{D}^+(\mathscr{A}) &= \{X^\bullet \in \mathsf{D}(\mathscr{A}) \mid H^i(X^\bullet) = 0 \text{ for } i \ll 0\} \\
\mathsf{D}^-(\mathscr{A}) &= \{X^\bullet \in \mathsf{D}(\mathscr{A}) \mid H^i(X^\bullet) = 0 \text{ for } i \gg 0\} \\
\mathsf{D}^b(\mathscr{A}) &= \mathsf{D}^+(\mathscr{A}) \cap \mathsf{D}^-(\mathscr{A})
\end{aligned}$$

Note that if $\mathscr{C}$ is a triangulated category, a full subcategory $\mathscr{C}' \hookrightarrow \mathscr{C}$ is a triangulated subcategory if

1. $\mathscr{C}'[1] = \mathscr{C}'$
2. If two objects in an exact triangle lie in $\mathscr{C}'$, then so does the third.

Thus, $\mathsf{D}^+(\mathscr{A})$, $\mathsf{D}^-(\mathscr{A})$, $\mathsf{D}^b(\mathscr{A})$ are triangulated subcategories of $\mathsf{D}^+(\mathscr{A})$ by the long exact sequence in cohomology.

Remark 5.12. If $\mathscr{C}' \hookrightarrow \mathscr{C}$ is a triangulated subcategory, then the triangulated structure on $\mathscr{C}$ induces a triangulated structure on $\mathscr{C}'$.

Remark 5.13. One can define $\mathsf{Ch}^*(\mathscr{A}), \mathsf{K}^*(\mathscr{A})$ for $* \in \{+, -, b\}$. For example,

$$\mathsf{Ch}^*(\mathscr{A}) = \{X^\bullet \in \mathsf{Ch}(\mathscr{A}) \mid X^i = 0 \text{ for } i \text{ in a suitable range}\}.$$

$\mathsf{K}^*(\mathscr{A})$ is a triangulated subcategory of $\mathsf{K}(\mathscr{A})$, and quasi-isomorphisms in $\mathsf{K}^*(\mathscr{A})$ form a localizing class. We can define the localization $\mathsf{K}^*(\mathscr{A})[\mathsf{qis}^{-1}]$, and get a functor $\mathsf{K}^*(\mathscr{A})[\mathsf{qis}^{-1}] \to \mathsf{D}^*(\mathscr{A})$. It is easy to see that this is an equivalence of categories. For example, for $-$, note that if $s\colon X^\bullet \to Y^\bullet$ is a quasi-isomorphism in $\mathsf{Ch}(\mathscr{A})$, and $Y^\bullet \in \mathsf{Ch}^-(\mathscr{A})$, then $\tau_{\le a}X^\bullet \to X^\bullet$ is a quasi-isomorphism for $a \gg 0$, and $\tau_{\le a}X^\bullet \in \mathsf{Ch}^-(\mathscr{A})$.

Now let $\mathscr{A}$ be an abelian category; then, we have the canonical inclusion $i\colon A \to \mathsf{D}(\mathscr{A})$ where M maps to the complex with M in the zeroth degree. We also have the functor $H^0\colon \mathsf{D}(\mathscr{A}) \to \mathscr{A}$, and $H^0 \circ i \cong \mathrm{id}_{\mathscr{A}}$.

Proposition 5.14. *i is fully faithful, hence we can identify $\mathscr{A}$ with its essential image*

$$\{X^\bullet \mid H^n(X^\bullet) = 0 \textit{ for } n \neq 0\}.$$

Proof. The only nontrivial thing is that i induces a surjection on Hom-sets. □

Thus, $\mathscr{A}$ sits inside $\mathsf{D}(\mathscr{A})$; t-structures give other interesting abelian categories which Bhargav will talk about later on.

Next time, we will define derived functors for bounded categories.

6 September 24, 2015

6.1 Derived functors

The plan today is to tell you everything there is about derived functors. In algebraic geometry, you can survive with derived functors in the classical settings. But for constructible sheaves, we need this new setting.

Old fashioned-wise, you can take an injective resolution, apply the functor term-wise, and then take cohomology to give you a sequence of functors. In derived categories, you keep more information by remembering the complex you applied the functor on. This also works better for taking cohomology of complexes, recovering hypercohomology. Composition also works better, since in the classical setting you need spectral sequences.

6.1.1 Setup

Let $F\colon \mathscr{A} \to \mathscr{B}$ be a functor between abelian categories. Then, for $* \in \{\emptyset, +, -, b\}$, we get a functor $\mathsf{Ch}^*(\mathscr{A}) \to \mathsf{Ch}^*(\mathscr{B})$ by applying F termwise. If $f \sim g$, then $F(f) \sim F(g)$, and so we get an induced functor $\mathsf{K}^*(\mathscr{A}) \xrightarrow{\mathsf{K}^*(F)} \mathsf{K}^*(\mathscr{B})$. We would like to also get an induced functor

$$\mathsf{D}^*(\mathscr{A}) \longrightarrow \mathsf{D}^*(\mathscr{B}).$$

If you do this naïvely, as in, if you want to do this directly by using universal properties, you need that $\mathsf{K}^*(F)$ takes quasi-isomorphisms to quasi-isomorphisms, that is, that F is exact. If this is the case you get $\mathsf{D}^*(\mathscr{A}) \to \mathsf{D}^*(\mathscr{B})$.

The interesting case is when you have F that is only left- or right-exact. Since we are mainly interested in sheaves, we will stick to the case of left-exactness for the most part.

There is a framework of Deligne in SGA4 to have a derived functor defined on complexes. We will take Verdier's more classical approach, in order to define derived functors $\mathbf{R}F\colon \mathsf{D}^+(\mathscr{A}) \to \mathsf{D}^+(\mathscr{B})$. First note that the diagram

$$\begin{tikzcd} \mathsf{K}^+(\mathscr{A}) \arrow[r, "\mathsf{K}^+(F)"] \arrow[d, "\tau_{\mathscr{A}}"'] & \mathsf{K}^+(\mathscr{B}) \arrow[d, "\tau_{\mathscr{B}}"] \\ \mathsf{D}^+(\mathscr{A}) \arrow[r, "\mathbf{R}F"] & \mathsf{D}^+(\mathscr{B}) \end{tikzcd}$$

is not commutative, unless F is exact! So we demand that $\mathbf{R}F$ is an exact functor and a natural transformation

$$\tau_{\mathscr{B}} \circ \mathsf{K}^+(F) \xrightarrow{\epsilon_F} \mathbf{R}F \circ \tau_{\mathscr{A}}$$

which is universal in the following way: given $G\colon \mathsf{D}^+(\mathscr{A}) \to \mathsf{D}^+(\mathscr{B})$, and $\epsilon_G\colon \tau_{\mathscr{B}} \circ \mathsf{K}^+(F) \to G \circ \tau_{\mathscr{A}}$, there exists a unique $\eta\colon \mathbf{R}F \to G$ such that $\epsilon_G(x) = \eta \circ \epsilon_F(x)$.

This construction depends on a choice! So we need to verify later that the construction $\mathbf{R}F$ does not depend on the choice.

One formulates everything in terms of a certain class of objects, like flasque, soft, etc.

6.1.2 Construction of $\mathbf{R}F$

Definition 6.1. A class of objects $\mathscr{R}$ in $\mathscr{A}$ is *adapted to F* if

1. $\mathscr{R}$ is closed under finite direct sums;
2. For every $X^\bullet \in \mathsf{Ch}^+(\mathscr{A})$ a complex of objects in $\mathscr{R}$ which is acyclic, we have $F(X^\bullet)$ is also acyclic;
3. There exists enough objects in $\mathscr{R}$ such that for all $M \in \mathrm{Obj}(\mathscr{A})$, there exists $M \hookrightarrow B \in \mathrm{Obj}(\mathscr{R})$.

We can define $\mathsf{Ch}^*(\mathscr{R})$ and $\mathsf{K}^*(\mathscr{R})$.

We will think of $\mathscr{R}$ as a full subcategory of $\mathscr{A}$, which is additive by 1.

Lemma 6.2. *If $\mathscr{R}$ is as above, then for every $X^\bullet \in \mathsf{Ch}^+(\mathscr{A})$, there is a quasi-isomorphism $X^\bullet \to M^\bullet \in \mathsf{Ch}^+(\mathscr{R})$.*

Proof. Exercise. You start by looking at the non-zero kernel, resolve that with objects in $\mathscr{R}$, and you kind of repeat this process. ☐

Proposition 6.3. *If $F, \mathscr{R}$ as above, then*

1. {Quasi-isomorphisms in $\mathsf{K}^+(\mathscr{R})$} *is a localizing class;*
2. $\mathscr{C} := \mathsf{K}^+(\mathscr{R})[\mathsf{qis}^{-1}] \to \mathsf{D}^+(\mathscr{A})$ *is an equivalence of categories.*

Proof. For 1, we argue as last time.

For 2, the Lemma 6.2 implies the functor is essentially surjective. We then need that $\mathrm{Hom}_{\mathscr{C}}(X^\bullet, Y^\bullet) \to \mathrm{Hom}_{\mathsf{D}^+(\mathscr{A})}(X^\bullet, Y^\bullet)$ is bijective. For surjectivity, consider a morphism

$$\begin{tikzcd} X^\bullet \arrow[dr] & & Y^\bullet \arrow[dl, "\mathsf{qis}"] \\ & W^\bullet \arrow[d, dashed, "\mathsf{qis}\text{ given by Lemma}"] & \\ & Z^\bullet \in \mathsf{Ch}^+(\mathscr{R}) & \end{tikzcd}$$

where $X^\bullet, Y^\bullet \in \mathsf{Ch}^+(\mathscr{R})$.

Now for injectivity, consider a morphism

$$\begin{tikzcd}
X^\bullet \arrow[dr, "g"] & & Y^\bullet \arrow[dl] \\
& Z^\bullet \arrow[d, "t"] & \\
& T^\bullet \arrow[d, dashed, "t'"', "\text{constructed by Lemma}"] & \\
& M^\bullet &
\end{tikzcd}$$

for $X^\bullet, Y^\bullet \in \mathrm{Hom}_{\mathscr{C}}(X^\bullet, Y^\bullet)$ that goes to 0 in $\mathsf{D}^+(\mathscr{A})$. Then, there exists a quasi-isomorphism t such that $t \circ g \sim 0$, and so $t' \circ t \circ g \sim 0$. $\square$

Now suppose F is left-exat, and there is $\mathscr{R}$ adapted to F. Then, consider the diagram

$$\begin{tikzcd}
& & \mathsf{K}^+(\mathscr{A}) \arrow[r, "\mathsf{K}^+(F)"] & \mathsf{K}^+(\mathscr{B}) \arrow[d] \\
\mathsf{K}^+(\mathscr{R}) \arrow[urr] \arrow[d] & & & \mathsf{D}^+(\mathscr{B}) \\
\mathsf{K}^+(\mathscr{R})[\mathsf{qis}^{-1}] \arrow[rrr, "\sim"] \arrow[urrr, dashed, "\substack{\text{given by}\\ \text{univ. prop.}}"] & & & \mathsf{D}^+(\mathscr{A}) \arrow[u, "\mathbf{R}F"']
\end{tikzcd}$$

We have that u is a quasi-isomorphism in $\mathsf{K}^+(\mathscr{R})$ if and only if $\mathrm{Cone}(u)$ is acyclic, but by property 2, we have that $F(\mathrm{Cone}(u)) = \mathrm{Cone}(F(u))$, and so $F(u)$ is a quasi-isomorphism as well. We therefore get a functor $\mathbf{R}F\colon \mathsf{D}^+(\mathscr{A}) \to \mathsf{D}^+(\mathscr{B})$.

Explicitly, for an object in $\mathsf{D}^+(\mathscr{A})$, we take $X^\bullet \to M^\bullet \in \mathsf{Ch}^+(\mathscr{R})$, and $\mathbf{R}F(X^\bullet) = F(M^\bullet)$.

Proposition 6.4. *This is a right-derived functor in the sense of Verdier.*

Sketch of Proof. Exactness of $\mathbf{R}F$ is easy once we have existence: it comes from $F(\mathrm{Cone}(u)) = \mathrm{Cone}(F(u))$.

For the other condition, consider

$$\begin{tikzcd}
\mathsf{K}^+(\mathscr{A}) \arrow[r, "\mathsf{K}^+(F)"] \arrow[d, "\tau_{\mathscr{A}}"'] & \mathsf{K}^+(\mathscr{B}) \arrow[d, "\tau_{\mathscr{B}}"] \\
\mathsf{D}^+(\mathscr{A}) \arrow[r, "\mathbf{R}F"] & \mathsf{D}^+(\mathscr{B})
\end{tikzcd}$$

Then, $\epsilon_F(X^\bullet)\colon F(X^\bullet) \to F(M^\bullet)$, where $u\colon X^\bullet \to M^\bullet \in \mathsf{Ch}^+(\mathscr{R})$ is a quasi-isomorphism. It is easy to check that this is universal. $\square$

Now suppose we have $F, \mathscr{R}$ as above. Then, we can define $\mathbf{R}^i F\colon \mathsf{D}^+(\mathscr{A}) \to \mathscr{A}$ as $H^i \circ \mathbf{R}F$. These take exact triangles to long exact sequences. Note that if $X \in \mathrm{Obj}(\mathscr{A})$, and we have a quasi-isomorphism $X \to I^\bullet \in \mathsf{Ch}(\mathscr{R})$ such that $I^m = 0$ for all $m < 0$, then we obtain that $\mathbf{R}^i F(X) = 0$ for all $i < 0$, and that $\mathbf{R}^0 F(X) \simeq F(X)$.

Important Example 6.5. If $\mathscr{A}$ has enough injectives, then $\{\text{Injectives in } \mathscr{A}\}$ is adapted to F for every left-exact F.

Recall that I is injective if and only if $\mathrm{Hom}_{\mathscr{A}}(-, I)$ is exact. You only have to check condition 2 for an adapted class: this is true since if a complex $I^\bullet \in \mathsf{Ch}^+(\mathsf{Inj})$ is acyclic, then it breaks up into *split* short exact sequences.

Remark 6.6. Suppose that $F\colon \mathscr{A} \to \mathscr{B}$ is left exact, and that there is an adapted class $\mathscr{R}$ for F. Then, there is a largest such class provided by F-acyclic objects:

$$\{M \in \mathrm{Obj}(\mathscr{A}) \mid \mathbf{R}^i F(M) = 0 \text{ for all } i \geq 1\}.$$

It is clear that $M \in \mathscr{R}$ implies M is F-acyclic. In particular, we have enough F-acyclic objects. $\mathbf{R}^i F$ is additive, which implies that finite direct sums of F-acyclic objects are F-acyclic. The only thing that requires a bit of thought is that if I have a complex $M^\bullet \in \mathsf{Ch}^+(\mathscr{A})$ such that each M^i is F-acyclic, then $F(M^\bullet)$ is acyclic: break $M^\bullet$ into short exact sequences, and use the long exact sequence for each short exact sequence.

The reason why we want this flexibility is because we want it in the context of sheaves.

Example 6.7. If $f\colon X \to Y$ is a continuous map of topological spaces and $F = f_*\colon \mathsf{Ab}(X) \to \mathsf{Ab}(Y)$, then $\mathscr{R} = \{\text{flasque sheaves on } X\}$ is adapted to $F = f_*$. Enough such sheaves exist since given $\mathscr{F}$, you can inject $\mathscr{F} \hookrightarrow (U \mapsto \prod_{x \in U} \mathscr{F}_x)$ (the *Godement resolution*).

6.1.3 The derived functor of a composition

Theorem 6.8. *Let $F\colon \mathscr{A} \to \mathscr{B}$ and $G\colon \mathscr{B} \to \mathscr{C}$ be left-exact functors. Suppose we have adapted classes $\mathscr{R}_{\mathscr{A}}$ for F and $\mathscr{R}_{\mathscr{B}}$ for G, such that $f(M) \in \mathscr{R}_{\mathscr{B}}$ if $M \in \mathscr{R}_{\mathscr{A}}$. Then, the natural transformation $\mathbf{R}G \circ \mathbf{R}F \to \mathbf{R}(G \circ F)$ is an isomorphism of functors.*

Proof. Follows from definition. ☐

Example 6.9. If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous maps, then $\mathbf{R}(g \circ f)_* \simeq \mathbf{R}g_* \circ \mathbf{R}f_*$ (use that f_* of a flasque sheaf is flasque).

This looks much nicer than the Leray spectral sequence! But of course if you want to compute things, then you use the hypercohomology spectral sequence to transform this isomorphism into the Leray spectral sequence.

6.1.4 $\mathbf{R}\,\mathrm{Hom}$ and Ext functors

Now I want to say a little about $\mathbf{R}\,\mathrm{Hom}$, in which case you see a bit more about what's nice about the derived category.

Suppose $\mathscr{A}$ is an abelian category with enough injectives. If $X \in \mathrm{Obj}(\mathscr{A})$, we can consider the functor $\mathrm{Hom}_{\mathscr{A}}(X, -)$, which is left-exact, and so we get a derived functor $\mathbf{R}\,\mathrm{Hom}_{\mathscr{A}}(X, -)$. What we want is to

1. extend this to the case where X is a complex; and
2. interpret $\mathrm{Ext}^i(X, Y) = H^i(\mathbf{R}\,\mathrm{Hom}(X, Y))$ as $\mathrm{Hom}_{\mathsf{D}(\mathscr{A})}(X, Y)$.

Let $X^\bullet \in \mathsf{Ch}(\mathsf{Ab})$, and $Y^\bullet \in \mathsf{Ch}^+(\mathscr{A})$. Then, we can define the Hom complex:

$$\mathrm{Hom}^n_{\mathscr{A}}(X^\bullet, Y^\bullet) = \prod_{i \in \mathbf{Z}} \mathrm{Hom}_{\mathscr{A}}(X^i, Y^{n+i}), \qquad d\left((f_i)_{i \in \mathbf{Z}}\right) = (d_Y \circ f_i + (-1)^{n+1} f_{i+1} \circ d_X)_{i \in \mathbf{Z}}$$

Basically, we're using both compositions in the diagram below:

$$\begin{tikzcd}
X^i \arrow[rr] \arrow[dr, "f_i"] \arrow[ddr, bend right, "d_X"'] & & Y^{n+1+i} \\
& Y^{n+i} \arrow[ur, "d_Y"] & \\
& X^{i+1} \arrow[uur, bend right, "f_{i+1}"'] &
\end{tikzcd}$$

Key observation is that $Z^0(\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)) = \mathrm{Hom}_{\mathsf{Ch}(\mathscr{A})}(X^\bullet, Y^\bullet)$, $B^0(\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)) = \{f \mid f \sim 0\}$, so

$$H^0(\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)) = \mathrm{Hom}_{\mathsf{K}(\mathscr{A})}(X^\bullet, Y^\bullet).$$

Lemma 6.10. *If $I^\bullet \in \mathsf{Ch}^+(\mathsf{Inj})$, and $X^\bullet \in \mathsf{Ch}^+(\mathscr{A})$ is acyclic, then $\mathrm{Hom}_{\mathsf{K}(\mathscr{A})}(X^\bullet, I^\bullet) = 0$.*

This is like the lemma you prove when defining injective resolutions about how a map from a complex of injectives to an acyclic complex is homotopic to the zero map.

Now suppose $X^\bullet \in \mathsf{Ch}(\mathscr{A})$, $Y^\bullet \in \mathsf{Ch}^+(\mathscr{A})$. Consider a quasi-isomorphism $Y^\bullet \to I^\bullet \in \mathsf{Ch}^+(\mathsf{Inj})$. Define $\mathbf{R}\operatorname{Hom}_{\mathscr{A}}(X^\bullet, Y^\bullet) \coloneqq \operatorname{Hom}^\bullet(X^\bullet, I^\bullet) \in \mathsf{D}(\mathscr{A})$. Consider the diagram

$$\begin{tikzcd} Y^\bullet \arrow[r, "u"] \arrow[d] & I^\bullet \arrow[d, dashed] \\ Z^\bullet \arrow[r] & J^\bullet \end{tikzcd}$$

Claim 6.11. *There exists $I^\bullet \to J^\bullet$, unique up to homotopy, such that it makes the diagram commute, up to homotopy.*

Proof. We claim the map

$$\operatorname{Hom}_{\mathsf{K}(\mathscr{A})}(I^\bullet, J^\bullet) \longrightarrow \operatorname{Hom}_{\mathsf{K}(\mathscr{A})}(Y^\bullet, J^\bullet)$$

is an isomorphism. This is by the long exact sequence for $\operatorname{Cone}(u)$, since $\operatorname{Cone}(u)$ is acyclic (use Lemma). $\square$

We thus get a for each $X^\bullet$ a functor $\mathbf{R}\operatorname{Hom}(X^\bullet, -)\colon \mathsf{D}^+(\mathscr{A}) \to \mathsf{D}(\mathscr{A})$. There is one more thing to check: that if $Y^\bullet$ is replaced by something quasi-isomorphic to it, then the result is isomorphic to the original one.

Exercise 6.12. Use the Lemma again to show that we get a bifunctor $\mathbf{R}\operatorname{Hom}\colon \mathsf{D}(\mathscr{A}) \times \mathsf{D}^+(\mathscr{A}) \to \mathsf{D}(\mathscr{A})$.

Exercise 6.13. Show that $\mathbf{R}\operatorname{Hom}$ is exact in each variable.

Proposition 6.14. *If $X^\bullet, Y^\bullet \in \mathsf{D}^+(\mathscr{A})$, then $\operatorname{Ext}^i(X^\bullet, Y^\bullet) \coloneqq H^i(\mathbf{R}\operatorname{Hom}(X^\bullet, Y^\bullet)) \simeq \operatorname{Hom}_{\mathsf{D}(\mathscr{A})}(X^\bullet, Y^\bullet[i])$.*

Proof. By shifting $Y^\bullet$, we can assume $i = 0$. Let $Y^\bullet \to I^\bullet \in \mathsf{Ch}^+(\mathsf{Inj})$ be a quasi-isomorphism. Then, $\operatorname{Ext}^0(X^\bullet, Y^\bullet) = H^0(\operatorname{Hom}^\bullet(X^\bullet, I^\bullet)) = \operatorname{Hom}_{\mathsf{K}(\mathscr{A})}(X^\bullet, I^\bullet)$. We need to show that $\operatorname{Hom}_{\mathsf{K}(\mathscr{A})}(X^\bullet, I^\bullet) \to \operatorname{Hom}_{\mathsf{D}(\mathscr{A})}(X^\bullet, I^\bullet)$ is an isomorphism, since the later Hom group is isomorphic to $\operatorname{Hom}_{\mathsf{D}(\mathscr{A})}(X^\bullet, Y^\bullet)$.

For surjectivity, suppose we have a map

$$\begin{tikzcd} & Y^\bullet \arrow[dl, "s", "\mathsf{qis}"'] \arrow[dr, "f"] & \\ X^\bullet & & I^\bullet \end{tikzcd}$$

We want a map $g\colon X^\bullet \to I^\bullet$ in $\mathsf{K}(\mathscr{A})$ such that $g \circ s = f$. The morphism $\operatorname{Hom}_{\mathsf{K}(\mathscr{A})}(X^\bullet, I^\bullet) \xrightarrow{\sim} \operatorname{Hom}_{\mathsf{K}(\mathscr{A})}(Y^\bullet, I^\bullet)$ is an isomorphism by the Lemma sine $\operatorname{Cone}(s)$ is acyclic, and by using the long exact sequence again.

Injectivity is proved similarly. $\square$

These are the basics of derived functor. Mark Haiman's notes on his webpage talk about derived categories and functors using the formalism of Deligne.

This allows you to talk about derived functors on the derived categories on the unbounded category. You need some results by Spaltenstein to talk about f_* extending to the full unbounded derived category.

7 September 29, 2015

Last time, Mircea talked about how to work with derived functors. The new problem set explains some examples of derived functors, especially those used in constructible sheaves, i.e., the derived functors of $f^!$ and $f_!$.

Today we will talk about t-structures. First, some motivation: if X is a smooth complex algebraic variety, the Riemann–Hilbert correspondence says

$$\mathsf{D}^b_{\text{reg,hol}}(\mathcal{D}_X) \cong \mathsf{D}^b_{\text{cons}}(X^{\text{an}}, \mathbf{C}).$$

The left is the subcategory of the bounded category of $\mathcal{D}_X$-modules, such that the cohomology is regular and holomonic; the right also has cohomological conditions, with respect to a certain stratification.

Now note that the left hand side has an abelian subcategory $\mathsf{Mod}_{\text{reg,holo}}(\mathcal{D}_X)$; it should correspond to an abelian one on the right hand side, but it's unclear what it should be until we talk about t-structures.

7.1 t-structures

Let $\mathscr{D}$ be a triangulated category (ex. $\mathscr{D} = \mathsf{D}(\mathsf{Ab}(X))$).

Definition 7.1. Let $\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0}$ be two full subcategories of $\mathscr{D}$. The pair $(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$ is/gives a *t-structure* on $\mathscr{D}$ if

(*i*) Set $\mathscr{D}^{\leq n} = \mathscr{D}^{\leq 0}[-n]$, $\mathscr{D}^{\geq n} = \mathscr{D}^{\geq 0}[-n]$. Then, $\mathscr{D}^{\leq -1} \subseteq \mathscr{D}^{\leq 0}$, $\mathscr{D}^{\geq 1} \subseteq \mathscr{D}^{\geq 0}$.

(*ii*) $\operatorname{Hom}(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 1}) = 0$, i.e., $\operatorname{Hom}_{\mathscr{D}}(X, Y) = 0$ if $X \in \mathscr{D}^{\leq 0}$ and $Y \in \mathscr{D}^{\geq 1}$.

(*iii*) For all $X \in \mathscr{D}$, there exists an exact triangle

$$Y \longrightarrow X \longrightarrow Z \longrightarrow Y[1]$$

where $Y \in \mathscr{D}^{\geq 0}$ and $Z \in \mathscr{D}^{\geq 1}$.

Set $\mathscr{D}^{\heartsuit} = \mathscr{D}^{\leq 0} \cap \mathscr{D}^{\geq 0}$. This is called the *heart* or the *core* of the t-structure.

For the rest of today, we will discuss properties of the t-structure and why the heart is a nice thing.

Remarks 7.2.

1. The notion is self-dual, in that $(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$ is a t-structure on $\mathscr{D}$ if and only if $((\mathscr{D}^{\leq 0})^{\mathsf{op}}, (\mathscr{D}^{\geq 0})^{\mathsf{op}})$ is a t-structure on $\mathscr{D}^{\mathsf{op}}$.
2. $(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$ is a t-structure if and only if $(\mathscr{D}^{\leq n}, \mathscr{D}^{\geq n})$ is a t-structure for all n, since t-structures are preserved under exact functors of triangulated categories.

Examples 7.3.

1. $\mathscr{A}$ an abelian category, $\mathscr{D} = \mathsf{D}(\mathscr{A})$. Then,

$$\mathscr{D}^{\leq 0} = \{K \in \mathscr{D} \mid H^i(K) = 0 \ \forall i > 0\}$$
$$\mathscr{D}^{\geq 0} = \{K \in \mathscr{D} \mid H^i(K) = 0 \ \forall i < 0\}$$

$(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$ is a t-structure on $\mathsf{D}(\mathscr{A})$.

Proof. $\mathscr{D}^{\leq -1} \subseteq \mathscr{D}^{\leq 0}$ is easy. $\operatorname{Hom}(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 1}) = 0$ (the point is that there is no interaction if everything were maps of chain complexes, although you have to be a bit careful about the quasi-isomorphisms involved; this is an exercise). Now let $X \in \mathscr{D}(\mathscr{A})$; we have that

$$\tau^{\leq 0}(X) \xrightarrow{\text{can}} X \xrightarrow{\text{can}} \tau^{\geq 1}(X) \longrightarrow \tau^{\leq 0}(X)[1]$$

is an exact triangle; in fact, you get a short exact sequence of chain complexes

$$0 \longrightarrow \tau^{\leq 0}(X) \xrightarrow{\text{can}} X \xrightarrow{\text{can}} \tau^{\geq 1}(X) \longrightarrow 0$$ □

2. Let $\mathscr{D}$ be arbitrary. Then, setting $\mathscr{D}^{\leq 0} = \mathscr{D}$ and $\mathscr{D}^{\geq 0} = 0$ gives a t-structure (trivially).

Definition 7.4. A t-structure is *non-degenerate* if $\bigcap_n \mathscr{D}^{\leq n} = 0 = \bigcap_n \mathscr{D}^{\geq n}$.

Most of the things we will encounter will be non-degenerate.

Proposition 7.5. *Let $(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$ be a t-structure on a triangulated category $\mathscr{D}$. Then,*

1. *The inclusion $\mathscr{D}^{\leq n} \to \mathscr{D}$ has a right adjoint $\tau^{\leq n} \colon \mathscr{D} \to \mathscr{D}^{\leq n}$. In particular, there is a canonical map $\tau^{\leq n}(X) \to X$, which is universal in the sense that if there is something in $\mathscr{D}^{\leq n}$ mapping to X, this map factors uniquely through $\tau^{\leq n}(X)$.*
2. *The inclusion $\mathscr{D}^{\geq n} \to \mathscr{D}$ has a left adjoint $\tau^{\geq n} \colon \mathscr{D} \to \mathscr{D}^{\geq n}$, and there is a canonical map $X \to \tau^{\geq n}(X)$.*
3. *For each object $X \in \mathscr{D}$, there is a unique morphism $\tau^{\geq n+1}(X) \xrightarrow{\delta} \tau^{\leq n}(X)[1]$ which makes the following sequence into an exact triangle:*

$$\tau^{\leq n}(X) \xrightarrow{\text{can}} X \xrightarrow{\text{can}} \tau^{\geq n+1}(X) \xrightarrow{\delta} \tau^{\leq n}(X)[1]$$

In fact, this gives a transformation of functors $\tau^{\geq n+1} \to \tau^{\leq n}((-)[1])$, i.e., δ is functorial in X.

Note that 3 is a robust generalization of the short exact sequence of chain complexes from before. It says that we can break up X into pieces in $\mathscr{D}^{\leq n}$ and $\mathscr{D}^{\geq n}$; the point is that with a t-structure, we do not have to worry about non-functoriality of cones, since we get canonical ones associated to the t-structure.

Corollary 7.6.

1. $\mathrm{Hom}_{\mathscr{D}}(X,Y) = \mathrm{Hom}_{\mathscr{D}^{\leq n}}(X, \tau^{\leq n}(Y))$ *if* $X \in \mathscr{D}^{\leq n}$, $Y \in \mathscr{D}$, *where the canonical map* $\tau^{\leq n}(Y) \to Y$ *is given by setting* $X = \tau^{\leq n}(Y)$*;*
2. $\mathrm{Hom}_{\mathscr{D}}(X,Y) = \mathrm{Hom}_{\mathscr{D}^{\geq n}}(\tau^{\geq n}(X), Y)$ *if* $Y \in \mathscr{D}^{\geq n}$, $X \in \mathscr{D}$.

To prove the Proposition, we need:

Lemma 7.7. *Suppose* $\mathscr{D}$ *is a triangulated category, and for* $i \in \{1,2\}$, *the sequences*

$$X \xrightarrow{a} Y \xrightarrow{b} Z \xrightarrow{d_i} X[1]$$

are two exact triangles. Assume $\mathrm{Hom}(X[1], Z) = 0$. *Then,* $d_1 = d_2$.

Proof of Lemma. By the Cone axiom (TR4), we get a commutative diagram

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "\mathrm{id}"] & Y \arrow[r, "b"] \arrow[d, "\mathrm{id}"] & Z \arrow[r, "d_1"] \arrow[d, "c"] & X[1] \arrow[d, "\mathrm{id}"] \\ X \arrow[r, "a"] & Y \arrow[r, "b"] & Z \arrow[r, "d_1"] & X[1] \end{tikzcd}$$

We know that $b = cb$, that is, $(\mathrm{id}_Z - c) \circ b = 0$, and $\mathrm{id}_Z - c \in \mathrm{Hom}(Z,Z)$. Applying the long exact sequence for $\mathrm{Hom}(-,Z)$ to the first row, there eixsts $e\colon X[1] \to Z$ such that $e \circ d_1 = \mathrm{id}_Z - c$. But $e = 0$ as $\mathrm{Hom}(X[1],Z) = 0$, and so $\mathrm{id}_Z = c$, and $d_1 = d_2$ from the commutative diagram. ☐

Proof of Proposition 7.5.

1. By shifting, we can assume $n = 0$. Fix $X \in \mathscr{D}$. Axiom (iii) of t-structures gives

   $$X_0 \longrightarrow X \longrightarrow X^1 \longrightarrow X_0[1]$$

   such that $X_0 \in \mathscr{D}^{\leq 0}$ and $X^1 \in \mathscr{D}^{\geq 1}$. Now fix some $X \in \mathscr{D}^{\leq 0}$. Apply $\mathrm{Hom}(Y,-)$ to get

   $$\begin{tikzcd} \mathrm{Hom}(Y, X^1[-1]) \arrow[r] \arrow[d, equal, "\mathrm{Hom}(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 2})=0"'] & \mathrm{Hom}(Y, X_0) \arrow[r, "\sim"] & \mathrm{Hom}(Y,X) \arrow[r] & \mathrm{Hom}(Y, X^1) \arrow[d, equal, "\mathrm{Hom}(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 1})=0"] \\ 0 & & & 0 \end{tikzcd}$$

   and so $X_0 \to X$ has the correct universal property for $\tau^{\leq 0}(X) \to X$, and so set $\tau^{\leq 0}(X) = X_0$. $\tau^{\leq 0}(X) \xrightarrow{\mathrm{can}} X$ is the given map $X_0 \to X$.
2. Similarly: $\tau^{\geq 1}(X) = X^1$.
3. Want a unique $\delta\colon X^1 \to X_0[1]$, where $X^1 = \tau^{\geq 1}(X)$ and $X_0 = \tau^{\leq 0}(X)$ making

   $$X_0 \xrightarrow{\mathrm{can}} X \xrightarrow{\mathrm{can}} X^1 \xrightarrow{\delta} X_0[1]$$

   exact. But the Lemma applies: $\mathrm{Hom}(X_0[1], X^1) = 0$ since $X_0[1] \in \mathscr{D}^{\leq -1} \subseteq \mathscr{D}^{\leq 0}$, while $X^1 \in \mathscr{D}^{\geq 1}$, and applying axiom (ii). ☐

If you are confused by the numbering, think about the canonical example of chain complexes.

I now want to show that the heart has nice properties; but first, we need some extra facts about the truncation functors:

Properties 7.8.

1. $\tau^{\leq n}(X[m]) \cong (\tau^{\leq n+m}(X))[n]$
2. $\tau^{\geq n}(X[m]) \cong (\tau^{\geq n+m}(X))[m]$
3. $X \in \mathscr{D}^{\leq n} \iff \tau^{\leq n}(X) \xrightarrow{\sim} X \iff \tau^{>n}(X) = 0.$

Notation: $\tau^{<a} = \tau^{\leq a-1}$, $\tau^{>b} = \tau^{\geq b+1}$.

In 3, the former equivalence is by formal nonsense about adjoint functors. The latter is by using the exact triangle from the Proposition.

Warning 7.9. $\tau^{\leq n}$ is *not* exact: ex. $\tau^{\leq 0}(\mathbf{Z}) = \mathbf{Z}$, while $\tau^{\leq 0}(\mathbf{Z}[-1]) = 0$.

Nevertheless:

Proposition 7.10. *Given an exact triangle*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

if $X, Z \in \mathscr{D}^{\leq n}$, then $Y \in \mathscr{D}^{\leq n}$.

Proof. We can assume $n = 0$ after shifting everything, and so $X, Z \in \mathscr{D}^{\leq 0}$. It suffices to show $\tau^{>0}(Y) = 0$ (by Property 3 above). First, $\mathrm{Hom}(X, \tau^{>0}(Y)) = 0$ by the fact that $\mathscr{D}^{\leq 0}$ and $\tau^{>0}(Y) \in \mathscr{D}^{\geq 1}$, and $\mathrm{Hom}(Z, \tau^{>0}(Y)) = 0$ for the same reason. The long exact sequence for $\mathrm{Hom}(-, \tau^{>0}(Y))$ then shows that $\mathrm{Hom}(Y, \tau^{>0}(Y)) = 0$. Thus, $\mathrm{Hom}(\tau^{>0}(Y), W) = 0$ for all $W \in \mathscr{D}^{>0}$, and so $\tau^{>0}(Y) = 0$. ☐

Remark 7.11. Similar reasoning shows $\mathscr{D}^{\leq 0} = \{X \in \mathscr{D} \mid \mathrm{Hom}(X, \mathscr{D}^{\geq 1}) = 0\}$, as well as the dual statement. Thus, a t-structure is just the data of a full triangulated subcategory $\mathscr{D}^{\leq 0}$ that satisfies some nice properties.

Exercises 7.12.
1. For $a < b \in \mathbf{Z}$, then $\tau^{\leq b} \circ \tau^{\leq a} = \tau^{\leq a} = \tau^{\leq a} \circ \tau^{\leq b}$, and $\tau^{\leq a} \circ \tau^{\geq b} = 0 = \tau^{\geq b} \circ \tau^{\leq a}$ (this is by adjointness with inclusions, and how that works with compositions).
2. For $a, b \in \mathbf{Z}$, show there is a canonical isomorphism

$$\tau^{\leq a} \circ \tau^{\geq b} \cong \tau^{\geq b} \circ \tau^{\leq a}$$

   The proof requires the octahedral axiom (TR5).

Definition 7.13. Define $H^0 \colon \mathscr{D} \to \mathscr{D}^{\heartsuit}$, where $X \mapsto (\tau^{\leq 0} \circ \tau^{\geq 0})(X) \cong (\tau^{\geq 0} \circ \tau^{\leq 0})(X)$, and also $H^n = H^0((-)[n]) \colon \mathscr{D} \to \mathscr{D}^{\heartsuit}$.

The following lemma is an analogue to the fact that a chain complexes with bounded cohomology can be reconstructed from that cohomology.

Lemma 7.14. *For all $X \in \mathscr{D}$, there is an exact triangle*

$$H^n(X)[-n] \longrightarrow \tau^{\geq n}(X) \longrightarrow \tau^{\geq n+1}(X) \longrightarrow H^n(X)[-n+1]$$

Proof. $\tau^{\leq n} \circ \tau^{\geq n}(X) = H^n(X)[-n]$. Use standard triangle for $\tau^{\geq n}(X)$:

$$\begin{array}{ccccccc}
\tau^{\leq n}(\tau^{\geq n}(X)) & \longrightarrow & \tau^{\geq n}(X) & \longrightarrow & \tau^{\geq n+1}(\tau^{\geq n}(X)) & \longrightarrow & \tau^{\leq n}(\tau^{\geq n}(X))[1] \\
\| & & & & \| & & \\
H^n(X)[-n] & & & & \tau^{\geq n+1}(X) & &
\end{array}$$

☐

Corollary 7.15. *Assume $X \in \mathscr{D}^{\geq a}$ for some $a \in \mathbf{Z}$. Then, $X \in \mathscr{D}^{\geq n} \iff H^i(X) = 0$ for all $i < n$.*

Proof. Use Lemma. ☐

Lemma 7.16. *If $X \to Y \to Z \to X[1]$ is exact, and $X, Z \in \mathscr{D}^{\heartsuit}$, then $Y \in \mathscr{D}^{\heartsuit}$.*

Proof. If $X, Z \in \mathscr{D}^{\leq 0}$, then $Y \in \mathscr{D}^{\leq 0}$ by a previous Lemma. Dually, if $X, Z \in \mathscr{D}^{\geq 0}$, then $Y \in \mathscr{D}^{\geq 0}$. Thus, $Y \in \mathscr{D}^{\heartsuit}$. ☐

Remark 7.17. Rotated version is false:

$$\mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{i_1} \mathbf{Z} \oplus \mathbf{Z}[1] \xrightarrow{\mathrm{pr}_2} \mathbf{Z}[1]$$

is an exact triangle in $\mathscr{D}(\mathsf{Ab})$. $\mathbf{Z} \in \mathscr{D}^{\heartsuit} = \mathsf{Ab}$, but $\mathbf{Z} \oplus \mathbf{Z}[1] \notin \mathscr{D}^{\heartsuit}$.

Theorem 7.18. *$\mathscr{D}^\heartsuit$ is an abelian category.*

This is kind of a miracle! Bhargav really likes this stuff because of how formal it is, in that everything just follows from the definitions.

Proof. $X, Y \in \mathscr{D}^\heartsuit$ implies $X \oplus Y \in \mathscr{D}^\heartsuit$ by the last Lemma, and so $\mathscr{D}^\heartsuit$ is additive (where we should check $X \oplus Y$ actually gives the direct sum in $\mathscr{D}^\heartsuit$).

Now say $f \colon X \to Y$ is a map in $\mathscr{D}^\heartsuit$. We want to find the kernel and the cokernel of this map in $\mathscr{D}^\heartsuit$. Extend to an exact triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$$

Check: $Z \in \mathscr{D}^{\leq 0} \cap \mathscr{D}^{\geq -1}$. We want to show that $H^0(Z) = \tau^{\geq 0}(Z) =: \operatorname{cok}(f)$, and that $H^{-1}(Z) = H^0(Z[-1]) = \tau^{\leq 0}(Z[-1]) =: \ker(f)$. Fix $W \in \mathscr{D}^\heartsuit$; we want to check the universal property for $\operatorname{cok}(f)$. Apply $\operatorname{Hom}(-, W)$ to the exact triangle above:

$$\operatorname{Hom}(X[1], W) \longrightarrow \operatorname{Hom}(Z, W) \longrightarrow \operatorname{Hom}(Y, W) \longrightarrow \operatorname{Hom}(X, W)$$

Now since $W \in \mathscr{D}^\heartsuit$, $\operatorname{Hom}(X[1], W) = 0$, since $X[1] \in \mathscr{D}^{\leq -1}$, and also $\operatorname{Hom}(Z, W) = \operatorname{Hom}(\tau^{\geq 0}(Z), W)$, by the universal property of truncations. So, we get a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}(\tau^{\geq 0}(Z), W) \arrow[r] \arrow[d, equal] & \operatorname{Hom}(Y, W) \arrow[r] & \operatorname{Hom}(X, W) \\
& \operatorname{Hom}(H^0(Z), W) & &
\end{tikzcd}$$

This implies $Y \to \tau^{\geq 0}(Z) = H^0(Z)$ is a cokernel for f. Dually,

$$\begin{tikzcd}
H^{-1}(Z) \arrow[r, "\mathrm{can}"] \arrow[rr, bend right] & Z[-1] \arrow[r, "\mathrm{bdy}"] & X
\end{tikzcd}$$

is a kernel for f. ◻

8 October 1, 2015

Last time, we defined what a t-structure on a triangulated category $\mathscr{D}$ is, i.e., a pair $(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$ of subcategories satisfying certain properties, and that they give rise to truncation functors $\tau^{\leq r}, \tau^{\geq r}$.

Example 8.1. If $\mathscr{D} = \mathsf{D}(\mathscr{A})$, where $\mathscr{A}$ is an abelian category, then

$$\mathscr{D}^{\leq 0} = \{K \mid H^i(K) = 0 \ \forall i > 0\}$$
$$\mathscr{D}^{\geq 0} = \{K \mid H^i(K) = 0 \ \forall i < 0\}$$

Claim. $\operatorname{Hom}(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 1}) = 0$.

Proof. Let $K \in \mathscr{D}^{\leq 0}$, $L \in \mathscr{D}^{\geq 1}$, and $f \colon K \to L$ in $\mathscr{D}$. We choose representatives of K and L such that $K \in \mathsf{Ch}^{\leq 0}(\mathscr{A})$ and $L \in \mathsf{Ch}^{\geq 1}(\mathscr{A})$. Then, f is given by a roof diagram

$$\begin{tikzcd}
& K' \arrow[dl, "\alpha"', "\mathsf{qis}"] \arrow[dr, "g"] & \\
K & & L
\end{tikzcd}$$

i.e., $f = g\alpha^{-1}$. We can then replace K' to get a new diagram

$$\begin{tikzcd}
& \tau^{\leq 0} K' \arrow[d, "\beta = \mathrm{can}"] \arrow[ddl, bend right, "\alpha'"'] \arrow[ddr, bend left, "g'"] & \\
& K' \arrow[dl, "\alpha"', "\mathsf{qis}"] \arrow[dr, "g"] & \\
K & & L
\end{tikzcd}$$

and α', β are quasi-isomorphisms, giving a new roof diagram

$$\begin{tikzcd} & \tau^{\leq 0} K' \arrow[dl, "\alpha'"', "\mathsf{qis}"] \arrow[dr, "g'"] & \\ K & & L \end{tikzcd}$$

and so $f = g' \circ (\alpha')^{-1}$. But $g' = 0$ as $\tau^{\leq 0} K' \in \mathsf{Ch}^{\leq 0}$ and $L \in \mathsf{Ch}^{\geq 1}$, and so $f = 0$. □

Now recall we were in the middle of proving the following

Theorem 8.2. *Let $\mathscr{D}$ be a triangulated category, with a t-structure $(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$. Then, $\mathscr{D}^{\heartsuit}$ is abelian.*

We will need to use the following

Corollary 8.3 (to the octahedral axiom (TR5))**.** *Let $\mathscr{D}$ be a triangulated category. Given*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z' \end{tikzcd}$$

with exact rows, there exists a new diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\ X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[d] \\ X'' \arrow[r] & Y'' \arrow[r] & Z'' \end{tikzcd}$$

such that all rows and columns are exact.

This formalizes the fact that cones are "weakly functorial but not too weakly functorial." This is the way we will be using the octahedral axiom.

Proof fo Theorem. Let $X, Y \in \mathscr{D}^{\heartsuit}$, and $f \colon X \to Y$. Extend to an exact triangle

$$X \xrightarrow{f} Y \longrightarrow Z \xrightarrow{\delta} X[1]$$

We know that $Z \in \mathscr{D}^{\leq 0} \cap \mathscr{D}^{\geq -1}$. We claim that the composite

$$\begin{tikzcd} Y \arrow[r] \arrow[rr, bend left] & Z \arrow[r, "\mathrm{can}"'] & H^0(Z) \end{tikzcd}$$

is a cokernel for f, and that the composite

$$\begin{tikzcd} H^{-1}(Z) = H^0(Z[-1]) \arrow[r, "\mathrm{can}"] \arrow[rr, bend right] & Z[-1] \arrow[r] & X \end{tikzcd}$$

is the kernel of f. Fix $W \in \mathscr{D}^{\heartsuit}$, and apply $\mathrm{Hom}(-, W)$ to get an exact sequence

$$\begin{tikzcd} \mathrm{Hom}(X[1], W) \arrow[r] \arrow[d, equal, "\text{orthogonality}"] & \mathrm{Hom}(Z, W) \arrow[r] \arrow[d, equal, "\text{univ. prop. of } \tau^{\geq 0}"] & \mathrm{Hom}(Y, W) \arrow[r] & \mathrm{Hom}(X, W) \\ 0 & \mathrm{Hom}(H^0(Z), W) & & \end{tikzcd}$$

and so $Y \to H^0(Z)$ is $\mathrm{cok}(f)$. Dually, $H^{-1}(Z) \to X$ is $\ker(f)$.

Now we want

$$\begin{tikzcd} \mathrm{coim}(f) \arrow[r, "\sim"] \arrow[d, equal] & \mathrm{im}(f) \arrow[d, equal] \\ \mathrm{cok}(\ker(f) \to X) & \ker(Y \to \mathrm{cok}(f)) \end{tikzcd}$$

First, $Y \xrightarrow{\alpha} \mathrm{cok}(f)$ is surjective, so $\ker(\alpha) \coloneqq \mathrm{Cone}(\alpha)[-1]$. Similarly, $H^{-1}(Z) = \ker(f) \xrightarrow{\beta} X$ is injective, so $\mathrm{cok}(f) \coloneqq \mathrm{Cone}(\beta)$. We want to show that $\mathrm{Cone}(\beta) \xrightarrow{\sim} \mathrm{Cone}(\alpha)[-1]$. Consider

$$\begin{tikzcd} H^{-1}(Z) \arrow[r, "\beta"] \arrow[d] & X \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{Cone}(\beta) \arrow[d, "\text{weak func. of cones}"] \\ Z[-1] \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ H^0(Z)[-1] \arrow[r] & 0 \arrow[r] & Q \end{tikzcd}$$

which exists by the Corollary to the octahedral axiom. You have to argue that the bottom left entry is canonically $H^0(Z)[-1]$; everything lives in the heart, however, so there is not much of a choice and $H^0(Z)[-1]$ "has" to be what we get there. By the exactness of the bottom triangle, we obtain that $Q \cong (H^0(Z)[-1])[1] = H^0(Z)$. You then need to check $(Y \to Q) \simeq (Y \xrightarrow{\alpha} H^0(Z))$. Then, the right column shows that $\mathrm{Cone}(\beta)[1] \cong \mathrm{Cone}(\alpha)$, and so $\mathrm{Cone}(\beta) \cong \mathrm{Cone}(\alpha)[-1]$. ☐

This concludes the proof, but note there are many compatibilities that require checking!

We note here that there can be no two t-structures that give the same heart, since the heart will define a H^0 functor, and so uniquely defines truncations via shifting.

In the world of stable ∞-categories, you actually *do* have functorial cones, and so the Corollary we used above is just a formal consequence of the new formalism. This remark also applies to *dg*-categories.

Corollary 8.4. *Let $X, Y, Z \in \mathscr{D}^\heartsuit$. Then, $0 \to X \xrightarrow{a} Y \xrightarrow{b} Z \to 0$ is exact in $\mathscr{D}^\heartsuit$ if and only if there exists a unique exact triangle $X \xrightarrow{a} Y \xrightarrow{b} Z \to X[1]$.*

Proof. $\Leftarrow$ is okay by the proof of the Theorem: $\mathrm{cok}(a) = H^0(Z) = Z$, and $\ker(a) = H^1(Z) = 0$, since $Z \in \mathscr{D}^\heartsuit$.

$\Rightarrow$. a is injective if and only if $\ker(a) = 0$, which is equivalent to $H^{-1}(\mathrm{Cone}(a)) = 0$. But this is equivalent to having $\mathrm{Cone}(a) \xrightarrow{\sim} H^0(\mathrm{Cone}(a)) = \mathrm{cok}(a) = Z$. The argument here is that $\mathrm{Cone}(a)$ is part of an exact triangle, and so $X, Y \in \mathscr{D}^\heartsuit$ restricts where $\mathrm{Cone}(a)$ has nonzero cohomology. We therefore get an exact triangle $X \to Y \to Z \xrightarrow{\delta} X[1]$. To get uniqueness of δ, we use that $\mathrm{Hom}(X[1], Z) = 0$ since $X[1] \in \mathscr{D}^{\leq -1}$ and $Z = \mathscr{D}^{\geq 0}$ by orthogonality, and using Lemma 7.7. ☐

Now we know what exact sequences in $\mathscr{D}^\heartsuit$ look like; this gives us information about Ext^1.

Corollary 8.5. *Let $X, Z \in \mathscr{D}^\heartsuit$. Then,*

$$\mathrm{Ext}^1_{\mathscr{D}^\heartsuit}(Z, X) \cong \mathrm{Hom}_{\mathscr{D}}(Z, X[1]) =: \mathrm{Ext}^1_{\mathscr{D}}(Z, X).$$

Often what happens is that we can calculate the right-hand side, since we often don't know what the heart looks like.

Proof. $\to$ is by the Corollary. $\leftarrow$ is by the fact that $\delta\colon Z \to X[1]$ gives an exact triangle

$$Z \longrightarrow X[1] \longrightarrow \mathrm{Cone}(\delta) \longrightarrow Z[1]$$

Shifting, we obtain another exact triangle

$$X \longrightarrow \mathrm{Cone}(\delta)[-1] \longrightarrow Z \xrightarrow{\delta} X[1]$$

Since $X, Z \in \mathscr{D}^\heartsuit$, we have that $\mathrm{Cone}(\delta)[-1] \in \mathscr{D}^\heartsuit$, and so you get a short exact sequence by the previous Corollary. ☐

You might get hopeful and expect this to generalize to general Ext's, but this is false!

Warnings 8.6.

1. This does not generalize to higher Ext, since we cannot use orthogonality properties from before.

   Example 8.7. Let $X = S^2$ (or any simply connected space which is not contractible). Then, let

   $$\mathscr{D} = \mathsf{D}_{\mathsf{loc}}(X) = \{K \in \mathsf{D}(\mathsf{Ab}(X)) \mid H^i(K) \text{ is locally constant (and therefore constant)}\}$$

   Check that $\mathscr{D}$ inherits a triangulated structure from $\mathsf{D}(\mathsf{Ab}(X))$ and a t-structure:

   $$\mathscr{D}^{\leq 0} = \{K \in \mathsf{D}_{\mathsf{loc}}(X) \mid H^i(K) = 0 \ \forall i > 0\}$$
   $$\mathscr{D}^{\geq 0} = \{K \in \mathsf{D}_{\mathsf{loc}}(X) \mid H^i(K) = 0 \ \forall i < 0\}$$

   Now

   $$\begin{aligned}\mathscr{D}^{\heartsuit} &= \{K \in \mathsf{D}(\mathsf{Ab}(X)) \mid H^i(K) = 0 \ \forall i \neq 0, \text{ and } H^0(K) \text{ locally constant}\} \\ &= \{\text{locally constant sheaves}/X\} \\ &\cong \mathsf{Ab}\end{aligned}$$

   since $\pi_1(X) = 0$. Therefore,

   $$\operatorname{Ext}^2_{\mathscr{D}^{\heartsuit}}(\underline{\mathbf{Z}}, \underline{\mathbf{Z}}) = \operatorname{Ext}^2_{\mathsf{Ab}}(\mathbf{Z}, \mathbf{Z}) = 0,$$

   while

   $$\operatorname{Ext}^2_{\mathscr{D}(\mathsf{Ab})}(\underline{\mathbf{Z}}, \underline{\mathbf{Z}}) = H^2(X, \underline{\mathbf{Z}}) = \mathbf{Z} \neq 0.$$

   In this way, passing to the hart is a "very lossy operation."
   You can ask when Ext's always agree. Something like being a $K(\pi, 1)$ is the condition.

2. In general, $\mathsf{D}(\mathscr{D}^{\heartsuit}) \neq \mathscr{D}$ (e.g. the previous example), but in fact, there is (probably) not even a natural functor between the two in general. This is also something that is fixed by stable ∞-categories. Beilinson was actually able to prove $\mathsf{D}(\mathscr{D}^{\heartsuit}) \cong \mathscr{D}$ for perverse sheaves/constructible sheaves, so this is not an issue in our case.

Theorem 8.8. *The functor $H^0 \colon \mathscr{D} \to \mathscr{D}^{\heartsuit}$ is a cohomological functor.*

Proof. Fix an exact triangle $X \to Y \to Z \to X[1]$ in $\mathscr{D}$. We want to show that $H^0(X) \to H^0(Y) \to H^0(Z)$ is exact in $\mathscr{D}^{\heartsuit}$. The proof follows in steps, à la dévissage.

1. Assume $X, Y, Z \in \mathscr{D}^{\geq 0}$. In this case, we prove a stronger statement

   Claim. *$0 \to H^0(X) \to H^0(Y) \to H^0(Z)$ is exact.*

   Proof. Let $A \in \mathscr{D}^{\heartsuit}$. Observe that $\operatorname{Hom}(A, X) = \operatorname{Hom}(A, H^0(X))$, since $A \in \mathscr{D}^{\leq 0}$ and $X \in \mathscr{D}^{\geq 0}$; the same applies to Y, Z. Applying $\operatorname{Hom}(A, -)$, we get the long exact sequence

   $$\begin{tikzcd} \operatorname{Hom}(A, Z[-1]) \arrow[r] \arrow[d, equal, "A \in \mathscr{D}^{\leq 0}", "{Z[-1] \in \mathscr{D}^{\geq 1}}"'] & \operatorname{Hom}(A, X) \arrow[r] \arrow[d, equal] & \operatorname{Hom}(A, Y) \arrow[r] \arrow[d, equal] & \operatorname{Hom}(A, Z) \arrow[d, equal] \\ 0 \arrow[r] & \operatorname{Hom}(A, H^0(X)) \arrow[r] & \operatorname{Hom}(A, H^0(Y)) \arrow[r] & \operatorname{Hom}(A, H^0(Z)) \end{tikzcd}$$

   and the Claim follows by Yoneda. ☐

2. Assume $Z \in \mathscr{D}^{\geq 0}$.

   Claim. *$0 \to H^0(X) \to H^0(Y) \to H^0(Z)$ is exact.*

Proof. We have an exact triangle $X \to Y \to Z \to X[1]$. Use $\mathrm{Hom}(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 1})$ to get $\tau^{<0}(X) = \tau^{<0}(Y)$. Then, we get the following diagram with exact rows

$$\begin{tikzcd}
\tau^{<0}(X) \arrow[r] \arrow[d] & \tau^{<0}(Y) \arrow[r] \arrow[d] & 0 \arrow[d] \\
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d, equal] \\
\tau^{\geq 0}(X) \arrow[r] & \tau^{\geq 0}(Y) \arrow[r] & Z
\end{tikzcd}$$

with the bottom row by the Corollary to the octahedral axiom from before. This implies the columns are also exact. In particular, the bottom row is exact. Now we use Step 1. $\square$

3. Assume $X \in \mathscr{D}^{\leq 0}$.

   Claim. *$H^0(X) \to H^0(Y) \to H^0(Z) \to 0$ is exact.*

   The proof follows by applying Step 2 to the opposite category.
4. In the general case, we again use the Corollary to the octahedral axiom to get

$$\begin{tikzcd}
\tau^{\leq 0}(X) \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, equal] & W \arrow[d] \\
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\
\tau^{>0}(X) \arrow[r] & 0 \arrow[r] & Q
\end{tikzcd}$$

The bottom row shows $Q \cong (\tau^{>0}(X))[1]$, and so we have two exact triangles obtained from the right column and top row:

$$W \longrightarrow Z \longrightarrow (\tau^{>0}(X))[1]$$

$$\tau^{\leq 0}(X) \longrightarrow Y \longrightarrow W$$

Step 2 and the first triangle give

$$0 \longrightarrow H^0(W) \longrightarrow H^0(Z) \longrightarrow H^0(\tau^{>0}(X)[1]) = H^1(X)$$

and Step 3 and the second triangle give

$$H^0(X) = H^0(\tau^{\leq 0}(X)) \longrightarrow H^0(Y) \longrightarrow H^0(W) \longrightarrow 0$$

Splicing together these two exact sequences, we have that $H^0(X) \to H^0(Y) \to H^0(Z)$ is exact. $\square$

Bhargav apologizes for all the diagram-chasing; there will be less of this from now on now that we've established our categorical foundations.

Definition 8.9. Let $F\colon \mathscr{D}_1 \to \mathscr{D}_2$ be an exact functor of triangulated categories, and assume that there are t-structures $(\mathscr{D}_1^{\leq 0}, \mathscr{D}_1^{\geq 0})$ on $\mathscr{D}_1$ and $(\mathscr{D}_2^{\leq 0}, \mathscr{D}_2^{\geq 0})$ on $\mathscr{D}_2$. Then, we say that
1. F is *t-left-exact* if $F(\mathscr{D}_1^{\geq 0}) \subseteq \mathscr{D}_2^{\geq 0}$.
2. F is *t-right-exact* if $F(\mathscr{D}_1^{\leq 0}) \subseteq \mathscr{D}_2^{\leq 0}$.
3. F is *t-exact* if it is both t-left- and t-right-exact.

Set ${}^pF = H^0 \circ F \circ (\mathrm{inc})\colon \mathscr{D}_1^{\heartsuit} \to \mathscr{D}_2^{\heartsuit}$.

Example 8.10. If $F\colon \mathscr{A} \to \mathscr{B}$ is a left exact functor between abelian categories, then the functor $\mathbf{R}F\colon \mathscr{D}(\mathscr{A}) \to \mathscr{D}(\mathscr{B})$ is t-left-exact. Note that $\mathbf{R}F$ is exact if and only if F is exact.

What happens in the world of perverse sheaves is that our standard notions of exactness get mixed up. For example, for abelian varieties, a morphism gives a push forward. On the level of perverse sheaves, you get a derived functor in the opposite direction. "It's weird, but it's kinda fun!"

What we wanted to do next was construct t-structures by tilting; this can show that $\mathsf{D}(\mathbf{P}^1_{\mathbf{C}})$ has infinitely many t-structures. Next week we will move on from general formalism and talk about the specific case of perverse sheaves.

9 October 6, 2015

Bhargav's office hours: Tuesday 3–5 and Friday 3–4.

Last time we finished talking about t-structures, but we had some questions about how many t-structures a given triangulated category would have. Since Bhargav prepared this anyway, he would like to discuss this first.

Example 1. There are lots of t-structures on $\mathscr{D} = \mathsf{D}(\mathsf{Coh}\mathbf{P}^1)$.

Let $\mathscr{A}$ be an abelian category, and $\mathscr{D} = \mathsf{D}(\mathscr{A})$. $\alpha = (T, F)$ is a *torsion pair* in $\mathscr{A}$ if $T, F \subseteq \mathscr{A}$ are both full additive subcategories, $\mathrm{Hom}(T, F) = 0$, and for all $X \in \mathscr{A}$, there exists a short exact sequence $0 \to Y \to X \to Z \to 0$ such that $Y \in T$ and $Z \in F$.

Examples 9.1.
1. Let $\mathscr{A} = \mathsf{Ab}$, T torsion groups, and F torsion-free groups.
2. Let $\mathscr{A} = \mathsf{Coh}(\mathbf{P}^1)$, $S \subseteq |\mathbf{P}^1|$ a subset of $\mathbf{P}^1$. Let $T = \{X \in \mathscr{A} \mid \mathrm{supp}(X) \subseteq S\}$, and $F = \{X \in \mathscr{A} \mid$ no sections of X supported at $S\}$.

Definition 9.2. ${}^{\alpha}\mathscr{D}^{\le 0} = \{K \in \mathsf{D}(\mathscr{A}) \mid H^i(K) = 0\ \forall i > 0,\ H^0(K) \in T\}$. Similarly, ${}^{\alpha}\mathscr{D}^{\ge 0} = \{K \in \mathsf{D}(\mathscr{A}) \mid H^i(K) = 0\ \forall i < -1,\ H^{-1}(K) \in F\}$.

For example, if $T = \mathscr{D}$ and $F = 0$, then you get the classical case of $\mathscr{D}^{\le 0}$ and $\mathscr{D}^{\ge 0}$.

Claim 9.3. $({}^{\alpha}\mathscr{D}^{\le 0}, {}^{\alpha}\mathscr{D}^{\ge 0})$ *is a t-structure on $\mathscr{D}$, with*

$${}^{\alpha}\mathscr{D}^{\heartsuit} = \{K \in \mathscr{D} \mid H^{-1}(K) \in F,\ H^0(K) \in T,\ H^i(K) = 0\ \forall i \neq 0\}$$

You can always write out this definition of the heart, but it would be unclear a priori that it is in fact an abelian category. Luckily, this follows from what we've discussed about t-structures. It's interesting to figure out what kernels and cokernels look like!

Proof.
1. $\mathrm{Hom}({}^{\alpha}\mathscr{D}^{\le 0}, {}^{\alpha}\mathscr{D}^{\ge 1}) = 0$: let $X \in {}^{\alpha}\mathscr{D}^{\le 0}$ and $Y \in {}^{\alpha}\mathscr{D}^{\ge 1}$. By definition, $H^0(X) \in T$, $H^i(X) = 0$ $\forall i > 0$, and also $H^0(Y) \in F$, and $H^i(Y) = 0$ $\forall i < 0$. Thus, $X \in \mathscr{D}^{\le 0}$ and $Y \in \mathscr{D}^{\ge 0}$, and so $\mathrm{Hom}_{\mathscr{D}}(X, Y) = \mathrm{Hom}_{\mathscr{D}}(H^0(X), H^0(Y))$ by using truncation functors. But this last Hom set is zero by assumption on (T, F).
2. ${}^{\alpha}\mathscr{D}^{\le 1} \supseteq {}^{\alpha}\mathscr{D}^{\le 0}$: obvious.
3. Existence of triangles "the fun one": say $X \in \mathscr{D}$. You have the canonical exact triangles

$$\tau^{\le 0}(X) \longrightarrow X \longrightarrow \tau^{>0}(X),$$

using the usual t-structure. We will modify the H^0 term on the left object to make it "smaller". We also have

$$\tau^{\le -1}(X) \longrightarrow \tau^{\le 0}(X) \longrightarrow H^0(X).$$

Now we break up $H^0(X)$: choose a short exact sequence

$$0 \longrightarrow M \longrightarrow H^0(X) \longrightarrow N \longrightarrow 0$$

where $M \in T$ and $N \in F$. If we were using ∞-categories, we could take fibre products, but we can't, so we instead consider Y defined via the "fibre product"

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & M \arrow[d] \\ \tau^{\leq 0}(X) \arrow[r] & H^0(X) \end{tikzcd}$$

by defining $Y = \mathrm{Cone}(\tau^{\leq 0}(X) \oplus M \to H^0(X))[-1]$. We then get an exact triangle $\tau^{\leq -1}(X) \to Y \to M$ where $\tau^{\leq -1}(X) \in \mathscr{D}^{\leq -1}$, and $M \in T$; this implies that $Y \in {}^\alpha\mathscr{D}^{\leq 0}$. Now,

Exercise 9.4. Check that setting $Z = \mathrm{Cone}(Y \to X)$ makes $Z \in {}^\alpha\mathscr{D}^{\leq 1}$.

We then get an exact triangle $Y \to X \to Z$, where $Y \in {}^\alpha\mathscr{D}^{\leq 0}$, and $Z \in {}^\alpha\mathscr{D}^{\geq 1}$, giving a t-structure. $\square$

It's interesting to run this construction for different subsets $S \subseteq |\mathbf{P}^1|$.

It is a lemma that if two elements are in the heart, there are no negative Ext group. So this still applies in this setting, telling us that in fact, the H^{-1}'s that we might expect to exist are in fact all trivial.

9.1 Some recollections on constructible sheaves

Let X be a topological space, and let

$$U \xhookrightarrow[\text{open}]{j} X \xhookleftarrow[\text{closed}]{i} Z$$

We can relate the sheaves on U, X, Z with the following functors:

$j_! \colon \mathsf{Ab}(U) \to \mathsf{Ab}(X)$	extension by 0	(exact)
$j^! = j^* \colon \mathsf{Ab}(X) \to \mathsf{Ab}(U)$	restriction	(exact)
$j_* \colon \mathsf{Ab}(U) \to \mathsf{Ab}(X)$	pushforward	(left-exact)
$i^* \colon \mathsf{Ab}(X) \to \mathsf{Ab}(Z)$	restriction	(exact)
$i_! = i_* \colon \mathsf{Ab}(Z) \to \mathsf{Ab}(Z)$	pushforward/extension by 0	(exact)
$i^! \colon \mathsf{Ab}(X) \to \mathsf{Ab}(Z)$	sections supported on Z	(left-exact)

These functors are related by adjointness properties: the sequences

$$(j_!, j^! = j^*, j_*), \qquad (i^*, i_*, i^!)$$

The adjointness of j^*, j_* is the most familiar in algebraic geometry; the one for $j_!, j^!$ pretty much follows by definition. The adjointness between $i_*, i^!$ is similar.

Let $F \in \mathsf{Ab}(X)$. Then, we have the short exact sequences

$$0 \longrightarrow j_! j^* F \xrightarrow{\text{can}} F \xrightarrow{\text{can}} i_* i^* F \longrightarrow 0$$

where the canonical maps are due to the adjointness properties above. You also have the short exact sequence

$$0 \longrightarrow i_* i^! F \xrightarrow{\text{can}} F \xrightarrow{\text{can}} j_* j^* F$$

But the last map is not surjective! It will be surjective on the right if F is injective. This is because j_* is only left-exact; before, we had good control on the stalks of $i_* i^* F$. You have to use the fact that $F(V) \to F(U \cap V)$ is surjective if F is injective, i.e., that injective sheaves are flasque. Note the sheaf on the left consists of sections which are supported on Z.

Now, the following composites vanish:

$$j^* i_* = 0, \qquad i^* j_! = 0 \text{ (left adjoints)}, \qquad i^! j_* = 0 \text{ (right adjoints)}$$

and the following identities:

$$i^* i_* \xrightarrow{\sim} \mathrm{id} \xrightarrow{\sim} i^! i_*, \qquad j^* j_* \xrightarrow{\sim} \mathrm{id} \xrightarrow{\sim} j^* j_!.$$

Formally, this implies that $i_*, j_*, j_!$ are fully faithful.

Now we have to talk about what happens on the derived level. What we will do is use these six functors to glue t-structures on the open part with t-structures on the closed part. We get

$$\begin{tikzcd} \mathsf{D}(Z) \arrow[r, "i_*"] & \mathsf{D}(X) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "\mathbf{R}i^!"] \arrow[r, "j^*"] & \mathsf{D}(U) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "\mathbf{R}j_*"] \end{tikzcd}$$

The arrows on top are the left-adjoint. We have the same adjunctions and formulas as before. We also have, for any $K \in \mathsf{D}(X)$, the following exact triangles:

$$j_!j^*K \longrightarrow K \longrightarrow i_*i^*K \xrightarrow{+1}$$

$$i_*i^!K \longrightarrow K \longrightarrow \mathbf{R}j_*j^*K \xrightarrow{+1}$$

where the first formally follows from the corresponding abelian case. The second follows by first replacing K with an injective resolution. "Thankfully, we don't have to left derive anything, yet."

Example 9.5. Let $X = S^2$ (the computation should work for any manifold), $Z = \{z\}$, and $K = \underline{\mathbf{Z}} \in \mathsf{D}(U)$. The goal is calculate $\mathbf{R}j_*\underline{\mathbf{Z}} \in \mathsf{D}(X)$.

First, we know that $j^*\mathbf{R}j_*\underline{\mathbf{Z}} = \underline{\mathbf{Z}} \in \mathsf{D}(U)$, i.e., nothing changes on the open part. However, something changes on the closed part: , that is

$$i^*\mathbf{R}j_*\underline{\mathbf{Z}} = \text{stalk of } \mathbf{R}j_*\underline{\mathbf{Z}} \text{ at } z \in X = \underset{\substack{V \ni z \\ \text{open neighborhood}}}{\operatorname{hocolim}} \mathbf{R}\Gamma(V, \mathbf{R}j_*\underline{\mathbf{Z}})$$

For us, we obtain

$$\underset{V \ni z}{\operatorname{hocolim}}\, \mathbf{R}\Gamma(V \cap U, \underline{\mathbf{Z}})$$

so we are looking at smaller and smaller punctured discs around z. But after shrinking far enough, the homology does not change, and so we just get that this is equal to $\mathbf{R}\Gamma(S^1, \underline{\mathbf{Z}}) = \mathbf{Z}[0] \oplus \mathbf{Z}[-1]$.

In particular, we obtain that this complex is an honest complex, and not a sheaf. We get an exact triangle:

$$\begin{tikzcd} j_!j^*(\mathbf{R}j_*\underline{\mathbf{Z}}) \arrow[r] \arrow[d, equal] & \mathbf{R}j_*\underline{\mathbf{Z}} \arrow[r] & i_*i^*(\mathbf{R}j_*\underline{\mathbf{Z}}) \arrow[d, equal] \\ j_!\underline{\mathbf{Z}}_U & & i_*(\mathbf{Z}[0] \oplus \mathbf{Z}[-1]) \end{tikzcd}$$

and so $\mathcal{H}^0(\mathbf{R}j_*\underline{\mathbf{Z}}) = j_*\underline{\mathbf{Z}} = \underline{\mathbf{Z}}$ and $\mathcal{H}^1(\mathbf{R}j_*\underline{\mathbf{Z}})) = i_*\underline{\mathbf{Z}}$. We therefore have an exact triangle

$$\begin{tikzcd} \underline{\mathbf{Z}} \arrow[r] \arrow[d, equal] & \mathbf{R}j_*\underline{\mathbf{Z}} \arrow[r] & i_*\mathbf{Z}[-1] \arrow[r, "\delta"] \arrow[d, equal] & \underline{\mathbf{Z}}[1] \\ \tau^{\leq 0}(\mathbf{R}j_*\underline{\mathbf{Z}}) & & \tau^{>0}(\mathbf{R}j_*\underline{\mathbf{Z}}) & \end{tikzcd}$$

But we don't know what δ does! All we can say for now is the following

Claim. *δ is non-zero (which implies $\mathbf{R}j_*\underline{\mathbf{Z}} \neq \underline{\mathbf{Z}} \oplus i_*\mathbf{Z}[-1]$).*

Proof. Apply $\mathbf{R}\Gamma(X, -)$, to obtain

$$\begin{tikzcd} \mathbf{R}\Gamma(S^2, \underline{\mathbf{Z}}) \arrow[r] \arrow[d, equal] & \mathbf{R}\Gamma(X, \mathbf{R}j_*\underline{\mathbf{Z}}) \arrow[r] \arrow[d, equal] & \mathbf{R}\Gamma(X, i_*\mathbf{Z}[-1]) \arrow[r, "\delta"] \arrow[d, equal] & {} \\ \mathbf{Z}[0] \oplus \mathbf{Z}[-2] & \mathbf{R}\Gamma(U, \underline{\mathbf{Z}}) \arrow[d, equal] & \mathbf{Z}[-1] & \\ & \mathbf{Z}[0] & & \end{tikzcd}$$

The boundary map δ therefore induces the isomorphism

$$\begin{tikzcd} H^1(\mathbf{Z}[-1]) \arrow[r, "\sim"] \arrow[d, equal] & H^2(\mathbf{Z}[0] \oplus \mathbf{Z}[-2]) \arrow[d, equal] \\ \mathbf{Z} \arrow[r, "\sim"] & \mathbf{Z} \end{tikzcd}$$

$\square$

9.2 Recollement (or glueing) of t-structures (BBD, §1.4)

The goal is to explain how to glue t-structures on $\mathsf{D}(U)$ and $\mathsf{D}(Z)$ to get one on $\mathsf{D}(X)$.

In order to do this, I will first explain it in the abstract setup, since in actual algebraic geometry you would have to work with more specific sheaves, like étale or ℓ-adic sheaves.

Set-up 9.6. Let $\mathscr{D}$, $\mathscr{D}_U$, and $\mathscr{D}_F$ be triangulated categories, with a sequence of exact functors

$$\mathscr{D}_F \xrightarrow{i_*} \mathscr{D} \xrightarrow{j^*} \mathscr{D}_U$$

Set $i_! = i_*$ and $j^! = j^*$. This is called a *glueing setup* if it satisfies:
(R1) i_* has a left adjoint i^* and a right adjoint $i^!$;
(R2) j^* has a left adjoint $j^!$ and a right adjoint j_*;
(R3) $j^*i_* = 0$ ($\Longrightarrow i^*j_! = 0$, $i^!j_* = 0$). These imply that if $A \in \mathscr{D}_F$ and $B \in \mathscr{D}_U$, then

$$\begin{array}{ccc} \mathrm{Hom}(j_!B, i_*A) & \qquad & \mathrm{Hom}(i_*A, j_*B) \\ \Big\| {\scriptstyle j^*i_*=0} & & \Big\| {\scriptstyle j^*i_*=0} \\ 0 & & 0 \end{array}$$

(R4) For $K \in \mathscr{D}$, there is a map $\delta\colon i_*i^*K \to j_!j^*K[1]$, and a map $\delta'\colon j_*j^*K \to i_*i^!K[1]$ such that there are exact triangles

$$j_!j^*K \longrightarrow K \longrightarrow i_*i^*K \xrightarrow{\delta}$$

$$i_*i^!K \longrightarrow K \longrightarrow j_*j^*K \xrightarrow{\delta}$$

Remark 9.7.
1. δ is unique, since $\mathrm{Hom}(j_!j^*K[1], i_*i^*K) = 0$ by (R3).
2. δ' is unique.

(R5) $j_!$, j_*, i_* are fully faithful.

We will show next time that we can glue together t-structures. The axioms can be used in almost mathematics and for l-adic sheaves.

Remark 9.8.
1. This is autodual: exchange $j_!, j_*$ with $i^*, i^!$;
2. The pair $(i_*\mathscr{D}_F, j^*\mathscr{D}_U)$ is a t-structure, using the fully faithful embedding of $\mathscr{D}_F, \mathscr{D}_U$ into the larger category $\mathscr{D}$. The lack of Hom's between the two follows from (R3), shifting works since $\mathscr{D}_F, \mathscr{D}_U$ are triangulated, and (R4) gives the exact sequences we want. But, the heart of this t-structure is 0: $j^*i_* = 0$, but $j^*j^* = \mathrm{id}$. This is therefore an example of an interesting t-structure whose heart is fairly stupid.
3. Dually, $(j_!\mathscr{D}_U, i_*\mathscr{D}_F)$ also gives a t-structure.
4. The sequences

$$0 \longrightarrow \mathscr{D}_U \xrightarrow{j_!} \mathscr{D} \xrightarrow{i^*} \mathscr{D}_F \longrightarrow 0$$

$$0 \longrightarrow \mathscr{D}_U \xrightarrow{j_*} \mathscr{D} \xrightarrow{i^!} \mathscr{D}_F \longrightarrow 0$$

$$0 \longrightarrow \mathscr{D}_F \xrightarrow{i_*} \mathscr{D} \xrightarrow{j^*} \mathscr{D}_U \longrightarrow 0$$

are "exact", that is, the right hand side is the Verdier quotient of the middle term by the left hand side.

Next time we will discuss why in this setup we can glue t-structures.

10 October 8, 2015

Last time, we were talking about glueing. Recall that we have three triangulated categories

$$\begin{tikzcd} \mathscr{D}_F \arrow[r, "i_*"] & \mathscr{D} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathscr{D}_U \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

with some formulas (what we call a "glueing setup"). What I want to explain today is how to take t-structures on each of the pieces, to get one in the middle. Note there is no compatibility condition!

Assume we are given t-structure $(\mathscr{D}_U^{\leq 0}, \mathscr{D}_U^{\geq 0})$ on $\mathscr{D}_U$ and $(\mathscr{D}_F^{\leq 0}, \mathscr{D}_F^{\geq 0})$ on $\mathscr{D}_F$.

Main Example 10.1. Translations of standard t-structures on $\mathsf{D}(U)$ and $\mathsf{D}(F)$.

Definition 10.2.

$$\begin{aligned}\mathscr{D}^{\leq 0} &= \{K \in \mathscr{D} \mid j^*K \in \mathscr{D}_U^{\leq 0},\ i^*K \in \mathscr{D}_F^{\leq 0}\}\\ \mathscr{D}^{\geq 0} &= \{K \in \mathscr{D} \mid j^*K \in \mathscr{D}_U^{\geq 0},\ i^!K \in \mathscr{D}_F^{\geq 0}\}\end{aligned}$$

Note for the standard t-structures on a topological space, we get back exactly the normal t-structure on $\mathsf{D}(X)$.

Theorem 10.3. *$(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$ is a t-structure on $\mathscr{D}$.*

Bhargav was going to dive right in and just prove this, but about half an hour ago, he realized that it would be better to give an example!

Example 10.4. Let X be a cone, i.e., $\mathbf{A}^1 \vee \mathbf{A}^1 \subseteq \mathbf{A}^2$, and let x be the vertex of this cone, i.e., $Z = \{x\}$, with $U = X \setminus Z$. The t-structures we will use are the following: on $\mathsf{D}(Z) = \mathsf{D}(\mathsf{Ab})$, we will use the standard t-structure, and on $\mathsf{D}(U)$, we will use $\mathsf{D}^{\leq -1}(U), \mathsf{D}^{\geq -1}(U))$. We therefore get a "perverse" (that is, one you get from the Theorem) t-structure on $\mathscr{D} = \mathsf{D}(X)$.

Claim.
1. *$j_!(\underline{\mathbf{Z}}[1]) \in \mathscr{D}^{\heartsuit}$;*
2. *$\underline{\mathbf{Z}}[1] \in \mathscr{D}^{\heartsuit}$;*
3. *$j_!\underline{\mathbf{Z}}[1] \to \underline{\mathbf{Z}}[1]$ is surjective on $\mathscr{D}^{\heartsuit}$.*

3 is surprising since the map normally would be injective but not surjective (look at stalks: the stalk at x is where the interesting things happen); in the perverse world this "flips around".

Proof of Claim. The definition says

$$\begin{aligned}\mathscr{D}^{\leq 0} &= \{K \in \mathsf{D}(X) \mid i^*K \in \mathscr{D}^{\leq 0}(Z),\ j^*K \in \mathscr{D}^{\leq -1}(U)\}\\ \mathscr{D}^{\geq 0} &= \{K \in \mathsf{D}(X) \mid i^!K \in \mathscr{D}^{\geq 0}(Z),\ j^*K \in \mathscr{D}^{\geq -1}(U)\}\end{aligned}$$

Usually if you have a good feel for what the sheaves are locally at stalks, these pullbacks won't be too interesting, except for j^* in which case you'll have to think about local cohomology.

First, for 1, we have the following standard exact triangle

$$\begin{tikzcd} j_!(\underline{\mathbf{Z}}[1]) \arrow[r] \arrow[d, equal] & \mathbf{R}j_*(\underline{\mathbf{Z}}[1]) \arrow[r] & i_*i^*\mathbf{R}j_*(\underline{\mathbf{Z}}[1]) \\ j_!(j^*\mathbf{R}j_*\underline{\mathbf{Z}}[1]) \end{tikzcd}$$

Now we need to compute stalks:

$$i^*(\mathbf{R}j_*\underline{\mathbf{Z}}[1]) = \text{stalk of } \mathbf{R}j_*\underline{\mathbf{Z}}[1] \text{ at } x = \underset{\substack{V \ni x \\ \text{open neighborhoods}}}{\operatorname{hocolim}} \mathbf{R}\Gamma(V \setminus \{x\}, \underline{\mathbf{Z}}[1]).$$

Now $V \setminus \{x\} \simeq S^1 \amalg S^1$ by excising a slightly larger neighborhood around the vertex x. Thus, we get

$${}^*(\mathbf{R}j_*\underline{\mathbf{Z}}[1]) = \mathbf{R}\Gamma(S^1 \amalg S^1, \mathbf{Z}[1]) = \mathbf{Z}^{\oplus 2}[1] \oplus \mathbf{Z}^{\oplus 2}[0]$$

We therefore get the exact triangle

$$j_!(\underline{\mathbf{Z}}[1]) \longrightarrow \mathbf{R}j_*(\underline{\mathbf{Z}}[1]) \longrightarrow i_*(\mathbf{Z}^{\oplus 2}[1] \oplus \mathbf{Z}^{\oplus 2}[0])$$

where the right element is a skyscraper sheaf. Now since $i^!\mathbf{R}j_* = 0$, applying $i^!$ to this exact triangle gives

$$i^! j_! \underline{\mathbf{Z}}[1] = i^! i_*(\mathbf{Z}^{\oplus 2}[1] \oplus \mathbf{Z}^{\oplus 2}[0])[-1] = \mathbf{Z}^{\oplus 2}[0] \oplus \mathbf{Z}^{\oplus 2}[-1] \in \mathsf{D}^{\geq 0}(Z).$$

Moreover, $j^*(j_! \mathbf{Z}[1]) = \mathbf{Z}[1] \in \mathsf{D}^{\geq -1}(U)$, and so $j_!(\mathbf{Z}[1]) \in \mathscr{D}^{\geq 0}$. We also have $i^*(j_! \mathbf{Z}[1]) = 0$, and also $j^*(j_! \mathbf{Z}[1]) = \mathbf{Z}[1] \in \mathsf{D}^{\leq -1}(U)$. Thus, $j_! \underline{\mathbf{Z}}[1] \in \mathscr{D}^{\heartsuit}$, that is, $j_! \underline{\mathbf{Z}}[1]$ is perverse.

For 2, we use the following exact triangle

$$i_* i^! \underline{\mathbf{Z}}[1] \longrightarrow \underline{\mathbf{Z}}[1] \longrightarrow \mathbf{R}j_*(\underline{\mathbf{Z}}[1])$$

We now calculate stalks:

$$i^! \underline{\mathbf{Z}}[1] \longrightarrow \mathbf{Z}[1] \longrightarrow \mathbf{Z}^{\oplus 2}[1] \oplus \mathbf{Z}^{\oplus 2}[0]$$

and so $H^{-1}(i^! \mathbf{Z}[1]) = 0$, $H^0(i^! \mathbf{Z}[1]) = \mathbf{Z}$, and $H^1(i^! \mathbf{Z}[1]) = \mathbf{Z}^{\oplus 2}$. We therefore have

$$i^! \mathbf{Z}[1] = \mathbf{Z}[0] \oplus \mathbf{Z}^{\oplus 2}[-1] \in \mathscr{D}^{\geq 0}(Z).$$

For the other condition, $j^*(\mathbf{Z}[1]) = \underline{\mathbf{Z}}[1] \in \mathsf{D}^{\geq -1}(U)$, and so $\underline{\mathbf{Z}}[1] \in \mathscr{D}^{\geq 0}$. For the other containment, $i^* \mathbf{Z}[1] = \mathbf{Z}[1] \in \mathsf{D}^{\leq 0}(Z)$, and $j^* \mathbf{Z}[1] = \underline{\mathbf{Z}}[1] \in \mathsf{D}^{\leq -1}(U)$, which implies $\underline{\mathbf{Z}}[1] \in \mathscr{D}^{\leq 0}$, and so $\underline{\mathbf{Z}}[1] \in \mathscr{D}^{\heartsuit}$.

Finally, we want to show 3. We first have a short exact sequence

$$0 \longrightarrow j_! \underline{\mathbf{Z}} \longrightarrow \underline{\mathbf{Z}} \longrightarrow i_* \underline{\mathbf{Z}} \longrightarrow 0$$

in $\mathsf{Ab}(X)$. This means we have an exact triangle

$$j_! \mathbf{Z}[1] \xrightarrow{\alpha} \underline{\mathbf{Z}}[1] \longrightarrow i_* \underline{\mathbf{Z}}[1],$$

where both the first two terms are in $\mathscr{D}^{\heartsuit}$ as above. Applying ${}^pH^0$, that is, H^0 for the new t-structure to get

$$\begin{array}{ccccccc} \cdots \longrightarrow {}^pH^{-1}(i_* \mathbf{Z}[1]) & \longrightarrow & {}^pH^0(j_! \mathbf{Z}[1]) & \xrightarrow{\alpha} & {}^pH^0(\underline{\mathbf{Z}}[1]) & \longrightarrow & {}^pH^0(i_* \mathbf{Z}[1]) \\ & & \| & & \| & & \\ & & j_! \mathbf{Z}[1] & & \mathbf{Z}[1] & & \end{array}$$

It suffices to show ${}^pH^0(i_* \mathbf{Z}[1]) = 0$. In general, i_* is t-exact, which implies ${}^pH^0(i_* F) = i_* {}^pH^0(F)$. In our case, ${}^pH^0(i_* \mathbf{Z}[1]) = i_*({}^pH^0(\mathbf{Z}[1])) = i_*(H^0(\mathbf{Z}[1])) = 0$. ☐

Now let's get back to the theoretic stuff. Note that Bhargav did this example since it will be a special case of Beilinson's "basic lemma" we will prove later, in which nice resolutions exist for perverse sheaves. Also, the intersection cohomology complex here would be $\mathbf{Z}[1]$, which is why Bhargav expected this to work!

Now we want to show the theorem from before.

Proof of Theorem. First, $\mathscr{D}^{\leq 0} \subseteq \mathscr{D}^{\leq 1}$ and $\mathscr{D}^{\geq 0} \supseteq \mathscr{D}^{\geq 1}$ is okay.

Now we want to show orthogonality: $\operatorname{Hom}(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 1}) = 0$. Say $X \in \mathscr{D}^{\leq 0}$ and $Y \in \mathscr{D}^{\geq 1}$. Using the exact triangle $j_! j^* X \to X \to i_* i^* X$ and applying $\operatorname{Hom}(-, Y)$, we get

$$\begin{array}{ccccc} & \operatorname{Hom}(j_! j^* X, Y) & \longrightarrow \operatorname{Hom}(X, Y) \longrightarrow & \operatorname{Hom}(j_! j^* X, Y) & \\ & \Big\| & & \Big\| & \\ 0 \overset{i^* X \in \mathscr{D}_F^{\leq 0}}{\underset{i^! Y \in \mathscr{D}_F^{\geq 1}}{=\!=}} & \operatorname{Hom}(i^* X, i^! Y) & & \operatorname{Hom}(j^* X, j^* Y) & \overset{j^* X \in \mathscr{D}_U^{\leq 0}}{\underset{j^* Y \in \mathscr{D}_U^{\geq 1}}{=\!=}} 0 \end{array}$$

Note the reason why we are even doing this is to glue together the only information we do know on the two subcategories separately.

We now prove the existence of triangles. We need to use the octahedral axiom, which is the "only interesting way to construct things in a triangulated category." We do what BBD(G) do, which is kind of clever. Fix $X \in \mathscr{D}$. We get a map $X \to j_* j^* X \to j_*(\tau^{\geq 1} j^* X)$, where both maps are canonical. τ exists on $\mathscr{D}_U$ since it has a t-structure. Now choose an exact triangle $Y \to X \to j_*(\tau^{\geq 1} j^* X)$. Also, we have a map

$Y \to i_* i^* Y \to i_*(\tau^{\geq 1} i^* Y)$. Again choose an exact triangle $A \to Y \to i_*(\tau^{\geq 1} i^* Y)$. The octahedral axiom for the composite $A \to Y \to X$ gives:

$$
\begin{tikzcd}
 & & & {} & & \\
 & & i_*(\tau^{\geq 1} i^* Y) \arrow[ur] \arrow[dr] & & & \\
 & Y \arrow[ur] \arrow[dr] & & B \arrow[r] \arrow[ddr] & {} & \\
 & & X \arrow[ur] \arrow[drr] & & & \\
A \arrow[uur] \arrow[urr] & & & & j_* \tau^{\geq 1}(j^* X) \arrow[dr] \arrow[ddr] & \\
 & & & & & {} \\
 & & & & & {}
\end{tikzcd}
$$

where the arrows going off into space denote that the previous three things along that sequence forms an exact triangle. We now

Claim. *$A \in \mathscr{D}^{\leq 0}$ and $B \in \mathscr{D}^{\geq 1}$.*

Proof. The proof follows by a sequence of "clever steps".

(a) $j^* B \cong \tau^{\geq 1}(j^* X) \in \mathscr{D}_U^{\leq 1}$ by looking at the exact triangle $i_*(\tau^{\geq 1} i^* Y) \to B \to j_* \tau^{\geq 1}(j^* X)$.

(b) Applying j^* to $A \to X \to B$, we get $j^* A \to j^* X \to j^* B = \tau^{\geq}(j^* X) \in \mathscr{D}_U^{\geq 1}$, where the last equality is from (a). This implies $j^* A \xrightarrow{\sim} \tau^{\leq 0}(j^* X)$.

(c) Applying i^* to $A \to Y \to i_* \tau^{\geq 1}(i^* Y)$, we get $i^* A \to i^* Y \to \tau^{\geq 1}(i^* Y)$, and so $i^* A \simeq \tau^{\leq 0} i^* Y$.

(d) Apply $i^!$ to $i_*(\tau^{\geq 1} i^* Y) \to B \to j_* \tau^{\geq 1}(j^* X)$ to get $\tau^{\geq 1}(i^* Y) \xrightarrow{\sim} i^! B$ since $i^! j_* = 0$.

Now we see (a) and (b) imply $B \in \mathscr{D}^{\geq 1}$, and that ($c$) and ($d$) imply $A \in \mathscr{D}^{\leq 0}$. □

We therefore have a t-structure on $\mathscr{D}$. □

Bhargav really has no idea how they came up with this.

Nowe have a recipe to glue together two t-structures into a larger one. Next time, we will do this inductively a ton of times by stratifying the variety, and then building up the entire t-structure by glueing step by step with parameters being the extra shifts, which we will call "pervisities".

Remark 10.5. There exists a converse to this theorem:

Proposition 10.6. *Given $\mathscr{D}_F \xrightarrow{i_*} \mathscr{D} \xrightarrow{j^*} \mathscr{D}_U$ a glueing step and a t-structure $(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$ on $\mathscr{D}$, the following are equivalent:*

1. *$j_! j^*$ is right t-exact;*
2. *$j_* j^*$ is left t-exact;*
3. *$(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$ is obtained by glueing.*

There should be a version with i's but it would be confusing to figure out how to dualize the statement.

Intermediate extensions and simple objects

Continuing in the same glueing setup, let $\mathscr{C}$ be the heart of the t-structure on $\mathscr{D}$, $\mathscr{C}_U$ that on $\mathscr{D}_U$, and $\mathscr{C}_F$ that on $\mathscr{D}_F$.

Goals 10.7.
1. Describe the "intermediate extension" $j_{!*}\colon \mathscr{C}_U \to \mathscr{C}$ also called the "Goresky–Macpherson extension."
2. Describe simple objects in $\mathscr{C}$

Some observations about exactness properties

Each "standard" functor induces one between the hearts.

Example 10.8. $j_!\colon \mathscr{D}_U \to \mathscr{D}$. This induces a functor $\mathscr{C}_U \to \mathscr{C}$ by including into $\mathscr{D}_U$, applying $j_!$, and then applying ${}^pH^0$, i.e.

$$\begin{tikzcd} \mathscr{C}_U \arrow[r, "{}^pj_!"] \arrow[d, hook] & \mathscr{C} \\ \mathscr{D}_U \arrow[r, "j_!"] & \mathscr{D} \arrow[u, "{}^pH^0"'] \end{tikzcd}$$

1. j^*, i_* are t-exact because of the formulas.

Bhargav will just do this next time.

11 October 13, 2015

The third problem set has been posted; it's about t-structures, and in particular has a problem classifying t-structures on the derived category of a DVR.

Set-up 11.1. $\mathscr{D}_F \xrightarrow{i_*} \mathscr{D} \xrightarrow{j_*} \mathscr{D}_U$. Fix t-structures $(\mathscr{D}_U^{\leq 0}, \mathscr{D}_U^{\geq 0})$ on $\mathscr{D}_U$, and $(\mathscr{D}_F^{\leq 0}, \mathscr{D}_F^{\geq 0})$ on $\mathscr{D}_F$. This induces a t-structure on $\mathscr{D}$, defined by

$$\mathscr{D}^{\leq 0} = \{K \in \mathscr{D} \mid i^*K \in \mathscr{D}_F^{\leq 0},\ j^*K \in \mathscr{D}_U^{\leq 0}\}$$
$$\mathscr{D}^{\geq 0} = \{K \in \mathscr{D} \mid i^!K \in \mathscr{D}_F^{\geq 0},\ j^*K \in \mathscr{D}_U^{\geq 0}\}$$

Associated hearts: $\mathscr{C} \subseteq \mathscr{D}$, $\mathscr{C}_U \subseteq \mathscr{D}_U$, $\mathscr{C}_F \subseteq \mathscr{D}_F$, with associated perverse functors ${}^pj_!$, ${}^pj_*$, etc.

Goals 11.2.
1. Construct $j_{!*}\colon \mathscr{C}_U \to \mathscr{C}$ "intermediate extension".
2. Classify simple objects.
3. Compute $j_{!*}$ in the example from last time.

$\mathscr{D}_U$ and $\mathscr{D}_F$ just start life off as triangulated categories, but they have several exactness properties.

11.1 Exactness Properties

1. j^*, i_* are t-exact:

   Proof for i_*. $i_*(\mathscr{D}_F^{\leq 0}) \subseteq \mathscr{D}^{\leq 0}$ (right t-exact): use $i^*i_* = \mathrm{id}$ to check $i^*(i_*\mathscr{D}_F^{\leq 0}) \subseteq \mathscr{D}_F^{\leq 0}$, and use $j^*i_* = 0$ to check $j^*(i_*\mathscr{D}_F^{\leq 0}) \subseteq \mathscr{D}_U^{\leq 0}$.
   $i_*(\mathscr{D}_F^{\leq 0}) \subseteq \mathscr{D}^{\geq 0}$: same argument using $i^!i_* = \mathrm{id}$, $j^*i_* = 0$. □

2. $j_!, i^*$ are right t-exact:

   Proof for $j_!$. $j_!(\mathscr{D}_U^{\leq 0}) \subseteq \mathscr{D}^{\leq 0}$. Use $i^*j_! = 0$, $j^*j_! = \mathrm{id}$. □

3. $j_*, i^!$ are left t-exact: same argument.
4. $({}^pj_!, {}^pj^*, {}^pj_*)$ form an adjoint sequence of functors between $\mathscr{C}_U$ and $\mathscr{C}$:

Proof. Say $X \in \mathscr{C}_U$ and $Y \in \mathscr{C}$. We are interested in

$$\mathrm{Hom}_{\mathscr{C}}({}^p j_! X, Y) = \mathrm{Hom}_{\mathscr{C}}({}^p H^0(j_! X), Y) \overset{\text{full}}{=} \mathrm{Hom}_{\mathscr{D}}({}^p H^0(j_! X), Y) = \mathrm{Hom}_{\mathscr{D}}(\tau^{\leq 0}(j_! X), Y)$$

since $Y \in \mathscr{D}^{\geq 0}$. Note that ${}^p H^0(j_! X) \neq \tau^{\leq 0}(j_! X)$; the only statement we are making here is that the terms in $\tau^{\leq 0}(j_! X)$ living to the left of 0 do not contribute any maps to Y, which lives to the right of 0. But then

$$\mathrm{Hom}_{\mathscr{D}}(\tau^{\leq 0}(j_! X), Y) = \mathrm{Hom}_{\mathscr{D}}(j_! X, Y)$$

since $j_!$ is right exact, and $X \in \mathscr{D}_U^{\leq 0}$. Now using adjointness of $j_!$ and j^*,

$$\mathrm{Hom}_{\mathscr{D}}(j_! X, Y) = \mathrm{Hom}_{\mathscr{D}_U}(X, j^* Y).$$

Now since j^* is exact, and $Y \in \mathscr{C}$, we get $j^* Y = {}^p H^0(j^* Y)$, and so

$$\mathrm{Hom}_{\mathscr{D}_U}(X, j^* Y) = \mathrm{Hom}_{\mathscr{D}_U}(X, {}^p H^0(j^* Y)) \overset{\text{full}}{=} \mathrm{Hom}_{\mathscr{C}_U}(X, {}^p j^* Y).$$

We note here that the only equality of objects are $\tau^{\leq 0}(j_! X) = j_! X$ and $j^* Y = {}^p H^0(j^* Y)$; every other "equality" is just an orthogonality statement, and is not actually an equality. $\square$

5. $({}^p i^*, {}^p i_*, {}^p i^!)$ form an adjoint sequence between $\mathscr{C}_F$ and $\mathscr{C}$.
6. ${}^p j^* \circ {}^p i_* = 0$ ($\overset{\text{adj}}{\Longrightarrow}$ ${}^p i^* \circ {}^p j_! = 0$, ${}^p i^! \circ {}^p j_* = 0$).

   Proof. j^*, i_* are exact, and so ${}^p j^* = j^*$ on $\mathscr{C}$, ${}^p i_* = i_*$ on $\mathscr{C}_F$. Use $j^* i_* = 0$. $\square$

7. For $A \in \mathscr{C}$, there is an exact sequence:

$$0 \longrightarrow {}^p i_* H^{-1}(i^* A) \longrightarrow {}^p j_! {}^p j^* A \longrightarrow A \longrightarrow {}^p i_* {}^p i^* A \longrightarrow 0$$

   [We explained the non-injectivity at A last time with the nodal cubic.]

   Proof. We have an exact triangle:

$$j_! j^* A \longrightarrow A \longrightarrow i_* i^* A$$

   Now taking the long exact sequence for $H^0(-)$:

$$\begin{tikzcd}
0 \arrow[r] & H^{-1}(i_* i^* A) \arrow[r] \arrow[d, equal, "i_* \text{ exact}"] & H^0(j_! j^* A) \arrow[r] \arrow[d, equal, "\text{def, } j^* \text{ exact}"] & A \arrow[r] & H^0(i_* i^* A) \arrow[r] \arrow[d, equal, "i_* \text{ exact}"] & H^1(j_! j^* A) \arrow[d, equal, "j^* A \in \mathscr{C}_U, \; j_! \text{ right exact}"] \\
& i_* H^{-1}(i^* A) \arrow[d, equal] & {}^p j_!(j^* A) \arrow[d, equal, "j^* \text{ exact}"] & & i_* H^0(i^* A) \arrow[d, equal, "\text{def, } i_* \text{ exact}"] & 0 \\
& {}^p i_* H^{-1}(i^* A) & {}^p j_! {}^p j^* A & & {}^p i_* {}^p i^* A &
\end{tikzcd}$$

   where the injectivity on the first line is by the fact that $H^{-1}(A) = 0$, since $A \in \mathscr{C}$. $\square$

8. Dually, for $A \in \mathscr{C}$, there is an exact sequence

$$0 \longrightarrow {}^p i_* {}^p i^! A \longrightarrow A \longrightarrow {}^p j_* {}^p j^* A \longrightarrow {}^p i_* H^1(i^! A) \longrightarrow 0$$

9. ${}^p i_*$, ${}^p j_*$, ${}^p j_!$ are fully faithful:

   Proof for ${}^p j_!$. Suffices to show ${}^p j^* {}^p j_! = \mathrm{id}$. Take $B \in \mathscr{C}_U$; them

$${}^p j^*({}^p j_! B) = j^*({}^p j_! B) \overset{\text{def}}{=} j^*(H^0(j_! B)) \overset{j^* \text{ exact}}{=} H^0(j^* j_! B) = H^0(B) = B. \qquad \square$$

10. ${}^p i_* \colon \mathscr{C}_F \to \mathscr{C}$ induces an equivalence $\mathscr{C}_F \xrightarrow{\sim} \overline{\mathscr{C}_F} := \{A \in C \mid {}^p j^* A = 0\}$.

Proof. $B \in \mathscr{C}_F$ implies ${}^p j^* {}^p i_* B = 0$ since ${}^p j^* {}^p i_* = 0$, and so ${}^p i_*(\mathscr{C}_F) \subseteq \overline{\mathscr{C}_F}$. Conversely, if $A \in \overline{C_F}$, then using 8, we have

$$0 \longrightarrow {}^p i_* {}^p i^! A \longrightarrow A \longrightarrow {}^p j_* {}^p j^* A \longrightarrow {}^p i_* H^1(i^! A) \longrightarrow 0$$

and so ${}^p i_*({}^p i^! A) \xrightarrow{\sim} A$, which implies $A \in \operatorname{im}({}^p i_*)$, i.e., the functor is essentially surjective. Using full faithfulness for 9, we get $\mathscr{C}_F \underset{{}^p i_*}{\xrightarrow{\sim}} \overline{\mathscr{C}_F}$. ☐

[We can probably prove this statement from the corresponding derived statement, since all the functors involved are exact.]
Note that this shows that $\overline{\mathscr{C}_F}$ is a thick abelian subcategory.

11. ${}^p j^*$ induces an equivalence $\mathscr{C}/\overline{\mathscr{C}_F} \xrightarrow{\sim} \mathscr{C}_U$.

Proof. We have the quotient functor $Q\colon \mathscr{C} \to \mathscr{C}/\overline{\mathscr{C}_F}$. Since ${}^p j^*(\overline{\mathscr{C}_F}) = 0$, we have a factorization

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "Q"] \arrow[dr, "{}^p j^*"'] & & \mathscr{C}/\overline{\mathscr{C}_F} \arrow[dl, "T"] \\ & \mathscr{C}_U & \end{tikzcd}$$

Claim 11.3. *T is faithful.*

Proof. Say $f\colon X \to Y$ in $\mathscr{C}$ such that ${}^p j^*(f) = 0$. We get a factorization

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[rr, bend left, "f"] & \operatorname{im}(f) \arrow[r, hook] & Y \end{tikzcd}$$

Since ${}^p j^*$ is exact, we get that

$$\begin{tikzcd} {}^p j^*(X) \arrow[r, two heads] \arrow[rr, bend right, "0"'] & {}^p j^*(\operatorname{im}(f)) \arrow[r, hook] & {}^p j^*(Y) \end{tikzcd}$$

Thus, ${}^p j^*(\operatorname{im}(f)) = 0$ in $\mathscr{C}_U$, and this is in fact equal to $j^*(\operatorname{im}(f))$ by exactness of j^*. Thus, $\operatorname{im}(f) \in \overline{\mathscr{C}_F}$, by definition. This implies that $Q(\operatorname{im}(f)) = 0$, and so $Q(f) = 0$. ☐

Claim 11.4. *T is essentially surjective.*

Proof. Obvious, since ${}^p j^*$ is essentially surjective (e.g., ${}^p j^* {}^p j_! = \mathrm{id}$). ☐

Claim 11.5. *T is fully faithful.*

Proof. For any $X \in \mathscr{C}$, we have the exact sequence

$$0 \longrightarrow {}^p i_* H^{-1}(i^* X) \longrightarrow {}^p j_! {}^p j^* X \longrightarrow X \longrightarrow {}^p i_* {}^p i^* X \longrightarrow 0$$

and so every object in $\mathscr{C}/\mathscr{C}_F$ is of the form $Q(j_! Y)$ for some $Y \in \mathscr{C}$ (since Q is exact by the fact that quotienting by a thick abelian subcategory is always exact, and Q kills ${}^p i_* \mathscr{C}_F$ giving an isomorphism in the middle).
Now pick $X, Y \in \mathscr{C}_U$; then, T induces

$$\begin{tikzcd} \operatorname{Hom}_{\mathscr{C}/\overline{\mathscr{C}_F}}(Q({}^p j_! X), Q({}^p j_! Y)) \arrow[d] & \\ \operatorname{Hom}_{\mathscr{C}_U}(TQ({}^p j_! X), TQ({}^p j_! Y)) \arrow[d, equal] & \\ \operatorname{Hom}_{\mathscr{C}_U}({}^p j^* {}^p j_! X, {}^p j^* {}^p j_! Y) \arrow[r, equal] & \operatorname{Hom}_{\mathscr{C}_U}(X, Y) \end{tikzcd}$$

Now use $f \mapsto Q({}^p j_! f)$ to get surjectivity of the arrow above. ☐

$\square$

The upshot is that this means even in this completely formal setting, you get analogues to the statements we know from topology. These statements will then apply automatically to perverse sheaves.

11.2 Extensions

Definition 11.6. An *extension* of $Y \in \mathscr{D}_U$ is some $X \in \mathscr{D}$ such that $j^*X = Y$.

Remark 11.7. These always exist, by letting $X = j_!Y$ or $X = j_*Y$.

Proposition 11.8. *For any $Y \in \mathscr{D}_U$, and $p \in \mathbf{Z}$, there exists a unique (up to unique isomorphism) extension X satisfying the following two properties: $i^*X \in \mathscr{D}_F^{\leq p-1}$, and $i^!X \in \mathscr{D}_F^{\geq p+1}$.*

The most relevant case for us is when $p = 0$. The hard part is showing existence; uniqueness is easier since these two properties are pretty strong.

Proof. Glue the degenerate t-structure $(\mathscr{D}_U, 0)$ on $\mathscr{D}_U$ to $(\mathscr{D}_F^{\leq 0}, \mathscr{D}_F^{\geq 0})$ on $\mathscr{D}_F$ to get a new t-structure on $\mathscr{D}$ (note that glueing does not require any compatibility of t-structures on $\mathscr{D}_U$ and $\mathscr{D}_F$!), giving truncation functors ${}^F\tau^{\leq i}\colon \mathscr{D} \to \mathscr{D}$ (intuitively, all the truncation does is change things on the closed part F).

Lemma 11.9. *i^* is exact for this t-structure.*

Assuming the Lemma for now, fix any extension X of Y. Then, we have an exact triangle $i_*i^!X \to X \to j_*j^*X = j_*Y$, where the last equality is by definition of what it means to be an extension. Applying i^* and rotating once, we get the exact triangle

$$i^*X \longrightarrow i_*^*Y \longrightarrow i^!X[1] \tag{1}$$

(1) implies that the following are equivalent:

1. $i^*X \in \mathscr{D}_F^{\leq p-1}$, $i^!X \in \mathscr{D}_F^{\geq p+1}$;
2. $i^*X \xrightarrow{\sim} \tau^{\leq p-1}(i^*j_*Y)$ via (1);
3. $i^!X[1] \cong \tau^{\geq 0}(i^*j_*Y)$ via (1).

So, set $X = {}^F\tau^{\leq p-1}(j_*Y)$. Note j^* is exact, and so $j^*X = {}^W\tau^{\leq p-1}(j^*j_*Y) = Y$, noting that ${}^W\tau^{\leq p-1} = \mathrm{id}$ on $\mathscr{D}_U$ since $\mathscr{D}_U^{\leq 0} = \mathscr{D}_U$. Now, applying i^* to this formula, we get $i^*X = \tau^{\leq p-1}(i^*j_*Y)$ by the Lemma, since i^* is exact for the weird t-structure. But this is the second condition listed above, and so this X resolves the existence problem.

Uniqueness is left as an exercise. $\square$

Next time we will discuss the Lemma and why the proposition gives intermediate extensions.

12 October 15, 2015

Last time, we proved the following

Proposition 12.1. *For any $Y \in \mathscr{D}_U$, and $p \in \mathbf{Z}$, there exists a unique extension $X \in \mathscr{D}$ satisfying the following two properties: $i^*X \in \mathscr{D}_F^{\leq p-1}$, and $i^!X \in \mathscr{D}_F^{\geq p+1}$.*

For the proof, we glued $(\mathscr{D}_U, 0)$ to $(\mathscr{D}_F^{\leq 0}, \mathscr{D}_F^{\geq 0})$ to obtain a new t-structure $({}^F\mathscr{D}^{\leq 0}, {}^F\mathscr{D}^{\geq 0})$ on $\mathscr{D}$. We also needed the following

Proof. Suffices to show $i^*(\mathscr{D}^{\geq 0}) \subseteq \mathscr{D}_F^{\geq 0}$. Fix $X \in \mathscr{D}$; we get an exact triangle ${}^F\tau^{<0}X \to X \to {}^F\tau^{\geq 0}X$. Apply $j_!j_*$ and i_*i^* to get:

$$\begin{tikzcd}
j_!j^{*F}\tau^{<0}X \arrow[r, "\sim"] \arrow[d] & j_!j^*X \arrow[r] \arrow[d] & j_!j^{*F}\tau^{\geq 0}X = 0 \text{ as } j^* \text{ exact and } {}^F\mathscr{D}_U^{\geq 0} = 0 \arrow[d] \\
{}^F\tau^{<0}X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & {}^F\tau^{\leq 0}X \arrow[d, "\alpha"', "\sim"] \\
i_*i^{*F}\tau^{<0}X \arrow[r] & i_*i^*X \arrow[r] & i_*i^{*F}\tau^{\geq 0}X
\end{tikzcd}$$

α is an isomorphism, and so

$$i^! {}^F\tau^{\geq 0} X \xrightarrow{\sim} i^! i_* i^* {}^F\tau^{\geq 0} X = i^* {}^F\tau^{\geq 0} X$$

Since $i^!$ is left t-exact and ${}^f\tau^{\geq 0} X \in {}^F\mathscr{D}^{\geq 0}$, this implies $i^* {}^F\tau^{\geq 0} X \in \mathscr{D}_F^{\geq 0}$. $\square$

We note in the 3×3 diagram above, all we really needed was the rightmost column.

12.1 Intermediate extensions

Recall that j_* is left t-exact, and $j_!$ is right t-exact. This implies that for all $B \in \mathscr{D}_U$, we get the following diagram:

$$\begin{tikzcd} j_! B \arrow[r, "\text{can}"] \arrow[d, "\substack{\text{pass to} \\ {}^pH^0}"'] & j_* B \\ {}^p j_! B \arrow[r, "\substack{\text{induced by can as} \\ j_! B \in \mathscr{D}^{\leq 0},\ j_* B \in \mathscr{D}^{\geq 0}}"'] & {}^p j_* B \arrow[u, "\substack{\text{from } {}^pH^0, \\ \text{as } j_* \text{ left } t\text{-exact}}"'] \end{tikzcd}$$

Definition 12.2. $j_{!*}(B) = \operatorname{im}({}^p j_! B \to {}^p j_* B) \in \mathscr{C}$.

Lemma 12.3. *$j_{!*}(B)$ is the unique extension X of B such that $i^* X \in \mathscr{D}_F^{\leq -1}$, $i^! X \in \mathscr{D}_F^{\geq 1}$.*

Note that ${}^p j^* \circ {}^p j_! = \mathrm{id} = {}^p j^* \circ {}^p j_*$, so $j_{!*} B$ extends B.

Proof. Need to show $i^* j_{!*} B \in \mathscr{D}_F^{\leq -1}$ (and dually: $i^!(j_{!*}(B)) \in \mathscr{D}_F^{\geq 1}$). We have a surjection ${}^p j_! B \twoheadrightarrow j_{!*} B$ in $\mathscr{C}$. Applying ${}^p i^*$, we have that the map ${}^p i^*({}^p j_! B) \twoheadrightarrow {}^p i^*(j_{!*} B)$ in $\mathscr{C}_F$, and since the former object ${}^p i^*({}^p j_! B) = 0$ by the fact that ${}^p i^* {}^p j_! = 0$, we have that ${}^p i^*(j_{!*} B) = 0$. Now, we have that ${}^p i^*(j_{!*} B) \overset{\text{def}}{=} {}^p H^0(i^*(j_{!*} B)) = \tau^{\geq 0}(i^* j_{!*} B)$ by right exactness, and the fact that $j_{!*} B \in \mathscr{D}^{\leq 0}$. This implies $\tau^{\geq 0}(i^* j_{!*} B) = 0$, and so $i^* j_{!*} B \in \mathscr{D}_F^{\leq -1}$. $\square$

This is the most useful way to check whether an extension X of B satisfies the definition of an intermediate extension. The following result, however, is a nice way to think about what intermediate extensions really mean intuitively in a geometric way.

Corollary 12.4. *$j_{!*} B$ is the unique extension of B with no non-trivial subobject or quotient in $\overline{\mathscr{C}_F}$ (that is, the image of ${}^p i_* \colon \mathscr{C}_F \to \mathscr{C}$).*

Proof. By adjointness $({}^p i^*, {}^p i_*)$, the natural map $X \twoheadrightarrow {}^p i_* {}^p i^* X$ in $\mathscr{C}$ for any $X \in \mathscr{C}$ is the largest quotient of X in $\overline{\mathscr{C}_F}$ (exercise in adjointness). It therefore suffices to show ${}^p i^*(j_{!*} B) = 0$. This is a consequence of the previous lemma.

Dually, for subobjects, we use the injection ${}^p i_* {}^p i^! X \hookrightarrow X$, and argue using adjointness as before. We then use that ${}^p i_* {}^p i^! B = 0$.

For uniqueness, we fix some X satisfying the Corollary. As X has no non-trivial quotients in $\overline{\mathscr{C}_F}$, we get ${}^p i_* {}^p i^*(X) = 0$ by the previous argument. Then, the long exact sequence (Property 7 from last time) collapses from four to three terms, giving the surjection ${}^p j_! {}^p j^* X \twoheadrightarrow X$, where the former object is ${}^p j_! B$, as X extends B. Dually, we get an injection $X \hookrightarrow {}^p j_* B$. Thus, we get that

$$\begin{tikzcd} {}^p j_! B \arrow[r, two heads] \arrow[rr, bend left, "\text{can}"] & X \arrow[r, hook] & {}^p j_* B \end{tikzcd}$$

which shows $X \cong j_{!*} B$. $\square$

Before we get to examples, we classify all simple objects:

Corollary 12.5. *All simple objects in $\mathscr{C}$ (that is, those that have no nontrivial subobjects or quotients) are of the form:*

1. *$j_{!*}(S_U)$ for simple $S_U \in \mathscr{C}_U$;*
2. *${}^p i_*(S_F)$for simple $S_F \in \mathscr{C}_F$.*

The point is that in this context, there are more simple objects than in the world of constructible sheaves, and they are in fact easy to classify.

Proof. Say $S \in \mathscr{C}$ is simple. If ${}^p i^* S \neq 0$, then $S \twoheadrightarrow {}^p i_* {}^p i^* S$ is a non-zero quotient of S, so $S \cong {}^p i_*({}^p i^* S)$ because of the simplicity of S. Similarly, if ${}^p i^! S \neq 0$, then ${}^p i_*({}^p i^! S) \xrightarrow{\sim} S$. Thus, if we are not in the case 2, then ${}^p i^* S = 0 = {}^p i^! S$. The previous corollary shows $S \cong j_{!*}(j^* S)$.

Now, to see $j^* S$ is simple if S is simple: if $j^* S \twoheadrightarrow Q$ is non-trivial, then

$$\begin{tikzcd} {}^p j_!(j^* S) \arrow[r, two heads, "{}^p j_! \text{ right exact}"] \arrow[d, two heads] & {}^p j_!(Q) \arrow[d, two heads] \\ S = j_{!*} j^* S \arrow[r, two heads] & j_{!*} Q \end{tikzcd}$$

and $j_{!*} Q$ is non-zero as it extends Q. This violates simplicity of S. $\square$

Remark 12.6. If $B \in \mathscr{C}_U$ is simple, then so is $j_{!*} B$:

Proof. If not, there exists a non-trivial quotient $j_{!*} B \overset{\alpha}{\twoheadrightarrow} Q$. As B is simple, ${}^p j^*(\alpha)$ is either 0 or an isomorphism. If ${}^p j^*(\alpha) = 0$, then ${}^p j^*(Q) = 0$ by exactness of ${}^p j^*$, and so the standard triangle (Property 7)

$${}^p j_! {}^p j^* Q \longrightarrow Q \xrightarrow{\sim} {}^p i_* {}^p i^* Q \longrightarrow 0$$

has an isomorphism in the middle. This is not possible since $j_{!*} B$ has no non-zero quotients.

"Is everyone here happy? I'm here to make everyone happy."

If ${}^p j^*(\alpha)$ is an isomorphism, then $\ker(\alpha)$ satisfies ${}^p j^*(\ker(\alpha)) = 0$, and so $\ker(\alpha) \in \overline{\mathscr{C}_F}$, and $\ker(\alpha) = {}^p i_*(\text{something})$. This is not possible, since $j_{!*} B$ has no such subobjects. $\square$

This is the end of this abstract nonsense. For the rest of class, we will talk about a special case of Deligne's formula.

12.2 Examples of $j_{!*}$ for isolated singularities

Example 12.7. Let X be a complex variety of dimension d with an isolated singularity at $x \in X$. Let $U = X \setminus \{x\}$, and $Z = \{x\}$. We want to do a non-trivial glueing to get a perverse t-structure on X. Glue $(\mathsf{D}^{\leq -d}(U), \mathsf{D}^{\geq -d}(U))$ to $(\mathsf{D}^{\leq 0}(Z), \mathsf{D}^{\geq 0}(Z))$ to get a "perverse" t-structure on $\mathsf{D}(X)$. Note: $\underline{\mathbf{Z}}[d] \in \mathsf{D}(U)^{\heartsuit}$.

Claim 12.8 (Deligne's formula)**.** $j_{!*}(\underline{\mathbf{Z}}[d]) = \tau^{\leq -1}(\mathbf{R}j_* \underline{\mathbf{Z}}[d])$.

The way Deligne's formula works in general is to take the largest pushforward $\mathbf{R}j_*$, then apply a certain truncation functor according to the numerics specified. There is a function that we would define on the stratification in general to say what the truncation should be.

Proof. Set $K = \tau^{\leq -1}(\mathbf{R}j_* \underline{\mathbf{Z}}[d])$. Suffices to show that (a) $i^* K \in \mathsf{D}^{\leq -1}(Z)$, and (b) $i^! K \in \mathsf{D}^{\geq 1}(Z)$.

We have an exact triangle

$$K \longrightarrow \mathbf{R}j_* \underline{\mathbf{Z}}[d] \longrightarrow \tau^{\geq 0}(\mathbf{R}j_* \underline{\mathbf{Z}}[d])$$

Apply i^*:

$$i^* K \longrightarrow i^* \mathbf{R}j_* \underline{\mathbf{Z}}[d] \longrightarrow \tau^{\geq 0}(i^* \mathbf{R}j_* \underline{\mathbf{Z}}[d])$$

where for the last term, we use that i^* is exact for the standard t-structure. This shows that $i^* K = \tau^{\leq -1}(\mathbf{R}j_* \underline{\mathbf{Z}}[d]) \in \mathsf{D}^{\leq -1}(Z)$, giving (a).

For (b), we apply $i^!$ to the same triangle:

$$i^! K \longrightarrow 0 \longrightarrow i^! \tau^{\geq 0}(\mathbf{R}j_* \underline{\mathbf{Z}}[d])$$

since $i^! \mathbf{R}j_* = 0$. This shows $i^! K = i^!(\tau^{\geq 0} \mathbf{R}j_* \underline{\mathbf{Z}}[d])[-1]$. Since $i^!$ is left exact, the right hand side of this equation is then in $\mathsf{D}^{\geq 1}(Z)$, as is the left hand side. $\square$

Example 12.9. Say $X = \mathbf{P}^1$, and $Z = \{x\}$. The previous example then shows that since $\dim X = d = 1$, we should have $j_{!*}(\underline{\mathbf{Z}}[1]) = \tau^{\leq -1}(\mathbf{R}j_*\underline{\mathbf{Z}}[1])$. First of all, we claim $\tau^{\leq -1}(\mathbf{R}j_*\underline{\mathbf{Z}}[1]) = j_*\underline{\mathbf{Z}}[1]$. Note that $j_*\underline{\mathbf{Z}}[1] = \underline{\mathbf{Z}}[1]$. Thus, the "constant sheaf" is the same as the "intersection cohomology sheaf" on a smooth curve.

Example 12.10. $X = \mathbf{P}^1 \vee \mathbf{P}^1$, the union of 2 lines in $\mathbf{P}^2$. Topologically, this is $S^2 \vee S^2$. We saw on the first day that we had Poincaré duality when we computed intersection cohomology.

Let $Z = \{x\} = S^2 \cap S^2$. Example 1 says that $j_{!*}(\underline{\mathbf{Z}}[1]) = \tau^{\leq -1}(\mathbf{R}j_*\underline{\mathbf{Z}}[1]) = (j_*\underline{\mathbf{Z}})[1]$. Note

1. Unlike the previous example, $j_*\underline{\mathbf{Z}}[1] \neq \underline{\mathbf{Z}}[1]$: the stalk of the right hand side at x is equal to $\mathbf{Z}[1]$, but for the left hand side, the stalk is $\operatorname{colim}_{V \ni x}(H^0(V \setminus \{x\}, \mathbf{Z}))[1] = (\mathbf{Z} \oplus \mathbf{Z})[1]$, since $V \setminus \{x\}$ has two connected components.
2. $\mathbf{R}\Gamma(X, j_{!*}\underline{\mathbf{Z}}[1])$ satisfies Poincaré duality:

$$\begin{tikzcd}
j_{!*}\mathbf{Z}[i] \arrow[d, equal] & & \\
j_*\underline{\mathbf{Z}}[1] \arrow[r] & \mathbf{R}j_*\underline{\mathbf{Z}}[1] \arrow[r] & (\mathbf{R}^1 j_*\mathbf{Z}[1]) \arrow[d, equal] \\
& & i_*(\mathbf{Z} \oplus \mathbf{Z})
\end{tikzcd}$$

Applying $\mathbf{R}\Gamma$, we obtain

$$\begin{tikzcd}
\mathbf{R}\Gamma(X, j_{!*}\mathbf{Z}[i]) \arrow[r] & \mathbf{R}\Gamma(X, \mathbf{R}j_*\underline{\mathbf{Z}}[1]) \arrow[r] \arrow[d, equal] & \mathbf{R}\Gamma(X, i_*(\mathbf{Z} \oplus \mathbf{Z})) \arrow[d, equal] \\
& \mathbf{R}\Gamma(X \setminus \{x\}, \mathbf{Z}[1]) \arrow[d, equal] & \mathbf{Z}^{\oplus 2} \\
& \mathbf{R}\Gamma(\mathbf{A}^1 \amalg \mathbf{A}^1, \mathbf{Z}[1]) \arrow[d, equal] & \\
& (\mathbf{Z}^{\oplus 2})[1] &
\end{tikzcd}$$

Thus, we have that $\mathbf{R}\Gamma(X, j_{!*}\mathbf{Z}[i]) = \mathbf{Z}^{\oplus 2}[1] \oplus \mathbf{Z}^{\oplus 2}[-1]$.

This is all I want to talk about two-step stratifications. We will work with multiple step stratifications from now on, and derive Deligne's formula in that case.

13 October 22, 2015

We are now done with most of the glueing abstract nonsense. We will start talking about actual algebraic geometry today.

13.1 Étale cohomology

The following few lectures will be a review of a semester's worth of étale cohomology. Bhargav will omit most proofs, but will try to give examples.

Let X be a scheme (always qcqs). Then, $X_{\text{ét}} = \{\text{all étale maps } j\colon U \to X\}$ is a *site* with *covers* given by families $\{f_i\colon U_i \to U\}$ such that $\coprod U_i \to U$ is surjective.

To a site, we can associate the categories of presheaves $\mathsf{PShv}(X_{\text{ét}}) = \mathsf{Fun}(X_{\text{ét}}^{\mathsf{op}}, \mathsf{Sets})$ and sheaves

$$\mathsf{Shv}(X_{\text{ét}}) = \left\{ F\colon X_{\text{ét}}^{\mathsf{op}} \to \mathsf{Sets} \;\middle|\; \begin{array}{c} \text{for all coverings } \{U_i \xrightarrow{f_i} U\} \text{ in } X_{\text{ét}}, \text{ the following sequence is exact:} \\ F(U) \longrightarrow \prod_i F(U)_i \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} \prod_{i,j} F(U_i \times_U U_j) \end{array} \right\}$$

Examples 13.1.

1. Any Zariski cover is an étale cover.

2. $E =$ elliptic curve$/k$, $n \in \mathbf{Z}$ invertible on k, $k = \overline{k}$. Then, $[n]\colon E \xrightarrow{[n]} E$ is finite étale and this is an étale cover. Note that $[n]$ is a finite *Galois* cover with Galois group $E[n](k)$.

   Exercise 13.2. The fibre product $E \times_E E$ given by $[n]\colon E \to E$ for both projections is isomorphic to $\coprod_{E[n](k)} E$, where this E is the source E.

Properties 13.3.

1. Criterion for being a sheaf:

   Proposition 13.4. *$F \in \mathsf{PShv}(X_{\acute{e}t})$. F is a sheaf if and only if it satsfies*
   (a) F is a Zariski sheaf;
   (b) F is a Nisnevich sheaf: for a cartesian square

$$\begin{array}{ccc} U' & \hookrightarrow & X' \\ \downarrow & & \downarrow f \\ U & \overset{j}{\hookrightarrow} & X \end{array}$$

   with f étale, j an open immersion, and $f^{-1}(X \setminus U) \xrightarrow{\sim} X \setminus U$. Then, we require that

$$F(X) \longrightarrow F(X') \times F(U) \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} F(U')$$

   is exact.
   (c) F has finite Galois descent: For $f\colon X' \to X$ being a finite étale Galois cover with group G, $F(X) \xrightarrow{\sim} F(X')^G$.
   Recall that $f\colon X \to Y$ is a finite Galois cover with group G or a G-torsor if
   i. f is finite, étale, surjective;
   ii. there exists a G-action on X commuting with f such that

$$\begin{array}{ccc} G \times X & \xrightarrow{(\mathrm{act},\mathrm{pr}_2)} & X \times_Y X \\ (g,x) & \longmapsto & (gx, x) \end{array}$$

   is an isomorphism.

   Examples 13.5.
   i. $E \xrightarrow{[n]} E$ as before.
   ii. $\mathbf{G}_m \xrightarrow{x \mapsto x^n} \mathbf{G}_m$ over $k = \overline{k}$, $n \in k^*$. Recall that $\mathbf{G}_m = \mathrm{Spec}(k[T, T^{-1}])$.

2. Canonical sheaves:

   Observation 13.6. All étale morphisms are flat, and so the étale topology is "subcanonical" (i.e., for all maps $f\colon Y \to X$, the functor $h_Y \in \mathsf{PShv}(X_{\acute{e}t})$, $h_Y(U) = \mathrm{Hom}_X(U, Y)$ is a sheaf).

   Examples 13.7.
   (a) $Y = X$ gives $h_Y = \{*\}$. Final object of $\mathsf{Shv}(X_{\acute{e}t})$.
   (b) $Y = \emptyset$. h_Y is the initial object ($\neq \emptyset$, since its value on the empty set is $\{*\}$).
   (c) $Y = \mathbf{A}^1 \times X$: $h_Y(U) = \mathrm{Hom}_X(U, \mathbf{A}^1 \times X) = \mathrm{Hom}(U, \mathbf{A}^1) = \Gamma(U, \mathcal{O}_U)$. Call this $\mathbf{G}_a$.
   (d) $Y = \mathbf{G}_m \times X$: $h_Y(U) = \Gamma(U, \mathcal{O}_U)^*$.
   (e) $Y = \mu_n \times X$, $\mu_n = \mathrm{Spec}(\mathbf{Z}[T, T^{-1}]/(T^n - 1))$. Then, $h_Y(U) = \Gamma(U, \mathcal{O}_U^*)[n]$. Note that if n is invertible on X, and $\mathcal{O}(U) \supset \mu_n$, then $\mu_n \cong \underline{\mathbf{Z}/n}$, where this isomorphism involves choosing a generator of μ_n.

3. Local rings and points:

   Observation 13.8. Let k be a separably closed field. Then, $\mathrm{Spec}(k)_{\acute{e}t} = \{\text{finite sets}\}$ via $X \mapsto X(k)$. $\mathsf{Shv}(X_{\acute{e}t}) = \{\mathsf{Sets}\}$, and the functor $F \mapsto F(k)$ is exact and commutes with colimits. More generally, if

$\bar{y}\colon \mathrm{Spec}(k) \to Y$ is a map, where k is separably closed, we get

$$I_{\bar{y}} = \left\{ \begin{tikzcd} \mathrm{Spec}(k) \arrow[rr] \arrow[dr, "\bar{y}"'] & & U \arrow[dl, "\text{étale}"] \\ & X & \end{tikzcd} \right\}$$

Check that $F \mapsto \mathrm{colim}_{U \in I_{\bar{y}}} F(U) =: F_{\bar{y}}$ is exact and commutes with all colimits, that is, you get a "point" of $\mathsf{Shv}(X_{\text{ét}})$. Call such $\bar{y}$ *geometric points* of Y.

Fact 13.9. All points arise in this fashion, and $F \to G \to H$ is exact if and only if $F_{\bar{y}} \to G_{\bar{y}} \to H_{\bar{y}}$ is exact.

Definition 13.10. If $\bar{y}\colon \mathrm{Spec}(k) \to Y$ is a geometric point, then set $\mathcal{O}^{\mathrm{sh}}_{Y,\bar{y}} := (\mathcal{O}_Y)_{\bar{y}} := \mathrm{colim}_{U \in I_{\bar{y}}} \mathcal{O}(U)$. This is a strictly henselian local ring.

Examples 13.11.

(a) Kummer sequence: if X is a scheme, and n is invertible on $\mathcal{O}(X)^*$, then we

Claim. $0 \to \mu_n \hookrightarrow \mathbf{G}_m \xrightarrow{(-)^n} \mathbf{G}_m \to 0$ *is exact.*

Proof. We want:

i. $0 \to \mu_n(R) \to \mathbf{G}_m(R) \to \mathbf{G}_m(R)$ exact for all rings R;

ii. for all $f \in \mathbf{G}_m(R)$, there exists an étale extension (finite flat) $R \to S$ such that f admits an nth root in S.

i is obvious. ii: take $R \to S$ to be the extension $R \to R[t]/(t^n - f)$. □

(b) Artin–Schreier sequence: Let $X/\mathbf{F}_p$. Then the following sequence is exact:

$$0 \longrightarrow \underline{\mathbf{F}_p} \xrightarrow{\text{can}} \mathbf{G}_a \xrightarrow{x \mapsto x^p - x} \mathbf{G}_a \longrightarrow 0$$

Proof. Exercise. □

Exercise 13.12. Let X, Y be varieties over k, and $\bar{x}$, $\bar{y}$ geometric points supported at closed points. Then,

$$\left(\mathcal{O}^{\mathrm{sh}}_{X,\bar{x}} \cong \mathcal{O}^{\mathrm{sh}}_{Y,\bar{y}}\right) \iff \text{exists a diagram} \quad \begin{tikzcd} & U \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd} \quad \text{where } f, g \text{ are étale neighborhoods of } \bar{x} \text{ and } \bar{y}$$

This means that if X is smooth, then $\mathcal{O}^{\mathrm{sh}}_{X,x} \cong \mathcal{O}^{\mathrm{sh}}_{\mathbf{A}^n,\{0\}}$ (since X is étale over $\mathbf{A}^n$ by smoothness (after shrinking)).

Fact 13.13. If $X \xrightarrow{f} Y$ is an étale map of varieties over $\mathbf{C}$, then $f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is a local isomorphism (and conversely).

4. Cohomology: Consider the functor $\mathsf{Ab}(X_{\text{ét}}) \to \mathsf{Ab}$ sending $F \mapsto H^0(X_{\text{ét}}, F) = \Gamma(X_{\text{ét}}, F)$. You then get derived functors $H^i(X_{\text{ét}}, -)\colon \mathsf{Ab}(X_{\text{ét}}) \to \mathsf{Ab}$.

Examples 13.14. Consider $A = \underline{\mathbf{Z}/n}$, the constant sheaf.

(a) $H^0(X_{\text{ét}}, \underline{\mathbf{Z}/n}) = (\mathbf{Z}/n)^{\pi_0(X)}$.

(b) $H^1(X_{\text{ét}}, \underline{\mathbf{Z}/n}) = ?$ More generally, for any finite abelian group A, formal arguments show that

$$H^1(X_{\text{ét}}, \underline{A}) = \{\text{all } \underline{A}\text{-torsors on } X_{\text{ét}}\}\,.$$

The latter set, by definition, is equal to the set

$$\left\{ \begin{array}{c} \text{all sheaves } Y \in \mathsf{Shv}(X_{\text{ét}}), \\ \text{with actions of } A \text{ on } Y \text{ over } X \end{array} \;\middle|\; \begin{array}{l} \text{i. } (\mathrm{act}, \mathrm{pr})\colon \underline{A} \times Y \to Y \times_X Y \text{ given by} \\ \quad (a, y) \mapsto (ay, y) \text{ is an isomorphism;} \\ \text{ii. there exists a cover } X' \to X \text{ in } X_{\text{ét}} \\ \quad \text{such that } Y(X') \neq \emptyset. \end{array} \right\}$$

Now since finite étale covers descend along étale covers, we have that this set is isomorphic to the set

$$\{\text{finite Galois cover } Y \to X \text{ with group } A\} = \text{“principal homogeneous spaces for } A\text{”}$$

where this terminology is that used in Milne's book. Computation:

i. $X = \mathbf{G}_m$ over $k = \overline{k}$, $n \in k^*$. Then, $H^1(X_{\text{ét}}, \underline{\mathbf{Z}/n}) \ni \mathbf{G}_m \xrightarrow{()^n} \mathbf{G}_m$ (after identifying $\underline{\mathbf{Z}/n} \cong \mu_n$), and $H^1(X_{\text{ét}}, \underline{\mathbf{Z}/n}) = (\mathbf{Z}/n) \cdot (\text{this cover})$, and so $H^1(\mathbf{G}_m, \underline{\mathbf{Z}/n}) = \mathbf{Z}/n$.

ii. If E/k is an elliptic curve, where $n \in k^*$ and $k = \overline{k}$, then $H^1(E, \underline{\mathbf{Z}/n})$ is non-canonically isomorphic to $H^1(E, \mu_n)$, which is isomorphic to $\text{Pic}(E)[n]$ via the map $L \mapsto \mathbf{Spec}_E(\mathcal{O}_E \oplus L \oplus L^2 \oplus \cdots \oplus L^{n-1})$, which is an algebra via $L^{\otimes n} \cong \mathcal{O}_E$. Now $\text{Pic}(E)[n] = \text{Pic}^0(E)[n] = E[n] \cong (\mathbf{Z}/n)^{\oplus 2}$, where this last isomorphism is non-canonical. Thus, $H^1(E, \underline{\mathbf{Z}/n}) \cong (\mathbf{Z}/n)^{\oplus 2}$.

iii. If X is any proper variety over $k = \overline{k}$, and $n \in k^*$, then $H^1(X, \mu_n) \cong \text{Pic}(X)[n]$.

Proof. First, use (fpqc) descent to show that $H^1(X_{\text{ét}}, \mathbf{G}_m) = H^1_{\text{Zar}}(X, \mathbf{G}_m) = \text{Pic}(X)$. The long exact sequence from the Kummer sequence gives

$$\begin{tikzcd} 0 \arrow[r] & \frac{H^0(X,\mathbf{G}_m)}{H^0(X,\mathbf{G}_m)^n} \arrow[r, hook] \arrow[d, equal] & H^1(X, \mu_n) \arrow[r] & H^1(X, \mathbf{G}_m)[n] \arrow[r] \arrow[d, equal] & 0 \\ & (k^*)/(k^*)^n = 0 & & \text{Pic}(X)[n] & \end{tikzcd}$$

Thus, $H^1(X, \mu_n) \cong \text{Pic}(X)[n]$, and if C is a smooth proper curve of genus g, then $H^1(C, \mu_n) \cong (\mathbf{Z}/n)^{2g}$. $\square$

14 October 27, 2015

Last time, we started talking about étale cohomology. We in particular computed $H^1_{\text{ét}}$ of a curve.

Properties 14.1. Continued from last time.

5. Local systems: let A be a finite ring, e.g. $\mathbf{Z}/\ell^n$.

Definition 14.2. $\mathsf{Loc}_A(X) = \{\underline{A}\text{-local systems}\} = \{\text{vector bundles on } (X_{\text{ét}}, \underline{A})\}$, that is

$$\left\{ \mathscr{E} \in \mathsf{Mod}(X_{\text{ét}}, \underline{A}) \;\middle|\; \exists X' \to X \text{ étale cover, such that } \mathscr{E}|_{X'_{\text{ét}}} \cong \underline{A}^{\oplus n} \right\}$$

where $n \in H^0(X'_{\text{ét}}, \mathbf{Z})$ (note in general we would need $\mathscr{E}$ to be locally projective, not locally free—these coincide since in our case A is finite).

Example 14.3. Let $X = \text{Spec}(k)$, k a field, and

$$X_{\text{ét}} \cong \{\text{finite étale } k\text{-algebras}\}^{\mathsf{op}} = \{\text{products of finite separable field extensions}\}.$$

If we choose $k \hookrightarrow k^{\text{sep}}$ be an inclusion into the separable closure of k, and then $G = \text{Gal}(k^{\text{sep}}/k)$. Then,

$$X_{\text{ét}} \cong \{\text{finite sets with continuous } G\text{-action}\}, \quad \text{given by } U \mapsto U(k^{\text{sep}}) \circlearrowleft G$$

which implies

$$\mathsf{Loc}_A(X_{\text{ét}}) \cong \{\text{finite free } A\text{-modules with continuous } G\text{-action}\}$$

There is a important example of this that will come up over and over again. When $X = \text{Spec}(\mathbf{Q})$, then $G \circlearrowright \mathbf{Z}/n(1) = \mu_n \subseteq \overline{\mathbf{Q}}^* \circlearrowleft G$, where μ_n are the nth roots of unity. Then, $\mathbf{Z}/n(1) \in \mathsf{Loc}_{\mathbf{Z}/n}(X)$. This trivializes over $\mathbf{Q}(\mu_n)$. You can think of $\mathbf{Z}/n(1)$ as a non-trivial line bundle on the site $\text{Spec}(\mathbf{Q})_{\text{ét}}$.

Facts 14.4.

(a) $\{\text{rank } n \text{ local systems in } \mathsf{Loc}_A(X)\} \cong \{\underline{\mathrm{GL}_n(A)}\text{-torsor on } X_{\text{ét}}\}$, where the map is given by $\mathscr{E} \mapsto \mathrm{Frame}(\mathscr{E}) \coloneqq \underline{\mathrm{Isom}}_A(A^{\oplus n}, \mathscr{E})$.

Corollary 14.5. *Any $\mathscr{E} \in \mathsf{Loc}_A(X)$ trivializes over a finite étale cover.*

(b) If $X/\mathbf{C}$ is a connected variety, and $x \in X(\mathbf{C})$, then $\mathsf{Loc}_A(X_{\text{ét}}) \cong \mathsf{Rep}_A(\pi_1^{\mathsf{top}}(X^{\text{an}}, x))$, where $F \mapsto (F^{\text{an}})_x$. (Note that the functor $\leftarrow$ is a bit more involved: it requires using the Riemann existence theorem. This gives a nice purely algebraic way to define local systems; note, however, this does not work for A infinite since the Riemann existence theorem does not hold.)

6. Derived categories: "This is not how it happened historically, but we don't need to respect history." Let A be a finite ring, and $\mathsf{D}(X_{\text{ét}}, A) = \mathsf{D}(\mathsf{Mod}(X_{\text{ét}}, A))$. We have the following 2 functors:
   (a) $\Gamma(X_{\text{ét}}, -)\colon \mathsf{Mod}(X_{\text{ét}}, A) \to \mathsf{Mod}_A$, which is left-exact, and so this gives the derived functor
   $$\mathbf{R}\Gamma(X_{\text{ét}}, -)\colon\ \mathsf{D}(X_{\text{ét}}, A) \longrightarrow \mathsf{D}(A),$$
   so that $H^i(X_{\text{ét}}, F) = H^i(\mathbf{R}\Gamma(X_{\text{ét}}, F))$.
   (b) Given a geometric point $\bar{x}$, we get a stalk functor $\mathsf{Mod}(X_{\text{ét}}, A) \longrightarrow \mathsf{Mod}_A$, which is exact, and so we get a functor
   $$\mathsf{D}(X_{\text{ét}}, A) \longrightarrow \mathsf{D}(A), \quad F \mapsto F_{\bar{x}}$$
   where
   $$F_{\bar{x}} = \operatorname*{hocolim}_{\substack{\bar{x} \longrightarrow U \\ \searrow \quad \swarrow_{\text{étale}} \\ X}} \mathbf{R}\Gamma(U, F)$$

7. Functoriality: étale maps.
   Let X, A as before, and let $j\colon U \to X$ be étale. Then, there is a restriction functor $j^*\colon \mathsf{Mod}(X_{\text{ét}}, A) \to \mathsf{Mod}(U_{\text{ét}}, A)$, which is exact, giving a functor
   $$j^*\colon\ \mathsf{D}(X_{\text{ét}}, A) \longrightarrow \mathsf{D}(U_{\text{ét}}, A)$$
   where if, for example, $\bar{u}$ is a geometric point of U, then $(j^*K)_{\bar{u}} = K_{j(\bar{u})}$.
   General fact: j^* has a right adjoint $j_*\colon \mathsf{D}(U_{\text{ét}}, A) \to \mathsf{D}(X_{\text{ét}}, A)$, and a left adjoint $j_!\colon \mathsf{D}(U_{\text{ét}}, A) \to \mathsf{D}(X_{\text{ét}}, A)$ such that j_* is t-left exact, and $j_!$ is t-exact, where t refers to the standard t-structure (later on, these statements will not be true for the perverse t-structure).

   Examples 14.6.
   (a) If $j\colon U \hookrightarrow X$ is an open immersion, then $j_!$ is extension by 0, i.e.,
   $$(j_!K)_{\bar{x}} = \begin{cases} 0 & \text{if } \bar{x} \notin U \\ K_{\bar{x}} & \text{if } \bar{x} \in U \end{cases}$$
   Thus, $j^*j_! \cong \mathrm{id}$, and so $j_!$ is fully faithful.

   This will *not* be true if j is not an open immersion!

   (b) If $j\colon U \hookrightarrow X$ is an open immersion, then j_* can be complicated (and involves some local cohomology), but $(j_*K)_{\bar{x}} = K_{\bar{x}}$ if $\bar{x} \in U$, and so $j^*j_* \cong \mathrm{id}$, ad so j_* is fully faithful.
   In both of these examples, most of what we'd expect to be true from topology is true, but we need to use the topology-to-étale dictionary to convert these notions to the étale setting, and then we need to think more carefully about how étale things work.
   (c) Let X be a smooth curve over $\mathbf{C}$, $x \in X(\mathbf{C})$, $U = X \setminus \{x\}$, and $A = \underline{\mathbf{Z}/n}$. Question: what is $j_*\underline{A}$?

   Claim 14.7. *$H^0\left(\left(j_*\underline{\mathbf{Z}/n}\right)_x\right) = \mathbf{Z}/n$, and $H^1\left(\left(j_*\underline{\mathbf{Z}/n}\right)_x\right) = \mathbf{Z}/n(-1) = \mu_n^{\otimes -1}$. All others vanish (Tsen).*

   Note this is similar to the cohomology of the circle, except (if we worked over $\mathbf{Q}$ instead) there is a Tate twist involved in H^1, corresponding to a different Galois action.

Proof. We first compute

$$\left(j_*\underline{\mathbf{Z}/n}\right)_x = \operatorname*{hocolim}_{\substack{\bar{x} \longrightarrow V \\ \searrow\ \swarrow_{\text{étale}} \\ X}} \mathbf{R}\Gamma(V, j_*\mathbf{Z}/n) = \operatorname*{hocolim}_{\substack{\bar{x} \longrightarrow V \\ \searrow\ \swarrow_{\text{étale}} \\ X}} \mathbf{R}\Gamma(V \setminus \{x\}, \underline{\mathbf{Z}/n})$$

Note,

$$\lim_{\substack{\bar{x} \longrightarrow V \\ \searrow\ \swarrow_{\text{étale}} \\ X}} V = \operatorname{Spec}(R), \quad R = \mathcal{O}_{X,x}^{\text{sh}} \text{ a strictly henselian DVR.}$$

Inside of this limit, we have $M_x = \operatorname{Spec}(K) = \operatorname{Spec}(\mathcal{O}_{X,x}^{\text{sh}}) \setminus \{x\}$, where $K = \operatorname{Frac}(R)$. A small argument shows

$$\left(j_*\underline{\mathbf{Z}/n}\right)_x = \mathbf{R}\Gamma(M_x, \underline{\mathbf{Z}/n})$$

Now $H^0(M_x, \mathbf{Z}/n) = \mathbf{Z}/n$, since M_x is connected. There interesting one is H^1, and we will use the Kummer sequence

$$0 \longrightarrow \mu_n \longrightarrow \mathbf{G}_m \xrightarrow{(\ )^n} \mathbf{G}_m \longrightarrow 0$$

to get

$$\begin{array}{ccccccc}
0 \longrightarrow & \frac{\mathbf{G}_m(M_x)}{\mathbf{G}_m(M_x)^n} & \hookrightarrow & H^1(M_x, \mu_n) & \longrightarrow & H^1(M_x, \mathbf{G}_m)[n] & \longrightarrow 0 \\
& \| & & & & \Big\|\text{Hilbert Theorem 90} & \\
& K^*/(K^*)^n & & & & \operatorname{Pic}(M_x)[n] = 0 &
\end{array}$$

where $\operatorname{Pic}(M_x) = 0$ since M_x is the spectrum of a field, and so $H^1(M_x, \mu_n) \cong K^*/(K^*)^n$. As R is a DVR, we have an exact sequence

$$0 \longrightarrow R^* \longrightarrow K^* \xrightarrow{\text{val}} \mathbf{Z} \longrightarrow 0$$

We apply $\otimes^{\mathbf{L}} \mathbf{Z}/n$ to this:

$$0 \longrightarrow R^*/(R^*)^n \longrightarrow K^*/(K^*)^n \xrightarrow{\text{val}} \mathbf{Z}/n \longrightarrow 0$$

As R is strictly henselian, every element in R^* has an nth root, so $R^* = (R^*)^n$. We conclude that $H^1(M_x, \mu_n) \cong \mathbf{Z}/n$. By twisting, this is equivalent to saying $H^1(M_x, \mathbf{Z}/n) \cong \mu_n^{\otimes -1}$. If everything is defined over $\mathbf{Q}$, the same result holds for a smooth curve over $\mathbf{Q}$ by base changing to $\overline{\mathbf{Q}}$, and noticing that our calculations would be Galois-equivariant. ☐

Remark 14.8 (Tsen's theorem). More generally, we can show $H^i(X, \mathbf{G}_m) = 0$ for all $i \geq 2$ if X is a smooth affine curve over $k = \bar{k}$.

8. Functoriality: closed immersions.
   Let $i\colon Y \hookrightarrow X$ be a closed immersion, with complement $j\colon U \hookrightarrow X$. By formal nonsense, you get $i_*\colon \mathsf{D}(Y_{\text{ét}}, A) \rightleftarrows \mathsf{D}(X_{\text{ét}}, A) : i^*$, such that i^* is left-adjoint to i_*.
   Check that then,

$$(i_*K)_{\bar{x}} = \begin{cases} 0 & \text{if } \bar{x} \notin Y \\ K_{\bar{x}} & \text{if } \bar{x} \in Y \end{cases}$$

This implies that $i^*i_* \cong \text{id}$, and so i_* is fully faithful. This means i_* gives an equivalence

$$\mathsf{D}(Y_{\text{ét}}, A) \xrightarrow{\sim} \{K \in \mathsf{D}(X_{\text{ét}}, A) \mid j^*K = 0\}$$

(This works with the Zariski topology for perfect schemes, as is on the problem set.) This gives that $j^*i_* = 0$, and by adjunction this implies a few other vanishings as in the glueing setup from before.
Likewise, we can show that i_* has a right-adjoint $i^!\colon \mathsf{D}(X_{\text{ét}}, A) \to \mathsf{D}(Y_{\text{ét}}, A)$, defined by an exact triangle

$$i_*i^!K \longrightarrow K \longrightarrow j_*j^*K \xrightarrow{+1} i_*i^!K[1]$$

if you are in a setting where cones are functorial. You can check:

$$\mathsf{D}(Y_{\text{ét}}, A) \xrightarrow{i_*} \mathsf{D}(X_{\text{ét}}, A) \xrightarrow{j^*} \mathsf{D}(U_{\text{ét}}, A)$$

gives a glueing setup.

9. Functoriality: locally closed immersions.

   Combining 7 and 8, we get: $Z \overset{k}{\hookrightarrow} X$ locally closed, then have

$$\begin{tikzcd} \mathsf{D}(Z_{\text{ét}}) \arrow[r, shift left=3, "k_!"] \arrow[r, shift left=9, "k_*"'] & \mathsf{D}(X_{\text{ét}}) \arrow[l, shift left=3, "k^!"'] \arrow[l, shift left=9, "k^*"] \end{tikzcd}$$

   such that (k^*, k_*) are adjoint, $(k_!, k^!)$ are adjoint, k open implies $k^! = k^*$, k closed implies $k_! = k_*$, and all are compatible with composition.

   Exercise 14.9. Check that $k_!$ is "extension by 0", and is t-exact.

10. Constructible sheaves: X scheme, A finite ring.

    Definition 14.10. A sheaf $F \in \mathsf{Mod}(X_{\text{ét}}, A)$ is *constructible* if there exists a decomposition of X into locally closed subsets $X = \coprod_i X_i$ with X_i locally closed, such that $K|_{X_i} \in \mathsf{Loc}_A(X_i)$.

    Note this definition works verbatim for X noetherian; for non-noetherian scheme you need to insert the word "constructible" a few times.

    Notation 14.11. $\mathsf{Cons}_A(X) = \{\text{all constructible sheaves}\}$

    Examples 14.12.
    (a) Any local system is constructible.
    (b) If $k\colon Y \hookrightarrow X$ is locally closed, and $L \in \mathsf{Loc}_A(Y)$, then $k_!L$ is constructible.

    Facts 14.13.
    (a) Any constructible sheaf $F \in \mathsf{Cons}_A(X)$ has a finite filtration, with graded pieces of the form $k_!L$ (as in Example 14.12(b)).
    (b) If $A = \mathbf{Z}/n$ and X is noetherian, then

$$\mathsf{Cons}_A(X_{\text{ét}}) = \{\text{compact objects in } \mathsf{Mod}(X_{\text{ét}}, \mathbf{Z}/n)\}.$$

    Lemma 14.14. $F \in \mathsf{Cons}_A(X)$. *The following are equivalent:*
    (a) $F \in \mathsf{Loc}_A(X)$;
    (b) For any specialization $x \rightsquigarrow x'$ *of geometric points,* $\mathrm{sp}\colon F_x \xleftarrow{\sim} F_{x'}$, *i.e., there exists a diagram*

$$\begin{tikzcd} \mathrm{Spec}(k(x')) \arrow[d, hook, "\text{closed}"] & \\ \mathrm{Spec}(\mathcal{O}^{sh}_{X,x'}) \arrow[d, "x'"] & \mathrm{Spec}(L) \arrow[l, dashed, "\exists \iff x \rightsquigarrow x'"'] \arrow[dl, "x"] \\ X & \end{tikzcd}$$

15 October 29, 2015

Last time we discussed local systems and constructible sheaves. For X noetherian, we associated to it the étale site $X_{\text{ét}}$, and for any finite ring A, we constructed local systems $\mathsf{Loc}_A(X)$, from which we also defined $\mathsf{Cons}_A(X)$.

Properties 15.1. Continued from before.

11. Constructible complexes. Let A be a finite field.

Definition 15.2.
(a) $\mathsf{D}^b_{\mathsf{Cons}}(X,A) = \{K \in \mathsf{D}^b(X_{\text{ét}},A) \mid H^i(K) \in \mathsf{Cons}_A(X)\}$.
(b) $\mathsf{D}^b_{\mathsf{Loc}}(X,A) = \{K \in \mathsf{D}^b(X_{\text{ét}},A) \mid H^i(K) \in \mathsf{Loc}_A(X)\}$.

There is another reasonable definition by constructing the former from the latter in the same way we did it for the corresponding abelian categories.

Lemma 15.3.
(a) $\mathsf{D}^b_{\mathsf{Cons}}(X,A) =$ *smallest triangulated subcategory of* $\mathsf{D}(X_{\acute{e}t},A)$ *containing* $k_!L$ *for* $k\colon Z \hookrightarrow X$ *locally closed, and* $L \in \mathsf{Loc}_A(X)$.
(b) $\mathsf{D}^b_{\mathsf{Cons}}(X,A) = \left\{ K \in \mathsf{D}(X_{\acute{e}t},A) \;\middle|\; \begin{array}{c} \exists \text{ a decomposition } X = \coprod_i X_i \text{ with } X_i \hookrightarrow X \text{ locally closed,} \\ \text{and } K|_{X_i} \in \mathsf{D}^b_{\mathsf{Loc}}(X_i,A) \end{array} \right\}$.

Theorem 15.4 (Folklore, proof in Peter and Bhargav's pro-étale paper)**.** *Let X be a variety over $k = \overline{k}$, and A a finite field. Then,*
(a) $\mathsf{D}^b_{\mathsf{Cons}}(X_{\acute{e}t},A) = \{$*compact objects in* $\mathsf{D}(X_{\acute{e}t},A)\} \subseteq \mathsf{D}(X_{\acute{e}t},A)$, *where recall that compact objects in a triangulated category are those K such that* $\operatorname{Hom}(K,-)$ *commutes with* $\bigoplus_{i\in I}$.
(b) $\mathsf{D}(X_{\acute{e}t},A)$ *is compactly generated, i.e.,* $\mathsf{D}(X_{\acute{e}t},A)$ *is the smallest triangulated subcategory of itself containing the compact objects* $\mathsf{D}^b_{\mathsf{Cons}}(X,A)$ *and having all direct sums.*
[Aside: in ∞-categories, you can write down a formula for this: $\mathsf{D}(X_{\acute{e}t},A) = \operatorname{Ind}(\mathsf{D}^b_{\mathsf{Cons}}(X_{\acute{e}t},A))$.*]*

Remark 15.5.
(a) For more general finite rings A, we have

$$\left\{ \begin{array}{c} \text{compact objects} \\ \text{in } \mathsf{D}(X_{\text{ét}},A) \end{array} \right\} = \left\{ K \in \mathsf{D}^b(X_{\text{ét}},A) \;\middle|\; \begin{array}{c} \exists \text{ a decomposition } X = \coprod_i X_i \\ \text{such that } K|_{X_i} \text{ is, locally on } X_{\text{ét}}, \\ \text{isomorphic to } \underline{L} \text{ where } L \in \mathsf{D}_{\mathsf{perf}}(A) \end{array} \right\}$$

For example, for $X = \operatorname{Spec}(\overline{k})$,

$$\{\text{compact objects in } \mathsf{D}(X_{\text{ét}},A)\} = \{\text{compact objects in } \mathsf{D}(A)\} = \mathsf{D}_{\mathsf{perf}}(A)$$

So, $\mathbf{Z}/\ell \notin \mathsf{D}_{\mathsf{perf}}(\mathbf{Z}/\ell^2)$, i.e., $\underline{\mathbf{Z}/\ell} \notin \{$compact objects in $\mathsf{D}^b(X_{\text{ét}},\mathbf{Z}/\ell^2\}$. We can therefore define

$$\mathsf{D}^b_{\mathsf{Cons}}(X_{\text{ét}},A) = \{\text{compact objects in } \mathsf{D}(X_{\text{ét}},A)\} =: \mathsf{D}^b_{\mathsf{ftd,c}}(X,A)$$

where the last notation is from SGA4.
(b) With this definition, $\mathsf{D}(X_{\text{ét}},A)$ is compactly generated by $\mathsf{D}^b_{\mathsf{Cons}}(X_{\text{ét}},A)$.
For example, let $X = \operatorname{Spec}(k)$, $G = \operatorname{Gal}(\overline{k}/k)$, A a finite field. Then,

$$\mathsf{D}(X_{\text{ét}},A) = \mathsf{D}\left(\left\{ \begin{array}{c} \text{all } A\text{-modles } M \text{ with} \\ \text{a continuous } G\text{-action} \end{array} \right\}\right)$$

$$\mathsf{D}^b_{\mathsf{Cons}}(X_{\text{ét}},A) = \left\{ K \text{ as above} \;\middle|\; \begin{array}{c} H^i(K) \text{ is a finitely generated free } A\text{-module} \\ \text{with a continuous } G\text{-action and } 0 \text{ if } |i| \gg 0 \end{array} \right\}$$

12. Functoriality and finiteness: Let A be a finite ring. If $f\colon X \to Y$, get $f^*\colon \mathsf{D}(Y_{\text{ét}},A) \rightleftarrows \mathsf{D}(X_{\text{ét}},A) : f_*$.

Theorem 15.6 (Gabber)**.** *If $f\colon X \to Y$ is a finite type map of noetherian quasi-excellent schemes,*
(a) f_* has finite cohomological dimension: there exists $N \gg 0$ such that $f_*(\mathscr{D}^{\leq 0}) \subseteq \mathscr{D}^{\leq N}$;
(b) f_* preserves $\mathsf{D}^b_{\mathsf{Cons}}$, provided $\#A \in \mathcal{O}(Y)^*$.

Remark 15.7.
(a) If X, Y are varieties over k, then this is proven in SGA.

(b) Easy if f is a closed immersion (the functor is just extension by zero, and we already know this is preserves contructibility), but is *hard* for f an open immersion (which we saw somewhat by the computation for $\mathbf{R}^1 f_*$ we did before for the sphere).
(c) We need $\# A \in \mathcal{O}(Y)^*$ for (b): If $R/\overline{F}_p$ is of finite type, then

$$H^1(\mathrm{Spec}(R)_{\text{ét}}, \mathbf{F}_p) = \frac{R}{(\mathrm{Frob}-1)R},$$

which is quite large. Use the Artin–Schreier sequence:

$$0 \longrightarrow \mathbf{F}_p \longrightarrow \mathcal{O}_X \xrightarrow{\mathrm{Frob}-1} \mathcal{O}_X \longrightarrow 0.$$

It is also useful to know:

Lemma 15.8. *Given a cartesian square*

$$\begin{tikzcd} V \arrow[r, "j'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ U \arrow[r, "j"] & Y \end{tikzcd}$$

with j being étale, then $j^* f_*(F) \xrightarrow{\sim} f'_* j'^*(F)$ for all $F \in \mathsf{D}(X_{\acute{e}t}, A)$.

Corollary 15.9. *Let $f\colon X \to Y$ be finite étale, and suppose Y is connected. Then, $f_*\underline{A} \in \mathsf{Loc}_A(Y)$.*

Proof. May work locally on $Y_{\text{ét}}$. By the Lemma, may reduce to the case when $X = \coprod_{i=1}^n Y$. Then, $f_*\underline{A} \cong \bigoplus_{i=1}^n \underline{A} \in \mathsf{Loc}_A(Y)$. ☐

For example, using the multiplication by n map on an elliptic curve, we get a rank n^2 local system when pushing forward.

Proposition 15.10. *If f is a finite map, then*
(a) f_* is exact;
(b) f_* preserves constructibility.

Proof. For (a), it suffices to show f_* preserves surjections, i.e., if $F \twoheadrightarrow G$ is surjective in $\mathsf{Ab}(X_{\text{ét}})$, then $(f_*F)_{\bar{y}} \twoheadrightarrow (f_*G)_{\bar{y}}$ is surjective for all $\bar{y}$ geometric points on Y. Now consider the cartesian square

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(\mathcal{O}^{\mathsf{sh}}_{Y,\bar{y}}) \arrow[r, "j"] & Y \end{tikzcd}$$

We want: $j^*f_*F \to j^*f_*G$ is surjective. Now by the Lemma, we may assume $Y = \mathrm{Spec}(R)$ is strictly henselian. Since f is finite, we also get that $X = \mathrm{Spec}(\prod_{i=1}^n R_i)$ with R_i strictly henselian. Thus, $(f_*F)(Y) = F(f^{-1}(Y)) = F(X)$. We need that $F(X) \to G(X)$ is surjective. But $X = \coprod_{i=1}^n \mathrm{Spec}(R_i)$, so $\Gamma(X, -)$ is exact, which implies $F(X) \twoheadrightarrow G(X)$.
Note this proof also works for the Nisnevich topology (where local rings are henselian, not strictly henselian).
For (b), if f is finite étale, then we may work locally on $Y_{\text{ét}}$, to reduce to the split case $X = \coprod_{i=1}^n Y$. Then, $\mathsf{Cons}_A(X) = \mathsf{Cons}_A(Y)^n$, and f_* corresponds to direct sums.
In general, there exists a decomposition $Y = \coprod_i Y_i$, $Y_i \hookrightarrow Y$ locally closed, such that $X_i \coloneqq X \times_Y Y_i$, and $X_i^{\mathrm{red}} \to Y_i$ is finite étale (in characteristic zero). We therefore get the following picture:

$$\begin{tikzcd} X_i^{\mathrm{red}} \arrow[dr, hook] \arrow[ddr, bend right, "\text{finite étale}"'] & & \\ & X_i \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & Y_i \arrow[r, "j_i"] & Y \end{tikzcd}$$

It is enough to show: $F \in \mathsf{Cons}_A(X)$, $j_i^* f_* F \in \mathsf{Cons}_A(Y_i)$. *If* we had base change, then we reduce to the previous case. But in this case, we get base change anyway by calculating stalks as in (a). $\square$

Example 15.11. $f\colon E \to \mathbf{P}^1$ over $k = \overline{k}$, a degree 2 cover, which ramifies at 4 points as always. So then, $f_* \underline{\mathbf{Z}/\ell} \in \mathsf{Cons}_A(\mathbf{P}^1)$. Over $\mathbf{P}^1 \setminus \mathrm{Ram}(f)$, we get a local system of rank 2. Over $\mathrm{Ram}(f)$, we get a local system of rank 1.

13. Proper base change.

Theorem 15.12. *Let $f\colon X \to Y$ is a proper map of noetherian schemes, and let A be a finite ring. Then, f_* preserves $\mathsf{D}^b_{\mathsf{Cons}}$, and commutes with arbitrary base change, i.e., given a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

then we have the isomorphism $g^* f_*(F) \xrightarrow{\sim} f'_* g'^*(F)$ if $F \in \mathsf{D}^b_{\mathsf{Cons}}(X_{\acute{e}t})$.

Corollary 15.13. *If $f\colon X \to \mathrm{Spec}(R)$ is proper, and R is strictly henselian, then*

$$\mathbf{R}\Gamma(X, K) \xrightarrow{\sim} \mathbf{R}\Gamma(X_0, K|_{X_0})$$

for all $K \in \mathsf{D}^b_{\mathsf{Cons}}(X, A)$, where $_0$ denotes passage to the closed point.

Proof. Let $Y' \to Y$ be the inclusion of the closed point in the Theorem. $\square$

The corresponding topological statement is that if $X \to D$ is a proper map to a disc, then there exists a deformation retraction $X \simeq X_0$.

Remark 15.14. In the situation of the Corollary, if $\overline{\eta}$ is another geometric point of $\mathrm{Spec}(R)$, we then get

$$\begin{tikzcd} \mathbf{R}\Gamma(X_0, K|_{X_0}) \arrow[rr, bend right, "\text{cospecialization}"'] & \mathbf{R}\Gamma(X, K) \arrow[l, "\sim"', "\text{PBC}"] \arrow[r] & \mathbf{R}\Gamma(X_{\overline{\eta}}, K|_{X_{\overline{\eta}}}) \end{tikzcd}$$

Example 15.15. Take $f\colon X \to Y$ be a degenerating family of elliptic curves. Set $K = \underline{\mathbf{Z}/n}$, where n is invertible on the base. Assume X_0 is an irreducible nodal cubic. Then,

$$\begin{tikzcd} H^1(X, \underline{\mathbf{Z}/n}) \cong H^1(X_0, \underline{\mathbf{Z}/n}) \cong \mathbf{Z}/n \arrow[d, hook] \\ H^1(X_{\overline{\eta}}, \underline{\mathbf{Z}/n}) = (\mathbf{Z}/n)^{\oplus 2} \arrow[d] \\ \mathrm{cok} \cong \mathbf{Z}/n \end{tikzcd}$$

16 November 3, 2015

Last time we discussed more properties about étale cohomology. In particular, we discussed propert base change.

The key special case is the following: let $X/\mathrm{Spec}(R)$ be a proper flat curve, and R a strictly henselian DVR. Then, we consider the diagram

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & X \arrow[d] \\ 0 \arrow[r, hook] & \mathrm{Spec}(R) \end{tikzcd}$$

Then, proper base change implies that $H^1(X_{\text{ét}}, \mu_n) \cong H^1(X_{0,\text{ét}}, \mu_n)$, so the Kummer sequence gives that $\mathrm{Pic}(X)[n] \cong \mathrm{Pic}(X_0)[n]$ (where we have to make sure $n \in R^*$).

Exercise 16.1.

1. Check this when R is complete by deformation theory.
2. Find a counterexample when $n \notin R^*$.

This implies the general statement of proper base change, that will be useful in the next properties we will discuss.

Properties 16.2.

14. Compactly supported cohomology.

Definition 16.3. Let $f\colon X \to Y$ be a separated, finite type map of noetherian schemes. Choose a Nagata compactification

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "f"'] & & \overline{X} \arrow[dl, "\bar{f}"] \\ & Y & \end{tikzcd}$$

where $\bar{f}$ is proper, and j is an open immersion. We define

$$f_!\colon \mathsf{D}^b_{\mathsf{Cons}}(X_{\text{ét}}, A) \longrightarrow \mathsf{D}^b_{\mathsf{Cons}}(Y_{\text{ét}}, A)$$

where $f_!K := \bar{f}_*(j_!K)$. Note $j_!$ preserves constructibility, and the fact that $\bar{f}_*$ does is proper base change.

Theorem 16.4. *$f_!$ is independent of choice of j.*

Proof. To show independence of choices, we reduce to the following case:

$$\begin{tikzcd} & \overline{X}' \arrow[d, "h"] \arrow[dd, bend left=60, "\bar{g}"] \\ X \arrow[ur, hook, "j'"] \arrow[r, hook, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\bar{f}"] \\ & Y \end{tikzcd}$$

with $\bar{f}, \bar{g}$ proper, and j, j' dense open immersions. We want: $\bar{f}_* \circ j_! \cong \bar{g}_* \circ j'_!$. The right hand side is $\bar{f}_* \circ h_* \circ j'_!$. So it suffices to show $j_! = h_* \circ j'_!$. So properness implies that h is an isomorphism over $X \hookrightarrow \overline{X}$, and therefore $j_! = h_* \circ j_!$ is "extension by zero," that is, computing stalks for both gives you what you would expect extension by zero to look like. ☐

Remark 16.5. If f is étale, then this agrees with the previous definition of $f_!$.

Corollary 16.6. *The statement of proper base change is true for $f_!$ without assuming f is proper.*

Definition 16.7. If X is a variety over $k = k^{\text{sep}}$, then $\mathbf{R}\Gamma_c(X, F) := \mathbf{R}\Gamma(\operatorname{Spec}(k), f_!F)$.

Example 16.8. Let X be a smooth curve over $k = \bar{k}$, and

$$X \xhookrightarrow{j} \overline{X} \xhookleftarrow[i]{} Z$$

where $\overline{X}$ is smooth and proper, and Z is a finite set. To calculate $H^i_c(X_{\text{ét}}, \underline{\mathbf{Z}/n})$, we have the short exact sequence

$$0 \longrightarrow j_!\underline{\mathbf{Z}/n} \longrightarrow \underline{\mathbf{Z}/n} \longrightarrow i_*\underline{\mathbf{Z}/n} \longrightarrow 0$$

Then, the long exact sequence says

$$H^0_c(X, \mathbf{Z}/n) := H^0(\overline{X}, j_!\underline{\mathbf{Z}/n}) = \begin{cases} 0 & \text{if } Z \neq \emptyset \\ \mathbf{Z}/n & \text{if } Z = \emptyset \end{cases}$$

For H^1_c,

$$\begin{tikzcd}
 & & H^1_c(X,\mathbf{Z}/n) \arrow[d, equal] & & \\
0 \arrow[r] & \dfrac{H^0(\overline{X}, i_*\mathbf{Z}/n)}{H^0(\overline{X},\mathbf{Z}/n)} \arrow[r] & H^1(\overline{X}, j_!\underline{\mathbf{Z}/n}) \arrow[r] & H^1(\overline{X},\mathbf{Z}/n) \arrow[r] & 0
\end{tikzcd}$$

where the last zero map is because $i_*\underline{\mathbf{Z}/n}$ is a (direct sum of) skyscraper sheaves. This gives the short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \dfrac{\bigoplus_{x\in Z}\mathbf{Z}/n}{\mathbf{Z}/n} \arrow[r] \arrow[d, equal] & H^1(\overline{X}, j_!\underline{\mathbf{Z}/n}) \arrow[r] & (\mathbf{Z}/n)^{2g} \arrow[r] & 0 \\
 & (\mathbf{Z}/n)^{\#Z-1} & & &
\end{tikzcd}$$

For H^2_c, we get

$$\begin{tikzcd}
0 \arrow[r] & H^2(\overline{X}, j_!\mathbf{Z}/n) \arrow[r, "\sim"] & H^2(\overline{X},\mathbf{Z}/n) \arrow[r] \arrow[d, equal] & 0 \\
 & & \mathbf{Z}/n = \dfrac{\operatorname{Pic}(\overline{X})}{n\cdot\operatorname{Pic}(\overline{X})} &
\end{tikzcd}$$

Assume $Z \neq \emptyset$. We then get

$$H^i_c(X,\mathbf{Z}/n) = \begin{cases} 0 & \text{if } i=0,\ i>2 \\ (\mathbf{Z}/n)^{2g+\#Z-1} & \text{if } i=1 \\ \mathbf{Z}/n2 & \text{if } i=2 \end{cases}$$

15. Deligne's generic base change.

Theorem 16.9. *Let $f\colon X\to Y$ be a finite type map of noetherian schemes, A a finite ring such that $\#A\in\mathcal{O}(Y)^*$, and fix $K\in \mathsf{D}^b_{\mathsf{Cons}}(X,A)$. Then, there exists $U\subseteq Y$ open and dense such that for all cartesian squares of the form*

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

with $g(Y')\subseteq U$, then we have $g^*(f_*(K))\cong f'_*(g'^*(K))$.

Example 16.10. Let $Y=\operatorname{Spec}(\mathbf{C}[[t]])$, and $X=\mathbf{A}^1\times Y\setminus\{(0,0)\}$, so that all fibres look like $\mathbf{A}^1$ except at the closed point in Y, whose fibre looks like $\mathbf{G}_m$. Then, we

Claim 16.11. *$H^1(X,\mathbf{Z}/n)=0$ and $H^1(X_0,\mathbf{Z}/n)=\mathbf{Z}/n$, and so there is no base change for f_*.*

Proof of Claim. $X_0=\mathbf{G}_m=\mathbf{A}^1\setminus\{0\}$. Then, $H^1(X_0,\mathbf{Z}/n)=\mathbf{Z}/n$ (proof uses Kummer sequence and $\operatorname{Pic}(\mathbf{G}_m)=0$). Then, recall we have

$$H^1(X,\mathbf{Z}/n)=\{\text{isomorphism classes of } \mathbf{Z}/n\text{-torsors over } X\}$$

But

$$\{\text{finite étale covers of } \mathbf{A}^1\times Y\}\xrightarrow{\sim}\{\text{finite étale covers of } X\},$$

where this equivalence is by the purity of the branch locus (the branch locus must be a divisor, but here all we are losing is a point, which is codimension 2, and so the branch locus must be empty). Now $\mathbf{A}^1\times Y$ has no interesting finite étale covers (at least topologically, this space is contractible). Thus, $H^1(X,\mathbf{Z}/n)=0$. ∎

16. Smooth base change.

Theorem 16.12. *Given a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

such that g is smooth, f is finite type, X, Y noetherian and quasi-excellent, and $\#A \in \mathcal{O}(Y)^*$. Then, $g^* f_*(K) \cong f'_* g'^*(K)$ for all $K \in \mathsf{D}^b_{\mathsf{Cons}}(X, A)$.

Remarks 16.13.

(a) If g is étale, then we saw this earlier.

(b) Topological analogue: if X is a topological space, and $I = [0,1]$, then $H^*(X, \mathbf{Z}/n) \cong H^*(X \times I, \mathbf{Z}/n)$.

Corollary 16.14. *If $f\colon X \to Y$ is proper and smooth, and $\#A \in \mathcal{O}(Y)^*$, then*

(a) $f_*(\underline{A}) \in \mathsf{D}^b_{\mathsf{Loc}}(Y, A)$, and so each $\mathbf{R}^i f_* \underline{A}$ is a local system. (Bhargav wrote an $\mathbf{R}$ in front of f_*: "Maybe this shows that I'm not fully derived myself.")

(b) If $\bar\eta$, $\bar y$ are geometric points of Y such that $\bar\eta \rightsquigarrow \bar y$, then we get

$$\mathrm{cosp}\colon \mathbf{R}\Gamma(X_{\bar y}, \underline{A}) \xrightarrow{\sim} \mathbf{R}\Gamma(X_{\bar\eta}, \underline{A})$$

Compare the family of degenerating elliptic curves we had last time, in which case the corresponding statement did not hold, and cospecialization had a cokernel.

17. Nearby and vanishing cycles: Let S be a strictly henselian DVR, for example $\mathbf{C}[[t]]$ or $\mathbf{Z}_p$. Let $s \in S$ be the closed point, and $\bar t \to s$ is the geometric generic point (t is the actual point in the scheme). Fix $f\colon X \to S$, a map of finite type. Then, we get the diagram

$$\begin{tikzcd} X_{\bar t} \arrow[r, "\bar\jmath"] \arrow[d, "f_{\bar t}"] & X \arrow[d, "f"] & X_s \arrow[l, hook', "i"'] \arrow[d, "f_s"] \\ \bar t \arrow[r, "\bar\jmath"] & S & s \arrow[l, hook', "i"'] \end{tikzcd}$$

Then, for any $K \in \mathsf{D}(X)$, we get $K \xrightarrow{\text{can}} \bar\jmath_* \bar\jmath^* K$, and thus $i^* K \xrightarrow{\text{can}} i^* \bar\jmath_* \bar\jmath^* K$ is a map in $\mathsf{D}(X_s)$.

Definition 16.15.

(a) $\psi_f = i^* \bar\jmath_* \bar\jmath^* K$ gives the "nearby cycles" functor $\psi_f\colon \mathsf{D}(X) \to \mathsf{D}(X_s)$.

(b) $\phi_f(K) = \mathrm{Cone}(\mathrm{can})$ (this is cheating, but the point is that we can lift this to an actual construction of sheaves, and in this category this makes sense and is functorial). This gives the "vanishing cycles" functor $\phi_f\colon \mathsf{D}(X) \to \mathsf{D}(X_s)$.
We then get the exact sequence in $\mathsf{D}(X_s)$:

$$i^* K \longrightarrow \psi_f(K) \longrightarrow \phi_f(K)$$

Remarks 16.16.

(a) $K = \underline{\mathbf{Z}/n}$, and $\bar x \to X$ a geometric point lying over s. Then,

$$(\psi_f(K))_{\bar x} = i^* \left(\bar\jmath_* \left(\bar\jmath^* \mathbf{Z}/n\right)\right)_{\bar x} = i^* \left(\bar\jmath_* \mathbf{Z}/n\right)_{\bar x} = \mathbf{R}\Gamma\left(X^{\mathrm{sh}}_x \times_s \bar t, \mathbf{Z}/n\right)$$

where $X^{\mathrm{sh}}_x \times_s \bar t$ is the Milnor fibre.

(b) Say $G = \mathrm{Gal}(k(t)) = \mathrm{Aut}(k(\bar t)/k(t))$. Then, the fundamental triangle triangle lifts to the "G-equivariant derived category of X_s."

(c) If f is proper, then proper base change says that $\mathbf{R}\Gamma(X_s, \psi_f(K)) = \mathbf{R}\Gamma(X, \bar\jmath_* \bar\jmath^*(K))$, which is the same as $\mathbf{R}\Gamma(X_{\bar t}, \bar\jmath^*)$. Thus, if we consider the canonical map can: $i^* K \to \psi_f(K)$, passing to $\mathbf{R}\Gamma$

gives the commutative diagram

$$\begin{tikzcd} \mathbf{R}\Gamma(X_s, i^*K) \arrow[r] \arrow[dr, "\mathrm{cosp}"'] & \mathbf{R}\Gamma(X_s, \psi_f(K)) \arrow[d, equal] \\ & \mathbf{R}\Gamma(X_{\bar{t}}, \bar{\jmath}^*(K)) \end{tikzcd}$$

Thus, if $\phi_f(K) = 0$, then cospecialization is an isomorphism.

Therefore, to prove the Corollary about proper smooth maps from before, it suffices to show

(d) If f is smooth, then $\phi_f(K) = 0$ for all $K \in \mathsf{D}^b_{\mathsf{Cons}}(X_{\text{ét}}, A)$ where $\#A \in \mathcal{O}(S)^*$.

Sketch of Proof. We work locally on $X_{\text{ét}}$, so we may assume $X = X_x^{\text{sh}}$ is a strictly henselian local scheme, for $x \to X$ a geometric point. Then, dévissage implies that we may assume $K = \underline{A}$. Now consider

$$\begin{tikzcd} X_{\bar{t}} \arrow[r, "\bar{\jmath}"] \arrow[d, "f_{\bar{t}}"'] & X \arrow[d, "f"] \\ \bar{t} \arrow[r, "\bar{\jmath}"] & S \end{tikzcd}$$

Now smooth base change shows that $f^*\bar{\jmath}_*\underline{A} \simeq \bar{\jmath}_* f_{\bar{t}}^*\underline{A} \cong \bar{\jmath}_*\underline{A}$. Now apply i^*:

$$\begin{tikzcd} i^*f^*\bar{\jmath}_*\underline{A} \cong i^*\bar{\jmath}_*\underline{A} \arrow[d, equal] \\ f_s^* i^*\bar{\jmath}_*\underline{A} \end{tikzcd}$$

Check that (along S) $i^*\bar{\jmath}_*\underline{A} \cong \underline{A}$. Thus,

$$\begin{tikzcd} f_s^*\underline{A} \arrow[r, "\sim"] \arrow[d, equal] & \psi_f(\underline{A}) \\ \underline{A} \arrow[ur, "\mathrm{cosp}"'] & \end{tikzcd}$$

which implies $\phi_f(\underline{A}) = 0$. $\square$

17 November 5, 2015

No class next week. Bhargav might want to give some makeup lectures at the end of this semester, or some informal lectures next semester about the Riemann–Hilbert correspondence, or some other topic. . .

Last time, we discussed nearby cycles. Suppose we have a map $f\colon X \to Y$ of curves that is generically étale, but not at y. We find the Milnor fibre at x' by taking a point t' nearby $y' = f(x')$, and $f^{-1}(t') = *$ in the branch of the curve still containing x' is our Milnor fibre. At x, instead we have two points in our Milnor fibre $f^{-1}(t) = * \amalg *$, which is not contractible.

Properties 17.1.

18. Vanishing theorems. Consider varieties over k, and A a finite ring such that $\#A \in k^*$.

Theorem 17.2. *Let $f\colon X \to Y$ be a map of varieties, and let*

$$d = \sup\left\{\dim\left(f^{-1}(y)\right) \;\middle|\; y \in Y\right\}.$$

Then, both f_* and $f_!$ have cohomological dimension $\le 2d$, i.e., $f_*(\mathsf{D}^{\le 0}_{\mathsf{Cons}}) \subseteq \mathsf{D}^{\le 2d}_{\mathsf{Cons}}$, and similarly for $f_!$.

Theorem 17.3 (Artin vanishing)**.** *Let $f\colon X \to Y$ be an affine morphism of varieties. Fix $n \in \mathbf{N}$, and $F \in \mathsf{Cons}_A(X)$. Assume for all $a \in X$ that if $\dim\left(\overline{\{a\}}\right) > n$, we have $F_{\overline{a}} = 0$. Then, for all $b \in Y$, $\dim \overline{\{b\}} > n - q$, we have $(\mathbf{R}^q f_* F)_{\overline{b}} = 0$.*

Corollary 17.4. *If X is an affine variety, and $F \in \mathsf{Cons}_A(X)$, $d = \dim(X)$, then $H^q(X, F) = 0$ for all $q > \dim(X)$.*

Proof. Apply Theorem 2 with $Y = *$, and $n = d$. $\square$

Remarks 17.5.

(a) Theorem 2 is an analogue of Andreatti–Frankel: Any complex affine variety of dimension n has the homotopy type of a CW complex of dimension n.

Example 17.6. $X = E \setminus \{0\}$, E an elliptic curve. $E \cong S^1 \times S^1$ (topologically), and $X \simeq E \setminus D \simeq S^1 \vee S^1$, where $D \ni 0$ is a small neighborhood.

(b) Esnault's "Variations on Artin vanishing" gives a de Rham version of this result.

Proof of Theorem 17.2 (vague idea). Dévissage down to $Y = \mathrm{Spec}(k)$, k separably closed, X a smooth affine curve, and $F = \mu_n$. This dévissage would involve the base change theorems from before.
Using the Kummer sequence and Tsen's theorem, we saw the following:

$$H^i_c(X, \mu_n) = \begin{cases} 0 & \text{if } i = 0 \\ (\mathbf{Z}/n)^{2g+\#Z-1} & \text{if } i = 1 \\ \mathbf{Z}/n & \text{if } i = 2 \end{cases}$$

where $Z = \overline{X} \setminus X$. Thus, $\mathbf{R}\Gamma_c(X, \mu_n) \in \mathscr{D}^{\leq 2}$. Likewise, for usual cohomology,

$$H^i(X, \mu_n) = \begin{cases} \mathbf{Z}/n & \text{if } i = 0 \\ (\mathbf{Z}/n)^{2g+\#Z-1} & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases}$$

(using $H^i(X, \mathbf{G}_m) = 0$ for all $i \geq 2$). Also need: $\mathrm{Pic}(X) \xrightarrow{n} \mathrm{Pic}(X)$ is surjective. Note

$$\mathrm{Pic}(X) = \frac{\mathrm{Pic}(\overline{X})}{\mathrm{im}(\mathbf{Z}^{\#Z})}$$

Since Z is nonempty, this kills the "degree" part, so $\mathrm{Pic}^0(\overline{X}) \twoheadrightarrow \mathrm{Pic}(X)$. Then we use the fact that the Jacobian is divisible. $\square$

If you want to see proofs of this, you should look at Lei Fu's book on étale cohomology, where all of the SGA stuff is redone in modern language. "It's kinda great."

19. Internal Hom. Recall that if X is a topological space, $K, L \in \mathsf{D}(X)$, then we get $\underline{\mathbf{R}\mathrm{Hom}}(K, L) \in \mathsf{D}(X)$, such that for all $V \subseteq X$ open subsets,

$$\mathbf{R}\Gamma(V, \underline{\mathbf{R}\mathrm{Hom}}(K, L)) = \mathbf{R}\,\mathrm{Hom}(K|_V, L|_V).$$

Example 17.7. Say $j\colon U \hookrightarrow X$ is an open subset. Consider $K = j_!\underline{A}$, $L \in \mathsf{D}(X, A)$, and fix $k\colon V \hookrightarrow X$ another open subset. This gives a cartesian diagram

$$\begin{tikzcd} U \cap V \arrow[r, hook, "h"] \arrow[d, hook, "i"] & V \arrow[d, hook, "k"] \\ U \arrow[r, hook, "j"] & X \end{tikzcd}$$

We then calculate:

$$\begin{aligned} \mathbf{R}\,\mathrm{Hom}(k_!\underline{A}, \underline{\mathbf{R}\mathrm{Hom}}(j_!A, L)) &= \mathbf{R}\Gamma(V, \underline{\mathbf{R}\mathrm{Hom}}(j_!\underline{A}, L)) \\ &= \mathbf{R}\mathrm{Hom}((j_!A)|_V, L|_V) \\ &\cong \mathbf{R}\mathrm{Hom}(h_!(\underline{A}|_{U\cap V}), L|_V) \\ &= \mathbf{R}\,\mathrm{Hom}(h_!\underline{A}, L|_V) \\ &\cong \mathbf{R}\Gamma(U \cap V, (L|_V)|_{U\cap V}) \\ &\cong \mathbf{R}\Gamma(V, \mathbf{R}j_*(j^*L)) \\ &= \mathbf{R}\mathrm{Hom}(k_!\underline{A}, \mathbf{R}j_*(j^*L)) \end{aligned}$$

This implies $\underline{\mathbf{R}\mathrm{Hom}}(j_!\underline{A}, L) \cong j_*(j^*L)$, using the fact that if $\mathbf{R}\Gamma(V, M) = 0$ for all $V \subseteq X$, then $M \cong 0$. We rewrite this suggestively as follows:

$$\underline{\mathbf{R}\mathrm{Hom}}(j_!\underline{A}, L) \cong j_* \underline{\mathbf{R}\mathrm{Hom}}(\underline{A}, j^! L)$$

which is the relevant statement of Verdier duality.

We may define $\underline{\mathbf{R}\mathrm{Hom}}$ in the étale setting, and moreover, there is some finiteness:

Theorem 17.8. *Let X be a noetherian quasi-excellent scheme, let $L \in \mathcal{O}(X)^*$, $A = \underline{\mathbf{Z}/\ell^n}$, and $K, L \in \mathsf{D}^b_{\mathsf{Cons}}(X, A)$. Then, $\underline{\mathbf{R}\mathrm{Hom}}(K, L) \in \mathsf{D}^b_{\mathsf{Cons}}(X, A)$.*

This is the fifth of the "six functors."

Proof. Reduce to $K = j_!\underline{A}$ for $j\colon U \to X$ in $X_{\text{ét}}$. Here we are not giving an argument as to why the local systems that are $j_!$'d to X can be assume to be, in fact, trivial: you have to take an étale cover, and there is some argument there. Now we have the same computation as before: $\underline{\mathbf{R}\mathrm{Hom}}(j_!\underline{A}, L) \cong j_*(j^*L)$. We are done by Gabber. □

Note the proof seems to work if A is arbitrary. At some point you have to think about whether A is Gorenstein, because if it isn't, then the dualizing complex changes.

20. Verdier duality.

Theorem 17.9 (Verdier (for varieties), Gabber (this statement))**.** *If $f\colon X \to Y$ is a separated map of noetherian quasi-excellent schemes, and $\ell \in \mathcal{O}(Y)^*$, $A = \underline{\mathbf{Z}/\ell^n}$, then*

$$f_!\colon \mathsf{D}^b_{\mathsf{Cons}}(X, A) \longrightarrow \mathsf{D}^b_{\mathsf{Cons}}(Y, A)$$

has a right adjoint

$$f^!\colon \mathsf{D}^b_{\mathsf{Cons}}(Y, A) \longrightarrow \mathsf{D}^b_{\mathsf{Cons}}(X, A)$$

We only need that A is a finite local ring with residue field of characteristic ℓ. The same thing is true for topological spaces, for proper maps (see Kashiwara–Schapira).

Remarks 17.10.
(a) If f is étale, then $f^! = f^*$.
(b) If f is a closed immersion, then $f^! = f^!$ from before.
(c) We have transitivity: $g^! f^! = (fg)^!$ (and also $f_! g_! = (fg)_!$).
(d) If $K \in \mathsf{D}^b_{\mathsf{Cons}}(Y, A)$, then we get the "counit" of adjunction

$$f_!(f^! K) \xrightarrow{\mathrm{Tr}_K} K$$

Corollary 17.11. *For f as above, $K \in \mathsf{D}^b_{\mathsf{Cons}}(X, A)$ and $L \in \mathsf{D}^b_{\mathsf{Cons}}(Y, A)$, we get*

$$\underline{\mathbf{R}\mathrm{Hom}}(f_! K, L) \cong f_* \underline{\mathbf{R}\mathrm{Hom}}(K, f^! L).$$

Proof of Corollary. $\mathbf{R}\Gamma(Y, \underline{\mathbf{R}\mathrm{Hom}}(f_! K, L)) = \mathbf{R}\mathrm{Hom}(f_! K, L)$ by definition of $\underline{\mathbf{R}\mathrm{Hom}}$. By Verdier duality, this is equal to $\mathbf{R}\mathrm{Hom}(K, f^! L)$ by using the Trace map, and the statement becomes the previous version of Verdier duality to check we have isomorphisms on cohomology. Then, by defintion this is equal to $\mathbf{R}\Gamma(X, \underline{\mathbf{R}\mathrm{Hom}}(K, f^! L)) = \mathbf{R}\Gamma(Y, f_* \underline{\mathbf{R}\mathrm{Hom}}(K, f^! L))$. The same argument over étale maps $V \to Y$ gives the claim. □

Idea of Proof of Theorem. If $f^!$ exists, then we know for all $j\colon U \to X$ étale,

$$\mathbf{R}\Gamma(U, f^! L) = \mathbf{R}\mathrm{Hom}(j_!\underline{A}, f^! L) = \mathbf{R}\mathrm{Hom}(f_! j_! \underline{A}, L) = \mathbf{R}\mathrm{Hom}((fj)_! \underline{A}, L)$$

Therefore, for any $L \in \mathsf{D}(Y, A)$, we define a presheaf of chain complexes on X via:

$$\begin{pmatrix} j\colon U \to X \\ \text{étale} \end{pmatrix} \longmapsto \mathbf{R}\mathrm{Hom}((fj)_! \underline{A}, L)$$

by choosing representatives of L and using a functorial K-injective resolution of $(fj)_!\underline{A}$; this defines $f^!L \in \mathsf{D}(X, A)$.
The hard part is showing $f^!$ preserves $\mathsf{D}^b_{\mathsf{Cons}}$ (consider looking at Verdier's original paper (in English!), where Verdier does everything from scratch, including the reduction to curves, which is nice). $\square$

21. Dualizing complexes. We work with varieties over $k = k^{\mathrm{sep}}$. Assume $A = \underline{\mathbf{Z}/\ell^n}$, $\ell \in k^*$, the "Gorenstein case" that is in use in the literature.

Definition 17.12. Let X be a variety.
(a) The "dualizing complex" is $\mathcal{D}_X := f^!\underline{A}$, where $f\colon X \to \operatorname{Spec}(k)$ is the structure map.
(b) For $K \in \mathsf{D}^b_{\mathsf{Cons}}(X, A)$, $\mathcal{D}_X(K) = \underline{\mathbf{R}\mathrm{Hom}}(K, \mathcal{D}_X) \in \mathsf{D}^b_{\mathsf{Cons}}(X, A)$.

Observation 17.13. Given $g\colon X \to Y$, $g^!\mathcal{D}_Y \cong \mathcal{D}_X$ since the left hand side is $g^!f^!\mathcal{D}_Y = (fg)^!\mathcal{D}_Y$, and

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr] & & Y \arrow[dl, "f"] \\ & \operatorname{Spec}(k) & \end{tikzcd}$$

commutes.

Lemma 17.14. *For $g\colon X \to Y$, we have $\mathcal{D}_Y \circ g_! \cong g_* \circ \mathcal{D}_X$.*

Proof. Let $K \in \mathsf{D}^b_{\mathsf{Cons}}(X, A)$. The right hand side is:

$$(\mathcal{D}_y \circ g_!)(K) = \mathcal{D}_Y(g_!K) = \underline{\mathbf{R}\mathrm{Hom}}(g_!K, \mathcal{D}_Y)$$

But by Verdier duality,

$$\underline{\mathbf{R}\mathrm{Hom}}(g_!K, \mathcal{D}_Y) = g_*\,\underline{\mathbf{R}\mathrm{Hom}}(K, g^!\mathcal{D}_Y) = g_*\,\underline{\mathbf{R}\mathrm{Hom}}(K, \mathcal{D}_X) = (g_*\mathcal{D}_X)(K)$$

by our observation. $\square$

Theorem 17.15. *The natural map*

$$K \longrightarrow \mathcal{D}_X(\mathcal{D}_X(K))$$

is an isomorphism (so $\mathcal{D}_X^2 \cong \mathrm{id}$).

Corollary 17.16. *For a map $g\colon X \to Y$, we get*
(a) $\mathcal{D}_Y \circ g_* = g_! \circ \mathcal{D}_X$;
(b) $g^* \circ \mathcal{D}_Y = \mathcal{D}_X \circ g^!$;
(c) $g^! \circ \mathcal{D}_Y \cong \mathcal{D}_X \circ g^*$.

Proof. Consider (a). We know $\mathcal{D}_Y g_! = g_*\mathcal{D}_X$. Then, $\mathcal{D}_Y g_! \mathcal{D}_X = g_*\mathcal{D}_X^2 = g_*$, and so $\mathcal{D}_Y^2 g_! \mathcal{D}_X \cong \mathcal{D}_Y g_*$ gives $g_!\mathcal{D}_X \cong \mathcal{D}_Y g_*$. For (b) and (c), pass to adjoints. $\square$

Example 17.17. Let X be a smooth variety, and then $\mathcal{D}_X = A(d)[2d]$, where $d = \dim(X)$, and where (d) is a Tate twist. Verdier proves the theorem by reducing to the case of a smooth affine curve, $A = \mathbf{Z}/\ell$, and $A(1) = \mu_\ell$. To show $\mu_\ell[2]$ is dualizing, need:
(a) A canonical map trace: $\mathbf{R}\Gamma_c(X, \mu_\ell[2]) \longrightarrow \mathbf{Z}/\ell$ (corresponds to $\mathbf{R}f_!f^!\mathbf{Z}/\ell \to \mathbf{Z}/\ell$).
(b) $\mathbf{R}\mathrm{Hom}(\mathbf{R}\Gamma_c(X, \mathbf{Z}/\ell), \mathbf{Z}/\ell) \simeq \mathbf{R}\mathrm{Hom}(\mathbf{Z}/\ell, \mu_\ell[2]) = \mathbf{R}\Gamma(X, \mu_\ell[2])$, where the first $\simeq$ is the statement of Verdier duality for $K = L = \mathbf{Z}/\ell$.

For (a), recall

$$H^i_c(X, \mu_\ell[2]) = \begin{cases} 0 & \text{if } i = -2 \\ (\mathbf{Z}/\ell)^{2g+\#(\overline{X}\setminus X)-1} & \text{if } i = -1 \\ \mathbf{Z}/\ell & \text{if } i = 0 \end{cases}$$

where this equality is an abstract isomorphism. We need a *natural* isomorphism $H^0_c(X, \mu_\ell[2]) \xrightarrow{\sim} \mathbf{Z}/\ell$. The left hand side is

$$H^2_c(X, \mu_\ell) \cong H^2(\overline{X}, \mu_\ell)$$

which we found by using a compactification $j\colon X \hookrightarrow \overline{X}$, and using the exact sequence $j_!\mu_\ell \to \mu \to i_*\mu_\ell$. The Kummer sequence and Tsen's theorem says that

$$H^2(\overline{X}, \mu_\ell) \xrightarrow{\sim} \frac{\operatorname{Pic}(\overline{X})}{\ell \operatorname{Pic}(\overline{X})} \xrightarrow[\deg]{\sim} \mathbf{Z}/\ell$$

and everything here was canonical.

18 November 17, 2015

18.1 Perverse sheaves on algebraic varieties

Set-up 18.1. Work with varieties over $k = \overline{k}$, ℓ prime and invertible on k. If X is a variety, we denote $\mathsf{D}(X) = \mathsf{D}^b_{\mathsf{cons}}(X, \mathbf{Z}/\ell)$.

Remarks 18.2.

1. One may pass to the limit and work with

$$\text{“}\mathsf{D}^b_{\mathsf{cons}}(X, \mathbf{Q}_\ell) = \left(\lim_n \mathsf{D}^b_{\mathsf{cons}}(X, \mathbf{Z}/\ell^n)\right)[\tfrac{1}{\ell}]\text{”}$$

to get characteristic 0 coefficients.

2. Most things work well with coefficients in $\{\mathbf{Z}/\ell, \mathbf{Z}_\ell, \mathbf{Q}_\ell, \overline{\mathbf{Q}}_\ell\}$. However, $\mathbf{Z}/\ell^n$ for $n \geq 2$ is a problem because $\tau^{\leq n}$ does not preserve $\mathsf{D}^b_{\mathsf{cons}}(X, \mathbf{Z}/\ell^n)$.

   Example 18.3. If X is a point, then $\mathsf{D}^b_{\mathsf{cons}}(X, \mathbf{Z}/\ell^n) = \mathsf{D}_{\mathsf{perf}}(\mathbf{Z}/\ell^n)$, Then,

$$K = \left(\mathbf{Z}/\ell^n \xrightarrow{\ell} \mathbf{Z}/\ell^n\right) \in \mathsf{D}_{\mathsf{perf}}(\mathbf{Z}/\ell^n)$$

   but $H^0(K) = \mathbf{Z}/\ell \notin \mathsf{D}_{\mathsf{perf}}(\mathbf{Z}/\ell^n)$ for $n \geq 2$.

   These problems go away in characteristic zero by Serre's theory of homological algebra over a regular ring.

Definition 18.4. Let X be a variety. The *perverse t-structure* on $\mathsf{D}(X)$ (for “middle perversity”) is given by:

- ${}^p\,\mathsf{D}^{\leq 0} = \{K \in \mathsf{D}(X) \mid K \text{ semiperverse}\} \coloneqq \{K \in \mathsf{D}(X) \mid \dim \operatorname{Supp}(H^i(K)) \leq -i\ \forall i\}$
- ${}^p\,\mathsf{D}^{\geq 0} = \{K \in \mathsf{D}(X) \mid \mathcal{D}K \text{ semiperverse}\} \coloneqq \{K \in \mathsf{D}(X) \mid \dim \operatorname{Supp}(H^i(\mathcal{D}K)) \leq -i\ \forall i\}$

where $\mathcal{D}K$ denotes the Verdier dual of K.

- $\mathsf{Perv}(X) = {}^p\,\mathsf{D}^{\leq 0}(X) \cap {}^p\,\mathsf{D}^{\geq 0}(X)$.

Theorem 18.5 (BBD)**.** *This gives a t-structure on* $\mathsf{D}(X)$*.*

Remark 18.6. There exist analogues for other “perversity” functions. Middle perversity is nicest, though, because the definitions are Verdier dual to each other.

Examples 18.7.

1. If X is a point, ${}^p\,\mathsf{D}^{\leq 0}(X) = \mathsf{D}^{\leq 0}(X)$. Likewise, ${}^p\,\mathsf{D}^{\geq 0}(X) = \mathsf{D}^{\geq 0}(X)$. This is the usual t-structure.
2. Let X be a curve; then, $K = \underline{\mathbf{Z}/\ell}[0]$ is in the heart of the regular t-structure. However, it is *not* in ${}^p\,\mathsf{D}^{\leq 0}(X)$, since $\dim \operatorname{Supp}(H^0(K)) = 1$, not ≤ 0.
3. Let X be a smooth curve, $x \in$, and $j\colon X \setminus \{x\} \to X$. Denote $U = X \setminus \{x\}$. Then, we

   Claim 18.8. $K = j_*(\underline{\mathbf{Z}/\ell}[1]) \in \mathsf{Perv}(X)$.

   Recall that $K|_U = \underline{\mathbf{Z}/\ell}[1]$. On the other hand,

$$H^i(K_x) = \begin{cases} \mathbf{Z}/\ell & \text{if } i = -1 \\ \mathbf{Z}/\ell(-1) & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

where (-1) denotes a Tate twist. Thus, $H^0(X)$ is supported at x, $H^{-1}(X)$ is supported on all of X, and $H^i(K) = 0$ for all $i \neq 0, -1$. Thus, $K \in {}^p\mathsf{D}^{\leq 0}(X)$.

Also, $\mathcal{D}K = \mathcal{D}(j_*(\underline{\mathbf{Z}/\ell}[1])) \cong j_!\mathcal{D}(\underline{\mathbf{Z}/\ell}[1])$. On U, $\mathcal{D}_U = \underline{\mathbf{Z}/\ell}(1)[2]$. Thus,

$$\mathcal{D}(\underline{\mathbf{Z}/\ell}[1]) = \underline{\mathbf{R}\mathrm{Hom}}_U(\mathbf{Z}/\ell[1], \mathbf{Z}/\ell(1)[2]) \cong \mathbf{Z}/\ell(1)[1]$$

where we remember that $\underline{\mathbf{R}\mathrm{Hom}}_U(\underline{\mathbf{Z}/\ell}, -) = \mathrm{id}$. Thus, $\mathcal{D} = j_!\mathbf{Z}/\ell(1)[1]$. Thus, $H^0(\mathcal{D}K) = 0$ and $H^{-1}(\mathcal{D}K) = j_!\underline{\mathbf{Z}/\ell}(1)$ is supported on U. $H^i(\mathcal{D}K) = 0$ for all $i \neq -1$. Thus, $\mathcal{D}K \in {}^p\mathsf{D}^{\leq 0}(X)$, and so $K \in \mathsf{Perv}(X)$.

Some easy observations.

1. ${}^p\mathsf{D}^{\leq 0}(X) \subseteq \mathsf{D}^{\leq 0}(X)$: if $K \in {}^p\mathsf{D}^{\leq 0}(X)$, then $\dim \mathrm{Supp}(H^i(K)) \leq -i$ for all $i > 0$, and so $H^i(K) = 0$ for all $i > 0$, i.e. $K \in \mathsf{D}^{\leq 0}(X)$. Note the dual statement is not true!
2. ${}^p\mathsf{D}^{\leq 0}(X)[n] \subseteq {}^p\mathsf{D}^{\leq 0}(X)$ for all $n > 0$. Say $K \in {}^p\mathsf{D}^{\leq 0}(X)$; we want $K[n] \in {}^p\mathsf{D}^{\leq 0}(X)$. $H^i(K[n]) = H^{i+n}(K)$, and so

$$\begin{aligned} \dim \mathrm{Supp}\, H^i(K[n]) = \dim \mathrm{Supp}\, H^{i+n}(K) &\leq -i - n \ \forall i \\ &\leq -i \ \forall i \text{ (since } n \geq i) \end{aligned}$$

   Thus, $K[n] \in {}^p\mathsf{D}^{\leq 0}(X)$. Set ${}^p\mathsf{D}^{\leq n}(X) = {}^p\mathsf{D}^{\leq 0}(X)[-n]$.
3. $K \in \mathsf{D}(X)$, then $K[n] \in {}^p\mathsf{D}^{\leq 0}(X)$ for all $n \gg 0$. Since K is bounded, $K[n] \in \mathsf{D}^{\leq -\dim(X)}(X)$ for all $n \gg 0$. But $\mathsf{D}^{\leq -\dim(X)}(X) \subseteq {}^p\mathsf{D}^{\leq 0}(X)$. So, $K[n] \in {}^p\mathsf{D}^{\leq 0}(X)$ for all $n \gg 0$.
4. $K \in \mathsf{D}(X)$, then $K[-m] \in {}^p\mathsf{D}^{\geq 0}(X)$ for all $m \gg 0$ [dualize (3)].
5. $K \in \mathsf{D}(X)$ implies $K \in {}^p\mathsf{D}^{\leq n}(X) \cap {}^p\mathsf{D}^{\geq -m}(X)$ for all $m, n \gg 0$ [combie (3) and (4)]. "Everything is bounded with respect to the perverse t-structure," that is, "the perverse t-structure is bounded."

18.1.1 Lisse complexes

Recall, if X is a variety, we have

$$\mathsf{D}_{\mathsf{loc}}(X) = \{K \in \mathsf{D}(X) \mid H^i(K) \text{ locally constant}\} \subseteq \mathsf{D}(X).$$

Such K are called *lisse*.

Lemma 18.9. *If $K \in \mathsf{D}_{\mathsf{loc}}(X)$, the $K^\vee \coloneqq \underline{\mathbf{R}\mathrm{Hom}}(K, \underline{\mathbf{Z}/\ell}) \in \mathsf{D}_{\mathsf{loc}}(X)$.*

Note it is not true that the Verdier dual of a lisse sheaf is lisse; this only works for the naïve dual as defined above.

Proof. Using étale descent (i.e., using the étale local nature of $\mathsf{D}_{\mathsf{loc}}(X)$), we reduce to the case where each $H^i(K) = \underline{\mathbf{Z}/\ell}^{\oplus n_i}$. But $\underline{\mathbf{R}\mathrm{Hom}}(\underline{\mathbf{Z}/\ell}, \underline{\mathbf{Z}/\ell}) = \underline{\mathbf{Z}/\ell}$, and so $H^i(K^\vee) = H^{-i}(K)^\vee$. In particular, $K^\vee \in \mathsf{D}_{\mathsf{loc}}(X)$. ∎

This is the sheaf-theoretic version of the fact that for a regular noetherian ring R, we have two dualities: Hom-ing into the dualizing complex (Grothendieck duality), or naïvely Hom-ing into R. In this case the latter operation preserves perfect complexes.

Lemma 18.10. *If X is smooth of dimension d, then $\mathcal{D}_X = \mathbf{Z}/\ell(d)[2d]$.*

Corollary 18.11. *If X is smooth of dimension d, and $K \in \mathsf{D}_{\mathsf{loc}}(X)$, then*

1. *$\mathcal{D}K \in \mathsf{D}_{\mathsf{loc}}(X)$;*
2. *$H^i(\mathcal{D}K) = H^{-i-2d}(K)^\vee(d)$.*

Proof. We have

$$\begin{aligned} \mathcal{D}K &= \underline{\mathbf{R}\mathrm{Hom}}(K, \mathcal{D}_X) \\ &= \underline{\mathbf{R}\mathrm{Hom}}(K, \mathbf{Z}/\ell(d)[2d]) \\ &= K \vee (d)[2d] \end{aligned}$$

and so $H^i(\mathcal{D}K) = H^i(K^\vee(d)[2d]) = H^{i+2d}(K^\vee)(d) = H^{-i-2d}(K)^\vee(d)$. ∎

Proposition 18.12. *Let X be smooth (connected) of dimension d, and $K \in \mathsf{D}_{\mathsf{loc}}(X)$. Then,*

1. *$K \in {}^p\mathsf{D}^{\leq 0}(X)$ if and only if $H^i(K) = 0$ for all $i > -d$*
2. *$K \in {}^p\mathsf{D}^{\geq 0}(X)$ if and only if $H^i(K) = 0$ for all $i < -d$.*
3. *$K \in \mathsf{Perv}(X)$ if and only if $H^i(K) = 0$ for all $i \neq -d$.*

Proof.

1. Since $K \in \mathsf{D}_{\mathsf{loc}}(X)$, each non-zero $H^i(K)$ is supported on all of X. Therefore, $\dim \mathrm{Supp}(H^i(K)) = d$ for such i. Thus, if $K \in {}^p\mathsf{D}^{\leq 0}(X)$, then $H^i(K) = 0$ for all $i > -d$.
2. $K \in \mathsf{D}_{\mathsf{loc}}(X)$ implies $\mathcal{D}K = K^\vee(d)[2d] \in \mathsf{D}_{\mathsf{loc}}(X)$. Then, $K \in {}^p\mathsf{D}^{\geq 0}(X) \iff \mathcal{D}K \in {}^p\mathsf{D}^{\leq 0}(X)$, which by (1) is equivalent to $H^i(\mathcal{D}K) = 0$ for all $i > -d$. Using our formula for $\mathcal{D}K$, this is equivalent to $H^i(K^\vee(d)[2d]) = 0$ for all $i > -d$, i.e., $H^{-i-2d}(K)^\vee(d) = 0$ for all $i > -d$, which is the same as $H^j(K) = 0$ for all $j < -d$.
3. (1) and (2). ☐

Proof of existence of t-structure. Let X be a variety. ${}^p\mathsf{D}^{\leq 0}(X) = \{K \in \mathsf{D}(X) \mid \dim \mathrm{Supp}\, H^i(K) \leq -i\ \forall i\}$, ${}^p\mathsf{D}^{\geq 0}(X) = \mathcal{D}({}^p\mathsf{D}^{\leq 0}(X))$. Our goal is to show this gives a t-structure on $\mathsf{D}(X)$.

Key Lemma 18.13. *Fix $j\colon U \hookrightarrow X$ open, and $i\colon Z \hookrightarrow X$ the closed complement. Fix $K \in \mathsf{D}(X)$. Then, we claim:*

1. *$K \in {}^p\mathsf{D}^{\leq 0}(X) \iff j^*K \in {}^p\mathsf{D}^{\leq 0}(U),\ i^*K \in {}^p\mathsf{D}^{\leq 0}(Z)$.*
2. *$K \in {}^p\mathsf{D}^{\geq 0}(X) \iff j^!K = j^*K \in {}^p\mathsf{D}^{\geq 0}(U),\ i^!K \in {}^p\mathsf{D}^{\geq 0}(Z)$.*

Thus, we have a glueing setup.

Proof.

1. Using exactness of j^* and i^* (for the usual t-structure), get: $\mathrm{Supp}(H^i(K)) = \mathrm{Supp}(H^i(j^*K)) \cup \mathrm{Supp}(H^i(i^*K))$. Thus, $\dim \mathrm{Supp}(H^i(K)) = \max(\dim \mathrm{Supp}(H^i(j^*K)), \dim \mathrm{Supp}(i^*K))$. This immediately gives (1).
2. We need to use that $i^*\mathcal{D}K = \mathcal{D}(i^!K)$, and $j^*\mathcal{D}K = \mathcal{D}(j^*K)$, and so get (2) from (1) by duality. ☐

Corollary 18.14. *K, U, Z as before. Assume U is smooth of dimension d, and $K|_U \in \mathsf{D}_{\mathsf{loc}}(U)$. Then,*

1. *$K \in {}^p\mathsf{D}^{\leq 0}(X) \iff K|_U \in \mathsf{D}^{\leq -d}_{\mathsf{loc}}(U)$, $i^*K \in {}^p\mathsf{D}^{\leq 0}(Z)$.*
2. *$K \in {}^p\mathsf{D}^{\geq 0}(X) \iff K|_U \in \mathsf{D}^{\geq -d}_{\mathsf{loc}}(U)$, $i^!K \in {}^p\mathsf{D}^{\geq 0}(Z)$.*

Proof. Combine Key Lemma with previous Corollary on lisse complexes on smooth varieties. ☐

Proof of the Goal. Work by induction on $d = \dim(X)$.

1. If $d = 0$, then X is the union of finitely many points. Reduce to $X = \{\mathrm{pt}\}$. But then, ${}^p\mathsf{D}^{\geq 0}(X) = \mathsf{D}^{\leq 0}(X)$, and ${}^p\mathsf{D}^{\geq 0}(X) = \mathsf{D}^{\geq 0}(X)$, and so we are done.
2. Assume $({}^p\mathsf{D}^{\leq 0}(Z), {}^p\mathsf{D}^{\geq 0}(Z))$ is a t-structure on $\mathsf{D}(Z)$ for all $Z \hookrightarrow X$ of smaller dimension. Fix $U \subseteq X$ dense open, U smooth. Set $\mathsf{D}(X, U) = \{K \in \mathsf{D}(U) \mid K|_U \in \mathsf{D}_{\mathsf{loc}}(U)\} \subseteq \mathsf{D}(X)$, a full triangulated subcategory. Then, $({}^p\mathsf{D}^{\leq 0}(X) \cap \mathsf{D}(X,U), {}^p\mathsf{D}^{\geq 0}(X) \cap \mathsf{D}(X,U))$ is a t-structure on $\mathsf{D}(X,U)$, obtained by glueing $({}^p\mathsf{D}^{\leq 0}(Z), {}^p\mathsf{D}^{\geq 0}(Z))$ on $\mathsf{D}(Z)$ to $(\mathsf{D}^{\leq -d}_{\mathsf{loc}}(U), \mathsf{D}^{\geq -d}_{\mathsf{loc}}(U))$ on $\mathsf{D}(U)$ (glueing as before).
3. Now
$$\mathsf{D}(X) = \bigcup_{\substack{U \subseteq X \\ \text{as above}}} \mathsf{D}(X, U)$$
and so we get a t-structure on $\mathsf{D}(X)$. ☐

☐

19 November 19, 2015

Let X be a variety, $\mathsf{D}(X) = \mathsf{D}^b_{\mathsf{cons}}(X, \mathbf{Z}/\ell)$. Then, we defined the semiperverse sheaves,

$$\begin{aligned} {}^p\,\mathsf{D}^{\leq 0}(X) &= \left\{K \in \mathsf{D}(X) \mid \dim(\operatorname{Supp} H^i(K)) \leq i\right\} \\ {}^p\,\mathsf{D}^{\geq 0}(X) &= \mathcal{D}({}^p\,\mathsf{D}^{\leq 0}(X)) \end{aligned}$$

Theorem 19.1. *This is a t-structure.*

Strategy: given $U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z$ open/closed, a decomposition of X; then you get an "exact sequence"

$$\begin{tikzcd} \mathsf{D}(Z) \arrow[r, "i_*"] & \mathsf{D}(X) \arrow[r, "j^*"] & \mathsf{D}(U) \\ \mathsf{D}(Z) \arrow[r] \arrow[u, equal] & \mathsf{D}(X,U) \arrow[r] \arrow[u] & \mathsf{D}_{\mathsf{loc}}(U) \arrow[u, hook] \end{tikzcd}$$

Step 1. If U is smooth of dimension d, then

$$\left({}^p\,\mathsf{D}^{\leq 0}(U) \cap \mathsf{D}_{\mathsf{loc}}(U), {}^p\,\mathsf{D}^{\geq 0}(U) \cap \mathsf{D}(U)\right) = \left(\mathsf{D}^{\leq -d}_{\mathsf{loc}}(U), \mathsf{D}^{\geq -d}_{\mathsf{loc}}(U)\right)$$

so we get a t-structure on $\mathsf{D}_{\mathsf{loc}}(U)$.
Step 2. $({}^p\,\mathsf{D}^{\leq 0}(Z), {}^p\,\mathsf{D}^{\geq 0}(Z))$ is a t-structure on $\mathsf{D}(Z)$ (by induction).
Step 3. $({}^p\,\mathsf{D}^{\leq 0}(X) \cap \mathsf{D}(X,U), {}^p\,\mathsf{D}^{\geq 0}(X) \cap \mathsf{D}(X,U))$ is a t-structure and obtained by glueing the ones in Steps 1 and 2.
Step 4. If $U \subseteq V$, open subset, V smooth, then we get $D(X,V) \subseteq \mathsf{D}(X,U)$. Thus,

$$\mathsf{D}(X) = \bigcup_{U \text{ open smooth dense}} \mathsf{D}(X,U)$$

and we therefore have a t-structure on $\mathsf{D}(X)$ from the ones on $\mathsf{D}(X,U)$.

19.1 Intermediate extensions

Let X be a variety, $U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z$ an open/closed decomposition. We saw that $(\mathsf{D}(X), \text{perverse } t\text{-structure})$ is obtained by glueing $(\mathsf{D}(U), \text{perverse } t\text{-structure})$ and $(\mathsf{D}(Z), \text{perverse } t\text{-structure})$. By the glueing formalism, we get:

1. **Exactness**: j^*, i^* are t-exact; $j_!, i^*$ are t-right exact, $j_*, i^!$ are t-left exact. Note now that we are talking about exactness with respect to the perverse t-structures.
2. **Adjunctions**: $\left({}^pH^0(j_!), {}^pH^0(j^*) = j^*, {}^pH^0(j_*)\right)$ and $\left({}^pH^0(i^*), {}^pH^0(i_*) = i_*, {}^pH^0(i^!)\right)$ are adjoint sequences.
3. **Standard exact triangles**: Let $K \in \mathsf{Perv}(X)$.
   (a) $0 \longrightarrow i_*{}^pH^0(i^!K) \longrightarrow K \longrightarrow {}^pH^0(j_*(j^*K))$, where we recall that $i_*{}^pH^0(i^!K)$ is the *largest* non-trivial subobject of K supported on Z.
   (b) ${}^pH^0(j_!(j^*K)) \longrightarrow K \longrightarrow i_*{}^pH^0(i^*K) \longrightarrow 0$, where we recall that $i_*{}^pH^0(i^*K)$ is the *largest* non-trivial quotient of K supported on Z.
4. **Intermediate extensions**: $j_{!*}\colon \mathsf{Perv}(U) \to \mathsf{Perv}(X)$ characterized by:
   (a) $j_{!*}A = \operatorname{im}({}^pH^0(j_!A), {}^pH^0(j_*A))$
   (b) $j_{!*}A =$ unique extension $\overline{A}$ of A such that $\overline{A}$ has no non-trivial subobjects or quotients supported on Z.
   (c) $j_{!*}A =$ unique extension $\overline{A}$ of A such that ${}^pH^0(i^!\overline{A}) = 0$ and ${}^pH^0(i^*\overline{A}) = 0$ (by the exact sequences).
   (d) $j_{!*}A =$ unique extension $\overline{A}$ of A such that $i^!\overline{A} \in {}^p\,\mathsf{D}^{\geq 1}(Z)$ and $i^*\overline{A} \in {}^p\,\mathsf{D}^{\leq -1}(Z)$.

We want to characterize simple objects. We already did this in the abstract setting, but here we can say something more precise. But first, a

Lemma 19.2. *$\mathcal{D}(j_{!*}A) = j_{!*}\mathcal{D}(A)$, that is, $j_{!*}$ is self-dual.*

Proof. We use characterization (d) above. Let $\overline{B} = \mathcal{D}(j_{!*}A)$. Then, $j^*\overline{B} = j^*\mathcal{D}(j_{!*}A) = \mathcal{D}(j^*j_{!*}A) = \mathcal{D}(A)$, that is, $\overline{B}$ is an extension of $\mathcal{D}(A)$.

Now $i^*\overline{B} = i^*\mathcal{D}(j_{!*}A) = \mathcal{D}(i^!j_{!*}A)$. Now $i^!j_{!*}A \in {}^p\mathsf{D}^{\geq 1}(Z)$, and so $i^*\overline{B} = \mathcal{D}(i^!j_{!*}A) \in {}^p\mathsf{D}^{\leq -1}(Z)$.

Similarly: $i^!\overline{B} \in {}^p\mathsf{D}^{\geq 1}(Z)$. Thus, $\overline{B} = j_{!*}\mathcal{D}(A)$. ☐

Lemma 19.3. *Let X be a smooth variety of dimension $d \geq 1$. Say $\overline{A} = L[d]$, where $L \in \mathsf{Loc}(X)$ (this is a perverse sheaf by the discussion by last time). Then, $\overline{A} \cong j_{!*}(L[d]|_U)$ for any $j\colon U \hookrightarrow X$ open and dense.*

Proof. Let $i\colon Z \hookrightarrow X$ be the complement of U, and $d' = \dim(Z) < d$. Now $i^*\overline{A} = i^*L[d] = (i^*L[d'])[d-d']$. Now $i^*L[d'] \in {}^p\mathsf{D}^{\leq 0}(Z)$ (only non-zero group in deg $= \dim(Z)$). Thus, $i^*\overline{A} = (i^*L[d'])[d-d'] \in {}^p\mathsf{D}^{\leq -1}(Z)$ as $d - d' > 0$. Duality implies that $i^!\overline{A} \in {}^p\mathsf{D}^{\geq 1}(Z)$ (use smoothness of X here). Thus, $\overline{A} \cong j_{!*}(L[d]|_U)$. ☐

Definition 19.4. Let X be a variety of dimension d. Then, $\mathrm{IC}_X \coloneqq j_{!*}(\mathbf{Z}/\ell[d])$ for any $j\colon U \hookrightarrow X$ open and dense, where U is smooth.

Remark 19.5. $\mathcal{D}(\mathrm{IC}_X) \cong \mathrm{IC}_X$, which implies that if X is proper, then $\mathbf{R}\Gamma(X, \mathrm{IC}_X)$ is self-dual, where $\mathbf{R}\Gamma(X, \mathrm{IC}_X)$ is intersection cohomology (we use X proper to get $\mathbf{R}\Gamma = \mathbf{R}f_* = \mathbf{R}f_!$). This is the statement of Poincaré duality.

19.2 Simple objects

Recall: the glueing formalism implies the following

Corollary 19.6. *Given an open/closed decomposition $U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z$ any simple $A \in \mathsf{Perv}(X)$ is one of the following two types:*

1. *i_*B for simple $B \in \mathsf{Perv}(Z)$;*
2. *$j_{!*}C$ or simple $C \in \mathsf{Perv}(U)$.*

Moreover, all such objects are simple.

For a more explicit description, we need:

Lemma 19.7. *Let U is smooth of dimension d, and $L \in \mathsf{Loc}(U)$. Assume L is irreducible. Then, $L[d] \in \mathsf{Perv}(U)$ is also simple.*

Note this isn't true for constructible sheaves!

Proof. Set $B = L[d]$. Assume there exists $A \hookrightarrow B$ in $\mathsf{Perv}(U)$. Then, there exists $V \subseteq U$ an open dense subset such that $A|_V = M[d]$ for some $M \in \mathsf{Loc}(V)$. Then, restriction is exact, so $A|_V \hookrightarrow Brvert_V$, and so we get $M \hookrightarrow L|_V$ ($\mathsf{Loc}_V \to \mathsf{Perv}(V)$, $M \mapsto M[d]$ is exact). We use the following

Fact 19.8. $L|_V$ is irreducible (use: $\pi_1(V) \to \pi_1(U)$ is surjective).

Thus, either $M = 0$ or $M = L|_V$.

Now, if $M = 0$, then $A \hookrightarrow B$ gives a subobject of B supported on $U \setminus V$. But $B = j_{!*}(L[d]|_V)$, and so B has no subobjects supported on $U \setminus V$. Then, $A = 0$.

On the other hand, if $M = L|_V$, then $B \to B/A$ gives a quotient of B supported on $U \setminus V$. Again, this is not possible. ☐

Corollary 19.9. *Let X be a variety, and $A \in \mathsf{Perv}(X)$. Then, A is simple (as an object in the abelian category) if and only if $A = i_*j_{!*}(L[d])$, where $i\colon Y \hookrightarrow X$ is a closed irreducible subset, and $j\colon U \hookrightarrow Y$ is open and dense, where U is smooth of dimension d, and $L \in \mathsf{Loc}(U)$ is irreducible.*

Proof. Call $\mathrm{Supp}(A) = Y$. If $Y \neq X$, then replace X with Y, to assume A supported everywhere. Now if $X = X_1 \cup X_2$, where $X_1 \subseteq X$ is irreducible, and $U_1 = X \setminus X_1$, then either: $A = i_*A'$, $i\colon X_1 \hookrightarrow X$, or $A = j_{!*}B'$, $j\colon U_1 \hookrightarrow X$. This way, we reduce to the case where $\mathrm{Supp}(A) = X$ is an irreducible variety (using that $i_* = i_!$ implies $i_* = i_{!*}$). Now, there exists a dense open $U \subseteq X$ such that $A|_U = L[d]$, where $L \in \mathsf{Loc}(U)$, and U is smooth. Thus, $A = j_{!*}(j^*A) = j_{!*}(L[d])$ for $j\colon U \hookrightarrow X$. Now show that L is irreducible. ☐

Proposition 19.10. *The category $\mathsf{Perv}(X)$ is artinian and noetherian, i.e., any $A \in \mathsf{Perv}(X)$ has finite length.*

Remark 19.11. This is completely false for constructible sheaves. For example, if you take $X = \mathbf{A}^1 \supset U_0 \supset U_1 \supset U_2 \supset \cdots$ a descending sequence of open subsets, and let $j_i \colon U_i \hookrightarrow \mathbf{A}^1$. Then, there is a compatible sequence of subobjects: $\cdots \subset j_{1!}\underline{\mathbf{Z}/\ell} \subset j_{0!}\underline{\mathbf{Z}/\ell} \subset \underline{\mathbf{Z}/\ell}$ is an infinite descending sequence of subobjects, and does not stabilize.

Proof. By induction, may assume any $B \in \mathsf{Perv}(Z)$, for $Z \subsetneq X$ of smaller dimension has finite length (this is because for points, we are in the category of finite dimensional vector spaces, which all have finite length). Now fix $A \in \mathsf{Perv}(X)$, and choose $U \subseteq X$ a dense open subset such that $B := A|_U = L[d]$ for $L \in \mathsf{Loc}(U)$. Now we have a short exact sequence:

$$0 \longrightarrow i_* {}^pH^0(i^! A) \longrightarrow A \longrightarrow {}^pH^0(j_* B),$$

where $i \colon Z \hookrightarrow X$ is a complementary closed subset to U. By induction, $i_* {}^pH^0(i^! A)$ has finite length, so we need to show ${}^pH^0(j_* B)$. Now B has finite length (since every local system has finite length: it's just a finite dimensional vector space with some extra data, since it's a representation of π_1). Now use the following exact sequence:

$$0 \longrightarrow j_{!*} B \longrightarrow {}^pH^0(j_* B) \longrightarrow i_* C \longrightarrow 0$$

for some $C \in \mathsf{Perv}(Z)$, since the quotient of the first inclusion is supported on Z. Now $j_{!*}$ preserves simple objects, so therefore preserves finite length objects (this requires a slight argument, which we will prove next time) and $i_* C$ is controlled by induction. $\square$

20 November 24, 2015

Last time, we proved the following

Theorem 20.1. *Any $M \in \mathsf{Perv}(X)$ has finite length.*

We proved this by induction on $\dim X$. We had the following

Claim 20.2. *If $j \colon U \hookrightarrow X$ is an open subset, then $j_{!*}$ preserves finite length objects.*

Proof of Claim. We know already that $j_{!*}$ preserves simple objects. So, it suffices to show that given a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathsf{Perv}(U)$, then if $j_{!*}A$ and $j_{!*}C$ have finite length, so does $j_{!*}B$. We then have the following diagram:

$$\begin{tikzcd}
 & j_{!*}A \arrow[r, "\alpha"] \arrow[d, hook] & j_{!*}B \arrow[r, "\beta"] \arrow[d, hook] & j_{!*}C \arrow[d, hook] \\
0 \arrow[r] & {}^pH^0(j_*A) \arrow[r] & {}^pH^0(j_*B) \arrow[r] & {}^pH^0(j_*C)
\end{tikzcd}$$

where the first row is not necessarily exact, but the second row is. Now we know that α is injective by the commutativity of the diagram, and so if $K = \ker(\beta)$, then we get a filtration

$$j_{!*}A \subseteq K \subseteq j_{!*}B.$$

Now, $j_{!*}B/K \hookrightarrow j_{!*}C$, so $j_{!*}B$ has finite length. Also, $j_{!*}A \subseteq K$ is an isomorphism over U (as $A = \ker(B \to C)$). We therefore have that $K/j_{!*}A$ is supported on $X \setminus U$, which has finite length by induction. Thus, $j_{!*}B$ has finite length. $\square$

There is a new problem set online about intersection cohomology. There are two problems: the first deals with how the intersection cohomology complex restricts well to hyperplane sections, and the second has to do with the intersection cohomology of an isolated singularity.

20.1 Affine maps

Recall the following

Theorem 20.3 (Artin)**.** *Fix $f\colon X \to Y$ an affine morphism of varieties. Fix $n \in \mathbf{N}$ and a constructible sheaf $F \in \mathsf{Cons}(X)$, such that $F_{\bar a} = 0$ for all $a \in X$ with $\dim \overline{\{a\}} > n$. Then, $(\mathbf{R}^q f_* F)_{\bar b} = 0$ for all $b \in Y$ with $\dim \overline{\{b\}} > n - q$.*

Theorem 20.4. *If $f\colon X \to Y$ is affine, then*
1. *f_* is right t-exact ($f_*({}^p\,\mathsf{D}^{\le 0}) \subseteq {}^p\,\mathsf{D}^{\le 0}$);*
2. *$f_!$ is left t-exact.*

Next time I will try to explain the Beilinson "Basic Lemma" to say that this result can let us define compactly supported cohomology as a derived functor of the global sections of one of these functors, which you cannot do in ordinary algebraic geometry.

Proof. Fix $K \in {}^p\,\mathsf{D}^{\le}(X)$. You would normally prove this by chasing spectral sequences, but we don't want to do that. K is filtered with graded pieces $H^i(K)[-i]$ such that $\dim \operatorname{Supp}(H^i(K)) \le -i$. This implies that $\mathbf{R}f_*K$ is filtered with graded pieces $\mathbf{R}f_*(H^i(K)[-i])$. We then

Claim. *It suffices to show: if $F \in \mathsf{Cons}(X)$ with $\dim \operatorname{Supp}(F) \le k$, then $\dim \operatorname{Supp}(\mathbf{R}^{i+k} f_* F) \le -i$.*

Artin says that if $b \in Y$ such that $\dim \overline{\{b\}} > k - (i + k) = -i$, then $(\mathbf{R}^{i+k} f_* F)_{\bar b} = 0$. Therefore, all generic points b of $\operatorname{Supp}(R^{i+k} F_* F)$ satisfy $\dim \overline{\{b\}} \le -i$, and so $\dim(\operatorname{Supp} R^{i+k} f_* F) \le -i$. ☐

This theorem is the opposite of the constructible world: there, f_* is left-exact, and $f_!$ is right-exact. You would imagine there are situations in which these two worlds overlap, and that is the content of the following

Corollary 20.5. *Assume $f\colon X \to Y$ is affine and quasi-finite. Then, f_* and $f_!$ are t-exact.*

Example 20.6. Consider f an affine open immersion. Then f_* kind of detects "nearby cycles." It is not exactly that, since in the latter case, you need to take a universal cover and then come back.

Proof. **Step 1**. Assume f is finite. Then, $f_! = f_*$ since f is proper, and so Artin says that $f_* = f_!$ is t-exact.

Step 2. f is an affine open immersion. Then, f_* is left t-exact, since it is a right adjoint of f^*. Artin then says that f_* is t-exact.

Step 3. Combine Steps 1 and 2 using the factorization

$$\begin{tikzcd} X \arrow[rr, hook, "j"] \arrow[dr, "f"'] & & \overline{X} \arrow[dl, "\bar f"] \\ & Y & \end{tikzcd}$$

that comes from Zariski's Main Theorem, where f is affine, j is an affine open immersion, and $\bar f$ is finite. Steps 1 and 2 imply that $\bar f_*, j_*$ are t-exact, and so f_* is t-exact. ☐

Example 20.7. If X is a local complete intersection variety of dimension d, then $\mathbf{Z}/\ell[d] \in \mathsf{Perv}(X)$. We know this is already true in the smooth case.

Step 1. Show that for any X, ${}^p\,\mathsf{D}^{\ge 0}(X) \subseteq \mathsf{D}^{\ge -d}(X)$. This doesn't just follow from duality from the statement we had before about the anti-semi-perverse category, since $\mathsf{D}^{\ge -d}(X)$ is not self-dual.

Step 2. Given $j\colon U \hookrightarrow Y$ an affine open immersion, with closed complement $i\colon Z = Y \setminus U \hookrightarrow Y$, then

$$i^*\mathsf{Perv}(Y) \subseteq {}^p\,\mathsf{D}^{[-1,0]}(Z) = \{K \in \mathsf{D}(Z) \mid {}^pH^i(K) = 0 \ \forall i \ne 0, -1\}$$

Proof. Fix $A \in \mathsf{Perv}(Y)$. Then, we have an exact triangle

$$j_!(j^*A) \longrightarrow A \longrightarrow i_*i^*A$$

where the first object is in $\mathsf{Perv}(Y)$ by Artin, and the second is by assumption. We therefore get that ${}^pH^i(i_*i^*A) = 0$ for all $i \ne 0, -1$ by using the long exact sequence associated to the exact triangle above. t-exactness shows that ${}^pH^i(i_*i^*A) = i_*{}^pH^i(i^*A)$. ☐

Step 3. Let j, i as in Step 2, and assume $\dim(Z) \leq d-1$, where $d = \dim Y$. Fix $A \in \mathsf{Perv}(Y)$ and assume $A \in \mathsf{D}^{\leq -d}(Y)$. Then, $i^*A[-1] \in \mathsf{Perv}(Z)$.

Proof. Step 2 says that $i^*A \in {}^p\mathsf{D}^{[-1,0]}(Z)$. It therefore suffices to show $\mathrm{Hom}_{\mathsf{D}(Z)}(i^*A, B) = 0$ for all $B \in \mathsf{Perv}(Z)$, where we note

$$\mathrm{Hom}_{\mathsf{D}(Z)}(i^*A, B) = \mathrm{Hom}_{\mathsf{Perv}(Z)}({}^pH^0(i^*A), B)$$

by adjunction. Now, our assumption on A implies that $i^*A \in \mathsf{D}^{\leq -d}(Z)$. But Step 1 implies that $B \in \mathsf{D}^{\geq -(d-1)}(Z) = \mathsf{D}^{\geq -d+1}(Z)$, and so $\mathrm{Hom}_{\mathsf{D}(Z)}(i^*A, B) = 0$ as desired. $\square$

Step 4. We may assume that $i\colon X \hookrightarrow \mathbf{A}^n$ is cut out by $f_1, \ldots, f_{n-d}$ a regular sequence in $\mathbf{A}^n$, where $\dim X = d$. Induction on $n-d$ and Step 3 implies that $i^*(\mathbf{Z}/\ell[n])[-(n-d)] = \mathbf{Z}/\ell[d] \in \mathsf{Perv}(X)$.

20.2 Beilinson's "Basic Lemma"

Setup: let X be a variety, $\mathsf{D}(X) = \mathsf{D}^b_{\mathsf{cons}}(X, \mathbf{Z}/\ell)$ such that $\ell \in k^*$.

Theorem 20.8 (Beilinson)**.**

1. *The canonical functor* $\mathrm{can}_X\colon \mathsf{Perv}(X) \to \mathsf{D}(X)$ *admits a natural extension to an exact functor* $\widetilde{\mathrm{can}}_X\colon \mathsf{D}^b(\mathsf{Perv}(X)) \to \mathsf{D}(X)$.
2. $\widetilde{\mathrm{can}}_X$ *is an equivalence.*
3. $\widetilde{\mathrm{can}}_X$ *is compatible with pushforwards along affine maps:* *If* $f\colon X \to Y$ *is an affine map, then the right* t*-exact functor*

$${}^pH^0(f_*)\colon \mathsf{Perv}(X) \to \mathsf{Perv}(Y)$$

admits a left-derived functor $\mathbf{L}{}^pH^0(f_*)\colon \mathsf{D}^b(\mathsf{Perv}(X)) \to \mathsf{D}^b(\mathsf{Perv}(X))$ *such that the following diagram commutes:*

$$\begin{tikzcd} \mathsf{D}^b(\mathsf{Perv}(X)) \arrow[r, "\widetilde{\mathrm{can}}_X"] \arrow[d, "\mathbf{L}{}^pH^0(f_*)"'] & \mathsf{D}(X) \arrow[d, "f_*"] \\ \mathsf{D}^b(\mathsf{Perv}(Y)) \arrow[r, "\widetilde{\mathrm{can}}_Y"] & \mathsf{D}(Y) \end{tikzcd}$$

4. (3) *holds for* $f_!$.

Remarks 20.9.

1. The analogue for $\mathsf{Cons}(X) \subseteq \mathsf{D}(X)$ is true in characteristic 0 (by Nori).
2. Beilinson gives analogues for $\mathsf{D}^b_{\mathsf{cons}}(X, \mathbf{Q}_\ell)$, $\mathsf{D}^b_{\mathsf{cons}}(X^{\mathrm{an}}, F)$ where $X/\mathbf{C}$ and F is a field, $\mathsf{D}_{\mathsf{hol}}(\mathscr{D}_X)$ where X in in characteristic 0.
3. Analogous statement is *false* if we fix the stratification, since we need to be able to refine the stratification in the proof.
4. This allows us to define $f_!$ as a derived functor.

20.2.1 Ingredients

For (1), use filtered derived categories: there is a $\mathsf{DF}(X) \to \mathsf{D}(X)$.

Question 20.10. If $\mathscr{D}$ is a triangulated dg-category with a t-structure, is there a canonical functor $\mathsf{D}^b(\mathscr{D}^\heartsuit) \to \mathscr{D}$?

You can prove this with enough projectives or injectives in the heart (Lurie's *Higher Algebra*) but perverse sheaves do not have enough injectives.

For (2), use vanishing cycles and t-exactness to reduce to showing

$$\mathrm{Ext}^i_{\mathsf{Perv}(X)}(M, N) \cong \mathrm{Ext}^i_{\mathsf{D}(X)}(M, N)$$

for $M, N \in \mathsf{Perv}(X)$ simple and supported everywhere. You can then reduce further to the generic point of X.

For (3), use "Basic Lemma":

Lemma 20.11. *Let $f\colon X \to Y$ be an affine map and fix $M \in \mathsf{Perv}(X)$. Then, there exists $j\colon U \hookrightarrow X$ a dense open affine immersion such that $N := j_!(M|_U) \in \mathsf{Perv}(X)$ (it is perverse by Artin), then $N \to M$ is surjective in $\mathsf{Perv}(X)$, and ${}^pH^i f_* N = 0$ for all $i \neq 0$.*

Remark 20.12. Nori proved an analogous statement for constructible sheaves in characteristic 0. This gives "cellular decompositions" of varieties.

21 December 1, 2015

21.1 Beilinson's basic lemma

We will only be proving the case for quasi-projective varieties.

Theorem 21.1 (BBL)**.** *If $f\colon X \to Y$ is an affine map, and $M \in \mathsf{Perv}(X)$, then there exists $j\colon U \hookrightarrow X$ a dense affine open immersion such that if $N = j_!(M|_U)$, then*
1. $\mathrm{can}\colon N \to M$ *is surjective in* $\mathsf{Perv}(X)$;
2. ${}^pH^i(f_*N) = 0$ *for all* $i \neq 0$.

Remark 21.2.
1. This is already interesting for $Y = \mathrm{pt}$:
   If X is an affine variety, and $M \in \mathsf{Perv}(X)$, then there exists $j\colon U \hookrightarrow X$ as above, such that $H^i(X, j_!(M|_U)) = 0$ for all $i \neq 0$.
2. The proof shows that we may choose U to be $X \setminus H$, where $H \subseteq X$ is a general hyperplane section for $X \hookrightarrow \mathbf{P}^n$ (as long as k is infinite; if k is finite, then you might have to increase the degree of H).

21.2 Nori's basic lemma

Theorem 21.3 (NBL)**.** *Let X be an affine variety of dimension d and let $F \in \mathsf{Cons}(X)$. Then, there exists $j\colon U \hookrightarrow X$ a dense affine open such that $H^i(X, j_!(F|_U)) = 0$ for all $i \neq d$.*

Nori proved this for characteristic zero by using resolution of singularities in one proof, and the analytic topology in another. Huber has some notes explaining why BBL $\Rightarrow$ NBL, hence NBL holds in arbitrary characteristic.

Corollary 21.4. *If X is an affine variety of dimension d, then there exists a filtration by closed subsets*

$$\emptyset = Y_{-1} \subset Y_0 \subset Y_1 \subset \cdots \subset Y_d = X,$$

such that
1. $\dim(Y_j) = j$
2. *If* $Y_{k-1} \overset{i_k}{\hookrightarrow} Y_k \overset{j_k}{\hookleftarrow} U_k = Y_k \setminus Y_{k-1}$ *are the natural maps, then* $H^i(Y_k, j_{k\,!}(\mathbf{Z}/\ell)) = 0$ *for all* $i \neq k$.
3. $\mathbf{R}\Gamma(X, \mathbf{Z}/\ell)$ *is isomorphic to the complex*

$$H^0(Y_0, \mathbf{Z}/\ell) \longrightarrow H^1(Y_1, j_{1\,!}\mathbf{Z}/\ell) \longrightarrow \cdots \longrightarrow H^k(Y_k, j_{k\,!}\mathbf{Z}/\ell) \longrightarrow \cdots \longrightarrow H^d(Y_d, j_{d\,!}\mathbf{Z}/\ell)$$

where

$$\begin{tikzcd} H^k(Y_k, j_{k\,!}\mathbf{Z}/\ell) \arrow[rr, "d"] \arrow[dr, "\text{forget}"'] & & H^{k+1}(Y_{k+1\,!}\mathbf{Z}/\ell) \\ & H^k(Y_k, \mathbf{Z}/\ell) \arrow[ur, "\text{boundary}"'] & \end{tikzcd}$$

where the boundary map comes from the sequence $j_{k+1\,!}\underline{\mathbf{Z}/\ell} \to \underline{\mathbf{Z}/\ell} \to i_{k+1\,*}\underline{\mathbf{Z}/\ell}$.

This is similar to how in algebraic topology if you have a CW complex, you can compute homology using the chain complexes given by the n-skeletons of the space, where in each step you introduce new homology only in dimension n (correpsonding to glueing in new copies of n-spheres).

Remark 21.5.

1. Get explicit chain complexes calculating $\mathbf{R}\Gamma(X, \mathbf{Z}/\ell)$.
2. There exists a version with $\mathbf{Q}$-coefficients for varieties over $\mathbf{C}$, so we get a lift of $\mathbf{R}\Gamma(X, \mathbf{Q})$ to $\mathsf{D}(\mathrm{MHS})$, the derived category of mixed hodge structures. The construction seems to depend on the Y_k, but this only changes the representative of the quasi-isomorphism class.
   Nori refined this to produce a *functor*
$$\begin{Bmatrix} \text{varieties} \\ \text{over } \mathbf{C} \end{Bmatrix} \longrightarrow \mathsf{D}(\mathrm{MHS})$$
   lifting $X \mapsto \mathbf{R}\Gamma(X^{\mathrm{an}}, \mathbf{Q})$. This is part of the formalism of "Nori motives."
3. There exists a version for $\mathbf{Q}_\ell$-coefficients for any field k. Using the formalism of Nori motives from above, you get a functor
$$\begin{Bmatrix} \text{varieties} \\ \text{over } k \end{Bmatrix} \longrightarrow \mathsf{D}\begin{pmatrix} \text{finite dimensional } \mathbf{Q}_\ell\text{-representations} \\ \text{of } G_k = \mathrm{Gal}(\overline{k}/k) \end{pmatrix}$$
   where $X \mapsto \mathbf{R}\Gamma_{\text{ét}}(X_{\overline{k}}, \mathbf{Q}_\ell) \circlearrowleft G_k$ (as opposed to $\mathsf{D}_{\mathsf{fd}}(\text{all } \mathbf{Q}_\ell\text{-representationsof } G_k)$).
4. Similarly, if $k/\mathbf{Q}_p$ is a finite extension, this formalism gives
$$\begin{Bmatrix} \text{varieties} \\ \text{over } k \end{Bmatrix} \longrightarrow \mathsf{D}\begin{pmatrix} \text{finite dimensional "pst" representations} \\ \text{of } G_k \text{ on } \mathbf{Q}_p\text{-vector spaces} \end{pmatrix}$$
   (where "pst" = potentially semistable) lifting
$$\begin{Bmatrix} \text{varieties} \\ \text{over } k \end{Bmatrix} \longrightarrow \mathsf{D}_{\mathsf{fd,pst}}\begin{pmatrix} \text{all representations} \\ \text{of } G_k \text{ on } \mathbf{Q}_p\text{-vector spaces} \end{pmatrix}$$
   where $X \mapsto \mathbf{R}\Gamma(X_{\overline{k}}, \mathbf{Q}_p) \circlearrowleft G_k$.

Proof of NBL $\Rightarrow$ Corollary. If $X = Y_d$, and $F = \underline{\mathbf{Z}/\ell}$, then NBL implies there exists $j\colon U \hookrightarrow X$ a dense affine open such that $H^i(X, j_!\underline{\mathbf{Z}/\ell}) = 0$ for all $i \neq d$. Therefore, $Y_{d-1} = X \setminus U$, so $\dim(Y_{d-1}) = d-1$. By induction, there exists a flag
$$\emptyset = Y_{-1} \subset Y_0 \subset \cdots \subset Y_{d-1}$$
such that
1. $H^i(Y_k, j_{k\,!}\mathbf{Z}/\ell) = 0$ for all $i \neq k$ and $\dim(Y_j) = j$, and
2. $\mathbf{R}\Gamma(Y_{d-1}, \mathbf{Z}/\ell)$ is isomorphic to
$$H^0(Y_0, \mathbf{Z}/\ell) \longrightarrow H^1(Y_1, j_{1\,!}\mathbf{Z}/\ell) \longrightarrow \cdots \longrightarrow H^k(Y_k, j_{k\,!}\mathbf{Z}/\ell) \longrightarrow \cdots \longrightarrow H^{d-1}(Y_{d-1}, j_{d-1\,!}\mathbf{Z}/\ell) \quad (*)$$

We have $j_!\underline{\mathbf{Z}/\ell} \to \underline{\mathbf{Z}/\ell} \to i_*\underline{\mathbf{Z}/\ell}$, where $Y_{d+1} \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U$. Also, $\mathbf{R}\Gamma(X, j_!\underline{\mathbf{Z}/\ell}) \cong H^d(X, j_!\mathbf{Z}/\ell)[-d]$. We therefore get an exact triangle
$$\begin{array}{ccccc} \mathbf{R}\Gamma(X, j_!\mathbf{Z}/\ell) & \longrightarrow & \mathbf{R}\Gamma(X, \mathbf{Z}/\ell) & \longrightarrow & \mathbf{R}\Gamma(Y_{d-1}, \mathbf{Z}/\ell) \\ \| & & & & \| \\ H^d(X, j_!\mathbf{Z}/\ell)[-d] & & & & (*) \end{array}$$

Rotating this triangle, we get
$$(*)[-1] \xrightarrow{\delta} H^d(X, j_!\mathbf{Z}/\ell)[-d] \longrightarrow \mathbf{R}\Gamma(X, \mathbf{Z}/\ell)$$
where the first term is in $\mathsf{D}^{\leq d}$ and the middle term is in $\mathsf{D}^{=d}$. Thus, δ has to be the map
$$H^{d-1}(Y_{d-1}, j_{d-1\,!}\mathbf{Z}/\ell) \xrightarrow{\text{bdy}} H^d(X, j_!\mathbf{Z}/\ell).$$

Taking cones gives you the claim. $\square$

Proof of BBL $\Rightarrow$ NBL.[1] Let X be an affine variety, and $F \in \mathsf{Cons}(X)$. The goal is to find $j\colon U \hookrightarrow X$ a dense affine open such that $H^i(X, j_!(F|_U)) = 0$ for all $i \neq d$. Choose some $k\colon W \hookrightarrow X$ an affine open immersion such that $F|_W$ is locally constant, and W is smooth of dimension d. This implies that $F[d]|_W \in \mathsf{Perv}(W)$ since W is smooth. Set $M = k_!(F[d]|_W) \in \mathsf{Perv}(X)$ by Artin. Now BBL implies there exists $h\colon V \hookrightarrow X$ an affine open immersion such that $H^i(X, h_!(M|_V)) = 0$ for all $i \neq 0$. We therefore get

$$\begin{tikzcd} U \coloneqq W \cap V \arrow[r, hook] \arrow[d, hook] \arrow[dr, "j"] & V \arrow[d, hook, "h"] \\ W \arrow[r, hook, "k"] & X \end{tikzcd}$$

Now $h_!(M|_V) = h_!((k_!(F[d]|_W))|_V) \cong j_!(F[d]|_U)$ by proper base change. Thus, $H^i(X, j_!(F[d]|_U)) = 0$ for all $i \neq 0$, and so $H^i(X, j_!(F|_U)) = 0$ for all $i \neq d$. $\square$

21.3 Proof of Beilinson's basic lemma

We need the following

Lemma 21.6. *If $f\colon X \to Y$ is a smooth morphism with fibres of equidimension $d > 0$, then $f^*[d]$ is t-exact for the perverse t-structure.*

Proof of BBL. Let $f\colon X \to Y$ be an affine map, and let $M \in \mathsf{Perv}(X)$.
Goal. There exists $j\colon U \hookrightarrow X$ a dense affine open immersion such that
1. $N \coloneqq j_!(M|_U) \to M$ is surjective, and
2. ${}^pH^i(f_*N) = 0$ for all $i \neq 0$.

Choose an embedding

$$\begin{tikzcd} X \arrow[r, hook, "k_{\overline{X}}"] \arrow[d, "f"'] & \overline{X} \arrow[d, "\bar{f}"] \\ Y \arrow[r, hook, "k_{\overline{Y}}"] & \overline{Y} \end{tikzcd}$$

with $\overline{X}, \overline{Y}$ projective, and $k_{\overline{X}}, k_{\overline{Y}}$ are affine open immersions. We can therefore define

$$\overline{M} = k_{\overline{X}*}(M) \in \mathsf{Perv}(\overline{X})$$

since $k_{\overline{X}}$ is affine.
Goal′. There exists $\overline{H} \subseteq \overline{X}$ a hyperplane section such that

$$\begin{tikzcd} V = \overline{X} \setminus \overline{H} \arrow[r, hook, "j_{\overline{X}}"] & \overline{X} & \overline{H} \arrow[l, hook', "i_{\overline{X}}"'] \\ U = X \setminus H \arrow[r, hook, "j_X"] \arrow[u, hook, "k_V"'] & X \arrow[u, hook, "k_{\overline{X}}"'] & H = \overline{H} \cap X \arrow[l, hook', "i_X"'] \arrow[u, hook, "k_{\overline{H}}"'] \end{tikzcd}$$

such that
1_H. $j_{\overline{X}!}(\overline{M}|_V) \twoheadrightarrow \overline{M}$ is surjective
2_H. The canonical map $j_{\overline{X}!}k_{V*}(\overline{M}|_U) \to k_{\overline{X}*}j_{X!}(\overline{M}|_U)$ is an isomorphism.

Proof that Goal′ $\Rightarrow$ Goal. Set $N = j_{X!}(M|_U)$.
$1_H \Rightarrow 1$ since restriction along $k_{\overline{X}}$ is exact.
$2_H \Rightarrow 2$. We have

$$\begin{aligned} f_*N &= f_*j_{X!}(M|_U) \\ &= \bar{f}_*k_{\overline{X}*}(j_{X!}(M|_U))|_Y \\ &= \bar{f}_*(j_{\overline{X}!}k_{V*}(\overline{M}|_U))|_Y && \text{(by } 2_H\text{)} \\ &= (\bar{f} \circ j_{\overline{X}})_!(k_V(\overline{M}|_U))|_Y && \text{(by definition)} \end{aligned}$$

[1] Look at Motives notes from TIFR.

But $(\bar{f} \circ j_{\overline{X}})$ is affine as V is so. Thus, $(\bar{f} \circ j_{\overline{X}})_!(k_{V*}(\overline{M}|_U)) \in {}^p\mathsf{D}^{\geq 0}(\overline{Y})$. Thus, $f_*N \in {}^p\mathsf{D}^{\geq}(Y)$. But f was affine, so $f_*N \in {}^p\mathsf{D}^{\geq 0}(Y)$, and therefore $f_*N \in \mathsf{Perv}(Y)$. $\square$

Now we want to show 1_H and 2_H. One can check:
$1_H \Leftrightarrow 1'_H$. ${}^pH^i(i^*_{\overline{X}}(\overline{M})) = 0$ for all $i \neq -1$.
$2_H \Leftrightarrow 2'_H$. $i^*_{\overline{X}} k_{X*}(M) \simeq k_{\overline{H}*} i^*_X(M)$. $\square$

22 December 10, 2015

We will meet next Monday, 1PM–2PM for a makeup session, and on Tuesday at a time to be determined.

22.1 Beilinson's basic lemma

Recall we are working with only X, Y quasi-projective.

Theorem 22.1. *Let $f\colon X \to Y$ be an affine map, and $M \in \mathsf{Perv}(X)$. Then, there exists $j\colon U \hookrightarrow X$ an affine open dense immersion such that*
1. *$j_!(M|_U) \to M$ is surjective;*
2. *${}^pH^i(f_*(j_!(M|_U))) = 0$ for all $i \neq 0$.*

You can think about how the perverse t-structure looks like on the level of cohomology—there is a nice paper about this with "perverse" and "Leray" in the title.

Choose $k_{\overline{X}}\colon X \hookrightarrow \overline{X}$, where $\overline{X}$is projective, and $k_{\overline{X}}$ is a dense affine open immersion. Last time, we showed that the theorem is proved by:

Goal 22.2. There exists $\overline{H} \hookrightarrow \overline{X}$ an ample divisor such that $H = \overline{H} \cap X \subseteq X$.

$$\begin{tikzcd} \overline{H} \arrow[r, hook, "i_{\overline{X}}"] & \overline{X} \\ H \arrow[u, hook, "k_{\overline{H}}"] \arrow[r, hook, "i_X"] & X \arrow[u, hook, "k_{\overline{X}}"'] \end{tikzcd} \tag{2}$$

Set $\overline{M} = k_{\overline{X}*}(M) \in \mathsf{Perv}(\overline{X})$; we have:
1_H. ${}^pH^i(i^*_{\overline{X}}(\overline{M})) = 0$ for all $i \neq -1$.
2_H. $i^*_{\overline{X}} k_{X*}(M) \simeq k_{\overline{H}*} i^*_X(M)$.

Proof for 1_H. Fix some $\overline{H} \subseteq \overline{X}$. We know that $i^*_{\overline{X}}(\overline{M}) \in {}^p\mathsf{D}^{[-1,0]}(\overline{H})$. Then,

$$\begin{aligned} 1_H \text{ is false} &\iff {}^pH^0(i^*_{\overline{X}}(\overline{M})) \neq 0 \\ &\iff i_{\overline{X}*}{}^pH^0(i^*_{\overline{X}}(\overline{M})) \neq 0 \end{aligned}$$

Now recall that $\overline{M} \xrightarrow{\text{can}} i_{\overline{X}*}{}^pH^0(i^*_{\overline{X}}(M))$ is the largest quotient of $\overline{M}$ supported on $\overline{H}$. Now,

$$\begin{aligned} 1_H \text{ is false} &\iff \exists \text{ non-zero quotient } \overline{M} \to Q \text{ with } \operatorname{Supp}(Q) \subseteq \overline{H} \\ &\iff \exists \text{ non-zero quotient } \overline{M} \to Q,\ Q \textit{ simple}, \text{ such that } \operatorname{Supp}(Q) \subseteq \overline{H} \end{aligned}$$

Now $\overline{M}$ has only finitely many simple subquotients S_i, and each $S_i = \alpha_{i*}(\beta_{i*}(L))$, where $\alpha_i\colon Z_i \hookrightarrow X$ is closed, and $\beta_i\colon V_i \hookrightarrow Z_i$ is a dense open, smooth of dimension d, and L is a simple perverse sheaf supported on all of V_i. Thus, $Q = S_i$ for some i, and this implies $\operatorname{Supp}(Q) = \operatorname{Supp}(S_i)$, and so $\overline{\operatorname{Supp}(Q)} = \overline{\operatorname{Supp}(S_i)} = Z_i$, and $Z_i \subseteq \overline{H}$. For general H, this will not happen. [Mircea points out this can be made easier by choosing a point in the support and making sure H doesn't go through that point, which is true for a general H]. $\square$

Proof for 2_H. Work over the universal family:

$$P = \text{projective space of all hyperplanes in } \overline{X}, \qquad N := \dim(P)$$

and the universal hyperplane

$$\begin{tikzcd} \overline{\mathcal{H}} \arrow[r, hook] \arrow[dr] & \overline{X} \times P \arrow[d] \\ & P \end{tikzcd}$$

Note $\dim(\mathcal{H}) = N + \dim(X) - 1$.

All objects have an analogue "relative to P", that is,

$$\begin{tikzcd} \overline{H} \arrow[r, squiggly] \arrow[d, hook] & \overline{\mathcal{H}} \arrow[d, hook] \\ \overline{X} & \overline{X} \times P \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, squiggly] \arrow[d, hook] & \mathcal{H} \arrow[d, hook] \\ X & X \times P \end{tikzcd}$$

$X \subseteq \overline{X}$ gives an inclusion $X \times P \subseteq \overline{X} \times P$. If $\overline{M} \in \mathsf{Perv}(\overline{X})$, this gives $\overline{M}_P := \overline{\mathrm{pr}}_1^*(\overline{M})[N] \in \mathsf{Perv}(\overline{X} \times P)$, where $\overline{\mathrm{pr}}_1 \colon \overline{X} \times P \to \overline{X}$. $M \in \mathsf{Perv}(X)$ also induces $M_P \in \mathsf{Perv}(X \times P)$.

We get:

$$\begin{tikzcd} \overline{\mathcal{H}} \arrow[r, hook, "i_{\overline{X}\times P}"] & \overline{X} \times P \arrow[r, "\overline{\mathrm{pr}_1}"] & \overline{X} \\ \mathcal{H} \arrow[r, hook, "i_{X\times P}"] \arrow[u, hook, "k_{\overline{H}}"] & X \times P \arrow[r, "\mathrm{pr}_1"] \arrow[u, hook, "k_P"] & X \arrow[u, hook, "k_X"] \end{tikzcd} \tag{3}$$

such that the fibre of the left square over $[H] \in P$ in (2).

Key Observations 22.3. $\overline{\mathcal{H}} \to \overline{X}$ is projective and smooth

1. of dimension $N - 1$;
2. same for $\mathcal{H} \to X$.

Proof. It's a projective space bundle! □

Claim 22.4. *We have base change in* (3)*:*

2_P: $i^*_{\overline{X}\times P} k_{P\,*}(M_P) \cong k_{\overline{H}\,*} i^*_{X\times P}(M_P)$.

Proof. Note: $M_P = \mathrm{pr}_1^*(M)[N]$. Smooth base change for the square on the right:

$$k_{P\,*}\mathrm{pr}_1^*(M) \cong \overline{\mathrm{pr}}_1^* k_{\overline{X}\,*}(M)$$
$$i^*_{\overline{X}\times P} k_{P\,*}\mathrm{pr}_1^*(M) \cong i^*_{\overline{X}\times P}\overline{\mathrm{pr}}_1^* k_{\overline{X}\,*}(M)$$

by applying $i^*_{\overline{X}\times P}$ to both sides in the first line. By smooth base change for the outer square, because $\overline{\mathcal{H}} \to \overline{X}$ is smooth, we have that this last object is isomorphic to $k_{\overline{H}\,*} i^*_{X\times P}\mathrm{pr}_1^*(M)$. Now shifting both by $[N]$ gives the statement 2_P. □

Now recall the following

Theorem 22.5 (Deligne, "Generic Base Change")**.** *Fix a variety S, and let $f \colon X \to Y$ be any map of S-varieties. Fix $K \in \mathsf{D}(X)$. Then, there exists a dense open subset $U \subseteq S$ such that the formation of f_*K commutes with any base change along $T \to U \subseteq S$.*

In our setup, with $P = S$, this gives the following:

There exists $V \subseteq P$ a dense open subset such that for all $[H] \in V$, we have 2_H (from 2_P). □

22.2 Back to homological algebra

Motivating question. Let $\mathscr{D}$ be a triangulated category, and assume $\mathscr{D}$ is equipped with a t-structure whose heart is $\mathscr{D}^\heartsuit$. Is there a canonical exact functor $\mathsf{D}^b(\mathscr{D}^\heartsuit) \to \mathscr{D}$ extending $\mathscr{D}^\heartsuit \hookrightarrow \mathscr{D}$?

In this generality, it's unclear whether this is true. But if we put some additional structure on $\mathscr{D}$, then the answer is yes.

- filtered derived categories (Beilinson);
- ∞-categorical enhancement (Lurie).

Today, we will check this for $\mathscr{A}$ an abelian category (e.g., étale sheaves on a scheme) and $\mathscr{D} = \mathsf{D}^b(\mathscr{A})$ (e.g., the derived category of étale sheaves with the perverse t-structure).

22.2.1 Filtered derived categories

Let

$$F\mathscr{A} = \mathsf{Fun}'(\mathbf{Z}^{\mathsf{opp}}, \mathscr{A}) = \left\{ \{\cdots \to M^{i+1} \to M^i \to \cdots\} =: \{M^i\} \;\middle|\; \begin{matrix} M^i = 0 \ \forall i \gg 0, \\ M^{i+1} \xrightarrow{\sim} M^i \ \forall i \ll 0 \end{matrix} \right\}$$

Lemma 22.6.

1. *$F\mathscr{A}$ is abelian.*
2. *$\{M^i\} \mapsto M^i$ gives an exact functor $F^i \colon F\mathscr{A} \to \mathscr{A}$.*
3. *$\{M^i\} \mapsto \operatorname{colim}_i M^i$ is an exact functor $F\mathscr{A} \xrightarrow{\omega} \mathscr{A}$.*
4. *$\{i \mapsto M^i\} \mapsto \{i \mapsto M^{i-1}\}$ is an equivalence $s \colon F\mathscr{A} \xrightarrow{\sim} F\mathscr{A}$. There is a canonical map $\mathrm{id} \to s$.*
5. *$M \mapsto \{\cdots \to 0 \to 0 \to M \xrightarrow{\sim} M \xrightarrow{\sim} M \xrightarrow{\sim} \cdots\}$, where the first nonzero term is in degree 0, gives an exact functor $\mathscr{A} \overset{i}{\hookrightarrow} F\mathscr{A}$.*

Definition 22.7. $\mathsf{D}\,F = \mathsf{D}^b(F\mathscr{A})$ is the *filtered derived category of $\mathscr{A}$*. We get exact functors

1. $\omega \colon \mathsf{D}\,F \to \mathscr{D}$
2. $F^i \colon \mathsf{D}\,F \to \mathscr{D}$
3. $s \colon \mathsf{D}\,F \xrightarrow{\sim} \mathsf{D}\,F$, $\mathrm{id} \to s$
4. $i \colon \mathscr{D} \hookrightarrow \mathsf{D}\,F$
5. $\mathrm{gr}^i \colon \mathsf{D}\,F \to \mathscr{D}$ fitting into an exact triangle:

$$F^{i+1}(-) \longrightarrow F^i(-) \longrightarrow \mathrm{gr}^i(-) \longrightarrow F^{i+1}(-)[1]$$

Example 22.8. Let $K \in \mathsf{Ch}^b(\mathscr{A})$. We can then get $\underline{K} \in \mathsf{D}\,F$ such that $F^i(\underline{K}) = 0 \to \cdots \to 0 \to K^i \to K^{i+1} \to K^{i+2} \to \cdots$ (by using the stupid filtration), such that $\mathrm{gr}^i(K) = K^i[-i]$.

Definition 22.9. $\mathsf{D}\,F(\geq n) = \{K \in \mathsf{D}\,F \mid \mathrm{gr}^i(K) = 0 \ \forall i < n\}$; $\mathsf{D}\,F(\leq n) = \{K \in \mathsf{D}\,F \mid \mathrm{gr}^i(K) = 0 \ \forall i > n\}$.

Lemma 22.10.

1. *The inclusion $\mathsf{D}\,F(\geq n) \hookrightarrow \mathsf{D}\,F$ has a right adjoint $\sigma^{\geq n}$.*
2. *The inclusion $\mathsf{D}\,F(\leq n) \hookrightarrow \mathsf{D}\,F$ has a left adjoint $\sigma^{\leq n}$.*
3. *$\mathrm{gr}^n \cong i^{-1} s^{-n} \sigma^{\leq n} \sigma^{\geq n}$.*
4. *There exists a* unique *exact triangle (for each $X \in \mathsf{D}\,F$)*

$$\sigma^{\geq 1}(X) \longrightarrow X \longrightarrow \sigma^{\leq 0}(X) \xrightarrow{d} \sigma^{\geq 1}(X)[1]$$

Proof (intuitive). Let

$$\underline{K} = \{\cdots \to F^{n+1}(K) \to F^n(K) \to F^{n-1}(K) \to \cdots\} \in \mathsf{D}\,F$$
$$\sigma^{\geq n}(\underline{K}) = \{\cdots \to F^{n+1}(K) \to F^n(K) \xrightarrow{\sim} F^{n-1}(K) \xrightarrow{\sim} \cdots\}$$

Proposition 22.11. *Fix a t-structure $(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$ on $\mathscr{D}$. Then, there exists a unique t-structure $(\mathsf{D}\,F^{\leq 0}, \mathsf{D}\,F^{\geq 0})$ on $\mathsf{D}\,F$ such that*

1. *$i \colon \mathscr{D} \hookrightarrow \mathsf{D}\,F$ is t-exact,*
2. *$s(\mathsf{D}\,F^{\geq 0}) = \mathsf{D}\,F^{\leq -1}$.*

Explicitly,

$$\begin{aligned}
\mathsf{D}\,F^{\leq 0} &= \left\{ K \in \mathsf{D}\,F \;\middle|\; \mathrm{gr}^i(K) \in \mathscr{D}^{\leq i} \right\} \\
\mathsf{D}\,F^{\geq 0} &= \left\{ K \in \mathsf{D}\,F \;\middle|\; \mathrm{gr}^i(K) \in \mathscr{D}^{\geq i} \right\} \\
\mathsf{D}\,F^{\heartsuit} &= \left\{ K \in \mathsf{D}\,F \;\middle|\; \mathrm{gr}^i(K) \in \mathscr{D}^{\leq i} \cap \mathscr{D}^{\geq i} \right\} \\
&= \left\{ K \in \mathsf{D}\,F \;\middle|\; \mathrm{gr}^i(K) = M_i[-i],\ M_i \in \mathscr{D}^{\heartsuit} \right\}
\end{aligned}$$

We have the (unique) exact triangle

$$\begin{tikzcd}
\dfrac{F^{i+1}(K)}{F^{i+2}(K)} \arrow[r] \arrow[d, equal] & \dfrac{F^{i}(K)}{F^{i+2}(K)} \arrow[r] & \dfrac{F^{i}(K)}{F^{i+1}(K)} \arrow[d, equal] \\
\mathrm{gr}^{i+1}(K) & & \mathrm{gr}^{i}(K)
\end{tikzcd}$$

The boundary map $\mathrm{gr}^i(K) \xrightarrow{d} \mathrm{gr}^{i+1}(K)[1]$. Now, for $K \in \mathsf{D}\,F^{\heartsuit}$, we get $\mathrm{gr}^i(K) \cong M_i[-i]$, $M_i \in \mathscr{D}^{\heartsuit}$. The boundary map gives $M_i \xrightarrow{d} M_{i+1}$. We therefore get a chain complex

$$\{\cdots \longrightarrow \cdots \longrightarrow M_{i+1} \longrightarrow M_{i+2} \longrightarrow \cdots\} \in \mathsf{Ch}^b(\mathscr{D}^{\heartsuit}).$$

Theorem 22.12. *This gives an equivalence* $\mathsf{Ch}^b(\mathscr{D}^{\heartsuit}) \xleftarrow{\sim} \mathsf{D}\,F^{\heartsuit}$.

We therefore obtain

$$\begin{tikzcd}
\mathsf{Ch}^b(\mathscr{D}^{\heartsuit}) \arrow[r] \arrow[ddr, "\widetilde{\mathrm{real}}"'] & \mathsf{D}\,F^{\heartsuit} \arrow[d, hook] \\
& \mathsf{D}\,F \arrow[d, "\omega"] \\
& \mathscr{D}
\end{tikzcd}$$

Need to check: $H^i \circ (\widetilde{\mathrm{real}})$ is cohomological. We therefore get

$$\begin{tikzcd}
\mathsf{Ch}^b(\mathscr{D}^{\heartsuit}) \arrow[r] \arrow[d, "\mathrm{can}"] & \mathsf{D}\,F \arrow[d, "\omega"] \\
\mathsf{D}^b(\mathscr{D}^{\heartsuit}) \arrow[r, "\mathrm{real}"] & \mathscr{D}
\end{tikzcd}$$

□