D-Modules and Mixed Hodge Modules

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Contents

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1 September 19 (Harold Blum)

Today we will discuss basic properties of D-modules. Note X will always be a smooth complex variety.

1.1 Definitions [HTT08, §1.1]

Let Θ_X be the sheaf of derivations on X, that is,

$$\Theta_X = \mathcal{D}er_{\mathbf{C}_X}(\mathcal{O}_X) = \{\theta \in \mathcal{E}nd_{\mathbf{C}_X} \mid \theta(fg) = \theta(f)g + f\theta(g) \text{ for all } f, g \in \mathcal{O}_X\},$$

where "$s \in \mathscr{F}$" for a sheaf $\mathscr{F}$ denotes a local section s of $\mathscr{F}$. Both Θ_X and the structure sheaf $\mathcal{O}_X$ are subsheaves of $\mathcal{E}nd(\mathcal{O}_X)$, where $f \in \mathcal{O}_X$ corresponds to $[\mathcal{O}_X \ni g \mapsto fg \in \mathcal{O}_X] \in \mathcal{E}nd_{\mathbf{C}_X}(\mathcal{O}_X)$.

Definition 1.1. The *sheaf of differential operators* on X is defined as

$$D_X \coloneqq \langle \Theta_X, \mathcal{O}_X \rangle \subseteq \mathcal{E}nd(\mathcal{O}_X),$$

that is, D_X is the sub-$\mathbf{C}$-algebra of $\mathcal{E}nd(\mathcal{O}_X)$ generated by Θ_X and $\mathcal{O}_X$.

Remark 1.2. Over singular varieties, you can still study this ring of differential operators, but it is pretty crazy (e.g., it is not finitely generated), and so the notion of a D-module is defined differently. Let Z be a singular variety, and suppose that it can be embedded into a smooth variety X. Then, we define D_Z-modules as those D_X-modules with support on Z. By Kashiwara's theorem, this definition is independent of the embedding $Z \hookrightarrow X$. Even if there isn't such an embedding, you can define D_Z-modules locally by embedding Z locally, and patching together.

We can also describe D_X locally using coordinates. Let U be an affine open, and let $\{x_i\}$ be a local coordinate system, which we recall is a set $\{x_i, \partial_i\}$ where $x_i \in \mathcal{O}_X(U)$ and $\partial_i \in \Theta_X(U)$, such that

$$[\partial_i, \partial_j] = 0, \quad \partial_i(x_j) = \delta_{ij}, \quad \Theta_U = \bigoplus_{i=1}^{n} \mathcal{O}_U \partial_i,$$

and which exists by [HTT08, Thm. A.5.1]. Then, we have

$$D_U = \bigoplus_{\alpha \in \mathbf{N}^n} \mathcal{O}_U \partial^\alpha$$

where $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. You can check that these ∂^α indeed generate D_U over $\mathcal{O}_U$, and that they are also independent over $\mathcal{O}_U$ as is done for $U \cong \mathbf{A}^n$ in [Cou95, Ch. 1].

Definition 1.3. We define the *order filtration* F on D_U locally by

$$F_\ell D_U \coloneqq \sum_{|\alpha| \le \ell} \mathcal{O}_U \partial^\alpha.$$

You can also define the filtration globally by

$$F_\ell D_X(V) \coloneqq \{P \in D_X(V) \mid P|_U \in F_\ell D(U) \text{ for all open affine } U \subseteq X\}.$$

Remark 1.4. Since this definition requires a choice of coordinates, you need to check that using commutator relations does not increase the order, and that the order is independent of coordinate systems. On the other hand, you can also define the order filtration more intrinsically as in [Cou95, Ch. 3] as follows:

$$F_0 D_X = \mathcal{O}_X, \quad F_\ell D_X = \{P \in \mathcal{E}nd_{\mathbf{C}}(\mathcal{O}_X) \mid [P, g] \in F_{\ell-1} D_X \text{ for all } g \in \mathcal{O}_X\}.$$

Coutinho in [Cou95, Thm. 3.2.3] shows that these two definitions are equivalent, at least for $\mathbf{A}^n$.

Remark 1.5. Recall the Bernstein filtration defined in [Cou95, Ch. 1]. This filtration doesn't make sense globally, in particular because it doesn't even exhaust everything: not all functions can be written as polynomials. However, the Bernstein filtration is easier to prove things with than the order filtration when it can be defined, especially when studying holonomicity.

Note 1.6. If $P \in F_m D_X$ and $Q \in F_n D_X$, then $P \cdot Q \in F_{m+n} D_X$ by [Cou95, Prop. 3.1.2]. We can also show $[P,Q] \in F_{m+n-1} D_X$ by induction on $m+n$. If $m+n=0$, then the claim is clear; for $m+n>0$, we use Jacobi's identity to obtain

$$[[P,Q],g] = [Q,[g,P]] + [P,[Q,g]],$$

and by induction, $[g,P] \in F_{m-1} D_X$ and $[Q,g] \in F_{n-1} D_X$, so $[Q,[g,P]]$ and $[P,[Q,g]]$ are in $F_{m+n-2} D_X$.

Definition 1.7. The *graded ring* associated to the filtration F on D_X is defined as

$$\operatorname{gr}^F D_X \coloneqq \bigoplus_{\ell=0}^{\infty} \operatorname{gr}_\ell^F(D_X), \quad \operatorname{gr}_\ell^F(D_X) \coloneqq F_\ell D_X / F_{\ell-1} D_X.$$

By Note 1.6, we have

Key Property 1.8. $\operatorname{gr}^F D_X$ is commutative.

Note 1.9. Key Property 1.8 implies that there exists a map $\operatorname{Sym} \Theta_X \to \operatorname{gr}^F D_X$ of $\mathcal{O}_X$-algebras, by the universal property of the symmetric algebra. This map is in fact an isomorphism: locally, if $U \subseteq X$ is an affine open set, with local coordinates $\{x_i, \partial_i\}$, then

$$\operatorname{gr}^F D_U = \mathcal{O}_U[\xi_1, \ldots, \xi_n],$$

where the ξ_i are the images of ∂_i in $\operatorname{gr}^F D_U$. Thus, we have an isomorphism $\pi_* \mathcal{O}_{T^*X} \to \operatorname{gr}^F D_X$, where $\pi\colon T^*X \to X$ is the cotangent bundle on X.

Definition 1.10. A *left D-module* is a sheaf M on X such that $M(U)$ is a left $D_X(U)$-module for each open $U \subseteq X$. A *right D-module* is defined similarly.

We denote by $\mathsf{Mod}(D_X)$ the collection of left D-modules, and $\mathsf{Mod}(D_X^{\mathsf{op}})$ the collection of right D-modules.

Example 1.11. Consider s differential operators $P_1, \ldots, P_s \in D_{\mathbf{A}^n}$. Then, consider

$$M \coloneqq D_{\mathbf{A}^n} \big/ D_{\mathbf{A}^n} P_1 + \cdots + D_{\mathbf{A}^n} P_s$$

Claim [Cou95, Thm. 6.1.2]**.** $\operatorname{Hom}_{D_{\mathbf{A}^n}}(M, \mathcal{O}_{\mathbf{A}^n}) = \{f \in \mathcal{O}_{\mathbf{A}^n} \mid P_i f = 0 \text{ for all } i\}$.

Proof. If $\varphi \in \operatorname{Hom}_{D_{\mathbf{A}^n}}(M, \mathcal{O}_{\mathbf{A}^n})$, then we can look at $\varphi(1) = f$. We know

$$0 = \varphi(P_i \cdot 1) = P_i \varphi(1) = P_i(f).$$

In the other direction, you can send 1 to f.

Alternatively, this is really something general about maps $R/I \to R$. □

1.2 The correspondence between D-modules and connections [HTT08, §1.2]

Proposition 1.12. *Suppose M is an $\mathcal{O}_X$-module. Then, giving a (left) D-module structure on M which extends its $\mathcal{O}_X$-module structure is equivalent to giving a $\mathbf{C}$-linear map*

$$\begin{aligned} \nabla\colon \Theta &\longrightarrow \mathcal{E}nd_{\mathbf{C}}(M) \\ \theta &\longmapsto \nabla_\theta \end{aligned}$$

satisfying the following properties:

(1) $\nabla_{f\theta}(s) = f\nabla_\theta(s)$
(2) $\nabla_\theta(fs) = \theta(f)s + f\nabla_\theta(s)$
(3) $\nabla_{[\theta_1,\theta_2]}(s) = [\nabla_{\theta_1}, \nabla_{\theta_2}](s)$

where $f \in \mathcal{O}_X$, $\theta, \theta_1, \theta_2 \in \Theta_X$, and $s \in M$.

There isn't too much to show, since D_X is generated by $\mathcal{O}_X$ and Θ_X, and satisfies the key relation $[\theta, f] = \theta(f)$ from [HTT08, Exc. 1.1.1(4)].

Note 1.13. When M is locally free, a map ∇ satisfying (1) and (2) is called a *connection*, and is called an *integrable* or *flat* connection if it satisfies (3) as well.

Proposition 1.12 also has an adjoint description: a map $\nabla\colon \Theta \to \mathcal{E}nd_{\mathbf{C}}(M)$ is equivalent to a map $\nabla^*\colon M \to \Omega \otimes M$, since in the forward direction, you can define

$$\begin{aligned}\nabla^*\colon M &\longrightarrow \Omega \otimes_{\mathbf{C}} M\\ u &\longmapsto \sum dx_i \otimes \nabla_{\partial_i}(u)\end{aligned}$$

and in the opposite direction, you can define

$$\begin{aligned}\nabla\colon \Theta &\longrightarrow \mathcal{E}nd(M)\\ D &\longmapsto [u \mapsto (D \otimes 1)\nabla^* u]\end{aligned}$$

The conditions (1) and (2) above then translate to

(1*) The map ∇^* in fact descends to a map $M \to \Omega \otimes_{\mathcal{O}_X} M$;

(2*) $\nabla^*(fu) = f\nabla^*(u) + df \otimes u$.

To formulate (3*), we note that (1*) and (2*) imply that there exist unique maps

$$\Omega^p \otimes M \xrightarrow{\nabla^*} \Omega^{p+1} \otimes M \tag{1.1}$$

for each p, such that for every $\omega_1 \in \Omega^q$ and $\alpha \in \Omega^{p-q} \otimes M$, we have

$$\nabla^*(\omega_1 \wedge \alpha) = d\omega_1 \wedge \alpha + (-1)^q \omega_1 \wedge \nabla^* \alpha.$$

Now we can formulate the integrability condition:

(3*) The composition

$$M \xrightarrow{\nabla^*} \Omega \otimes_{\mathcal{O}_X} M \xrightarrow{\nabla^*} \Omega^2 \otimes M$$

is zero, and using the map ∇^* in (1.1) to extend this sequence of maps, the resulting chain is in fact a complex.

In this way, the adjoint description of Proposition 1.12 gives the de Rham complex for free. This is often used by people who study dg-algebras, since the de Rham complex forms a dg-algebra with multiplication given by the map ∇^*.

Since vector bundles with integrable connection have vanishing Chern classes, we have

Claim 1.14. *Let X be a smooth projective curve, and $\mathscr{L}$ a line bundle on X. Then, $\mathscr{L}$ has a D_X-module structure if and only if* $\deg \mathscr{L} = 0$.

1.3 The relationship between left and right D-modules [HTT08, §1.2]

We want to take $M \in \mathsf{Mod}(D_X)$, and associate to it a right D-module $M_R \in \mathsf{Mod}(D_X^{\mathsf{op}})$. Locally, this is simple and is done in [Cou95, §16.2]: if $\{x_i, \partial_i\}$ are local coordinates, and $P = \sum_\alpha a_\alpha(x)\partial^\alpha \in D_X$, then

$${}^tP = \sum (-1)^{|\alpha|} \partial^\alpha a_\alpha(x),$$

which satisfies ${}^tPQ = {}^tQ\,{}^tP$, that is, ${}^t-$ gives a ring anti-automorphism of D_X. If $M \in \mathsf{Mod}(D_X)$, we can get a right action by setting

$$m \cdot P := {}^tP \cdot m, \quad m \in M,\ P \in D_X.$$

Since $m \cdot (PQ) = {}^tQ\,{}^tP \cdot m = (m \cdot {}^tP)Q$, this definition gives a right D_X-module structure on M.

Now we globalize this construction. It turns out we need to use the sheaf of n-forms to make this work.

Claim 1.15. *Ω_X^n has a natural right D_X-module structure, given by*

$$\omega \cdot \theta = -(\operatorname{Lie}\theta)\omega, \tag{1.2}$$

where Lie *is the Lie derivative, defined below.*

Definition 1.16. The *Lie derivative* of a differential operator $\theta \in D_X$ is the map

$$\begin{aligned}\operatorname{Lie}\theta\colon \Omega_X^n &\longrightarrow \Omega_X^n\\ \omega &\longmapsto \theta(\omega(\theta_1,\dots,\theta_n)) - \sum_{i=1}^n \omega(\theta_1,\dots,[\theta,\theta_i],\dots,\theta_n)\end{aligned}$$

The Lie derivative comes from differential geometry, and satisfies the following properties:
(1) $(\operatorname{Lie}[\theta_1,\theta_2])\omega = [\operatorname{Lie}\theta_1, \operatorname{Lie}\theta_2]\omega$;
(2) $(\operatorname{Lie}\theta)(f\omega) = f((\operatorname{Lie}\theta)\omega) + \theta(f)\omega$;
(3) $(\operatorname{Lie}(f\theta))\omega = (\operatorname{Lie}\theta)(f\omega)$.
Properties (1) and (3) turn the definition in (1.2) into a right D_X-action. Property (1) also explains the sign in (1.2): without the negative sign, we would have a left D_X-action, instead of a right D_X-action.

Locally, we have $f\,dx_1 \wedge \cdots \wedge dx_n \cdot P = ({}^tPf)dx_1 \wedge \cdots \wedge dx_n$; also, the right D-module structure on Ω_X^n gives a map

$$D_X^{\mathsf{op}} \longrightarrow \mathcal{E}nd_{\mathbf{C}}(\Omega_X^n).$$

Proposition 1.17 (Tensoring D-modules)**.** *Let $M, N \in \mathsf{Mod}(D_X)$ and $M' \in \mathsf{Mod}(D_X^{\mathsf{op}})$. Then,*
(i) $M \otimes_{\mathcal{O}_X} N \in \mathsf{Mod}(D_X)$, with the left action given by $\theta\cdot(m\otimes n) = (\theta m)\otimes n + m \otimes (\theta n)$;
(ii) $M' \otimes_{\mathcal{O}_X} N \in \mathsf{Mod}(D_X^{\mathsf{op}})$, with the right action given by $(m'\otimes n)\theta = m'\theta\otimes n - m'\otimes\theta n$.
Here, $\theta \in \Theta_X$.

We give an example of how to prove this, using the connection formulation 1.12:

Proof of (i). Condition (3) says that $\theta(f(m\otimes n)) = \theta(f)(m\otimes n) + f(\theta(m\otimes n))$, which we can check:

$$\begin{aligned}\theta(f(m\otimes n)) &= \theta(fm\otimes n)\\ &= (\theta fm)\otimes n + fm\otimes\theta n\\ &= (\theta f)m\otimes n + f\theta(m)\otimes n + fm\otimes \theta n\\ &= \theta(f)m\otimes n + f\theta(m)\otimes n + fm\otimes\theta(n)\\ &= \theta(f)(m\otimes n) + f(\theta(m\otimes n))\end{aligned}$$

□

Note it's not clear that the formulas in Proposition 1.17 that the actions are balanced.

Proposition 1.18. *We have an equivalence of categories*

$$\Omega_X^n \otimes_{\mathcal{O}_X} -\colon \mathsf{Mod}(D_X) \to \mathsf{Mod}(D_X^{\mathsf{op}}),$$

with quasi-inverse given by $(\Omega_X^n)^\vee \otimes_{\mathcal{O}_X} -$.

1.4 *D*-modules that are coherent over $\mathcal{O}$ [HTT08, Thm. 1.4.10]

Proposition 1.19. *If M is a D_X-module, and M is coherent over $\mathcal{O}_X$, then M is locally free.*

Proof. The idea is to work locally; we want to show that for all closed points $x \in X$, the module M_x is free. Choose local coordinates $\{x_i, \partial_i\}$. By Nakayama's lemma, there exist sections $s_1,\dots,s_m \in M_x$ that generate M_x, such that $\overline{s}_1,\dots,\overline{s}_m \in M_x/\mathfrak{m}_x M_x$ form a basis for this vector space. We want to show that $s_1,\dots,s_m$ have no non-trivial relation. Assume $\sum f_i s_i = 0$, where $f_i \in \mathcal{O}_{X,x}$. Define the *order* $\operatorname{ord}_x f$ of $f \in \mathcal{O}_{X,x}$ as $\max\{\ell \mid f \in \mathfrak{m}^\ell\}$. If the minimal order of the f_i is zero, then the residue of $\sum f_i s_i = 0$ in $\mathcal{O}_{X,x}/\mathfrak{m}_x$ gives a non-trivial relation on the $\overline{s}_i$, which is a contradiction. Otherwise, let i_0 be the index such that $\operatorname{ord}_x(f_{i_0})$ is minimal, and choose j such that $\partial f_i/\partial x_j \neq 0$. Then, we can act on the relation $\sum f_i s_i = 0$ by ∂_j to obtain

$$\sum(\partial_j f_i)s_i = \sum\left(\frac{\partial f_i}{\partial x_j}\right)s_i + \sum f_i\partial_j s_i = 0.$$

Since

$$\operatorname{ord}_x\left(\frac{\partial f_{i_0}}{\partial x_j}\right) < \operatorname{ord}_x(f_{i_0}) \leq \operatorname{ord}_x(f_i)$$

for all i by choice of i_0, writing $\partial_j s_i = \sum_k h^i_k s_k$, we can combine terms to get a non-trivial relation

$$\sum g_i s_i = 0$$

where

$$\min_i \{\operatorname{ord}_x(g_i)\} \le \operatorname{ord}_x\left(\frac{\partial f_{i_0}}{\partial x_j}\right).$$

Thus, repeating this process gives relations $\sum f_i s_0 = 0$ with ever-decreasing minimal orders, and so eventually we have a non-trivial relation $\sum h_i s_i = 0$ with $h_i \in \mathcal{O}_{X,x}/\mathfrak{m}_x \cong \mathbf{C}$, contradicting our choice of s_i. □

2 September 26: The Classical Riemann–Hilbert Correspondence (Mircea Mustaţă)

The basic framework for this theorem is the analytic category. We will explain how to get back to the algebraic case using GAGA, as long as the variety we are working with is complete.

Let X be a connected complex manifold. Then, recall the following definition:

Definition 2.1. A *local system* on X is a sheaf L of $\mathbf{C}$-vector spaces on X, such that locally, $L \simeq \mathbf{C}^r$, where r is called the *rank* of $\mathscr{L}$. These form a category Loc, where morphisms are morphisms as sheaves.

The basic fact about local systems is the following:

Fact 2.2. There is an equivalence of categories

$$\left\{\begin{matrix}\text{Local systems}\\ \text{on } X\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{Finite-dimensional}\\ \text{representations of } \pi_1(X,x)\end{matrix}\right\}$$

where a local system L is defined by looking at the monodromy action of an element of $\pi_1(X,x)$ on a stalk L_x to get a representation $\pi(X,x) \to \mathrm{GL}(L_x)$.

Remark 2.3. A local system is different from a vector bundle, which is locally $\mathbf{C}^r \times U$.

Before stating the Theorem, we define morphisms in the other category involved:

Definition 2.4. A *morphism* $\varphi\colon (E,\nabla) \to (E',\nabla')$ of vector bundles with integrable connection is a morphism $\varphi\colon E \to E'$ of sheaves (hence, of vector bundles by Proposition 1.19) such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\nabla} & \Omega \otimes E \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle 1\otimes\varphi} \\ E' & \xrightarrow{\nabla'} & \Omega \otimes E' \end{array}$$

commutes. In particular, this implies both $\operatorname{cok}\varphi$ and $\ker\varphi$ carry integrable connections.

Theorem 2.5 (Riemann–Hilbert Correspondence)**.** *There is an equivalence of categories*

$$\left\{\begin{matrix}\textit{Local systems}\\ \textit{on } X\end{matrix}\right\} \simeq \left\{\begin{matrix}\textit{Vector bundles with}\\ \textit{integrable connection}\end{matrix}\right\}$$

where the functor F *from local systems to vector bundles with integrable connection is given by*

$$\mathsf{F}\colon L \longmapsto (L \otimes_{\mathbf{C}} \mathcal{O}_X, 1_L \otimes d)$$

where $\mathcal{O}_X$ *is the sheaf of holomorphic functions, and the connection* $1_L \otimes d$ *is defined to be the composition*

$$L \otimes_{\mathbf{C}} \mathcal{O}_X \xrightarrow{1_L \otimes d} (L \otimes_{\mathbf{C}} \mathcal{O}_X) \otimes_{\mathcal{O}_X} \Omega_X = L \otimes_{\mathbf{C}} \Omega_X,$$

and the functor G *in the other direction is given by*

$$\mathsf{G}\colon (E,\nabla) \longmapsto (\ker\nabla \subseteq E).$$

One part of the proof is clear:

Proof that $\mathsf{G}\circ\mathsf{F}\simeq \mathrm{id}$. It's clear that $(L\otimes_{\mathbf{C}}\mathcal{O}_X, 1_L\otimes d)$ is a vector bundle with integrable connection, and since $\ker(\mathcal{O}_X\to\Omega_X)=\mathbf{C}$ because sections of $\mathcal{O}_X$ with vanishing derivative are constant, we have that

$$(\mathsf{G}\circ\mathsf{F})(L)=\ker(1\otimes d)=L\otimes_{\mathbf{C}}\ker(\mathcal{O}_X\to\Omega_X)\simeq L. \qquad \square$$

The subtle part is to show that if (E,∇) is a vector bundle with integrable connection, then $\ker\nabla$ is a local system, and that the canonical morphism

$$(\mathsf{F}\circ\mathsf{G})(E,\nabla)=(\ker\nabla\otimes_{\mathbf{C}}\mathcal{O}_X)\to E$$

is an isomorphism. This is a local assertion, so you can check this locally with a system of coordinates (that is, we can assume $X\subset\mathbf{C}^n$ is open with coordinates $x_1,\dots,x_n$), and assume that $E=\mathcal{O}_X^{\oplus n}$ is moreover trivial, with basis $e_1,\dots,e_r$.

Given these simplifications, a connection ∇ on E is described by saying where each e_i goes:

$$\nabla(e_j)=\sum_{i=1}^{n}\sum_{k=1}^{r}\Gamma_{ij}^{k}dx_i\otimes e_k,$$

where the Γ_{ij}^k are Christoffel coefficients. Our goal will be to describe the integrability condition into a geometric condition.

Definition 2.6. A *flat section* is a section $s=(s_1,\dots,s_r)\in\ker\nabla\subset\Gamma(E)$.

The condition that $s\in\ker\nabla$ can be written as $\nabla(\sum s_je_j)=0$, which is equivalent to

$$\frac{\partial s_k}{\partial x_i}+\sum_{j=1}^{r}\Gamma_{ij}^k s_j=0$$

for all $i\le n$, $k\le r$, which is a linear system of partial differential equations.

The condition that ∇ is integrable is that $\nabla(\nabla(e_j))=0$, which means

$$\nabla\Big(\sum_{i,k}\Gamma_{ij}^k dx_i\otimes e_k\Big)=0\quad\text{for all } j.$$

The left-hand side is

$$\sum_{i,k}\left(d\big(\Gamma_{ij}^k\,dx_i\big)\otimes e_k-\Gamma_{ij}^k\,dx_i\wedge\nabla(e_k)\right).$$

The condition is therefore

$$\frac{\partial\Gamma_{ij}^q}{\partial x_p}-\frac{\partial\Gamma_{pj}^q}{\partial x_i}+\sum_{k=1}^{r}\left(\Gamma_{ij}^k\Gamma_{pk}^q-\Gamma_{pj}^k\Gamma_{ik}^q\right)=0$$

for all $i,p\le n$, $j,q\le r$.

Now consider the total space E, which is just $Y=X\times\mathbf{C}^r\xrightarrow{\pi}X$ with coordinates $(x_1,\dots,x_n,y_1,\dots,y_r)$, and consider the following vector fields on Y:

$$v_i=\partial_{x_i}-\sum_{k=1}^{r}\left(\sum_{j=1}^{r}\Gamma_{ij}^k y_j\right)\partial_{y_k}\quad\text{for } 1\le i\le n.$$

These $v_1,\dots,v_n$ span a subbundle $F\subset TY$ of rank n, since these v_i's are independent at every point. We first note that $[v_i,v_j]=0$ for all i,j, and so $[F,F]\subseteq F$, since if $u_1=\sum f_iv_i$ and $u_2=\sum g_iv_i$, then $[u_1,u_2]$ is a linear combination of the v_i:

$$\begin{aligned}f_iv_i(g_jv_j)(h)-g_jv_j(f_iv_i)(h)&=f_iv_i(g_jv_j(h))-g_jv_j(f_iv_i(h))\\&=f_iv_i(g_j)v_j(h)+f_ig_jv_i(v_j(h))-g_jv_j(f_i)v_i(h)-g_jf_iv_j(v_i(h))\\&=\left(f_iv_i(g_j)v_j-g_jv_j(f_i)v_i\right)(h)\end{aligned}$$

The integrability condition can be translated into the condition that $[v_i, v_j] = 0$ for all i, j.

Frobenius theorem asks for an integrable submanifold in TY such that the tangent bundle is precisely F. More precisely, the integrability condition implied $[F, F] \subseteq F$ above, which implies

Theorem 2.7 (Frobenius)**.** *For all $y \in Y$, there exists a submanifold $W_y \hookrightarrow Y$ containing y such that $T_z W_y = F_z$ for all $z \in W_y$. Moreover, this is unique in a neighborhood of y.*

Now consider a section $s\colon X \to X \times \mathbf{C}^r$ of the bundle projection $\pi\colon Y = X \times \mathbf{C}^r \to X$, given by

$$x \overset{s}{\longmapsto} (x, s_1(x), \ldots, s_n(x)).$$

The condition for integrability is that $T_{s(p)} s(X) \subseteq F_{s(p)}$ for all $p \in X$. But

$$T_{s(p)} s(X) = ds(T_p X) = \operatorname{im}\left(\begin{aligned} T_x X &\longrightarrow T_x(X \times \mathbf{C}^r) \\ u &\longmapsto \left(u, \left(\textstyle\sum_{j=1}^{n} \frac{\partial s_i}{\partial x_j} u_j\right)_{1 \le i \le r}\right)\end{aligned}\right),$$

and so the condition $T_{s(p)} s(X) \subseteq F_{s(p)}$ is equivalent to

$$\frac{\partial s_i}{\partial x_j} = -\sum_{k=1}^{r} \Gamma_{ij}^k s_k$$

for all i, j. This is exactly the condition $\nabla(s) = 0$.

Given a point $x \in X$, consider the following map:

$$(\ker \nabla)_x \longrightarrow E_x \longrightarrow E_x / \mathfrak{m}_x E_x = \mathbf{C}^r. \tag{$*$}$$

Step 1. The map $(*)$ is injective.

Proof. Suppose $s \in \ker \nabla \mapsto 0 \in \mathbf{C}^r$, i.e., this says that $s_1(x) = \cdots = s_r(x) = 0$. Since $s(X)$ is an integrable submanifold for $F \subseteq TY$, and the same holds for the zero section, this implies that $s = 0$ in a neighborhood of x by local uniqueness. □

Step 2. The map $(*)$ is also surjective, i.e., "there are enough flat sections."

Proof. Consider $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbf{C}^r = \pi^{-1}(x)$, and let W an integrable submanifold of Y through the point (x, α). Then,

$$\begin{array}{ccc} W & \hookrightarrow & Y \\ & \searrow & \downarrow \\ & & X \end{array}$$

By construction, this diagonal map is a local diffeomorphism at x. This means that W is isomorphic to X, i.e., X is the image of a section s of $E = \mathcal{O}_X^{\oplus r}$ in a neighborhood U of x such that $W = s(U)$ around (x, α). This implies that $\nabla(s) = 0$, and $s(x) = (x, \alpha)$. □

Step 3. Fix $x \in X$, We know that $(\ker \nabla)_x \simeq \mathbf{C}^r$, and so there exists a neighborhood U of x and $V \subseteq \Gamma(U, \ker \nabla)$ such that $V \simeq (\ker \nabla)_x$, and so we have a morphism

$$V \otimes_{\mathbf{C}} \otimes_U \to E,$$

such that at x, it induces $V \simeq E(x)$. In particular, after replacing U by a smaller subset, we may assume that this is an isomorphism of vector bundles on U. If $x' \in U$, then

$$\begin{array}{ccc} V & \longrightarrow & (\ker \nabla)_{x'} \\ & \overset{\sim}{\searrow} & \downarrow \wr \\ & & E(x') \end{array}$$

implies that $V \simeq (\ker \nabla)_{x'}$. This implies that $L|_U \simeq V$. □

Corollary 2.8. *If* (E, ∇) *is a vector bundle with integrable connection, then the de Rham complex*

$$0 \longrightarrow E \xrightarrow{\nabla} \Omega \otimes E \xrightarrow{\nabla} \wedge^2\Omega \otimes E \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \wedge^n\Omega \otimes E \longrightarrow 0$$

is a resolution of $\ker \nabla$.

Proof. By Riemann–Hilbert, this is $L \otimes_{\mathbf{C}}$ the usual de Rham complex on X. A version of the Poincaré lemma (for holomorphic forms) says the usual de Rham complex is a resolution of $\mathcal{O}_X$. □

This is in fact an analytic story; you want to say that analytic vector bundles with connection are the same as algebraic vector bundles with connection. But the connection is not a linear object, so you have to be a bit careful, and can't just apply GAGA!

Corollary 2.9. *If* X *is a complete complex algebraic variety, then*

$$\left\{\begin{matrix}\textit{Algebraic vector bundles}\\ \textit{with integrable connection}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\textit{Analytic vector bundles}\\ \textit{with integrable connection}\end{matrix}\right\}$$

Proof. We want to use GAGA, but for this we need to interpret ∇ as an $\mathcal{O}_X$-linear map, i.e., we need to "linearize" the integrable connection.

Consider $\Delta\colon X \hookrightarrow X \times X$ the diagonal embedding, and let $\mathscr{I}$ be the ideal defining X. Then, $\mathscr{P} := \mathcal{O}_{X\times X}/\mathscr{I}^2$. This is a sheaf supported on X, with two $\mathcal{O}_X$-module structures: one coming from left, induced by the first projection $p_1\colon X \times X \to X$, and the other coming from the right, induced by the second projection $p_2\colon X \times X \to X$. We have an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{I}/\mathscr{I}^2 & \longrightarrow & \mathscr{P} & \xrightarrow{\pi} & \mathcal{O}_X & \longrightarrow & 0 \\ & & \| & & & & & & \\ \mathcal{O} & \xrightarrow{d} & \Omega_{X/\mathbf{C}} & & & & & & \end{array}$$

and $d(f) = \overline{f \otimes 1 - 1 \otimes f} \in \mathscr{I}/\mathscr{I}^2$.

Claim 2.10. *A connection on* E *is the same as giving an* $\mathcal{O}_X$*-linear map* $\varphi\colon E \to \mathscr{P} \otimes_{\mathcal{O}_X} E$*, such that* $(\pi \otimes 1) \circ \varphi = \mathrm{id}$*, where you use the right* $\mathcal{O}_X$*-module structure on* $\mathscr{P}$ *for* $\otimes$ *then the left* $\mathcal{O}_X$*-module structure on* E *to make* $\mathscr{P} \otimes_{\mathcal{O}_X} E$ *into an* $\mathcal{O}_X$*-module.*

Proof. Such φ is given by $\varphi(e) = (1 \otimes 1) \otimes e + \nabla(e)$, where $\nabla(e) \in \mathscr{I}/\mathscr{I}^2 \otimes_{\mathcal{O}_X} E$. $\nabla(fe) = f\nabla(e) + (f \otimes 1 - 1 \otimes f) \otimes e$ if and only if $\varphi(fe) - (1 \otimes 1) \otimes fe = (f \cdot \varphi(e) - f \cdot (1 \otimes 1 \otimes e)) + (f \otimes 1 - 1 \otimes f) \otimes e$, which is equivalent to $\varphi(fe) = f\varphi(e)$ since $(1 \otimes 1) \otimes fe = (1 \otimes f) \otimes e$, and $f \cdot (1 \otimes 1 \otimes e) = (f \otimes 1) \otimes e$.

We therefore have that {algebraic vector bundles with connection} $\leftrightarrow$ {analytic vector bundles with connection} by GAGA. For integrability: the curvature

$$\theta = E \xrightarrow{\nabla} \Omega \otimes E \xrightarrow{\nabla} \wedge^2\Omega \otimes E$$

is always $\mathcal{O}_X$-linear, hence it vanishes in the algebraic category if and only if it does in the analytic category. □

The issue is much more subtle if you are in the quasi-projective case, in which case you naturally get to the world of regular singularities. Typically, if you have a quasi-projective variety, you compactify so that the complement is snc, and you have a condition of regular singularities, i.e., it has log poles on the boundary. We will come back to this later.

We can for example check compatibility will pullbacks:

$$\nabla(e_j) = \sum_{i,k} \Gamma_{ij}^k dx_i \otimes e_k,$$

and so

$$\nabla(f^*e_j) = \sum_{i,j} (\Gamma_{ij}^k \circ f) d(x_k \circ f) \otimes f^*e_i$$

On the other hand, pushforwards don't make sense except for smooth maps. If you replace categories to have constructible systems and holonomic D-modules you do get compatibility.

3 September 26 (Harold Blum)

3.1 Good Filtrations [HTT08, §2.1]

Recall that X denotes a smooth variety over $\mathbf{C}$, and that D_X denotes the ring of differential operators. We have defined the order filtration $F_\ell D_X$ on D_X (Definition 1.3), which locally given by saying that on an open affine subset $U \subseteq X$, with a trivialization $\{x_i, \partial_i\}$ of the tangent bundle, we have

$$F_\ell D_X(U) = \bigoplus_{|\alpha| \leq \ell} \mathcal{O}_U \partial^\alpha,$$

where $\alpha \in \mathbf{N}^n$ and $|\alpha| = \sum \alpha_i$. We can look at the graded ring

$$\operatorname{gr}^F D_X \coloneqq \bigoplus_\ell F_\ell D_X / F_{\ell-1} D_X,$$

which is commutative, and on open affines U of the form above, $\operatorname{gr}^F D_X(U) = \mathcal{O}_U[\xi_1, \ldots, \xi_n]$, where $\xi_i = \overline{\partial_i}$.

Now what we want to do is to take a D-module $M \in \mathsf{Mod}_{\mathsf{qc}}(D_X)$ that is quasi-coherent over $\mathcal{O}_X$, and associate to it a graded module over $\operatorname{gr}^F D_X$.

Definition 3.1. Let $M \in \mathsf{Mod}_{\mathsf{qc}}(D_X)$. Then, we say (M, F) is a *filtration of D_X-modules* if

- For every integer i, $F_i M$ is a quasi-coherent $\mathcal{O}_X$-submodule of M;
- $F_i M \subset F_{i+1} M$;
- $F_i M = 0$ for $i \ll 0$;
- $M = \bigcup F_i M$;
- $F_i D_X \cdot F_j M \subseteq F_{i+j} M$.

Once we have one of these filtrations, we can define

$$\operatorname{gr}^F(M) \coloneqq \bigoplus_{i \in \mathbf{Z}} F_i M / F_{i-1} M.$$

Proposition 3.2. *The following conditions are equivalent:*

(1) $\operatorname{gr}^F M$ *is coherent over* $\operatorname{gr}^F D_X$*;*
(2) $F_i M$ *is coherent over* $\mathcal{O}_X$ *for all* i*, and for* $i_0 \gg 0$*,* $F_j D_X F_{i_0} M = F_{j+i_0} M$*.*

Proof of (1) $\Rightarrow$ (2). Work locally on an open affine $U \subseteq X$. Choose generators $\overline{m}_1, \ldots, \overline{m}_s \in \operatorname{gr}^F M(U)$ over $\operatorname{gr}^F D_X(U)$, where $m_i \in F_{k_i} M \setminus F_{k_{i-1}} M$. Then, for $k \geq \max\{k_i\}$, the map

$$F_{k-k_1} D_X \oplus \cdots \oplus F_{k-k_s} D_x \longrightarrow F_k M$$

mapping $e_i \mapsto m_i$ is surjective. Letting $\ell = \max\{k_i\}$, the map $F_{k-\ell} D_X \cdot F_\ell M \twoheadrightarrow F_k M$ is surjective. □

Definition 3.3. If (M, F) is a filtration, where M is a D-module, we say it is a *good filtration* if one of the equivalent conditions (1) and (2) hold in the previous Proposition.

The theorem below describes when good filtrations exist, and how different good filtrations are related to each other. Recall that a coherent D_X is one whose local sections on every open affine set U are of finite type over $D_X(U)$.

Theorem 3.4.

(1) *If* $M \in \mathsf{Mod}_{\mathsf{Coh}}(D_X)$*, then there exists a good filtration.*
(2) *If* (M, F) *and* (M, F') *are two filtrations on* M*, and* F *is good, then there exists* i_0 *such that* $F_i M \subset F'_{i+i_0} M$ *for all* i*. In particular, if* F, F' *are both good, then choosing the maximum of the two* i_0 *values you get, you have*

$$F'_{i-i_0} M \subset F_i M \subset F'_{i+i_0} M$$

for all i*, i.e., any two good filtrations are not too far apart.*

Proof of (1). Choose generators $m_1, \dots, m_s$ locally on an open affine U. Then, let $F_i M = F_i D_X \cdot (m_1, \dots, m_s)$ for $i \geq 0$, and $F_i M = 0$ for $i < 0$. The associated graded module $\operatorname{gr}^F M$ is generated in degree 0. Now using [HTT08, Cor. 1.4.17], these sections on U extend to global sections of some coherent $\mathcal{O}_X$-sheaf $\mathscr{F}^U$, which generate $M|_U$ as a D_X-module. The direct sum $\bigoplus_U \mathscr{F}^U$ for a finite cover of X gives a coherent $\mathcal{O}_U$ module, which globally generates M as a D_X-module. Now we can define the global good filtration in the same way as in the local description above.

(2) is from Property (2) from Proposition 3.2. □

3.2 Characteristic varieties [HTT08, §2.2]

Let X be a smooth complex variety, and consider again the order filtration $F_\ell D_X$. We can consider the associated graded ring

$$\operatorname{gr}^F D_X \coloneqq \bigoplus_{\ell \in \mathbf{N}} F_\ell D_X / F_{\ell-1} D_X \simeq \pi_* \mathcal{O}_{T^*X},$$

which is commutative. Now let $M \in \mathsf{Mod}_{\mathsf{qc}}(D_X)$. If M has a filtration F, then we get a graded module

$$\operatorname{gr}^F M \coloneqq \bigoplus_{\ell \in \mathbf{Z}} F_\ell M / F_{\ell-1} M$$

over $\operatorname{gr}^F D_X$, where we note that $\operatorname{gr}^F M$ is $\mathbf{Z}$-graded in general, in contrast to $\operatorname{gr}^F D_X$ which is $\mathbf{N}$-graded. Recall that (M, F) is a *good filtration* if $\operatorname{gr}^F M$ is finitely generated over $\operatorname{gr}^F D_X$; this is equivalent to saying that each $F_\ell M$ is coherent by Proposition 3.2.

We can then define the following:

Definition 3.5 (Characteristic variety)**.** Let $M \in \mathsf{Mod}_{\mathsf{c}}(D_X)$, and let (M, F) be a good filtration. Then, let

$$\widetilde{\operatorname{gr}^F M} = \mathcal{O}_{T^*X} \otimes_{\pi^{-1}(\pi_* \mathcal{O}_{T^*X})} \pi^{-1} \operatorname{gr}^F M,$$

which we note is the associated module of $\operatorname{gr}^F M$ under relative **Spec**. Then, we set

$$\operatorname{ch}(M) \coloneqq \operatorname{Supp} \widetilde{\operatorname{gr}^F M}.$$

We can make this more explicit as follows: if we assume X is affine with local coordinates $\{x_i, \partial_i\}$, then

$$\operatorname{gr}^F D_X = \mathcal{O}_X[\xi_1, \dots, \xi_n],$$

and so $\operatorname{ch}(M)$ is given by the ideal $\sqrt{\operatorname{Ann}(\operatorname{gr}^F M)}$.

Note 3.6. $\operatorname{ch}(M)$ is preserved by scalar multiplication on the fibers of $T^*X \to X$, since the annihilator is a homogeneous ideal in $\mathcal{O}_X[\xi_1, \dots, \xi_n]$.

In the sequel, we say that $f \in \operatorname{Ann}(\operatorname{gr}^F)$ is homogeneous of degree p when

$$f \cdot F_\ell M \subseteq F_{\ell+p-1} M.$$

Theorem 3.7 (Basic properties of the characteristic variety)**.**
(1) $\operatorname{ch}(M)$ *does not depend on the choice of good filtration;*
(2) *If* $0 \to M \to N \to L \to 0$ *is a short exact sequence,* $\operatorname{ch}(N) = \operatorname{ch}(M) \cup \operatorname{ch}(L)$*.*

The second statement essentially follows by choosing a filtration on N, which induces filtrations on the others, and gives a relationship between annihilators.

We prove the first statement.

Proof of (1). The idea is to use our result from last time about how good filtrations are related. We work locally. Suppose (M, F) and (M, F') are good filtrations, and suppose $f \in \operatorname{Ann}(\operatorname{gr}^F M)$ is homogeneous of degree p, and so $f^m \cdot F_\ell M \subseteq F_{\ell+mp-m} M$. Now let $i_0 \gg 0$ such that

$$F'_{i-i_0} M \subseteq F_i M \subseteq F'_{i+i_0} M,$$

which exists by Theorem 3.4. Then,

$$f^m \cdot F'_\ell M \subseteq f^m \cdot F_{\ell+i_0} M \subseteq F_{i+i_0} M \subseteq F_{\ell+i_0+mp-m} M \subseteq F'_{\ell+2i_0+mp-m} M$$

which is contained in $F'_{\ell+mp-1}$ as long as $2i_0 - 1m \leq -1$. Thus, $f^m \in \operatorname{Ann} \operatorname{gr}^{F'} M$. □

Now that the characteristic variety is well-defined, we will talk about a special case.

Proposition 3.8. *$M \in \mathsf{Mod}_{\mathsf{coh}}(D_X)$, coherent over $\mathcal{O}_X$. This is equivalent to $\operatorname{ch}(M) = \operatorname{im}\{X \overset{0}{\hookrightarrow} T^*X\}$.*

Proof. Choose the filtration $F_iM = 0$ for $i < 0$, and $F_iM = M$ for $i \geq 0$, in which case $\operatorname{gr}^F M \cong M$. For the other direction, assume that $\operatorname{ch}(M)$ has a zero section. Suppose that X is affine, with coordinates $\{x_i, \partial_i\}$. Then,

$$(\xi_1, \ldots, \xi_n)^{m_0} \subset \operatorname{Ann}(\operatorname{gr}^F M)$$

for $m_0 \gg 0$. Using properties of the filration as before, you see that $F_{i-1}M = F_iM = M$ for $i \gg 0$. Since the F_iM are coherent, this shows that M is coherent. □

Example 3.9. Let $M = D_X/D_X \cdot P$. Then, there is a short exact sequence

$$0 \longrightarrow \operatorname{gr}(D_X \cdot P) \longrightarrow \operatorname{gr} D_X \longrightarrow \operatorname{gr}(D_X/D_X \cdot P) \longrightarrow 0$$

Write $P = P_0 + \cdots + P_r$, such that each P_i is homogeneous of order i, and $P_r \neq 0$. Then, the left hand side is $\operatorname{gr} D_X \cdot P_r(x, \xi)$, and so $\operatorname{ch}(M)$ is defined by the radical of the annihilator of $P_r(x, \xi)$. Note that $P_r(x, \xi)$ is called the *symbol* of P.

This is more subtle when there is more than one differential operator. This is similar to tangent cone computations, in that taking the symbol of the entire ideal is not the same as taking the symbol of each generator.

4 October 3: Operations on D-modules (Takumi Murayama)

We will freely use intuition from the study of differential equations on manifolds. Much of the analogies and motivation for constructions have been taken from [Ber82]; to make them precise, we should probably mention that basic material on distributions, in particular distributions on manifolds, can be found in the book(s) by Hörmander [Hör03].

To keep track of what is going on, we recall the following motivation that Harold gave:

Example 1.11. Let $P_1, \ldots, P_s$ be s differential operators on $\mathbf{A}^n$. Then, letting

$$M := {D_{\mathbf{A}^n}} \big/ {D_{\mathbf{A}^n} P_1 + \cdots + D_{\mathbf{A}^n} P_s},$$

we had that

$$\operatorname{Hom}_{D_{\mathbf{A}^n}}(M, \mathcal{O}) \simeq \{f \in \mathcal{O}_{\mathbf{A}^n} \mid P_i f = 0 \text{ for all } i\}.$$

So a D-module M keeps track of solutions to a system of differential equations.

4.1 Inverse images [HTT08, §1.3]

Let $f : X \to Y$ be a morphism of smooth complex varieties. The map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ tells us how we can pullback functions. Since a D-module keeps track of a system of differential equations, we can ask:

Question 4.1. If we pullback a collection of functions that satisfy a system of differential equations, how does this affect the system that they satisfy?

This is what the inverse image functor will do.

Definition 4.2. Let M be a left D_Y-module. Then, its *inverse image* is defined by

$$f^\circ M := f^*M = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}M,$$

which is the formula for inverse images of $\mathcal{O}$-modules.

What we need to check is that this new sheaf $f^\circ M$ has a left D-module structure. Note that we use different notation to differentiate the fact that $f^\circ M$ has a D-module structure, following [Bor+87, IV,4.1].

4.1.1 Inverse images of left D-modules

Suppose M is a quasi-coherent $\mathcal{O}_Y$-module. Let $\{y_i, \partial_i\}$ be a local coordinate system on Y. For any $\psi \otimes f^{-1}s \in f^\circ M$ and $\psi' \in \mathcal{O}_Y$, $\theta \in \Theta_Y$, we can define a left action by

$$\psi'(\psi \otimes s) = \psi'\psi \otimes s$$
$$\theta(\psi \otimes s) = \theta(\psi) \otimes s + \psi \sum_{i=1}^{n} \theta(y_i \circ f) \otimes \partial_i s$$

The second term in $\theta(\psi \otimes s)$ can be thought of as a kind of chain rule, and allows the definition to transform well under change of coordinates. More precisely, we check that the definition is independent of choice of coordinates, following [Bor+87, VI,4.1]. Let $\{y'_j, \partial'_j\}$ be another local coordinate system. Then, we have

$$\begin{aligned}
\psi \sum_{j=1}^{n} \theta(y'_j \circ f) \otimes \partial'_j s &= \psi \sum_{i,j,k} \left(\frac{\partial y'_j}{\partial y_j} \circ f\right) \theta(y_i \circ f) \otimes \left(\frac{\partial y_k}{\partial y'_j} \partial_k s\right) \\
&= \psi \sum_{i,k} \theta(y_i \circ f) \otimes (\partial_k s) \cdot \left[\sum_j \frac{\partial y'_j}{\partial y_j} \frac{\partial y_k}{\partial y'_j} \circ f\right] \\
&= \psi \sum_{i,k} \theta(y_i \circ f) \otimes (\partial_k s) \cdot \partial_{ik} \\
&= \psi \sum_{i} \theta(y_i \circ f) \otimes \partial_i s
\end{aligned}$$

You can also check that $[\partial_i, \partial_j] = 0$ implies that this defines a flat connection, and so you get a D_X-module structure as desired.

Instead of checking the definition transforms well under changes of coordinates, we can also just write down a global description of this action. First, there is a natural map $f^*\Omega^1_Y \to \Omega^1_X$, and so by dualizing on X, we get a morphism

$$\begin{aligned}
\Theta_X &\longrightarrow f^*\Theta_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} \Theta_Y \\
\theta &\longmapsto \tilde{\theta} = \sum_j \varphi_j \otimes \theta_j
\end{aligned}$$

and we can define

$$\theta(\psi \otimes s) = \theta(\psi) \otimes s + \psi\tilde{\theta}(s) = \theta(\psi) \otimes s + \psi \sum_j \varphi_j \otimes \theta_j(s).$$

We note that tracing this description above, you can show that the inverse image is well-behaved under composition, that is,

Proposition 4.3. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of morphisms of smooth varieties. Then, $(g \circ f)^\circ = f^\circ \circ g^\circ$.*

Proof following [Mil99, Thm. 10.3(i)]. This is already true on the level of $\mathcal{O}_X$-modules, and so it suffices to show the left D_X-module structures match locally, for differential operators of the form ∂_{x_k}:

$$\begin{aligned}
\partial_{x_k}(\psi \otimes s) &= \partial_{x_k}(\psi \otimes (1 \otimes s)) \\
&= \partial_{x_k}\psi \otimes (1 \otimes s) + \psi \sum_j \partial_{x_k}(y_j \circ f) \otimes \partial_{y_j}(1 \otimes s) \\
&= \partial_{x_k}\psi \otimes s + \psi \sum_j \partial_{x_k}(y_j \circ f) \otimes \sum_i \partial_{y_j}(z_i \circ g) \otimes \partial_{z_i} s \\
&= \partial_{x_k}\psi \otimes s + \psi \sum_{i,j} \partial_{x_k}(y_j \circ f)(\partial_{y_j}(z_i \circ g) \circ f) \otimes \partial_{z_i} s \\
&= \partial_{x_k}\psi \otimes s + \psi \sum_i \partial_{x_k}(z_i \circ g \circ f) \otimes \partial_{z_i} s.
\end{aligned}$$

Alternatively, it suffices to note that the commutative triangle

$$\begin{array}{ccc} \Theta_X & \longrightarrow & g^* f^* \Theta_Z \\ & \searrow \quad \nearrow & \\ & f^* \Theta_Y & \end{array}$$

implies that the D_X-module structures on $(g \circ f)^* M$ and $f^*(g^*(M))$ are the same. □

4.1.2 The sheaf $D_{X \to Y}$

We want an alternative description of the inverse image functor. Recall from Proposition 1.18 that the functor $\mathsf{Mod}(D_X) \to \mathsf{Mod}(D_X^{\mathsf{op}})$ was defined by $\omega_X \otimes_{\mathcal{O}_X} -$. We want a similar description of the inverse image as a tensor product with a suitable sheaf.

Definition 4.4. We define

$$D_{X \to Y} \coloneqq f^\circ D_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1} D_Y,$$

which has a $(D_X, f^{-1}D_Y)$-bimodule structure: the left D_X-module structure comes from before, and the right $f^{-1}D_Y$-module structure comes from acting on the right.

By associativity of the tensor product,

$$f^\circ M \simeq D_{X \to Y} \otimes_{f^{-1}D_Y} f^{-1} M.$$

Example 4.5. Let $i \colon \mathbf{A}^r_x \hookrightarrow \mathbf{A}^n_y$ be the embedding of $\mathbf{A}^r$ as $\{y_{r+1} = y_{r+2} = \cdots = y_n = 0\}$. Then,

$$\begin{aligned} D_{X \to Y} &= \mathbf{C}[\underline{x}] \otimes_{\mathbf{C}[\underline{y}]} \mathbf{C}[\underline{y}, \underline{\partial_y}] \\ &\simeq \mathbf{C}[\underline{x}] \otimes_{\mathbf{C}[\underline{y}]} \mathbf{C}[\underline{y}, \partial_{y_1}, \ldots, \partial_{y_r}] \otimes_{\mathbf{C}} \mathbf{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \\ &\simeq D_X \otimes_{\mathbf{C}} \mathbf{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}]. \end{aligned}$$

as a left D_X-module. This is locally true for an arbitrary closed embedding $X \to Y$ [HTT08, Ex. 1.3.2].

4.2 Direct images [HTT08, §1.3]

As before, let $f \colon X \to Y$ be a morphism of smooth algebraic varieties. There is no way to pushforward regular functions or systems of differential equations on X to those on Y, and so we seem stuck.

This is where the equivalence of left and right D-modules becomes useful. We first recall:

Proposition 1.18. *We have an equivalence of categories*

$$\omega_X \otimes_{\mathcal{O}_X} - \colon \mathsf{Mod}(D_X) \to \mathsf{Mod}(D_X^{\mathsf{op}}),$$

with quasi-inverse given by $\omega_X^\vee \otimes_{\mathcal{O}_X} -$.

We can think of this in terms of systems of differential equations as follows. Solution spaces of functions are left D-modules, but solution spaces of *distributions* are right D-modules. And indeed, the functor $\omega_X \otimes -$ for the special case $M = \mathcal{O}_X$ locally can be described as

$$f \longmapsto dx_1 \wedge \cdots \wedge dx_n \otimes f,$$

that is, it transforms a function into a distribution [HTT08, Lem. 1.2.6]. Thus, the functor $\omega_X \otimes -$ can be thought of as transforming a system of differential equations for functions, into one for distributions.

Recall that given a distribution (or maybe more correctly, a current) E (say, with compact support) and a map $f \colon X \to Y$, we can intergrate E to get a distribution on Y, by the formula $\langle \int_f E, \varphi \rangle = \langle E, f^* \varphi \rangle$. This suggests the following:

Question 4.6. If we integrate a collection of distributions that satisfy a system of differential equations, how does this affect the system that they satisfy?

This is what the direct image functor will do.

4.2.1 Direct images of right D-modules

Let N be a right D_X-module. We already have a direct image f_*N of $\mathcal{O}$-modules, but there's no natural right D_Y-module structure on f_*N:

Example 4.7 [Mil99, p. 46]**.** Consider the inclusion i of $X = \{0\}$ into $Y = \mathbf{A}^1$. Then, $D_X = \mathbf{C}$ and $D_Y = \mathbf{C}[x, \partial]$. Consider the module $i_*D_X = \mathbf{C}$. This is coherent as an $\mathcal{O}_Y$-module, but if it had a D_Y-module structure, it must also be locally free (Proposition 1.19), which it is not.

Alternatively, you can also compute the characteristic variety. Choose the filtration where $\operatorname{gr}^F(i_*D_X) = \mathbf{C}$ in degree 0. The annihilator of $\operatorname{gr}^F(i_*D_X)$ in $\operatorname{gr}^F D_Y \simeq \mathbf{C}[x, \xi]$ is some maximal ideal $\mathfrak{m}$, and so the characteristic variety $\operatorname{Ch}(i_*D_X)$ would have dimension 0. This contradicts Bernstein's inequality [HTT08, Cor. 2.3.2], since $\dim(\operatorname{Ch}(i_*D_X)) = 0 \not\geq 1 = \dim \mathbf{A}^1$.

Instead, consider the $\mathcal{O}_X$-module

$$N \otimes_{D_X} D_{X\to Y},$$

the right D_X-module structure on N and the left D_X-module structure on $D_{X\to Y}$ are eaten up by the tensor product, and so a right $f^{-1}D_Y$-module structure remains. Now if we apply the pushforward, we obtain

$$f_*(N \otimes_{D_X} D_{X\to Y}).$$

which has a right $f_*(f^{-1}D_Y)$-module structure. Finally, using the canonical map $D_Y \to f_*f^{-1}D_Y$, we can make the following definition:

Definition 4.8. Let N be a right D_X-module. Then, its *direct image* is defined by

$$f_\circ N \coloneqq f_*(N \otimes_{D_X} D_{X\to Y}),$$

which is a right D_Y-module.

Warning 4.9. This definition is not compatible with composition! We shouldn't expect it to be because it is the composition of a right-exact functor with a left-exact functor. This will hopefully be clearer when we define the derived versions of f° and $f_\circ$.

4.2.2 Direct images of left D-modules and the sheaf $D_{Y\leftarrow X}$

Since we prefer left D-modules, we want to rewrite this definition in terms of left D-modules. The trick is to use Proposition 1.18. What we want to do is to find a functor fitting into the commutative diagram below:

$$\begin{array}{ccc} \mathsf{Mod}(D_X) & \dashrightarrow & \mathsf{Mod}(D_Y) \\ {\scriptstyle \omega_X\otimes_{\mathcal{O}_X}-}\downarrow\wr & & \wr\downarrow{\scriptstyle \omega_Y\otimes_{\mathcal{O}_Y}-} \\ \mathsf{Mod}(D_X^{\mathsf{op}}) & \xrightarrow{f_\circ} & \mathsf{Mod}(D_Y^{\mathsf{op}}) \end{array}$$

Since $\omega_Y^\vee \otimes -$ is a quasi-inverse for the vertical functor on the right, we compose around the square to get a candidate for the direct image of a left D_X-module M:

$$\begin{aligned} \omega_Y^\vee \otimes_{\mathcal{O}_Y} f_\circ(\omega_X \otimes_{\mathcal{O}_X} M) &= \omega_Y^\vee \otimes_{\mathcal{O}_Y} f_*((\omega_X \otimes_{\mathcal{O}_X} M) \otimes_{D_X} D_{X\to Y}) \\ &\simeq \omega_Y^\vee \otimes_{\mathcal{O}_Y} f_*((\omega_X \otimes_{\mathcal{O}_X} D_{X\to Y}) \otimes_{D_X} M) \\ &\simeq f_*((\omega_X \otimes_{\mathcal{O}_X} D_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\omega_Y^\vee) \otimes_{D_X} M) \end{aligned}$$

where the second line is an isomorphism of the input of f_* by Lemma [HTT08, Lem. 1.2.11], and the third line is an isomorphism of $\mathcal{O}_Y$-modules by the projection formula.

Much like in the case for inverse images, we give the part of this formula that does not contain M a name, and restate the definition of the inverse image with this new sheaf.

Definition 4.10. We define

$$D_{Y\leftarrow X} \coloneqq \omega_X \otimes_{\mathcal{O}_X} D_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\omega_Y^\vee,$$

which has a $(f^{-1}D_Y, D_X)$-bimodule structure: $D_{X\to Y}$ has the opposite structure, and the leftmost and rightmost factors switch these.

Remark 4.11. $D_{X\to Y}$ and $D_{Y\leftarrow X}$ are called the *transfer bimodules* for $f\colon X \to Y$ by [HTT08].

Definition 4.12. Let M be a left D_X-module. Then, its *direct image* is defined by

$$f_\circ M \coloneqq f_*(D_{Y\leftarrow X} \otimes_{D_X} M)$$

Lemma 4.13. *We have the following alternate descriptions of $D_{Y\leftarrow X}$ as a $(f^{-1}D_Y, D_X)$-bimodule:*

$$D_{Y\leftarrow X} \simeq f^{-1}(D_Y \otimes_{\mathcal{O}_Y} \omega_Y^\vee) \otimes_{f^{-1}\mathcal{O}_Y} \omega_X \simeq D_{X\to Y} \otimes_{\mathcal{O}_X} \omega_{X/Y},$$

where the bimodule structure is described in [HTT08, Lem. 1.3.4]*.*

Proof. We just use the definition of $D_{X\to Y}$:

$$\begin{aligned}
D_{Y\leftarrow X} &= \omega_X \otimes_{\mathcal{O}_X} D_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\omega_Y^\vee \\
&= \omega_X \otimes_{\mathcal{O}_X} (\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y) \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\omega_Y^\vee \\
&\simeq \omega_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\omega_Y^\vee \\
&\simeq \omega_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(D_Y \otimes_{\mathcal{O}_Y} \omega_Y^\vee) \\
&\simeq \omega_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\omega_Y^\vee \otimes_{\mathcal{O}_Y} D_Y^{\mathsf{op}} \otimes_{\mathcal{O}_Y} \omega_Y \otimes_{\mathcal{O}_Y} \omega_Y^\vee) \\
&\simeq \omega_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\omega_Y^\vee \otimes_{\mathcal{O}_Y} D_Y^{\mathsf{op}}) \\
&\simeq f^{-1}(D_Y \otimes_{\mathcal{O}_Y} \omega_Y^\vee) \otimes_{f^{-1}\mathcal{O}_Y} \omega_X
\end{aligned}$$

where the third isomorphism is by [HTT08, Lem. 1.2.7], and the last isomorphism is given by

$$\omega \otimes \eta \otimes P^\circ \longleftrightarrow P \otimes \eta \otimes \omega.$$

The second description follows by definition of the inverse image of $\mathcal{O}$-modules:

$$D_{Y\leftarrow X} \simeq f^{-1}(D_Y \otimes_{\mathcal{O}_Y} \omega_Y^\vee) \otimes_{f^{-1}\mathcal{O}_Y} \omega_X \simeq f^*(D_Y \otimes_{\mathcal{O}_Y} \omega_Y^\vee) \otimes_{\mathcal{O}_X} \omega_X \simeq f^*D_Y \otimes_{\mathcal{O}_X} \omega_{X/Y}. \qquad \square$$

We return to the example of an embedding of affine spaces:

Example 4.14. Recall the situation of Example 4.5. We have an embedding $i\colon \mathbf{A}^r_x \hookrightarrow \mathbf{A}^n_y$ of $\mathbf{A}^r$ as $\{y_{r+1} = y_{r+2} = \cdots = y_n = 0\}$. Then, $i^{-1}\omega_Y^\vee \otimes_{i^{-1}\mathcal{O}_Y} \omega_X$ can be identified with $\mathcal{O}_X$ via the section

$$(dy_1 \wedge \cdots \wedge dy_n)^{\otimes -1} \otimes (dx_1 \wedge \cdots \wedge dx_r),$$

and so

$$D_{Y\leftarrow X} \simeq \mathbf{C}[\underline{y}, \underline{\partial_y}] \otimes_{\mathbf{C}[\underline{y}]} \mathbf{C}[\underline{x}] \simeq \mathbf{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \otimes_{\mathbf{C}} D_X.$$

Again, this is locally true for an arbitrary closed embedding $X \to Y$ [HTT08, Ex. 1.3.5].

Remark 4.15. For reasons of symmetry, we can also define the inverse image functor for right D-modules, following [Bor+87, p. 244]. Let $f\colon X \to Y$ be a morphism of smooth complex varieties, and let M be a right D_Y-module. Its inverse image is

$$f^\circ(M) = f^{-1}M \otimes_{f^{-1}D_Y} D_{Y\leftarrow X}.$$

4.3 The derived category of D-modules [HTT08, §§1.4–1.5]

We assume from now on that all algebraic varieties are quasiprojective.

We saw in §3.1 that it is nicest to work with *quasi-coherent* D-modules. To make the derived functor machinery work, we will see that this is also essential because we will need locally projective resolutions.

Notation 4.16. We denote $\mathsf{Mod}_{\mathsf{qc}}(D_X)$ to be the category of D_X-modules that are also quasi-coherent $\mathcal{O}_X$-modules, and we denote $\mathsf{Mod}_{\mathsf{c}}(D_X)$ to be the subcategory of $\mathsf{Mod}_{\mathsf{qc}}(D_X)$ consisting of modules that are coherent as D_X-modules.

We mainly work with the *bounded* derived category.

Definition 4.17. We denote by $\mathsf{D}^b_{\mathsf{qc}}(D_X)$ (resp. $\mathsf{D}^b_{\mathsf{c}}(D_X)$) the full subcategory of $\mathsf{D}^b(D_X)$ with cohomology sheaves in $\mathsf{Mod}_{\mathsf{qc}}(D_X)$ (resp. $\mathsf{Mod}_{\mathsf{c}}(D_X)$). We can get analogous definitions for $\mathsf{D}^+(D_X)$, the category of complexes that are bounded to the left.

Proposition 4.18 [HTT08, Props. 1.4.14, 1.4.18; Cor. 1.4.19]**.** *Let X be a quasi-projective variety, and let $M \in \mathsf{Mod}_{\mathsf{qc}}(D_X)$. Then,*

- (i) *M has a resolution by injective objects in $\mathsf{Mod}_{\mathsf{qc}}(D_X)$;*
- (ii) *M has a resolution by locally free objects in $\mathsf{Mod}_{\mathsf{qc}}(D_X)$;*
- (iii) *M has a* finite *resolution by locally* projective *objects in $\mathsf{Mod}_{\mathsf{qc}}(D_X)$ of length $\le 2\dim X$.*

Thus, any object in $\mathsf{D}^b_{\mathsf{qc}}(D_X)$ can be represented by a bounded complex of locally projective objects in $\mathsf{Mod}_{\mathsf{qc}}(D_X)$.

If $M \in \mathsf{Mod}_{\mathsf{c}}(D_X)$, then all sheaves in the resolutions in (ii) and (iii) can be taken to be of finite rank.

This will allow us to define the derived versions of functors of D-modules.

Remark 4.19. There are equivalences of categories $\mathsf{D}^b_{\mathsf{qc}}(D_X) \simeq \mathsf{D}^b(\mathsf{Mod}_{\mathsf{qc}}(D_X))$ and $\mathsf{D}^b_{\mathsf{c}}(D_X) \simeq \mathsf{D}^b(\mathsf{Mod}_{\mathsf{c}}(D_X))$ [HTT08, Thm. 1.5.7].

Note we will be defining derived functors for complexes in $D^b(D_X)$ which do not have quasicoherent cohomology (at least at first), and so we need resolutions for these objects as well:

Lemma 4.20 [HTT08, Lem. 1.5.2]**.** *Let R be a sheaf of rings on a topological space X, and let $M \in \mathsf{Mod}(R)$. Then,*

- (i) *M has a resolution by injective objects in $\mathsf{Mod}(R)$;*
- (ii) *M has a resolution by flat objects in $\mathsf{Mod}(R)$.*

4.4 Derived inverse images and shifted inverse images [HTT08, §1.5]

Definition 4.21. Let $f\colon X \to Y$ be a morphism of smooth quasi-projective complex varieties. We define the *derived inverse image functor*

$$\begin{aligned}\mathbf{L}f^\circ\colon \mathsf{D}^b(D_Y) &\longrightarrow \mathsf{D}^b(D_X)\\ M^\bullet &\longmapsto D_{X\to Y} \otimes^{\mathbf{L}}_{f^{-1}D_Y} f^{-1}M^\bullet\end{aligned}$$

by using a flat resolution as in Lemma 4.20.

Proposition 4.22. $\mathbf{L}f^\circ$ *descends to a functor $\mathsf{D}^b_{\mathsf{qc}}(D_Y) \to \mathsf{D}^b_{\mathsf{qc}}(D_X)$.*

Proof. A complex $M^\bullet \in \mathsf{D}^b_{\mathsf{qc}}(D_Y)$ is quasi-isomorphic to a bounded complex of locally projective objects $P^\bullet$. Now as complexes of D_X-modules,

$$\begin{aligned}\mathbf{L}f^\circ M^\bullet &= D_{X\to Y} \otimes^{\mathbf{L}}_{f^{-1}D_Y} f^{-1}P^\bullet\\ &\simeq (\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y) \otimes_{f^{-1}D_Y} f^{-1}P^\bullet\\ &\simeq \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}P^\bullet\\ &\simeq f^*P^\bullet,\end{aligned}$$

which has quasi-coherent cohomology. □

Warning 4.23. The functor $\mathbf{L}f^\circ$ does not necessarily descend to a functor $\mathsf{D}^b_{\mathsf{c}}(D_Y) \to \mathsf{D}^b_{\mathsf{c}}(D_X)$! First note

$$\mathbf{L}f^\circ D_Y = D_{X\to Y} \otimes^{\mathbf{L}}_{f^{-1}D_Y} f^{-1}D_Y = D_{X\to Y}.$$

In Example 4.5, we saw that $D_{X\to Y}$ was locally free of *infinite rank*, and so $\mathbf{L}f^\circ D_Y = D_{X\to Y} \notin \mathsf{D}^b_{\mathsf{c}}(D_X)$.

For convenience later on (especially when discussing Kashiwara's equivalence, adjointness, and the Riemann–Hilbert correspondence), we will introduce a shift in degree into the derived inverse image functor. To those of you who know about perverse sheaves, this amounts to preserving perversity.

Definition 4.24. Let $f\colon X \to Y$ be a morphism of smooth quasi-projective complex varieties. We define the *shifted inverse image functor*

$$\begin{aligned} f^\dagger\colon \mathsf{D}^b(D_Y) &\longrightarrow \mathsf{D}^b(D_X)\\ M^\bullet &\longmapsto \mathbf{L}f^\circ M^\bullet[\dim X - \dim Y] \end{aligned}$$

where $(M^\bullet[\dim X - \dim Y])^i = M^{i+\dim X - \dim Y}$.

Proposition 4.25. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of morphisms of smooth varieties. Then,*

$$\mathbf{L}(g\circ f)^\circ \simeq \mathbf{L}f^\circ \circ \mathbf{L}g^\circ, \qquad (g\circ f)^\dagger = f^\dagger \circ g^\dagger.$$

Proof. This follows by the Grothendieck spectral sequence, since f° preserves flat complexes [Bor+87, VI, Prop. 4.3], and any complex of modules has a flat resolution by Lemma 4.20. We also present the proof in [HTT08, Prop. 1.5.11]. First, we have a chain of isomorphisms of $(D_X, (g\circ f)^{-1}D_Z)$-bimodules

$$\begin{aligned} D_{X\to Y} \otimes^{\mathbf{L}}_{f^{-1}D_Y} f^{-1}D_{Y\to Z} &= (\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y) \otimes^{\mathbf{L}}_{f^{-1}D_Y} f^{-1}(\mathcal{O}_Y \otimes_{g^{-1}\mathcal{O}_Z} g^{-1}D_Z)\\ &\simeq (\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y) \otimes^{\mathbf{L}}_{f^{-1}D_Y} (f^{-1}\mathcal{O}_Y \otimes_{(g\circ f)^{-1}\mathcal{O}_Z} (g\circ f)^{-1}D_Z)\\ &= (\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y) \otimes^{\mathbf{L}}_{f^{-1}D_Y} (f^{-1}\mathcal{O}_Y \otimes^{\mathbf{L}}_{(g\circ f)^{-1}\mathcal{O}_Z} (g\circ f)^{-1}D_Z)\\ &\simeq \mathcal{O}_X \otimes^{\mathbf{L}}_{(g\circ f)^{-1}\mathcal{O}_Z} (g\circ f)^{-1}D_Z\\ &= D_{X\to Z} \end{aligned}$$

where we use repeatedly that D is a locally free $\mathcal{O}$-module. We therefore have

$$\begin{aligned} \mathbf{L}(g\circ f)^\circ(M^\bullet) &= D_{X\to Z} \otimes^{\mathbf{L}}_{(g\circ f)^{-1}D_Y} (g\circ f)^{-1}M^\bullet\\ &\simeq (D_{X\to Y} \otimes^{\mathbf{L}}_{f^{-1}D_Y} f^{-1}D_{Y\to Z}) \otimes^{\mathbf{L}}_{f^{-1}g^{-1}D_Y} f^{-1}g^{-1}M^\bullet\\ &\simeq D_{X\to Y} \otimes^{\mathbf{L}}_{f^{-1}D_Y} f^{-1}(D_{Y\to Z} \otimes^{\mathbf{L}}_{g^{-1}D_Y} g^{-1}M)\\ &= \mathbf{L}f^\circ(\mathbf{L}g^\circ(M^\bullet)). \end{aligned}$$

□

We now present an example of (shifted) inverse images.

Example 4.26 (Open embeddings)**.** Let $j\colon U \hookrightarrow X$ be an open embedding into a smooth algebraic variety X. Then, j^{-1} is just restriction to U, and so in particular, $D_{U\to X} = j^{-1}D_X = D_U$, and $j^\dagger = \mathbf{L}j^\circ = j^{-1}$.

5 October 10 and October 17: Operations on D-modules (Takumi Murayama)

Today, we want to describe the derived inverse image functor (in more detail) and the derived direct image. We first give some motivation for the specific examples we will be studying in both contexts.

Let $f\colon X \to Y$ be any morphism of smooth quasi-projective varieties. We can factor this map f as

$$X \xrightarrow{\Gamma_f} X \times Y \xrightarrow{p_2} Y, \tag{5.1}$$

where Γ_f is a closed immersion since it is the base change of the diagonal $\Delta\colon Y \to Y \times Y$, and $X \times Y \to Y$ is a projection map. Therefore, to understand inverse images and direct images of D-modules, it suffices to understand closed immersions and projections separately.

Along the way, we will point out that because all of our varieties are smooth, we can assume even more about the morphisms Γ_f and p_2 above. For example, Γ_f will realize X as local complete intersection in $X \times Y$, and p_2 will be smooth.

We start first by talking about tensor products and box products, since these will be useful when we study projections.

5.1 Tensor products and box products

First, the bifunctor

$$- \otimes_{\mathcal{O}_X} -\colon \mathsf{Mod}(D_X) \times \mathsf{Mod}(D_X) \longrightarrow \mathsf{Mod}(D_X)$$

is right-exact in both factors, we can define its left derived functor as

$$- \otimes^{\mathbf{L}}_{\mathcal{O}_X} -\colon \quad \mathsf{D}^b(D_X) \times \mathsf{D}^b(D_X) \quad \longrightarrow \mathsf{D}^b(D_X)$$

by using flat resolutions as D_X-modules. Since a flat D_X-module is flat over $\mathcal{O}_X$, we have a commutative diagram

$$\begin{array}{ccc} \mathsf{D}^b(D_X) \times \mathsf{D}^b(D_X) & \xrightarrow{-\otimes^{\mathbf{L}}_{\mathcal{O}_X}-} & \mathsf{D}^b(D_X) \\ \downarrow & & \downarrow \\ \mathsf{D}^b(\mathcal{O}_X) \times \mathsf{D}^b(\mathcal{O}_X) & \xrightarrow{-\otimes^{\mathbf{L}}_{\mathcal{O}_X}-} & \mathsf{D}^b(\mathcal{O}_X) \end{array}$$

and so the functor $- \otimes^{\mathbf{L}}_{\mathcal{O}_X} -$ descends to a functor

$$- \otimes^{\mathbf{L}}_{\mathcal{O}_X} -\colon \mathsf{D}^b_{\mathsf{qc}}(D_X) \times \mathsf{D}^b_{\mathsf{qc}}(D_X) \longrightarrow \mathsf{D}^b_{\mathsf{qc}}(D_X).$$

Now consider the smooth variety $X \times Y$, and consider the two projections

$$\begin{array}{ccc} & X \times Y & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ X & & Y \end{array}$$

Let M be a left D_X-module, and N a left D_Y-module. We want to describe how they pullback to $X \times Y$. To do so, we first make some general remarks about $X \times Y$. Since

$$\Theta_{X\times Y} \simeq p_1^*\Theta_X \oplus p_2^*\Theta_Y$$

we also expect that $D_{X\times Y}$ is related to D_X and D_Y somehow. In fact,

Lemma 5.1. *$D_{X\times Y} \simeq \mathcal{O}_{X\times Y} \otimes_{p_1^{-1}\mathcal{O}_X \otimes_{\mathbf{C}} p_2^{-1}\mathcal{O}_Y} p_1^{-1}D_X \otimes_{\mathbf{C}} p_2^{-1}D_Y$ as $\mathcal{O}_{X\times Y}$-modules.*

Proof. Choose local coordinates, and notice that

$$p_1^{-1}D_X \otimes_{\mathbf{C}} p_2^{-1}D_Y \simeq (p_1^{-1}\mathcal{O}_X \otimes_{\mathbf{C}} p_2^{-1}\mathcal{O}_Y)[\underline{\partial_x}, \underline{\partial_y}],$$

where $\underline{\partial_x}$ are the local partials on X, and similarly for Y. □

Remark 5.2. The statement of this Lemma in [Bor+87, IV,4.5] is a bit wrong: they identify $\mathcal{O}_{X\times Y}$ and $p_1^{-1}\mathcal{O}_X \otimes_{\mathbf{C}} p_2^{-1}\mathcal{O}_Y$ and so only the pullbacks of D_X and D_Y appear. I think their statement is correct if you ignore non-closed points, since the two sheaves $\mathcal{O}_{X\times Y}$ and $p_1^{-1}\mathcal{O}_X \otimes_{\mathbf{C}} p_2^{-1}\mathcal{O}_Y$ differ only at non-closed points.

We can now define the "box product":

Definition 5.3. Let $M \in \mathsf{Mod}(D_X)$ and $N \in \mathsf{Mod}(D_Y)$. Then, we define

$$\begin{aligned} M \boxtimes N &:= \mathcal{O}_{X\times Y} \otimes_{p_1^{-1}\mathcal{O}_X \otimes_{\mathbf{C}} p_2^{-1}\mathcal{O}_Y} (p_1^{-1}M \otimes_{\mathbf{C}} p_2^{-1}N) \\ &\cong D_{X\times Y} \otimes_{p_1^{-1}D_X \otimes_{\mathbf{C}} p_2^{-1}D_Y} (p_1^{-1}M \otimes_{\mathbf{C}} p_2^{-1}N) \end{aligned} \tag{5.2}$$

This description shows $M \boxtimes N$ is exact in both factors (since both projections are flat: they are base changes of the smooth structure morphisms for X and Y), so it defines functors

$$\begin{aligned} - \boxtimes -\colon \mathsf{Mod}(D_X) \times \mathsf{Mod}(D_Y) &\longrightarrow \mathsf{Mod}(D_{X\times Y}) \\ - \boxtimes -\colon \mathsf{D}^b(D_X) \times \mathsf{D}^b(D_Y) &\longrightarrow \mathsf{D}^b(D_{X\times Y}) \end{aligned}$$

Lemma 5.4. *This functor descends to functors*

$$\begin{aligned} - \boxtimes -\colon \mathsf{D}^b_{\mathsf{qc}}(D_X) \times \mathsf{D}^b_{\mathsf{qc}}(D_Y) &\longrightarrow \mathsf{D}^b_{\mathsf{qc}}(D_{X\times Y}) \\ - \boxtimes -\colon \mathsf{D}^b_{\mathsf{c}}(D_X) \times \mathsf{D}^b_{\mathsf{c}}(D_Y) &\longrightarrow \mathsf{D}^b_{\mathsf{c}}(D_{X\times Y}) \end{aligned}$$

Proof. This is clear by the isomorphisms (5.2). □

We now show that tensor products and inverse images are compatible, by first showing it for box products:

Proposition 5.5.

(*i*) *Let $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$ be morphisms of smooth algebraic varieties. Then for $M_1^\bullet \in \mathsf{D}^b(D_{Y_1})$, $M_2^\bullet \in \mathsf{D}^b(D_{Y_2})$, we have*

$$\mathbf{L}(f_1 \times f_2)^\circ(M_1^\bullet \boxtimes M_2^\bullet) \simeq \mathbf{L}f_1^\circ M_1^\bullet \boxtimes \mathbf{L}f_2^\circ M_2^\bullet.$$

(*ii*) *Let $f\colon X \to Y$ be a morphism of smooth algebraic varieties. Then, for $M^\bullet, N^\bullet \in \mathsf{D}^b(D_Y)$, we have*

$$\mathbf{L}f^\circ(M^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} N^\bullet) \simeq \mathbf{L}f^\circ M^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathbf{L}f^\circ N^\bullet.$$

Proof. For (*i*), since $- \boxtimes -$ is exact, it suffices to note that this is true on the level of modules.

We want to reduce (*ii*) to (*i*) by using the diagonal embedding $\Delta_Y\colon Y \to Y \times Y$. First, note

$$\begin{aligned} \Delta_Y^\circ(M \boxtimes N) &= \Delta_Y^\circ(\mathcal{O}_{Y\times Y} \otimes_{p_1^{-1}\mathcal{O}_Y \otimes_{\mathbf{C}} p_2^{-1}\mathcal{O}_Y} (p_1^{-1}M \otimes_{\mathbf{C}} p_2^{-1}N)) \\ &\simeq \mathcal{O}_Y \otimes_{\Delta_Y^{-1}\mathcal{O}_{Y\times Y}} \Delta_Y^{-1}\mathcal{O}_{Y\times Y} \otimes_{\Delta_Y^{-1}(p_1^{-1}\mathcal{O}_Y \otimes_{\mathbf{C}} p_2^{-1}\mathcal{O}_Y)} (M \otimes_{\mathbf{C}} N) \\ &\simeq \mathcal{O}_Y \otimes_{\mathcal{O}_Y \otimes_{\mathbf{C}} \mathcal{O}_Y} (M \otimes_{\mathbf{C}} N) \\ &\simeq M \otimes_{\mathcal{O}_Y} N \end{aligned}$$

where tracing the D_Y-module structure everywhere, we see that this isomorphism preserves the D_Y-module structure. Since $- \boxtimes -$ preserves flat modules, we have a canonical isomorphism

$$M^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} N^\bullet \simeq \mathbf{L}\Delta_Y^\circ(M^\bullet \boxtimes N^\bullet)$$

in $\mathsf{D}^b(D_X)$. Now can prove (*ii*):

$$\begin{aligned} \mathbf{L}f^\circ(M^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} N^\bullet) &\simeq \mathbf{L}f^\circ \mathbf{L}\Delta_Y^\circ(M^\bullet \boxtimes N^\bullet) \\ &\simeq \mathbf{L}\Delta_X^\circ \mathbf{L}(f \times f)^\circ(M^\bullet \boxtimes N^\bullet) \\ &\simeq \mathbf{L}\Delta_X^\circ(\mathbf{L}f^\circ M^\bullet \boxtimes \mathbf{L}f^\circ N^\bullet) \\ &\simeq \mathbf{L}f^\circ M^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathbf{L}f^\circ N^\bullet. \end{aligned}$$

□

5.2 Derived inverse images and shifted inverse images

5.2.1 Projections

Now we can write down what inverse images through projections look like:

Example 5.6. Given notation as before, we have

$$p_1^\circ M \simeq M \boxtimes \mathcal{O}_Y, \quad p_2^\circ \simeq \mathcal{O}_X \boxtimes N.$$

Since $- \boxtimes -$ was exact in both factors, we get that

$$p_1^\dagger M^\bullet \simeq \mathbf{L}p_1^\circ M^\bullet[\dim Y] = p_1^\circ M^\bullet[\dim Y], \quad p_2^\dagger N^\bullet \simeq \mathbf{L}p_2^\circ N^\bullet[\dim X] = p_2^\circ N^\bullet[\dim X]$$

and so

$$H^i(p_1^\dagger M^\bullet) = p_1^\circ(H^{i+\dim Y}(M^\bullet)), \quad H^i(p_2^\dagger N^\bullet) = p_2^\circ(H^{i+\dim X}(N^\bullet)).$$

We can use this description of projections for slightly more general maps:

Proposition 5.7. *Let $f\colon X \to Y$ be a smooth morphism of smooth varieties, and let $M \in \mathsf{Mod}(D_Y)$. Then, $H^i(f^\dagger M) = 0$ for all $i \neq \dim X - \dim Y$, and if $M \in \mathsf{Mod}_{\mathsf{c}}(D_Y)$, then $f^\dagger M \in \mathsf{D}^b_{\mathsf{c}}(D_X)$.*

Proof. The first statement follows from the flatness of $\mathcal{O}_X$ over $f^{-1}\mathcal{O}_Y$, and $\mathbf{L}f^\circ M \simeq \mathcal{O}_X \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_Y} f^{-1}M$.

For the second claim, this is a local question, and so we can assume that f factors as a composition

$$\begin{array}{ccc} X & \xrightarrow{\;g\;} & \mathbf{A}^n \times Y \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p_2} \\ & & Y \end{array}$$

where g is étale. Pulling back through p_2 preserves coherence by our description of $p_2^\dagger$ in terms of $\boxtimes$, and then by Lemma 5.4. To show $g^\dagger$ preserves coherence, it suffices to show that the canonical morphism $D_X \to D_{X\to\mathbf{A}^n\times Y}$ is surjective, since $g^\circ M = D_{X\to Y} \otimes^{\mathbf{L}}_{f^{-1}D_Y} f^{-1}M$. By étaleness, we can choose local coordinates on X and $Z \coloneqq \mathbf{A}^n \times Y$ such that ∂_{x_i} maps to ∂_{z_i}. Then, we have that

$$D_{X\to\mathbf{A}^n\times Y} \simeq \bigoplus_{i=1}^{\dim X} \mathcal{O}_X \partial_{z_i},$$

and so $D_X \to D_{X\to\mathbf{A}^n\times Y}$ is surjective. □

5.2.2 Closed immersions

Now let $i\colon X \to Y$ be a closed immersion of smooth quasi-projective varieties. Denote $d \coloneqq \dim Y - \dim X$. Since X is smooth, its image in Y is a local complete intersection [Har77, Ex. 8.22.1], and so locally I_X is defined by a regular sequence $y_1 = \cdots = y_d = 0$. Locally, we then have a Koszul resolution [FL85, p. 76]:

$$0 \longrightarrow K_d \longrightarrow K_{d-1} \longrightarrow \cdots \longrightarrow K_1 \longrightarrow K_0 \longrightarrow \mathcal{O}_X \longrightarrow 0, \tag{5.3}$$

where $K_p = \bigwedge^p \bigoplus_{i=1}^d i^{-1}\mathcal{O}_Y \cdot dy_i$, and the differential is given by contraction:

$$\begin{aligned} K_p &\longrightarrow K_{p-1} \\ dy_1 \wedge \cdots \wedge dy_p &\longmapsto \sum_{j=1}^p (-1)^{j-1} y_j \, dy_1 \wedge \cdots \wedge \widehat{dy_j} \wedge \cdots \wedge dy_p. \end{aligned}$$

Remark 5.8. [HTT08, p. 35] makes it sound like this resolution exists globally, but this seems too strong.

Now suppose we chose another coordinate system y'_j on Y such that $y'_1 = \cdots = y'_d = 0$ defines X, and the $y'_1, \ldots, y'_d$ form a regular sequence. Then, just by computing the changes of coordinates, we see the corresponding Koszul complexes glue together to form a global resolution with $K_d \simeq \bigwedge^d \Omega_{Y/X}$, and so we have the following result:

Lemma 5.9. *There exists a* global *locally free resolution of the right $i^{-1}D_Y$-module $D_{X\to Y}$:*

$$0 \longrightarrow K_d \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}D_Y \longrightarrow \cdots \longrightarrow K_0 \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}D_Y \longrightarrow D_{X\to Y} \longrightarrow 0, \tag{5.4}$$

and so $\mathbf{L}i^\circ M$ is represented by the complex

$$\cdots \longrightarrow 0 \longrightarrow K_d \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}M \longrightarrow \cdots \longrightarrow K_0 \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}M \longrightarrow 0 \longrightarrow \cdots .$$

Proof. Tensor the Koszul resolution (5.3) by $i^{-1}D_Y$, which is flat over $i^{-1}\mathcal{O}_Y$. ☐

One thing we did not discuss before is what the inverse image functor should be an analogue for in the theory of constructible sheaves. The following Proposition will make this a bit clearer, and motivate the next Definition as well.

Proposition 5.10. *Let $i\colon X \to Y$ be a closed embedding of smooth algebraic varieties. Set $d = \dim Y - \dim X$.*
(i) If $M \in \mathsf{Mod}(D_Y)$, then $H^j(i^\dagger M) = 0$ unless $0 \le j \le d$. Moreover,

$$H^0(i^\dagger M) \simeq \omega_{X/Y}^{-1} \otimes_{\mathcal{O}_Y} \{m \in M \mid I_X m = 0\}, \quad H^d(i^\dagger M) \simeq M/I_X M.$$

(ii) For $M^\bullet \in \mathsf{D}^+(\mathsf{D}_Y)$, we have a canonical isomorphism

$$i^\dagger M^\bullet \simeq \mathbf{R}\mathscr{H}\!om_{i^{-1}D_Y}(D_{Y\leftarrow X}, i^{-1}M^\bullet)[d] \simeq \mathbf{R}\mathscr{H}\!om_{i^{-1}D_Y}(D_{Y\leftarrow X}, i^{-1}\Gamma_X(M^\bullet))[d]$$

where $\Gamma_X(M)$ is the subsheaf of M consisting of sections with support in X.

Remark 5.11. (i) says that $H^0 i^\dagger$ is an analogue to the "exceptional inverse image" functor, which takes a sheaf to its sections with support in a closed subvariety, while $H^d i^\dagger$ is the analogue of the ordinary inverse image functor. According to [Bor+87, IV, Rem. 7.6], $i^\dagger$ should be thought to act more like $H^0 i^\dagger$, hence more like the exceptional inverse image. One reason to introduce the grade shift in $-^\dagger$, then, is to make sure that this important cohomology group actually lies in degree 0. (ii) makes this a bit more precise.

Proof. (i) follows by looking at the Koszul resolution (5.4): the vanishing follows by the length of the Koszul resolution, and the description of H^d follows from the fact that $\mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} M \simeq M/I_X M$. For H^0, it suffices to note that $K_d \simeq i^{-1}\mathcal{O}_Y \cdot dy_1 \wedge \cdots \wedge dy_d$, and so

$$K_d \simeq i^{-1}\mathcal{O}_Y \cdot dy_1 \wedge \cdots \wedge dy_d,$$

which close to being $\omega_{X/Y}^{-1}$. After putting everything into the complex in the Lemma, we see that the description of H^0 also holds.

For (ii), it is enough to show that

$$\mathbf{R}\mathscr{H}\!om_{i^{-1}D_Y}(D_{Y\leftarrow X}, i^{-1}D_Y) \simeq D_{X\to Y}[-d], \tag{5.5}$$

since then,

$$\begin{aligned} \mathbf{L}i^* M &= D_{X\to Y} \otimes^{\mathbf{L}}_{i^{-1}D_Y} i^{-1}M \\ &\simeq \mathbf{R}\mathscr{H}\!om_{i^{-1}D_Y}(D_{Y\leftarrow X}, i^{-1}D_Y) \otimes^{\mathbf{L}}_{i^{-1}D_Y} i^{-1}M[d] \\ &\simeq \mathbf{R}\mathscr{H}\!om_{i^{-1}D_Y}(D_{Y\leftarrow X}, i^{-1}M)[d]. \end{aligned}$$

By side-changing, the isomorphism (5.5) is equivalent to

$$\mathbf{R}\mathscr{H}\!om_{i^{-1}D_Y^{\mathsf{op}}}(D_{X\to Y}, i^{-1}D_Y) \simeq D_{Y\leftarrow X}[-d].$$

We have

$$\begin{aligned}\mathbf{R}\mathscr{H}om_{i^{-1}D_Y^{\mathsf{op}}}(D_{X\to Y}, i^{-1}D_Y) &\simeq \mathbf{R}\mathscr{H}om_{i^{-1}D_Y^{\mathsf{op}}}(\mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}D_Y, i^{-1}D_Y)\\ &\simeq \mathbf{R}\mathscr{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{O}_X, i^{-1}D_Y)\\ &\simeq i^{-1}D_Y \otimes_{i^{-1}\mathcal{O}_Y} \mathbf{R}\mathscr{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{O}_X, i^{-1}\mathcal{O}_Y).\end{aligned}$$

The complex $\mathbf{R}\mathscr{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{O}_X, i^{-1}\mathcal{O}_Y)$ can be described by

$$K_0^* \longrightarrow K_1^* \longrightarrow \cdots \longrightarrow K_d^*,$$

where $K_p^* = \mathscr{H}om_{i^{-1}\mathcal{O}_Y}(K_p, i^{-1}\mathcal{O}_Y)$. By using the duality of the Koszul complex, we have

$$\begin{aligned}\{K_0^* \longrightarrow K_1^* \longrightarrow \cdots \longrightarrow K_d^*\} &\simeq \{K_d \longrightarrow K_{d-1} \longrightarrow \cdots \longrightarrow K_0\} \otimes_{i^{-1}\mathcal{O}_Y} K_d^*\\ &\simeq \mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} K_d^*[-d]\\ &\simeq i^{-1}\omega_Y^{\otimes -1} \otimes_{i^{-1}\mathcal{O}_Y} \omega_X[-d]\end{aligned}$$

and so

$$\mathbf{R}\mathscr{H}om_{i^{-1}D_Y^{\mathsf{op}}}(D_{X\to Y}, i^{-1}D_Y) \simeq i^{-1}D_Y \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}\omega_Y^{\otimes -1} \otimes_{i^{-1}\mathcal{O}_Y} \omega_X[-d] \simeq D_{Y\leftarrow X}[-d].$$

We won't prove the last statement; see [HTT08, Prop. 1.5.16]. □

Definition 5.12. For a closed embedding $i\colon X \to Y$ of smooth algebraic varieties, we define a left exact functor

$$\begin{aligned}i^\natural\colon \mathsf{Mod}(D_Y) &\longrightarrow \mathsf{Mod}(D_X)\\ M &\longmapsto \mathscr{H}om_{i^{-1}D_Y}(D_{Y\leftarrow X}, i^{-1}M)\end{aligned}$$

5.3 Derived direct images

Definition 5.13. Let $f\colon X \to Y$ be a morphism of smooth quasi-projective complex varieties. We define the *derived direct image functor*

$$\begin{aligned}\int_f\colon \mathsf{D}^b(D_X) &\longrightarrow \mathsf{D}^b(D_Y)\\ M^\bullet &\longmapsto \mathbf{R}f_*(D_{Y\leftarrow X} \otimes^{\mathbf{L}}_{\mathsf{D}_X} M^\bullet)\end{aligned}$$

by using a flat resolution of $M^\bullet$ as in Lemma 4.20, and then an injective resolution of $D_{Y\leftarrow X} \otimes^{\mathbf{L}}_{\mathsf{D}_X} M^\bullet$.

We want to say that quasi-coherence and coherence are preserved (the latter when f is proper), but this is a bit difficult to prove, so we will return to this later.

Proposition 5.14. *The functor $\mathbf{R}f_*$ preserves $\mathcal{O}$-quasi-coherence, and if f is proper, it preserves $\mathcal{O}$-coherence as well.*

Definition 5.15. We also define for each $k \in \mathbf{Z}$

$$\int_f^k M^\bullet = H^k\left(\int_f M^\bullet\right),$$

and

$$\begin{aligned}\int_f\colon \mathsf{D}^b(D_X^{\mathsf{op}}) &\longrightarrow \mathsf{D}^b(D_Y^{\mathsf{op}})\\ M^\bullet &\longmapsto \mathbf{R}f_*(M^\bullet \otimes^{\mathbf{L}}_{D_X} D_{X\to Y})\end{aligned}$$

Just as for the non-derived version, we have a commutative diagram

$$\begin{array}{ccc}\mathsf{D}^b(D_X) & \xrightarrow{\int_f} & \mathsf{D}^b(D_Y)\\ {\scriptstyle \omega_X\otimes_{\mathcal{O}_X}-}\big\downarrow\wr & & \wr\big\downarrow{\scriptstyle \omega_Y\otimes_{\mathcal{O}_Y}-}\\ \mathsf{D}^b(D_X^{\mathsf{op}}) & \xrightarrow{\int_f} & \mathsf{D}^b(D_Y^{\mathsf{op}})\end{array}$$

5.3.1 Composition of direct images

Our first main goal will be to show that these derived direct images are well-behaved under composition. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a composition of morphisms of smooth varieties. Then, just as for the inverse image, we first need to describe the transfer bimodule $D_{Z\leftarrow X}$ in terms of the two others. We have

$$D_{Z\leftarrow X} \simeq f^{-1}D_{Z\leftarrow Y} \otimes_{f^{-1}D_Y} D_{Y\leftarrow X} \simeq f^{-1}D_{Z\leftarrow Y} \otimes^{\mathbf{L}}_{f^{-1}D_Y} D_{Y\leftarrow X}$$

We also need the following, which can be thought of as a projection formula for D-modules.

Lemma 5.16. *Let $F^\bullet \in \mathsf{D}^-_{\mathsf{qc}}(D_Y^{\mathsf{op}})$ and $G^\bullet \in \mathsf{D}^b(f^{-1}D_Y)$. The canonical morphism*

$$F^\bullet \otimes^{\mathbf{L}}_{D_Y} \mathbf{R}f_*(G^\bullet) \longrightarrow \mathbf{R}f_*(f^{-1}F^\bullet \otimes^{\mathbf{L}}_{f^{-1}D_Y} G^\bullet)$$

is an isomorphism.

Proof. We may replace $F^\bullet$ with a locally free resolution, and so $f^{-1}F^\bullet$ is locally free over $f^{-1}D_Y$, and we can turn the right-hand side into an ordinary tensor product. Moreover, we can replace $G^\bullet$ with an injective complex, in which case $f^{-1}F^\bullet \otimes_{f^{-1}D_Y} G^\bullet$ is locally a direct sum of injective complexes, hence is also injective. This implies we may turn all higher direct images into ordinary ones, and so we want to show

$$F^\bullet \otimes_{D_Y} f_*(G^\bullet) \longrightarrow f_*(f^{-1}F^\bullet \otimes_{f^{-1}D_Y} G^\bullet)$$

is an isomorphism; this is true since direct images commute with direct sums. □

We can now show

Proposition 5.17. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of morphisms of smooth varieties. Then,*

$$\int_{g\circ f} = \int_g \int_f .$$

Proof. We compute directly:

$$\begin{aligned}\int_g\int_f M^\bullet &\simeq \mathbf{R}g_*\mathbf{R}f_*(f^{-1}D_{Z\leftarrow Y} \otimes^{\mathbf{L}}_{f^{-1}D_Y} (D_{Y\leftarrow X} \otimes^{\mathbf{L}}_{D_X} M^\bullet))\\ &\simeq \mathbf{R}(g\circ f)_*((f^{-1}D_{Z\leftarrow Y} \otimes^{\mathbf{L}}_{f^{-1}D_Y} D_{Y\leftarrow X}) \otimes^{\mathbf{L}}_{D_X} M^\bullet)\\ &\simeq \mathbf{R}(g\circ f)_*(D_{Z\leftarrow X} \otimes^{\mathbf{L}}_{D_X} M^\bullet)\\ &= \int_{g\circ f} M^\bullet.\end{aligned}$$

□

Just as for inverse images, we work through three kinds of morphisms in detail.

Example 5.18 (Open embeddings)**.** Let $j\colon U \hookrightarrow X$ be an open embedding into a smooth algebraic variety X. Then, $D_{X\leftarrow U} = j^{-1}D_X = D_U$, and so

$$\int_j = \mathbf{R}j_*$$

5.3.2 Projections

We now come to the first special case of push forwards. Consider the following diagram:

$$\begin{array}{ccc} & X = Y\times Z & \\ {}^{f}\swarrow & & \searrow^{g} \\ Y & & Z \end{array}$$

We want to compute $\int_f M := \mathbf{R}f_*(D_{Y\leftarrow X} \otimes^{\mathbf{L}}_{D_X} M)$ for $M \in \mathsf{Mod}_{\mathsf{qc}}(D_X)$; to do so, we first need to find a locally free resolution of $D_{Y\leftarrow X}$. To do this, we introduce the *de Rham resolution*:

Lemma 5.19. *Let $n = \dim X$. We have the following locally free resolution of the right D_X-module $\omega_X = \Omega_X^n$:*

$$0 \longrightarrow \Omega_X^0 \otimes_{\mathcal{O}_X} D_X \longrightarrow \cdots \longrightarrow \Omega_X^n \otimes_{\mathcal{O}_X} D_X \xrightarrow{\epsilon} \omega_X \longrightarrow 0 \tag{5.6}$$

where $\Omega_X^k = \bigwedge^k \Omega_X^1$, the augmentation morphism is

$$\begin{aligned} \epsilon\colon \Omega_X^n \otimes_{\mathcal{O}_X} D_X &\longrightarrow \omega_X \\ \omega \otimes P &\longmapsto \omega P \end{aligned}$$

and the differentials are

$$\begin{aligned} d\colon \Omega_X^k \otimes_{\mathcal{O}_X} D_X &\longrightarrow \Omega_X^{k+1} \otimes_{\mathcal{O}_X} D_X \\ \omega \otimes P &\longmapsto d\omega \otimes P + \sum_i dx_i \wedge \omega \otimes \partial_i P \end{aligned}$$

where $\{x_i, \partial_i\}$ is a system of local coordinates on X.

We've already seen (§1.2) that the sequence (5.6) is a complex, and so it suffices to show that locally, this complex is exact. To do so, the main idea is that by applying the order filtration on the complex, it suffices to show that the associated graded complex (which is a complex of modules over the commutative ring $\operatorname{gr} D_X$) is exact. The associated graded complex is a particular case of the (dual of the) Koszul resolution [FL85, p. 76], which is exact.

One issue is that we haven't said what the order of a differential form should be. The proof in [HTT08, Lem. 1.5.27] gets around this issue by using side-changing operations (i.e., applying $- \otimes_{\mathcal{O}_X} \omega_X$) to reduce exactness to exactness of the *Spencer resolution* of the left D_X-module $\mathcal{O}_X$:

$$0 \longrightarrow D_X \otimes_{\mathcal{O}_X} \bigwedge^n \Theta_X \longrightarrow \cdots \longrightarrow D_X \otimes_{\mathcal{O}_X} \bigwedge^0 \Theta_X \longrightarrow \mathcal{O}_X \longrightarrow 0, \tag{5.7}$$

where the proof then proceeds as we outlined above. We get around this by just saying that the order of a differential form in Ω_X^k should be $-k$.

Proof of Lemma 5.19. We already saw (§1.2) that this sequence forms a complex. It therefore suffices to verify exactness locally. Temporarily denoting the complex (5.6) by $N^\bullet$, we consider its filtration $\{F_p N^\bullet\}$:

$$F_p N^\bullet = \Big\{0 \longrightarrow \Omega_X^0 \otimes_{\mathcal{O}_X} F_p D_X \longrightarrow \cdots \longrightarrow \Omega_X^n \otimes_{\mathcal{O}_X} F_{p+n} D_X \xrightarrow{\epsilon} F_p(\omega_X) \longrightarrow 0\Big\}$$

where

$$F_p(\omega_X) = \begin{cases} \omega_X & \text{if } p \geq -n \\ 0 & \text{if } p < -n \end{cases}$$

It then suffices to show that the associated graded complex $\operatorname{gr} N^\bullet$ is exact.

Denote $\pi\colon T^*X \to X$ and $i\colon X \to T^*X$ to be the projection from the cotangent bundle and the embedding by the zero-section into the cotangent bundle, respectively. Then, we claim we have $\operatorname{gr} N^\bullet \simeq \pi_* L^\bullet$, with

$$L^\bullet = \Big\{0 \longrightarrow \pi^{-1}\Omega_X^0 \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{O}_{T^*X} \longrightarrow \cdots \longrightarrow \pi^{-1}\Omega_X^n \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{O}_{T^*X} \xrightarrow{\epsilon} i_*\omega_X \longrightarrow 0\Big\},$$

where the augmentation morphism is given by

$$\begin{aligned} \epsilon\colon \pi^{-1}\Omega_X^n \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{O}_{T^*X} &\longrightarrow i_*\omega_X \\ \pi^{-1}\omega \otimes \varphi &\longmapsto \varphi \cdot i_*(\pi^{-1}\omega) \end{aligned}$$

and the differential is given by

$$\begin{aligned} d\colon \pi^{-1}\Omega_X^k \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{O}_{T^*X} &\longrightarrow \pi^{-1}\Omega_X^{k+1} \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{O}_{T^*X} \\ \omega \otimes \varphi &\longmapsto \sum_i dx_i \wedge \omega \otimes \partial_i \varphi \end{aligned}$$

The isomorphism $\operatorname{gr} N^\bullet \simeq \pi_* L^\bullet$ follows since each term in this complex is $\pi^*\Omega_X^k$, and $\pi_*\pi^*\Omega_X^k \simeq \Omega_X^k \otimes \operatorname{gr} D_X$ by the projection formula; moreover, the maps in $L^\bullet$ pushforward to the maps $\operatorname{gr} N^\bullet$.

Finally, $L^\bullet$ is exact since it can be formed by applying $\pi^*\omega_X^{-1} \otimes -$ to the Koszul resolution for $i_*\mathcal{O}_X$ on $\mathcal{O}_{T^*X}$ [FL85, p. 76]. Since π is affine, we see that $\operatorname{gr} N^\bullet \simeq \pi_* L^\bullet$ is also exact, and we conclude that (5.6) is exact as well. $\square$

This lets us easily describe direct images of projections, as follows. Recall our notation:

$$\begin{array}{ccc} & X = Y \times Z & \\ {}^{f}\swarrow & & \searrow^{g} \\ Y & & Z \end{array}$$

We want to compute $\int_f M \coloneqq \mathbf{R}f_*(D_{Y\leftarrow X} \otimes^{\mathbf{L}}_{D_X} M)$ for $M \in \mathsf{Mod}_{\mathsf{qc}}(D_X)$; we first compute $D_{Y\leftarrow X} \otimes^{\mathbf{L}}_{D_X} M$ by using the de Rham resolution in Lemma 5.19. First, notice that by Lemma 4.13, we have an isomorphism

$$\begin{aligned} D_{Y\leftarrow X} &\simeq f^{-1}(D_Y \otimes_{\mathcal{O}_Y} \omega_Y^\vee) \otimes_{f^{-1}\mathcal{O}_Y} \omega_X \\ &\simeq f^{-1}(D_Y \otimes_{\mathcal{O}_Y} \omega_Y^\vee) \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\omega_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \otimes_{g^{-1}\mathcal{O}_Z} g^{-1}\omega_Z \\ &\simeq f^{-1}D_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \otimes_{g^{-1}\mathcal{O}_Z} g^{-1}\omega_Z \\ &\simeq \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y \otimes_{\mathbf{C}} g^{-1}\mathcal{O}_Z} (f^{-1}D_Y \otimes_{\mathbf{C}} g^{-1}\omega_Z) \\ &= D_Y \boxtimes \omega_Z \end{aligned}$$

and since $- \boxtimes -$ is exact, the de Rham resolution from Lemma 5.19 gives a resolution of the right D_X-module $D_{Y\leftarrow X}$ as

$$0 \longrightarrow D_Y \boxtimes (\Omega_Z^0 \otimes_{\mathcal{O}_Z} D_Z) \longrightarrow \cdots \longrightarrow D_Y \boxtimes (\Omega_Z^n \otimes_{\mathcal{O}_Z} D_Z) \xrightarrow{\epsilon} D_{Y\leftarrow X} \longrightarrow 0.$$

By replacing $D_{Y\leftarrow X}$ with this locally free resolution, we can write down a concrete complex representing $D_{Y\leftarrow X} \otimes^{\mathbf{L}}_{D_X} M$ as follows.

Definition 5.20. Let $n = \dim Z = \dim X - \dim Y$, and let $\Omega^k_{X/Y} \coloneqq \mathcal{O}_Y \boxtimes \Omega_Z^k$ for $0 \le k \le n$. For $M \in \mathsf{Mod}_{\mathsf{qc}}(D_X)$, we define the *relative de Rham complex* by

$$(\mathrm{DR}^\bullet_{X/Y}(M))^k \coloneqq \begin{cases} \Omega^{n+k}_{X/Y} \otimes_{\mathcal{O}_X} M & \text{if } -n \le k \le 0 \\ 0 & \text{otherwise} \end{cases}$$

$$d(\omega \otimes s) = d\omega \otimes s + \sum_{i=1}^n (dz_i \wedge \omega) \otimes \partial_i s,$$

where $\{z_i, \partial_i\}$ is a local coordinate system on Z.

We note that $\mathrm{DR}^\bullet_{X/Y}(M)$ in fact is a complex of left-$f^{-1}D_Y$-modules, where the action on

$$\begin{aligned} \Omega^{n+k}_{X/Y} \otimes_{\mathcal{O}_X} M &= (\mathcal{O}_Y \boxtimes \Omega_Z^k) \otimes_{\mathcal{O}_X} M \\ &= (\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y \otimes_{\mathbf{C}} g^{-1}\mathcal{O}_Z} (f^{-1}\mathcal{O}_Y \otimes_{\mathbf{C}} g^{-1}\Omega_Z^k)) \otimes_{\mathcal{O}_X} M \\ &\simeq g^{-1}\Omega_Z^k \otimes_{g^{-1}\mathcal{O}_Z} M \end{aligned}$$

is given by $P(\omega \otimes s) = \omega \otimes ((P \otimes 1)s)$, where $P \in f^{-1}D_Y$, $\omega \in g^{-1}\Omega_Z^k$, and $s \in M$; this induces a quasi-isomorphism

$$D_{Y\leftarrow X} \otimes^{\mathbf{L}}_{D_X} M \simeq \mathrm{DR}^\bullet_{X/Y}(M)$$

of complexes of $f^{-1}D_Y$-modules. We then have

Proposition 5.21. *Let Y and Z be smooth algebraic varieties and let $f\colon Y \times Z \to Y$ be the projection. Then,*

(*i*) *For $M \in \mathsf{Mod}(D_X)$, we have $\int_f M \simeq \mathbf{R}f_*(\mathrm{DR}^\bullet_{X/Y}(M))$, at least as $\mathcal{O}_Y$-modules;*

(*ii*) *For* $M \in \mathsf{Mod}(D_X)$*, we have* $\int_f^j M = 0$ *unless* $-\dim Z \le j \le \dim Z$*;*

(*iii*) *The functor* $\int_f$ *sends* $\mathsf{D}^b_{\mathsf{qc}}(D_X)$ *to* $\mathsf{D}^b_{\mathsf{qc}}(D_Y)$*.*

Proof. (*i*) follows by the quasi-isomorphism $D_{Y\leftarrow X} \otimes^{\mathbf{L}}_{D_X} M \simeq \mathrm{DR}^\bullet_{X/Y}(M)$. (*ii*) follows since f_* has cohomological dimension $\dim Z$. (*iii*) follows since if M is quasi-coherent, then $\mathrm{DR}^\bullet_{X/Y}(M)$ is a complex of quasi-coherent $\mathscr{O}_X$-modules, and so its direct image would be as well. □

5.3.3 Closed immersions

Now we consider closed immersions $i\colon X \to Y$. These are actually easier to describe:

Proposition 5.22. *Let* $i\colon X \to Y$ *be a closed embedding of smooth varieties.*

(*i*) *For* $M \in \mathsf{Mod}(D_X)$*, we have* $\int_i^k M = 0$ *for* $k \neq 0$*. In particular,* $\mathsf{Mod}(D_X) \to \mathsf{Mod}(D_Y)$ *is an exact functor.*

(*ii*) $\int_i^0$ *sends* $\mathsf{Mod}_{\mathsf{qc}}(D_X)$ *to* $\mathsf{Mod}_{\mathsf{qc}}(D_Y)$*.*

Proof. Note that i_* is exact since i is affine, and so Example 4.14 says that locally, choosing coordinates $\{y_k, \partial_{y_k}\}_{1\le k\le n}$ such that $X = \{y_{r+1} = \cdots = y_n = 0\}$, we have that

$$\int_i^k M = \begin{cases} \mathbf{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \otimes_{\mathbf{C}} i_*M & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

where the left D_Y-action is given by ∂_{y_k} on the left factor if $k > r$, and

$$\varphi(1\otimes m) = 1 \otimes (\varphi|_X)m$$
$$\partial_{y_k}(1\otimes m) = 1 \otimes \partial_{x_k} m$$

for $\varphi \in \mathscr{O}_Y$ and $1 \le k \le r$. This description shows (*i*), and (*ii*) follows since $\mathbf{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \otimes_{\mathbf{C}} i_*M$ is a quasi-coherent $\mathscr{O}_Y$-module. □

We also have an adjunction property for closed immersions. Recall that we have defined the left-exact functor

$$i^\natural\colon \mathsf{Mod}(D_Y) \longrightarrow \mathsf{Mod}(D_X)$$
$$M \longmapsto \mathscr{H}om_{i^{-1}D_Y}(D_{Y\leftarrow X}, i^{-1}M)$$

Proposition 5.23. *Let* $i\colon X \to Y$ *be a closed embedding of smooth algebraic varieties.*

(*i*) *There exists a functorial isomorphism*

$$\mathbf{R}\mathscr{H}om_{D_Y}\left(\int_i M^\bullet, N^\bullet\right) \simeq i_*\,\mathbf{R}\mathscr{H}om_{D_X}(M^\bullet, \mathbf{R}i^\natural N^\bullet)$$

where $M^\bullet \in D^-(D_X)$ *and* $N^\bullet \in D^+(D_Y)$*, and similarly, if* $M \in \mathsf{Mod}(D_X)$ *and* $N \in \mathsf{Mod}(D_Y)$*,*

$$\mathscr{H}om_{D_Y}\left(\int_i^0 M, N\right) \simeq i_*\,\mathscr{H}om_{D_X}(M, i^\natural N)$$

(*ii*) *The functor* $\mathbf{R}i^\natural\colon \mathsf{D}^b(D_Y) \to \mathsf{D}^b(D_X)$ *is right adjoint to* $\int_i\colon \mathsf{D}^b(D_X) \to \mathsf{D}^b(D_Y)$*, and the functor* $i^\natural\colon \mathsf{Mod}(D_Y) \to \mathsf{Mod}(D_X)$ *is right adjoint to* $\int_i^0\colon \mathsf{Mod}(D_X) \to \mathsf{Mod}(D_Y)$*.*

Proof. (ii) follows from (i) by applying $H^0(\mathbf{R}\Gamma(Y,-))$. For (i), we have

$$\begin{aligned}\mathbf{R}\mathscr{H}om_{D_Y}\left(\int_i M^\bullet, N^\bullet\right) &\simeq \mathbf{R}\mathscr{H}om_{D_Y}(i_*(D_{Y\leftarrow X}\otimes^{\mathbf{L}}_{D_X} M^\bullet), N^\bullet)\\ &\simeq \mathbf{R}\mathscr{H}om_{D_Y}(i_*(D_{Y\leftarrow X}\otimes^{\mathbf{L}}_{D_X} M^\bullet), \mathbf{R}\Gamma_X(N^\bullet))\\ &\simeq \mathbf{R}\mathscr{H}om_{D_Y}(i_*(D_{Y\leftarrow X}\otimes^{\mathbf{L}}_{D_X} M^\bullet), i_*i^{-1}\mathbf{R}\Gamma_X(N^\bullet))\\ &\simeq i_*\,\mathbf{R}\mathscr{H}om_{i^{-1}D_Y}(i^{-1}i_*(D_{Y\leftarrow X}\otimes^{\mathbf{L}}_{D_X} M^\bullet), i^{-1}\mathbf{R}\Gamma_X(N^\bullet))\\ &\simeq i_*\,\mathbf{R}\mathscr{H}om_{i^{-1}D_Y}(D_{Y\leftarrow X}\otimes^{\mathbf{L}}_{D_X} M^\bullet, i^{-1}\mathbf{R}\Gamma_X(N^\bullet))\\ &\simeq i_*\,\mathbf{R}\mathscr{H}om_{D_X}(M^\bullet, \mathbf{R}\mathscr{H}om_{i^{-1}D_Y}(D_{Y\leftarrow X}, i^{-1}\mathbf{R}\Gamma_X(N^\bullet)))\\ &\simeq i_*\,\mathbf{R}\mathscr{H}om_{D_X}(M^\bullet, \mathbf{R}i^\natural N^\bullet).\end{aligned}$$

□

5.3.4 Properties of general direct images

Since any morphism can be factored as a closed immersion followed by a projection we have the following.

Proposition 5.24. *If $f\colon X\to Y$ is a morphism of smooth algebraic varieties, then $\int_f$ sends $\mathsf{D}^b_{\mathsf{qc}}(D_X)$ to $\mathsf{D}^b_{\mathsf{qc}}(D_Y)$.*

Proposition 5.25. *Let $f_1\colon X_1\to Y_1$ and $f_2\colon X_2\to Y_2$ be morphisms of smooth algebraic varieties. Then for $M_1^\bullet\in\mathsf{D}^b_{\mathsf{qc}}(D_{X_1})$, $M_2^\bullet\in\mathsf{D}^b_{\mathsf{qc}}(D_{X_2})$, the canonical morphism*

$$\left(\int_{f_1} M_1^\bullet\right)\boxtimes\left(\int_{f_2} M_2^\bullet\right)\longrightarrow\int_{f_1\times f_2}(M_1^\bullet\boxtimes M_2^\bullet)$$

is an isomorphism.

Proof. By factoring

$$X_1\times X_2\xrightarrow{f_1\times\mathrm{id}} Y_1\times X_2\xrightarrow{\mathrm{id}\times f_2} Y_1\times Y_2$$

it suffices to show that for $f\colon X\to Y$ and any smooth algebraic variety T, we have

$$\left(\int_f M^\bullet\right)\boxtimes N^\bullet\xrightarrow{\sim}\int_{f\times\mathrm{id}_T}(M^\bullet\boxtimes N^\bullet).$$

By factoring further we can assume f is a closed immersion or a projection. Moreover, we can assume $M^\bullet = M\in\mathsf{Mod}_{\mathsf{qc}}(D_X)$ and $N^\bullet = N\in\mathsf{Mod}_{\mathsf{qc}}(D_Y)$.

If f is a closed embedding, then we can just compute in local coordinates:

$$\begin{aligned}\left(\int_f M\right)\boxtimes N &\simeq (\mathbf{C}[\partial_{y_{r+1}},\dots,\partial_{y_n}]\otimes_{\mathbf{C}} i_*M)\boxtimes N\\ &\simeq \mathbf{C}[\partial_{y_{r+1}},\dots,\partial_{y_n}]\otimes_{\mathbf{C}}(i\times 1)_*(M\boxtimes N)\\ &\simeq\int_{i\times\mathrm{id}_T}(M\boxtimes N).\end{aligned}$$

If f is a projection, we use the de Rham complex:

$$\begin{aligned}\left(\int_f M\right)\boxtimes N &\simeq \mathbf{R}f_*(\mathrm{DR}^\bullet_{X/Y}(M))\boxtimes N\\ &\simeq \mathbf{R}(f\times\mathrm{id}_T)_*(\mathrm{DR}^\bullet_{X/Y}(M)\boxtimes N)\\ &\simeq \mathbf{R}(f\times\mathrm{id}_T)_*(\mathrm{DR}^\bullet_{X\times T/Y\times T}(M\boxtimes N))\\ &\simeq\int_{f\times\mathrm{id}_T}(M\boxtimes N),\end{aligned}$$

where we used base change in the second isomorphism. □

We promised to show that $\int_f$ preserves coherence if f is proper, so we sketch the result now, following [Bor+87, VII, Prop. 9.4; Mal93, III, Thm. 1]:

Theorem 5.26 [HTT08, Thm. 2.5.1]. *If $f\colon X \to Y$ is projective, then $\int_f$ maps $\mathsf{D}^b_{\mathsf{c}}(D_X)$ into $\mathsf{D}^b_{\mathsf{c}}(D_Y)$.*

Proof Sketch. If $f\colon X \to Y$ is projective, then it factors as

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & Y \times \mathbf{P}^n =: Z \\ & \underset{f}{\searrow} & \downarrow{\scriptstyle p_1} \\ & & Y \end{array} \tag{5.8}$$

where $i\colon X \hookrightarrow \mathbf{P}^n \times Y$ is a closed immersion. We will show in the proof of Kashiwara's theorem that $\int_i$ preserves D-coherence, and so we will show that $\int_{p_1}$ preserves D-coherence. Since this is a local condition, we assume that Y is affine.

We first show that $\int_{p_1} D_Z = D_Y[-n]$, where n is the dimension of the projective space in the factorization (5.8). By the proof of Lemma 5.19, and by using the projection formula for abelian sheaves,

$$\begin{aligned} \int_{p_1} D_Z &= \mathbf{R}p_{1*}(p_1^{-1}(D_Y) \otimes_{\mathbf{C}} p_2^{-1}(\omega_{\mathbf{P}^n})) \\ &\simeq D_Y \otimes_{\mathbf{C}} \mathbf{R}p_{1*}(p_2^{-1}(\omega_{\mathbf{P}^n})) \end{aligned}$$

By the cohomology of projective space [Har77, Thm. III.5.1], we have

$$\mathbf{R}^i p_{1*}(p_2^{-1}(\omega_{\mathbf{P}^n})) = \begin{cases} \mathbf{C} & \text{if } i = n \\ 0 & \text{if } i \neq n \end{cases}$$

and so $\int_{p_1} D_Z = D_Y[-n]$. This shows the Theorem for D_Z.

For general elements in $\mathsf{D}^b_{\mathsf{c}}(D_Z)$, [Bor+87, VII, Prop. 9.4] concludes by noting that D_Z is a generator for $\mathsf{D}^b_{\mathsf{c}}(D_Z)$. We will be a bit more explicit, following [Mal93, III, Thm. 1].

Let $M \in \mathsf{Mod}_{\mathsf{c}}(D_Z)$. Since M is coherent as a D_Z-module, it has a good filtration, hence we have $D_Z M_{\ell_0} = M$ for some ℓ_0. We therefore have a surjection

$$D_Z \otimes_{\mathcal{O}_Z} M_{\ell_0} \longrightarrow M \longrightarrow 0.$$

By repeating this process on the kernel (which also has a good filtration by restriction), we have a resolution

$$D_Z \otimes G^L \longrightarrow D_Z \otimes G^{L-1} \longrightarrow D_Z \otimes G^1 \longrightarrow D_Z \otimes G^0 \longrightarrow M \longrightarrow 0$$

of M. Call $N^\bullet$ this complex, excluding M. Letting $M' \coloneqq \ker(D_Z \otimes G^L \to D_Z \otimes G^{L-1})$, we have a short exact sequence of complexes:

$$0 \longrightarrow M'[L] \longrightarrow N^\bullet \longrightarrow M \longrightarrow 0.$$

This gives a long exact sequence

$$\cdots \longrightarrow \int_{p_1}^{i+L} M' \longrightarrow \int_{p_1}^{i} N^\bullet \longrightarrow \int_{p_1}^{i} M \longrightarrow \int_{p_1}^{i+L+1} M' \longrightarrow \cdots.$$

By our description in Proposition 5.21 that

$$\int_{p_1} M^j \simeq \mathbf{R}p_{1*}^j(\mathrm{DR}^\bullet_{Z/Y}(M)) = 0 \qquad \text{unless } -n \leq j \leq n$$

and similarly for M', we then an isomorphism $\int_{p_1}^j M \simeq \int_{p_1}^j N^\bullet$ for all $-n \leq j \leq n$, as long as L is large enough. Finally, we note that the $\int_{p_1}^j N^\bullet$ are coherent, since we can use the hypercohomology spectral sequence

$$E_2^{p,q} = R^p p_{1*} H^q(N^\bullet) \Rightarrow R^{p+q} p_{1*}(N^\bullet)$$

But since the objects on the E_2 page are coherent (by doing a similar argument as to the proof for D_Z above), they abut to coherent objects on E^∞. ∎

6 October 17: Kashiwara's Equivalence (Harold Blum)

Let $i\colon X \hookrightarrow Y$ be a closed immersion of smooth varieties.

Theorem 6.1. *$\int_i\colon \mathsf{Mod}_\#(D_X) \to \mathsf{Mod}_\#^X(D_Y)$, where $\# = \mathsf{qc}, \mathsf{c}$, and the superscript X says that the modules are supported on X as $\mathcal{O}_Y$-modules, gives an equivalence of categories with quasi-inverse $i^\natural$. Additionally, if $N \in \mathsf{Mod}_\#^X(D_Y)$, then $H^j(i^\dagger) = 0$ for $j \neq 0$.*

This last statement is useful for when we want to lift this to the derived category.

Proof. We have a map $M \to i^\natural \int_i^0 M$, and a map $\int_i^0 i^\natural N \to N$ from adjointness. It then suffices to show that these are isomorphisms locally.

We work locally on Y, in which case X is a nice complete intersection; we also assume that $X \hookrightarrow Y$ has codimension 1. Choose local coordinates $\{y_k, \partial_{y_k}\}_{k=1,\ldots,n}$, $X = \{y_n = 0\}$, and write $y \coloneqq y_n$, $\partial = \partial_{y_k}$. We have that

$$\int_i^0 M = \mathbf{C}[\partial] \otimes_{\mathbf{C}} i_* M$$

for $M \in \mathsf{Mod}(D_X)$, and

$$H^0(i^\dagger N) = \ker(y\colon i^{-1}N \to i^{-1}N) \qquad H^1(i^\dagger N) = \operatorname{cok}(y\colon i^{-1}N \to i^{-1}N)$$

for $N \in \mathsf{Mod}^X(D_Y)$.

The key idea is to understand multiplication of y, but to do so, it is easier to understand the following differential operator: $\theta \coloneqq y\partial$. Set $N^j \coloneqq \{n \in N \mid \theta \cdot n = jn\}$ for $j \in \mathbf{Z}$; this is the jth eigenspace for θ. Now, $\partial y = \theta + 1$ (a restatement of the Lie bracket property), so that $y \cdot N^j \subseteq N^{j+1}$, and $\partial \cdot N^j \subseteq N^{j-1}$, since, e.g.,if $n \in N^j$, then

$$\theta(y \cdot n) = y\partial y \cdot n = y(\theta + 1) \cdot n = y \cdot (j + 1)n = (j + 1)yn.$$

Note $\theta\colon N^j \xrightarrow{\sim} N^j$ for $j \neq 0$. Similarly, $\partial y = \theta + 1\colon N^j \xrightarrow{\sim} N^j$ for $j \neq -1$. Thus, if $j < -1$, then the maps $N^j \xrightarrow{y} N^{j+1} \xrightarrow{\partial} N^j$ are both isomorphisms. We now assume the following Claim, which we will show later:

Claim. $N = \bigoplus_{j=1}^\infty N^{-j}$.

It then follows that $H^1(i^\dagger N) = 0$, since $y \cdot N^{-j-1} = N^{-j}$, and that $N = \mathbf{C}[\partial] \otimes N^{-1}$, since $\partial^j\colon N^{-1} \xrightarrow{\sim} N^{-1-j}$. This implies that

$$M \longrightarrow i^\natural \int_i^0 M = i^\natural(\mathbf{C}[\partial] \otimes_{\mathbf{C}} M) = (\mathbf{C}[\partial] \otimes_{\mathbf{C}} M)^{-1} = M$$

The same argument shows that $\int_i^0 i^\natural N \to N$ is an isomorphism.

To show coherence of $\int_i^0 M$ and $i^\natural N$ for $M \in \mathsf{Mod}_{\mathsf{c}}(D_X)$ and $N \in \mathsf{Mod}_{\mathsf{c}}(D_Y)$, we just note that

$$\int_i^0 M = \mathbf{C}[\partial] \otimes_{\mathbf{C}} i_* M$$

is clearly coherent since both objects are locally finitely generated, and for $i^\natural N$, we can just look at submodules generated by subsets of elements.

We now give an idea for the proof of the Claim. We finally use the property that N is supported on X. If $n \in N$, there exists k such that $y^k n = 0$. We then want to show that $n \in \bigoplus_{j=1}^k N^{-j}$. This is true if $k = 1$, since $y \cdot n = 0$ implies $\theta \cdot n = (\partial y - 1)n = -n$. We then proceed by induction on k. □

7 October 24: Holonomicity (Harold Blum)

Recall 7.1. Let $M \in \mathsf{Mod_c}(D_X)$. We set $\mathrm{Ch}(M) = \mathrm{Supp}\,\widetilde{\mathrm{gr}^F M}$, where (F, M) is a good filtration. We write $\widetilde{\mathrm{gr}^F M}$ as a module on T^*X. We also define the *characteristic cycle*, which adds integer multiplicities to components of $\mathrm{Ch}(M)$:

$$\mathrm{CC}(M) \coloneqq \sum_{\substack{C \subseteq \mathrm{Ch}(M) \\ \text{irreducible component}}} m_C(M) C,$$

where

$$m_C(M) \coloneqq \ell(\widetilde{\mathrm{gr}^F M} \cdot \mathcal{O}_{T^*X,C}).$$

You can show that this independent of good filtration by doing a comparison argument like before.

Note that if $0 \to M \to N \to L \to 0$ in $\mathsf{Mod_c}(D_X)$, we have

$$\mathrm{CC}(N) = \mathrm{CC}(M) + \mathrm{CC}(L).$$

To see this, you choose a filtration on N, which induces filtrations (G, M) and (H, L), which gives rise to an exact sequence of graded modules

$$0 \longrightarrow \mathrm{gr}^G M \longrightarrow \mathrm{gr}^F N \longrightarrow \mathrm{gr}^H L \longrightarrow 0.$$

Multiplicities add in the correct way in short exact sequences, so you are done.

7.1 Bernstein's inequality

We need Kashiwara's equivalence to prove the following:

Theorem 7.2 (Bernstein's inequality)**.** *If $M \in \mathsf{Mod_c}(D_X)$ and Λ is an irreducible component of $\mathrm{Ch}(M)$, then $\dim(\Lambda) \geq \dim(X)$.*

Proof. We can reduce to the case where $\mathrm{Ch}(M)$ has pure dimension. This follows by homological algebra you can find in the appendix: there exists a filtration

$$0 = C^{2\dim X+1} \subset C^{2\dim X} \subset \cdots \subset C^0 = M,$$

with the property that $\dim(\mathrm{Ch}(C^s/C^{s+1}))$ has pure codimension s. You can do this locally in a canonical way, and so you can globalize this.

We now induce on $\dim X$. If $\dim X = 0$, then the statement is trivial, since $\dim T^*X = 0$. So now suppose $\dim X > 0$. If $\mathrm{Supp}(M) = \pi(\mathrm{Ch}(M)) = X$, where $\pi\colon T^*X \to X$ (this is closed since the characteristic variety is a conical variety in T^*X), and we note

$$V(\mathrm{Ann}_{\mathcal{O}_X}(M)) = \mathrm{Supp}(M) \supset \pi(\mathrm{Ch}(M))$$

and the reverse inclusion holds by considering the complement of $\pi(\mathrm{Ch}(M))$, where $\mathrm{Ch}(M) = \emptyset$, so that $M = 0$. This implies $\dim \mathrm{Ch}(M) \geq \dim X$. Otherwise, choose a hypersurface $H \overset{i}{\hookrightarrow} X$ such that $\mathrm{Supp}(M) \subseteq H$, where H is smooth after possibly restricting to an open subset of X. Then, by Kashiwara's equivalence, there exists $N \in \mathsf{Mod_c}(D_H)$ such that $\int_i N = M$. Now we claim that

$$\dim(\mathrm{Ch}(M)) = \dim(\mathrm{Ch}(N)) + 1.$$

This follows from the following Lemma:

Lemma 7.3. *Let $S \overset{i}{\hookrightarrow} X$ be a closed embedding,* $\mathrm{codim}(S) = 1$*, $N \in \mathsf{Mod_c}(D_S)$. Then, we have the diagram*

$$\begin{array}{ccc} & S \times_X T^*X & \\ {}^{p_i}\swarrow & & \searrow^{w_i} \\ T^*S & & T^*X \end{array}$$

Then, we have $\mathrm{Ch}(\int_i^0 N) = w_i p_i^{-1}\,\mathrm{Ch}(N)$.

Proof of Lemma. Work locally on X with coordinates $\{x_i, \partial_i\}$, $x \coloneqq x_1$, $\partial \coloneqq \partial_1$, and $S = \{x = 0\}$. Also, call $M \coloneqq \int_i^0 N$. Now let (F, N) be a good filtration. Locally,

$$M = \mathbf{C}[\partial] \otimes_{\mathbf{C}} i_* N.$$

We get a filtration (G, M) by setting

$$G_j M \coloneqq \sum_{\ell=0}^{j} \sum_{k \leq \ell} \mathbf{C} \cdot \partial^k \otimes i_* F_{j-\ell} N$$

which is the tensor product filtration; the idea is that ∂ should have weight 1. Now

$$G_j M / G_{j-1} M = \sum_{k=0}^{j} \mathbf{C} \cdot \partial^k \otimes i_* \left(\frac{F_{j-k} N}{F_{j-k-1} N} \right),$$

which implies $\mathrm{gr}^F M = \mathbf{C}[\xi] \otimes_{\mathbf{C}} \mathrm{gr}^F N$, where $\overline{\xi} = \partial$, and we think of $\mathrm{gr}^F N$ as living on $T^* X$. □

This concludes the proof of the Theorem by induction. □

7.2 Properties of holonomic *D*-modules

Definition 7.4. $M \in \mathsf{Mod}_{\mathsf{c}}(D_X)$ is *holonomic* if $\dim(\mathrm{Ch}(M)) = \dim X$, or $M = 0$. We then denote

$$\mathsf{Mod}_{\mathsf{h}}(D_X) \coloneqq \text{subcategory of } \mathsf{Mod}_{\mathsf{c}}(D_X) \text{ consisting of holonomic } D_X\text{-modules.}$$

We can also define holonomicity by saying $\dim(\mathrm{Ch}(M)) \leq \dim X$, which makes proofs a bit nicer.

Proposition 7.5.
(*a*) *Let* $0 \to M \to N \to L \to 0$ *be a short exact sequence in* $\mathsf{Mod}_{\mathsf{c}}(D_X)$. *Then,* $N \in \mathsf{Mod}_{\mathsf{h}}(D_X)$ *if and only if* $M, L \in \mathsf{Mod}_{\mathsf{h}}(D_X)$.
(*b*) *If* $M \in \mathsf{Mod}_{\mathsf{h}}(D_X)$, *then* M *has finite length.*

Proof. For (*a*), $\mathrm{Ch}(N) = \mathrm{Ch}(M) \cup \mathrm{Ch}(L)$. Also, we could write $\mathrm{CC}(N) = \mathrm{CC}(M) + \mathrm{CC}(L)$.

For (*b*), suppose $N \subsetneq M$, with $N \in \mathsf{Mod}_{\mathsf{c}}(D_X)$. Then, $\mathrm{CC}(N) < \mathrm{CC}(M)$, and since the coefficients are integers, this operation must stop eventually. □

Note that coherent D_X-modules do not have finite length: $A_1/A_1 \cdot x$ is holonomic, but $A_1 \cdot x$ is not, since we have a chain $\cdots \subseteq A_1 \cdot x^2 \subseteq A_1 \cdot x$.

Here are some nice categorical properties of holonomic D-modules.

Theorem 7.6. $\mathsf{Mod}_{\mathsf{h}}(D_X)$ *is abelian.*

Theorem 7.7. *The subcategory* $\mathsf{D}^b_{\mathsf{h}}(D_X)$ *of* $\mathsf{D}^b_{\mathsf{c}}(D_X)$ *consisting of complexes with holonomic cohomology is equivalent to* $\mathsf{D}^b(\mathsf{Mod}_{\mathsf{h}}(D_X))$.

Holonomic D-modules have nice finiteness properties. Here is a first example of this phenomenon:

Proposition 7.8. *If* $M \in \mathsf{Mod}_{\mathsf{h}}(D_X)$, *then there exists an open subset* $U \subseteq X$ *such that* $M|_U$ *is coherent over* $\mathcal{O}_U$, *that is,* M *restricts to a vector bundle with integrable connection.*

This sort of says that holonomic D-modules are vector bundles with integral connections, where the integrable connection can have singularities on the boundary; it is the limit of an integrable connection.

Proof. Set $S \coloneqq \mathrm{Ch}(M) \setminus T^*_X X$, the zero section, and choose $U \subseteq X$ such that $U \cap \pi(S) = \emptyset$. Then, $\mathrm{Ch}(M|_U) \subseteq T^*_U U$. Thus, $M|_U$ has an integrable connection. □

7.3 Holonomicity and Functors

We know that quasi-coherence is preserved by our functors, and direct images via proper morphisms preserve coherence.

Proposition 7.9. *If $M \in \mathsf{Mod_h}(D_X)$, and $N \in \mathsf{Mod_h}(D_Y)$, then $M \boxtimes N \in \mathsf{Mod_h}(D_{X\times Y})$.*

Proof. Show that $\mathrm{Ch}(M \boxtimes N) = \mathrm{Ch}(M) \times \mathrm{Ch}(N)$. □

Theorem 7.10 (∗∗)**.** *If $f\colon X \to Y$ is a map of smooth varieties, then*
(i) $\int_f$ sends $\mathsf{D}^b_\mathsf{h}(D_X)$ to $\mathsf{D}^b_\mathsf{h}(D_Y)$.
(ii) $f^\dagger$ sends $\mathsf{D}^b_\mathsf{h}(D_Y)$ to $\mathsf{D}^b_\mathsf{h}(D_X)$, where $f^\dagger = \mathbf{L}f^\circ[\dim Y - \dim X]$.

Corollary 7.11. *$- \otimes^{\mathbf{L}} -$ also preserves holonomicity.*

Proof. Set $\Delta\colon X \hookrightarrow X \times X$. Then, we showed before that $- \otimes^{\mathbf{L}} - = \mathbf{L}\Delta^\circ(- \boxtimes -)$. □

We want to head in the direction of explaining how Theorem (∗∗) works. One easy thing is to consider closed embeddings:

Lemma 7.12. *If $i\colon X \hookrightarrow Y$ is a closed embedding, and $M^\bullet \in \mathsf{D}^b_\mathsf{c}(D_X)$, then $M^\bullet \in \mathsf{D}^b_\mathsf{h}(D_X)$ if and only if $\int_i M^\bullet \in \mathsf{D}^b_\mathsf{h}(D_Y)$.*

Proof. We have shown this already for modules. Now $\int_i$ is exact, so it is sufficient to consider $M \in \mathsf{Mod_c}(D_X)$. Then, you can apply the previous Lemma, which says $\mathrm{codim}(X) + \dim \mathrm{Ch}(M) = \dim \mathrm{Ch}(\int_i^0 M)$. □

Proof of Theorem (∗∗), (i). Consider

$$\begin{array}{ccc} & X \times Y & \\ \nearrow & & \searrow \\ X & & Y \end{array}$$

By the Lemma, it is enough to consider projections. It is enough to consider the following projection $f\colon \mathbf{A}^n \to \mathbf{A}^{n-1}$; holonomicity is local on Y, and so you can assume Y is affine. You can also show that holonomicity is local on $X \times Y$ (using a Čech complex). Then, you can embed $Y \hookrightarrow \mathbf{C}^m$, $X \hookrightarrow \mathbf{C}^n$, so you have a diagram

$$\begin{array}{ccc} X \times Y & \hookrightarrow & \mathbf{C}^{m+n} \\ \downarrow & & \downarrow \\ Y & \hookrightarrow & \mathbf{C}^m \end{array}$$

We will finish this next time. □

Proof of Theorem (∗∗), (ii). This follows from Theorem (∗∗), (*i*).

Case 1. Suppose $X = Z \times Y \to Y$, and $M \in \mathsf{Mod_h}(D_Y)$ (suffices since $f^\dagger$ is exact in this case).

Then, $f^\dagger = \mathbf{L}f^\circ M[-\dim Z]$. But $\mathbf{L}f^\circ M = M \boxtimes \mathcal{O}_Z$, which is holonomic by the previous Lemma.

Case 2. $X \overset{i}{\hookrightarrow} Y$ is a closed embedding.

Consider the diagram $X \overset{i}{\hookrightarrow} Y \overset{j}{\hookleftarrow} U = Y \setminus X$. We then have the following distinguished triangle:

$$\int_i i^\dagger M \longrightarrow M \longrightarrow \int_j j^\dagger M \xrightarrow{[1]}$$

and so the assumption that $M \in \mathsf{Mod_h}(D_X)$ is holonomic implies $\int_i i^\dagger M \in \mathsf{Mod_h}(D_X)$ by the long exact sequence on cohomology, since $\int_j j^\dagger M$ is holonomic: $j^\dagger$ preserves holonomicity, and $\int_j$ preserves holonomicity by Theorem (∗∗), (*i*). Finally, $\int_i i^\dagger M \in \mathsf{Mod_h}(D_Y)$ if and only if $i^\dagger M \in \mathsf{Mod_h}(D_X)$. □

7.4 Finiteness property

We now show more finiteness properties of holonomic D-modules. First, the idea is that by strengthening the condition on intergable connection on anopen set, we can categorize holonomicity.

Theorem 7.13. *The following conditions on $M^\bullet \in \mathsf{D}^b_{\mathsf{c}}(D_X)$ are equivalent:*

(i) $M^\bullet \in \mathsf{D}^b_{\mathsf{h}}(D_X)$*;*

(ii) There exists a sequence of closed sets

$$X = X_0 \supset X_1 \supset \cdots \supset X_m = \emptyset$$

such that $X_r \setminus X_{r+1}$ is smooth, and $H^k(i_r^\dagger M)$ has an integrable connection (is $\mathcal{O}_{X_r \setminus X_{r+1}}$-coherent), where $i_r \colon X_r \setminus X_{r+1} \hookrightarrow X$.

(iii) For all $\{x\} \overset{i_x}{\hookrightarrow} X$, $H^k(i_x^\dagger M^\bullet)$ is finite dimensional over $\mathbf{C}$.

Example 7.14. Consider the following trivial example of a holonomic D-module: $A_1/A_1 \cdot x \simeq \mathbf{C}[\partial]$. Then,

$$\mathbf{C}[\partial] \otimes_{\mathbf{C}[x]} \mathbf{C}[x]/(x) \simeq \mathbf{C}$$

since

$$\begin{aligned} \partial \otimes 1 &= \partial \otimes (1 + x) \\ &= \partial \otimes 1 + x\partial \otimes 1 \\ &= \partial \otimes 1 + (-1)\partial \otimes 1 \\ &= 0 \end{aligned}$$

We briefly discuss two of the implications. The last one will need more work.

Proof of (i) ⇒ (ii). This is an easy application of the fact that inverse images preserve holonomicity: $\mathsf{Mod}_{\mathsf{h}}(D_{\{x\}}) = \{$finite-dimensional $\mathbf{C}$-vector spaces$\}$. □

Proof of (ii) ⇒ (i). Set $U_r = X \setminus X_r$. By induction on r, we show that $M|_{U_r}$ is holonomic. Note that $U_m = X$. First, $M|_{U_1}$ is holonomic, since $i_0^\circ M = M|_{U_1}$.

Now consider $U_r \overset{j}{\hookrightarrow} U_{r+1} \overset{i}{\hookleftarrow} X_r \setminus X_{r+1}$. We consider the distinguished triangle from before:

$$\int_i i^\dagger(M^\bullet|_{U_{r+1}}) \longrightarrow (M^\bullet|_{U_{r+1}}) \longrightarrow \int_j j^\dagger(M^\bullet|_{U_{r+1}}) \xrightarrow{[1]}$$

Note that $j^\dagger M^\bullet|_{U_{r+1}} = M^\bullet|_{U_r}$, which is holonomic by inductive hypothesis, so its pushforward $\int_j j^\dagger M^\bullet|_{U_{r+1}}$ is also holonomic. Now, $i^\dagger(M^\bullet|_{U_{r+1}}) = i_r^\dagger M^\bullet$, which is holonomic by assumption, and so $\int_i i^\dagger M^\bullet|_{U_{r+1}}$ is holonomic by the long exact sequence. □

8 October 31 (Harold Blum)

Last time, we stated and used the following:

Theorem 8.1. *If $f \colon X \to Y$ is a morphism of smooth algebraic varieties, then $\int_f$ sends $\mathsf{D}^b_{\mathsf{h}}(D_X)$ to $\mathsf{D}^b_{\mathsf{h}}(D_Y)$.*

We reduced to the case of a projection, and furthermore to the case $f \colon \mathbf{C}^n \to \mathbf{C}^{n-1}$. This also suffices for the pullback statement.

The goal today is to prove this theorem for this case.

8.1 Holonomic D-modules on $\mathbf{C}^n$

We will describe these by using the Hilbert function of D-modules. What's strange is that instead of just computing what this is, we will rephrase it in terms of open and closed embeddings.

Let $D_n = \Gamma(\mathbf{C}^n, D_{\mathbf{C}^n}) = \bigoplus_{\alpha,\beta} \mathbf{C}x^\alpha\partial^\beta$. We will use the *Bernstein filtration*

$$B_i D_n \coloneqq \sum_{|\alpha|+|\beta|\le i} \mathbf{C}x^\alpha\partial^\beta.$$

Note that

$$\operatorname{gr}^B D_n = \mathbf{C}[x,\xi], \quad \xi = \overline{\partial}.$$

We will explain why this is handy in a second, but for now, note that if $M \in \mathsf{Mod_c}(D_{\mathbf{C}^n})$ we can find a good filtration (F, M) with respect to $(B, D_{\mathbf{C}^n})$. Then, F_iM is finitely generated over $\mathbf{C}$. It's unclear whether computing characteristic varieties with respect to these different filtrations is the same.

We have a surjective map

$$B_{i-i_1}D_{\mathbf{C}^n} \oplus \cdots \oplus B_{i-i_m}D_{\mathbf{C}^n} \twoheadrightarrow F_iM.$$

Proposition 8.2. *Let (F, M) be a good filtration of $M \in \mathsf{Mod_c}(D_{\mathbf{C}^n})$. Then,*

(i) *There exists a polynomial*

$$\chi(M, F; T) \in \mathbf{Q}[T]$$

such that $\chi(M, F; i) = \dim_{\mathbf{C}} F_iM$ for $i \gg 0$.

(ii) *Setting $d(M) = \deg\chi$ and $m(M)$ the leading coefficient of $\chi \cdot d!$, we have that d, m are independent of our choice of good filtration,*

Proof. For (i), look at $\dim_{\mathbf{C}} F_i(M) - \dim_{\mathbf{C}} F_{i-1}(M) = \dim_{\mathbf{C}}[\operatorname{gr}^F M]_i$. We know $i \mapsto \dim_{\mathbf{C}}[\operatorname{gr}^F M]_i$ is a polynomial for $i \gg 0$.

For (ii), if F, F' are two good filtrations, then there exists an i_0 such that

$$F'_{i-i_0}M \subseteq F_iM \subseteq F'_{i+i_0}M.$$

So asymptotically you get the same answer. □

Proposition 8.3. *If $M \in \mathsf{Mod_C}(D_{\mathbf{C}^n})$, then*

$$\dim(\operatorname{Ch}(M)) = d_B(M).$$

Proof. $\dim(\operatorname{Ch}(M))$ can be computed using the Bernstein filtration or the order filtration (the proof uses the cohomological description of the dimension of the characteristic variety). We know $\deg(i \mapsto \dim_{\mathbf{C}}[\operatorname{gr}^F M]_i) = \dim \operatorname{Supp}_{\mathbf{P}^{2n-1}}(\widetilde{\operatorname{gr}^F M})$. But this equals $\dim \operatorname{Ch}(M) - 1$. We want to show the leftmost side is equal to $\deg(\chi(F, M; t)) - 1$. □

Proposition 8.4. *If $0 \to L \to M \to N \to 0$ in $\mathsf{Mod_c}(D_{\mathbf{C}^n})$, then $d(M) = \max\{d(L), d(N)\}$, and $m(M) = m(L) + m(N)$ when $d(L) = d(N)$.*

It is not true the Hilbert polynomials are additive, unless the filtrations are chosen compatibly in the sequence.

8.2 Fourier transform

If N is a $D_{\mathbf{C}^n}$-module, we set $\widehat{N}$ to be the D-module such that

- $\widehat{N} = N$ as additive groups;
- For $s \in \widehat{N}$, $x_i \circ s = -\partial_i s$, and $\partial_i \circ s = x_i s$.

Note that $\widehat{N}$ is a left $D_{\mathbf{C}^n}$-module.

Example 8.5. If $n = 1$, then $\widehat{\mathbf{C}[x]} \simeq \mathbf{C}[\partial] = \mathbf{C}[x, \partial]/\mathbf{C}[x, \partial]x$.

We now consider the case of our theorem. Now consider the projection $p\colon \mathbf{C}\times\mathbf{C}^{n-1}\to\mathbf{C}^{n-1}$ and the closed embedding $i\colon \{0\}\times\mathbf{C}^{n-1}\hookrightarrow\mathbf{C}^n$.

Proposition 8.6. *If $M\in\mathsf{Mod}_{\mathsf{qc}}(D_{\mathbf{C}^n})$, then*

$$H^k\widehat{\Big(\int_p M\Big)} = H^k(\mathbf{L}i^*\widehat{M}).$$

Proof. First compute the left-hand side:

$$\begin{aligned}\int_p M &= \mathbf{R}p_*(\mathrm{DR}_{\mathbf{C}^n/\mathbf{C}^{n-1}}(M))\\ &= [p_*M \xrightarrow{\partial_1} p_*M]\end{aligned}$$

since $\int_p M = \mathbf{R}p_*(D_{\mathbf{C}^{n-1}\leftarrow\mathbf{C}\times\mathbf{C}^{n-1}}\otimes^{\mathbf{L}}_{D_{\mathbf{C}\times\mathbf{C}^{n-1}}} M)$. But

$$D_{\mathbf{C}^{n-1}\leftarrow\mathbf{C}\times\mathbf{C}^{n-1}} = D_{\mathbf{C}^{n-1}}\boxtimes\omega_{\mathbf{C}}.$$

Now use the de Rham resolution of $\omega_{\mathbf{C}}$:

$$0\longrightarrow \mathscr{O}_{\mathbf{C}}\otimes D_{\mathbf{C}}\longrightarrow \omega_{\mathbf{C}}\otimes D_{\mathbf{C}}\longrightarrow\omega_{\mathbf{C}}\longrightarrow 0$$

and take the box product with $D_{\mathbf{C}^{n-1}}$:

$$0\longrightarrow D_{\mathbf{C}^{n-1}}\boxtimes(\mathscr{O}_{\mathbf{C}}\otimes D_{\mathbf{C}})\longrightarrow D_{\mathbf{C}^{n-1}}\boxtimes(\omega_{\mathbf{C}}\otimes D_{\mathbf{C}})\longrightarrow D_{\mathbf{C}^{n-1}}\boxtimes\omega_{\mathbf{C}}\longrightarrow 0$$

where the first map is $f\otimes P\mapsto df\otimes P + d(x_1\wedge f)\otimes\partial P$. Thus,

$$D_{\mathbf{C}^{n-1}\leftarrow\mathbf{C}\times\mathbf{C}^{n-1}}\otimes^{\mathbf{L}}_{D_{\mathbf{C}\times\mathbf{C}^{n-1}}} M\simeq \big[M\xrightarrow{\partial_1}M\big].$$

Now p_* is exact since p is affine, so we don't have to take a right-derived functor in the first equation. Thus,

$$H^k\Big(\int_p M\Big) = \begin{cases}\ker(M\xrightarrow{\partial_1}M) & k=-1\\ \operatorname{cok}(M\xrightarrow{\partial_1}M) & k=0\\ 0 & \text{otherwise}\end{cases}$$

The Fourier transform is then

$$H^k\widehat{\Big(\int_p M\Big)} = \begin{cases}\ker(\widehat{M}\xrightarrow{x_1}\widehat{M}) & k=-1\\ \operatorname{cok}(\widehat{M}\xrightarrow{x_1}\widehat{M}) & k=0\\ 0 & \text{otherwise}\end{cases}$$

On the other hand,

$$H^k(\mathbf{L}i^*N) = \begin{cases}\ker(N\xrightarrow{x}N) & k=-1\\ \operatorname{cok}(N\xrightarrow{x}N) & k=0\\ 0 & \text{otherwise}\end{cases}$$

We therefore have the stated isomorphism. □

8.3 Proof of main theorem

Claim 8.7. *If $M\in\mathsf{Mod}_{\mathsf{c}}(D_{\mathbf{C}^n})$, then M is holonomic if and only if $\widehat{M}$ is holonomic (this is by inducing a filtration on either side).*

Claim 8.8. $j\colon\mathbf{C}\setminus\{0\}\times\mathbf{C}^{n-1}\to\mathbf{C}^n$. $M\in\mathsf{Mod}_{\mathsf{h}}(\mathbf{C}^n)$ *implies $\int_j j^\dagger$ holonomic.*

Proof, assuming claim. Assume $M \in \mathsf{Mod_h}(D_{\mathbf{C}^n})$. It is sufficient to show $i^\dagger M$ is holonomic. Look at the excision sequence

$$\int_i i^\dagger M \longrightarrow M \longrightarrow \int_j j^\dagger M \xrightarrow{[1]} .$$

Now Claim 2 says that $\int_i i^\dagger M$ is holonomic, and so $i^\dagger M$ is holonomic. $\square$

What is left is the second claim. Note that this is a bit strange because we started talking about projections, turned the problem into one about closed immersions, and now we have turned it into a problem about open immersions instead.

Proposition 8.9. *If $M \in \mathsf{Mod_{qc}}(D_{\mathbf{C}^n})$ and (F, M) is a filtration such that*

$$\dim_{\mathbf{C}} F_i M \leq \frac{c}{n!} i^n + c' i^{n-1}$$

for some c, c'. Then, M is holonomic.

Proof. Assume $N \subseteq M$ and is coherent over $D_{\mathbf{C}^n}$. Let (G, N) be a good filtration. There exists i_0 such that

$$G_i N \subseteq N \cap F_{i+i_0} M \subseteq F_{i+i_0} M$$

since G is good. Thus, N is holonomic and $m(N) \leq c$.

We show M is finitely generated. If $N_1 \subseteq N_2 \subseteq \cdots \subseteq M$ where N_i is finitely generated for all i, then $m(N_1) \leq m(N_2) \leq \cdots \leq c$. Thus, $\{m(N_i)\}_i$ stabilizes, and so does $\{N_i\}_i$. $\square$

Proof of Claim 2. Recall that $j\colon \mathbf{C} \setminus \{0\} \times \mathbf{C}^{n-1} \to \mathbf{C}^n$ is an open embedding, and we want to show that M being a holonomic $D_{\mathbf{C}^n}$-module implies $\int_j j^\dagger M$ is holonomic. Now let (F, M) be a good filtration. We have that

$$\int_j j^\dagger M = \mathbf{C}[x, x_1^{-1}] \otimes_{\mathbf{C}[x]} M_{x_1}.$$

We set $F_i M_{x_1} = \operatorname{im}\{F_{2i} M \to M : m \mapsto \frac{1}{x_1^i} m\}$ (you should check this is a filtration). Note: $\dim_{\mathbf{C}} F_i M \leq \dim_{\mathbf{C}} F_{2i} M = \frac{m(M)}{n!}(2i)^n + O(i^{n-1})$. The previous Proposition tells us that M_{x_1} is holonomic. $\square$

8.4 Excision sequence

Consider

$$Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U = X \setminus Z.$$

If F is a flasque sheaf on X, then we get the following short exact sequence:

$$0 \longrightarrow \Gamma_Z F \longrightarrow F \longrightarrow j_* j^{-1} F \longrightarrow 0,$$

where $\Gamma_Z F$ is the sheaf of sections with support on Z.

Proposition 8.10. *$M^\bullet \in \mathsf{D}^b_{\mathsf{qc}}(D_X)$, then*

(*i*) *There exists a distinguished triangle*

$$\mathbf{R}\Gamma_Z(M^\bullet) \longrightarrow M^\bullet \longrightarrow \int_j j^\dagger M^\bullet \xrightarrow{[1]}$$

(*ii*) *If Z is smooth, and $N \in \mathsf{D}^b_{\mathsf{qc}}(D_U)$, then $i^\dagger \int_j N^\bullet = 0$.*

(*iii*) *If Z is smooth, then $\mathbf{R}\Gamma_Z M^\bullet = \int_i i^\dagger M^\bullet$.*

Proof. For (i), note j^{-1} and $j^\dagger$ are the same, as are j_* and $\int_j$. By replacing $M^\bullet$ with a flasque resolution, we are done by using the case for flasque sheaves.

Assuming (ii), we prove (iii). Note that $\mathbf{R}\Gamma_Z M \in \mathsf{D}^{b,Z}_{\mathsf{qc}}(D_X)$. Thus, Kashiwara's equivalence says that $\mathbf{R}\Gamma_Z M \simeq \int_i i^\dagger \mathbf{R}\Gamma_Z M$. It then suffices to show that $i^\dagger \mathbf{R}\Gamma_Z(M) = i^\dagger M$. To do this, apply $i^\dagger$ to the distinguished triangle in (i), and use vanishing of the third term:

$$i^\dagger \mathbf{R}\Gamma_Z M \longrightarrow i^\dagger M \longrightarrow 0 \xrightarrow{[1]} .$$

It remains to show (ii), which relates to the following

Proposition 8.11. *If* $N^\bullet \in \mathsf{D}^b_{\mathsf{qc}}(\mathcal{O}_U)$*, then* $\mathcal{O}_Z \otimes^{\mathbf{L}}_{i^{-1}\mathcal{O}_X} i^{-1}\mathbf{R}j_* N^\bullet = 0$ *in* $\mathsf{D}^b_{\mathsf{qc}}(\mathcal{O}_Z)$.

It suffices to show that $i_*(\mathcal{O}_Z \otimes^{\mathbf{L}}_{i^{-1}\mathcal{O}_X} i^{-1}\mathbf{R}j_* N^\bullet) = 0$. By the projection formula, the left-hand side is

$$i_*\mathcal{O}_Z \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathbf{R}j_* N^\bullet.$$

Using the projection formula again, this is

$$\mathbf{R}j_*(j^{-1}i_*\mathcal{O}_Z \otimes^{\mathbf{L}}_{\mathcal{O}_X} N^\bullet) = 0.$$

□

9 October 31 and November 7: Duality Functors (Takumi Murayama)

We want to define the "dual" of a left D-module. Let's first think about what this should be. If M is a left D_X-module, then $\mathscr{H}om_{D_X}(M, D_X)$ is a *right* D_X-module by right multiplication of D_X on D_X. To change this back into a right D_X-module, we use the side-changing operation $- \otimes_{\mathcal{O}_X} \omega_X^{-1}$ to get a preliminary definition:

$$\mathscr{H}om_{D_X}(M, D_X) \otimes_{\mathcal{O}_X} \omega_X^{-1}.$$

However, $\mathscr{H}om_{D_X}(-, D_X)$ is only left-exact, and so it is more natural in our derived setting to consider the *complex*

$$\mathbf{R}\mathscr{H}om_{D_X}(M, D_X) \otimes_{\mathcal{O}_X} \omega_X^{-1}.$$

Like some of the other functors, though, we will introduce a shift in the definition. To motivate, this consider the following example:

Example 9.1. Let $X = \mathbf{C}$ and consider a differential operator $P \in D_X$. Let $M = D_X/D_X P$. To compute $\mathbf{R}\mathscr{H}om_{D_X}(M, D_X)$ we use the following free resolution of M:

$$0 \longrightarrow D_X \xrightarrow{\cdot P} D_X \longrightarrow M \longrightarrow 0.$$

Applying $\mathscr{H}om_{D_X}(-, D_X)$, we get the exact sequence

$$0 \longrightarrow \mathscr{H}om_{D_X}(M, D_X) \longrightarrow D_X \xrightarrow{P\cdot} D_X.$$

In this case, we have that

$$\mathscr{E}xt^0_{D_X}(M, D_X) = \mathscr{H}om_{D_X}(M, D_X) = \ker(P\colon D_X \to D_X) = 0,$$

and so the only non-vanishing cohomology is

$$\mathscr{E}xt^1_{D_X}(M, D_X) \simeq D_X/PD_X,$$

which is a right D_X-module. Applying the side-changing functor gives

$$\mathscr{E}xt^1_{D_X}(M, D_X) \otimes_{\mathcal{O}_X} \omega_X^{-1} \simeq D_X/D_X P^*,$$

where P^* is the formal adjoint of P. This shows $\mathscr{E}xt^1$ seems more suited to be called the dual of M than $\mathscr{E}xt^0$.

Remark 9.2. If M is holonomic, then $\mathscr{E}xt^n$ will be the only non-vanishing $\mathscr{E}xt$.

We therefore make the following definition:

Definition 9.3. The *duality functor* $\mathbf{D} = \mathbf{D}_X\colon \mathsf{D}^-(D_X) \to \mathsf{D}^+(D_X)^{\mathsf{op}}$ is defined by

$$\mathbf{D}M^\bullet \coloneqq \mathbf{R}\mathscr{H}om_{D_X}(M^\bullet, D_X) \otimes_{\mathcal{O}_X} \omega^{-1}[\dim X] = \mathbf{R}\mathscr{H}om_{D_X}(M^\bullet, D_X \otimes_{\mathcal{O}_X} \omega^{-1}[\dim X]).$$

Example 9.4.

$$H^k(\mathbf{D}D_X) = \begin{cases} D_X \otimes_{\mathcal{O}_X} \omega_X^{-1} & k = -\dim X \\ 0 & k \neq -\dim X \end{cases}$$

Before proving some properties about $\mathbf{D}$, we state the following basic lemma about $\mathscr{E}xt$:

Lemma 9.5. *Let M be a coherent D_X-module. Then, for any open affine $U \subset X$,*

$$(\mathscr{E}xt^i_{D_X}(M, D_X))(U) = \operatorname{Ext}^i_{D_X(U)}(M(U), D_X(U)).$$

Proof. Take a resolution $P_\bullet \to M|_U$ of $M|_U$ by finite rank locally free D_U-modules. Since U is affine, taking sections is exact, and so we have a resolution $P_\bullet(U) \to M(U)$ by locally free $D_X(U)$-modules of finite rank. By definition,

$$\begin{aligned}(\mathscr{E}xt^i_{D_X}(M, D_X))(U) &= (H^i(\mathscr{H}om_{D_U}(P_\bullet, D_U)))(U) \\ &= H^i(\mathscr{H}om_{D_U}(P_\bullet(U), D_U(U))) \\ &= H^i(\mathscr{H}om_{D_X(U)}(P_\bullet(U), D_X(U))) \\ &= \operatorname{Ext}^i_{D_X(U)}(M(U), D_X(U)).\end{aligned}$$

□

Proposition 9.6.

(*i*) $\mathbf{D}$ *sends* $\mathsf{D}^b_{\mathsf{c}}(D_X)$ *to* $\mathsf{D}^b_{\mathsf{c}}(D_X)^{\mathsf{op}}$.

(*ii*) $\mathbf{D}^2 \simeq \mathrm{id}$ *on* $\mathsf{D}^b_{\mathsf{c}}(D_X)$. *In particular,* $\mathbf{D}$ *is fully faithful.*

Proof. (*i*) follows by the previous Lemma, and the fact that locally free resolutions are bounded for coherent complexes.

For (*ii*), we have the evaluation morphism

$$M^\bullet \otimes_{\mathbf{C}} \mathbf{R}\mathscr{H}om_{D_X}(M^\bullet, D_X) \longrightarrow D_X$$

which gives rise via tensor–Hom adjunction to a morphism

$$M^\bullet \longrightarrow \mathbf{R}\mathscr{H}om_{D_X^{\mathsf{op}}}(\mathbf{R}\mathscr{H}om_{D_X}(M^\bullet, D_X), D_X) = \mathbf{D}^2 M^\bullet.$$

To show this is an isomorphism, since the question is local we may assume that X is affine, in which case we can compute everything after replacing $M^\bullet$ with a complex of finite rank locally free D_X-modules, in which case the claim is clear. □

9.1 Duals and holonomicity

Taking duals preserves holonomicity:

Theorem 9.7. *Let X be a smooth algebraic variety and M a coherent D_X-module. Then,*

(*i*) $H^i(\mathbf{D}M) = 0$ *unless* $\operatorname{codim}_{T^*X} \operatorname{Ch}(X) - \dim X \leq i \leq 0$.

(*ii*) $\operatorname{codim}_{T^*X} \operatorname{Ch}(H^i(\mathbf{D}M)) \geq i + \dim X$.

(*iii*) *M is holonomic if and only if $H^i(\mathbf{D}M) = 0$ for all $i \neq 0$.*

(*iv*) *If M is holonomic, then $\mathbf{D}M \simeq H^0(\mathbf{D}M)$ is also holonomic.*

Proof. The proof is by general properties of Ext in the appendix, and by calculation of Ext on affine opens. There is one particularly interesting statement: the ⇐ direction in (*iii*). Suppose $\mathbf{D}M \simeq H^0(\mathbf{D}M)$. Then, we have $M \simeq \mathbf{D}^2 M \simeq \mathbf{D}H^0(\mathbf{D}M)$, and $H^0(\mathbf{D}H^0(\mathbf{D}M)) \simeq M$. On the other hand,

$$\operatorname{codim}_{T^*X} \operatorname{Ch}(H^0(\mathbf{D}H^0(\mathbf{D}M))) \geq \dim X,$$

and so by Bernstein's inequality $\mathbf{D}H^0(\mathbf{D}M) \simeq M$ is a holonomic D_X-module. □

General theory on filtrations gives the following description of characteristic varieties.

Proposition 9.8. *Let X be a smooth algebraic variety and M a coherent D_X-module. Then,*

$$\mathrm{Ch}(M) = \bigcup_{0\le i\le \dim X} \mathrm{Ch}(\mathscr{E}xt^i_{D_X}(M, D_X) \otimes_{\mathcal{O}_X} \omega_X^{-1}).$$

In particular, if M is holonomic, then the characteristic varieties of M and $\mathbf{D}M$ are the same.

Idea. A good filtration on M induces a good filtration on $\mathscr{E}xt$ groups whose associated graded module has support contained in the support of $\mathrm{gr}^F M$. See [HTT08, Prop. D.4.2]. □

9.2 Hom in terms of duality functors

Our next goal is to describe Hom in terms of duality functors. We first give an example, which shows why we might expect something like this to exist.

Example 9.9. Let M be a vector bundle with integrable connection. Then, we claim

$$\mathbf{D}M \simeq \mathscr{H}om_{\mathcal{O}_X}(M, \mathcal{O}_X).$$

The way to see this is as follows. Consider the Spencer resolution

$$0 \longrightarrow D_X \otimes_{\mathcal{O}_X} \bigwedge^{\dim X}\Theta_X \longrightarrow \cdots \longrightarrow D_X \otimes_{\mathcal{O}_X} \Theta_X \longrightarrow D_X \longrightarrow \mathcal{O}_X \longrightarrow 0$$

for $\mathcal{O}_X$; since M is locally free over $\mathcal{O}_X$, tensoring by $-\otimes_{\mathcal{O}_X} M$ gives a locally free resolution of M. We can then calculate $\mathscr{E}xt^{\dim X}_{D_X}(M, D_X)$ by the complex

$$\begin{array}{ccccc}
\cdots \longrightarrow \mathscr{H}om_D(D\otimes_{\mathcal{O}} \wedge^{\dim X-1}\Theta \otimes_{\mathcal{O}} M, D) & \longrightarrow & \mathscr{H}om_D(D\otimes_{\mathcal{O}} \wedge^{\dim X}\Theta \otimes_{\mathcal{O}} M, D) & \longrightarrow & 0\\
\downarrow\wr & & \downarrow\wr & & \\
\mathscr{H}om_{\mathcal{O}}(\wedge^{\dim X-1}\Theta \otimes_{\mathcal{O}} M, D) & \longrightarrow & \mathscr{H}om_{\mathcal{O}}(\wedge^{\dim X}\Theta \otimes_{\mathcal{O}} M, D) & \longrightarrow & 0\\
\downarrow\wr & & \downarrow\wr & & \\
\mathscr{H}om_{\mathcal{O}}(M, \Omega^{\dim X-1}\otimes_{\mathcal{O}} D) & \longrightarrow & \mathscr{H}om_D(M, \Omega^{\dim X}\otimes_{\mathcal{O}} D) & \longrightarrow & 0
\end{array}$$

On the other hand, since M is locally free over $\mathcal{O}_X$, we can apply $\mathscr{H}om_{\mathcal{O}}(M, -)$ to the de Rham sequence to get the exact sequence

$$\mathscr{H}om_{\mathcal{O}}(M, \Omega^{\dim X-1}\otimes_{\mathcal{O}} D) \longrightarrow \mathscr{H}om_D(M, \Omega^{\dim X}\otimes_{\mathcal{O}} D) \longrightarrow \mathscr{H}om_D(M, \Omega^{\dim X}) \longrightarrow 0$$

and so we see

$$\mathscr{E}xt^{\dim X}_D(M, D) \simeq \mathscr{H}om_{\mathcal{O}}(M, \omega).$$

Passing to left D_X-modules by applying a side-changing functor, we get

$$\mathbf{D}M \simeq \mathscr{H}om_{\mathcal{O}}(M, \omega) \otimes_{\mathcal{O}} \omega^{-1} \simeq \mathscr{H}om_{\mathcal{O}}(M, \mathcal{O}).$$

Lemma 9.10. *For $M^\bullet \in \mathsf{D}^b_{\mathsf{c}}(D_X)$ and $N^\bullet \in \mathsf{D}^b(D_X)$, we have*

$$\mathbf{R}\mathscr{H}om_{D_X}(M^\bullet, N^\bullet) \simeq \mathbf{R}\mathscr{H}om_{D_X}(M^\bullet, D_X) \otimes^{\mathbf{L}}_{D_X} N^\bullet.$$

Proof. There is a canonical morphism $\leftarrow$. By replacing $M^\bullet$ with a locally free resolution and restricting to an open set, we may assume $M^\bullet = D_X$. In that case both sides are locally isomorphic to $N^\bullet$. □

Proposition 9.11. *With the same hypotheses as before, we have isomorphisms*

$$\begin{aligned}
\mathbf{R}\mathscr{H}om_{D_X}(M^\bullet, N^\bullet) &\simeq (\omega_X \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathbf{D}_X M^\bullet) \otimes^{\mathbf{L}}_{D_X} N^\bullet[-\dim X]\\
&\simeq \omega_X \otimes^{\mathbf{L}}_{D_X} (\mathbf{D}_X M^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} N^\bullet)[-\dim X]\\
&\simeq \mathbf{R}\mathscr{H}om_{D_X}(\mathcal{O}_X, \mathbf{D}_X M^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} N^\bullet)
\end{aligned}$$

in $\mathsf{D}^b(\mathbf{C}_X)$. In particular, if $M^\bullet = \mathcal{O}_X$, then

$$\mathbf{R}\mathscr{H}om_{D_X}(\mathcal{O}_X, N^\bullet) \simeq \omega_X \otimes^{\mathbf{L}}_{D_X} N^\bullet[-\dim X]$$

Proof. We first show the last isomorphism. By the previous lemma, we may assume that $N^\bullet = D_X$, in which case we use the Spencer complex to resolve $\mathcal{O}_X$. Applying $\mathscr{H}om_{D_X}(-, D_X)$ then gives the de Rham complex (with a shift), and we get $\omega_X[-\dim X]$.

The first set of isomorphisms follow by writing down the definition of the dual, and applying the last isomorphism at the last step. □

Applying $\mathbf{R}\Gamma(X, -)$ gives the following:

Corollary 9.12. *Denoting* $p\colon X \to \{\mathrm{pt}\}$,

$$\mathbf{R}\mathrm{Hom}_{D_X}(M^\bullet, N^\bullet) \simeq \int_p (\mathbf{D}_X M^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} N^\bullet)[-\dim X] \simeq \mathbf{R}\mathrm{Hom}_{D_X}(\mathcal{O}_X, \mathbf{D}_X M^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} N^\bullet).$$

9.3 Relations with other functors

We now go on to explaining how taking duals commutes with inverse and (proper) direct images.

9.3.1 Inverse images

Theorem 9.13. *Let $f\colon X \to Y$ be a morphism of smooth algebraic varieties, and let M be a coherent D_Y-module.*

(i) Assume $\mathbf{L}f^\circ M \in \mathsf{D}^b_{\mathsf{x}}(D_X)$. Then there exists a canonical morphism

$$\mathbf{D}_X(\mathbf{L}f^\circ M) \longrightarrow \mathbf{L}f^\circ(\mathbf{D}_Y M).$$

(ii) Assume that f is smooth (in fact, "non-characteristic" is enough). Then, we have

$$\mathbf{D}_X(\mathbf{L}f^\circ M) \simeq \mathbf{L}f^\circ(\mathbf{D}_Y M).$$

Proof. For (i), we use the previous Corollary in two different ways:

$$\begin{aligned}\mathrm{Hom}_{\mathsf{D}^b(D_Y)}(M, M) \simeq \mathrm{Hom}_{\mathsf{D}^b(D_Y)}(\mathcal{O}_Y, \mathbf{D}_Y M \otimes^{\mathbf{L}}_{\mathcal{O}_Y} M) &\longrightarrow \mathrm{Hom}_{\mathsf{D}^b(D_X)}(\mathbf{L}f^\circ \mathcal{O}_Y, \mathbf{L}f^\circ(\mathbf{D}_Y M) \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathbf{L}f^\circ M)\\ &\simeq \mathrm{Hom}_{\mathsf{D}^b(D_X)}(\mathcal{O}_X, \mathbf{L}f^\circ M \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathbf{L}f^\circ(\mathbf{D}_Y M))\\ &\simeq \mathrm{Hom}_{\mathsf{D}^b(D_X)}(\mathbf{D}_X(\mathbf{L}f^\circ M), \mathbf{L}f^\circ(\mathbf{D}_Y M))\end{aligned}$$

and choose the image of the identity.

For (ii), we prove the smoothness statement following [Bor+87, Prop. 9.13]. We can restrict to smaller open sets since checking the morphism in (i) is an isomorphism is a local question. Since f is smooth, locally we have a decomposition of f into an étale morphism followed by a projection (of a relative affine space).

If f is étale, then $f^\circ D_Y = D_X$ and $f^{-1}\omega_Y = \omega_X$. After replacing M with a (locally) projective resolution, and possibly enlarging M to be free, we can reduce to the case $M = D_Y$, in which case

$$\begin{aligned}\mathbf{L}f^\circ(\mathbf{D}_Y D_Y) &\simeq \mathbf{L}f^\circ(\mathscr{H}om_{D_Y}(D_Y, D_Y) \otimes_{\mathcal{O}_Y} \omega_Y^{-1}[\dim Y])\\ &\simeq \mathbf{L}f^\circ D_Y \otimes_{\mathcal{O}_X} \omega_X^{-1}[\dim X]\\ &\simeq D_X \otimes_{\mathcal{O}_X} \omega_X^{-1}[\dim X]\\ &\simeq \mathscr{H}om_{D_X}(D_X, D_X) \otimes_{\mathcal{O}_X} \omega_X^{-1}[\dim X]\\ &\simeq \mathbf{D}_X(D_X)\\ &\simeq \mathbf{D}_X(\mathbf{L}f^\circ D_Y).\end{aligned}$$

If f is a projection $X = T \times Y \to Y$, it suffices to show the morphism in (i) is an isomorphism locally, and moreover assume that $M = D_Y$. In that case, we have

$$\begin{aligned}\mathbf{D}_X(\mathbf{L}f^\circ D_Y) &\simeq \mathbf{D}_X(\mathcal{O}_T \boxtimes D_Y)\\ &\simeq \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_T \boxtimes D_Y, \mathcal{O}_T \boxtimes \mathcal{O}_Y)\\ &\simeq \mathcal{O}_T \boxtimes \mathscr{H}om_{\mathcal{O}_Y}(D_Y, \mathcal{O}_Y)\\ &\simeq \mathcal{O}_T \boxtimes \mathbf{D}_Y(D_Y)\\ &\simeq \mathbf{L}f^\circ(\mathbf{D}_Y D_Y).\end{aligned}$$

□

9.3.2 Direct images

We now come to the proof that duality functors commute with proper direct images.

Let $f\colon X \to Y$ be a proper morphism of smooth algebraic varieties. We first construct the *trace map*

$$\mathrm{Tr}_f\colon \int_f \mathscr{O}_X[\dim X] \longrightarrow \mathscr{O}_Y[\dim Y].$$

In the analytic setting, you can construct this using resolutions by currents/Schwartz distributions, but we will construct it by decomposing f into $X \hookrightarrow \mathbf{P}^n \times Y \to Y$.

First, suppose $i\colon X \hookrightarrow Y$ is a closed embedding. By adjunction, we have a morphism

$$\int_i i^\dagger \mathscr{O}_Y \longrightarrow \mathscr{O}_Y.$$

But in this case, $i^\dagger \mathscr{O}_Y = i^\circ \mathscr{O}_Y[\dim X - \dim Y] = \mathscr{O}_X[\dim X - \dim Y]$, and so we have the required morphism after a shift.

Now consider the projection $X = \mathbf{P}^n \times Y \to Y$. Since $\mathscr{O}_X = \mathscr{O}_{\mathbf{P}^n} \boxtimes \mathscr{O}_Y$, and the external product commutes with direct images, the problem is reduced to the case $p\colon \mathbf{P}^n \to \{\mathrm{pt}\}$. In this case $\int_p \mathscr{O}_{\mathbf{P}^n}$ is given by

$$\mathbf{R}\Gamma\Big(\mathbf{P}^n, \big[\mathscr{O}_{\mathbf{P}^n} \to \Omega^1_{\mathbf{P}^n} \to \cdots \to \omega^n_{\mathbf{P}^n}\big]\Big).$$

Thus, we have isomorphisms

$$\int^0_p \mathscr{O}_{\mathbf{P}^n}[n] \simeq \tau^{\geq 0} \int_p \mathscr{O}_{\mathbf{P}^n}[n] \simeq H^n(\mathbf{P}^n, \omega_{\mathbf{P}^n}) \simeq \mathbf{C}$$

by Serre duality. Note the last isomorphism is by the trace morphism, hence the name. We therefore obtain a morphism

$$\int_p \mathscr{O}_{\mathbf{P}^n}[n] \longrightarrow \tau^{\geq 0} \int_p \mathscr{O}_{\mathbf{P}^n}[n] \simeq \mathbf{C} = \mathscr{O}_{\mathrm{pt}}.$$

In general, denoting $X \overset{i}{\hookrightarrow} \mathbf{P}^n \times Y \overset{p}{\to} Y$, we define the trace as

$$\int_f \mathscr{O}_x[\dim X] = \int_p \int_i \mathscr{O}_x[\dim X] \longrightarrow \int_p \mathscr{O}_{\mathbf{P}^n \times Y}[\dim Y + n] \longrightarrow \mathscr{O}_Y[\dim Y]$$

One can show that this definition does not depend on choice of decomposition and is functorial (really, this is the trace map that comes from Grothendieck duality).

Theorem 9.14. *Let $f\colon X \to Y$ be a proper morphism. Then, we have a canonical isomorphism*

$$\int_f \mathbf{D}_X \xrightarrow{\sim} \mathbf{D}_Y \int_f\colon \mathsf{D}^b_{\mathsf{c}}(D_X) \longrightarrow \mathsf{D}^b_{\mathsf{c}}(D_Y)$$

of functors.

Proof. We first construct the morphism. We have

$$\begin{aligned}
\int_f \mathbf{D}_X M^\bullet &= \mathbf{R}f_*(\mathbf{R}\mathscr{H}om_{D_X}(M^\bullet, D_X) \otimes^{\mathbf{L}}_{D_X} D_{X\to Y}) \otimes^{\mathbf{L}}_{\mathscr{O}_Y} \omega_Y^{-1}[\dim X] \\
&= \mathbf{R}f_*(\mathbf{R}\mathscr{H}om_{D_X}(M^\bullet, D_{X\to Y})) \otimes^{\mathbf{L}}_{\mathscr{O}_Y} \omega_Y^{-1}[\dim X] \\
\mathbf{D}_Y \int_f M^\bullet &= \mathbf{R}\mathscr{H}om_{D_Y}(\int_f M^\bullet, D_Y) \otimes^{\mathbf{L}}_{\mathscr{O}_Y} \omega_Y^{-1}[\dim Y]
\end{aligned}$$

and so it suffices to construct canonical morphisms without the twists by ω_Y^{-1}:

$$\Phi(M^\bullet)\colon \mathbf{R}f_*(\mathbf{R}\mathscr{H}om_{D_X}(M^\bullet, D_{X\to Y}[\dim X])) \longrightarrow \mathbf{R}\mathscr{H}om_{D_Y}(\int_f M^\bullet, D_Y[\dim Y])$$

in $\mathsf{D}^b_{\mathsf{c}}(D_Y^{\mathsf{op}})$.

By the projection formula, we have

$$\int_f D_{X\to Y}[\dim X] = \int_f \mathbf{L}f^* D_Y[\dim X] \simeq \int_f \mathcal{O}_X[\dim X] \otimes^{\mathbf{L}}_{\mathcal{O}_Y} D_Y$$

so the trace morphism gives a map

$$\int_f D_{X\to Y}[\dim X] \longrightarrow D_Y[\dim Y].$$

Now define $\Phi(M^\bullet)$ to be the composite

$$\begin{array}{c}
\mathbf{R}f_*(\mathbf{R}\mathscr{H}om_{D_X}(M^\bullet, D_{X\to Y}[\dim X])) \\
\downarrow \\
\mathbf{R}f_*\,\mathbf{R}\mathscr{H}om_{f^{-1}D_Y}(D_{Y\leftarrow X}\otimes^{\mathbf{L}}_{D_X} M^\bullet, D_{Y\leftarrow X}\otimes^{\mathbf{L}}_{D_X} D_{X\to Y}[\dim X]) \\
\downarrow \\
\mathbf{R}\mathscr{H}om_{D_Y}(\mathbf{R}f_*(D_{Y\leftarrow X}\otimes^{\mathbf{L}}_{D_X} M^\bullet), \mathbf{R}f_*(D_{Y\leftarrow X}\otimes^{\mathbf{L}}_{D_X} D_{X\to Y})[\dim X]) \\
\| \\
\mathbf{R}\mathscr{H}om_{D_Y}(\int_f M^\bullet, \int_f D_{X\to Y}[\dim X]) \\
\downarrow \\
\mathbf{R}\mathscr{H}om_{D_Y}(\int_f M^\bullet, D_Y[\dim Y]).
\end{array}$$

We now need to show that this composition is an isomorphism; we do this by checking for closed embeddings and projections separately. We will reduce to the case where $M^\bullet = D_X$ as before, although you have to be careful: you need that the resolution of $M^\bullet$ as locally projective D_X-modules exists locally *on* Y. This is obvious for a closed embedding, but is why we need the projectivity assumption: the product of an affine open in Y with projective space is D-affine.

For a closed embedding, the composite above is

$$\begin{array}{c}
i_*(\mathrm{Hom}_{D_X}(D_X, i^\circ D_Y))[\dim X] \\
\text{Kashiwara}\downarrow\wr \\
\mathbf{R}\mathscr{H}om_{D_Y}(\int_i D_X, \int_i i^* D_Y)[\dim X] \\
\| \\
\mathbf{R}\mathscr{H}om_{D_Y}(\int_i D_X, \int_i i^\dagger D_Y)[\dim Y] \\
\downarrow \\
\mathbf{R}\mathscr{H}om_{D_Y}(\int_i D_X, D_Y)[\dim Y]
\end{array}$$

so it suffices to show the last map is an isomorphism. Let $U = Y \setminus X$, and let $j\colon U \hookrightarrow Y$ be the open embedding of U in Y. The distinguished triangle

$$\int_i i^\dagger D_Y \longrightarrow D_Y \longrightarrow \int_j j^\dagger D_Y \rightsquigarrow$$

implies it suffices to show $\mathbf{R}\mathscr{H}om_{D_Y}(\int_i D_X, \int_j j^* D_Y) = 0$. But this follows from the fact that

$$\mathbf{R}\mathscr{H}om_{D_Y}(\int_i D_X, \int_j j^* D_Y) \simeq i_*\,\mathbf{R}\mathscr{H}om_{D_X}(D_X, i^\dagger \int_j j^* D_Y) = i_* i^! \int_j j^* D_Y = 0,$$

where the first isomorphism is one of the properties we showed about how pushforward for closed immersions interacts with $\mathscr{H}om$, and the last equality is by the fact that X is smooth in Y, so $i^! \int_j = 0$.

For a projection $X = \mathbf{P}^n \times Y \to Y$, since $D_X = D_{\mathbf{P}^n} \boxtimes D_Y$, we reduce to the case where Y is a point. Then, $D_{\mathbf{P}^n \to \mathrm{pt}} = \mathscr{O}_{\mathbf{P}^n}$ and $D_{\mathrm{pt} \leftarrow \mathbf{P}^n} = \omega_{\mathbf{P}^n}$, and so

$$\mathbf{R}p_*(\mathbf{R}\mathscr{H}om_{D_X}(D_X, D_{X\to Y}[\dim X])) = \mathbf{R}\mathrm{Hom}_{D_{\mathbf{P}^n}}(D_{\mathbf{P}^n}, \mathscr{O}_{\mathbf{P}^n})[n] = \mathbf{R}\Gamma(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n})[n] \simeq \mathbf{C}[n]$$

and

$$\mathbf{R}\mathscr{H}om_{D_Y}(\int_p D_X, D_Y[\dim Y]) = \mathbf{R}\mathrm{Hom}_{\mathbf{C}}(\mathbf{R}\Gamma(\mathbf{P}^n, \omega_{\mathbf{P}^n}), \mathbf{C}) \simeq \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}[-n], \mathbf{C}) = \mathbf{C}[n],$$

and so all you have to check is that the morphism is nontrivial. □

Just by writing out definitions and using this Theorem plus the description of Hom in terms of duality functors, we get

Corollary 9.15 (Adjunction formula)**.** *Let $f\colon X \to Y$ be a proper morphism. Then, we have an isomorphism*

$$\mathbf{R}\mathscr{H}om_{D_Y}(\int_f M^\bullet, N^\bullet) \simeq \mathbf{R}f_*\,\mathbf{R}\mathscr{H}om_{D_X}(M^\bullet, f^\dagger N^\bullet),$$

where $M^\bullet \in \mathsf{D}^b_{\mathsf{c}}(D_X)$ and $N^\bullet \in \mathsf{D}^b(D_Y)$.

This if the first of many!

9.4 Adjunction formulas and six functor formalism [HTT08, §3.2.3]

Let $f\colon X \to Y$ be a morphism of smooth algebraic varieties.

Definition 9.16. We define new functors by

$$\int_{f!} := \mathbf{D}_Y \circ \int_f \circ\, \mathbf{D}_X\colon\ \mathsf{D}^b_{\mathsf{h}}(D_X) \longrightarrow \mathsf{D}^b_{\mathsf{h}}(D_Y)$$
$$f^\star :=\ \mathbf{D}_X \circ f^\dagger \circ \mathbf{D}_Y\colon\ \mathsf{D}^b_{\mathsf{h}}(D_Y) \longrightarrow \mathsf{D}^b_{\mathsf{h}}(D_X)$$

Note that these make sense: we have checked all functors involved preserve holonomicity.

We collect various facts into one statement:

Theorem 9.17.

1. *The functor $\int_{f!}$ is left adjoint to $f^\dagger$.*
2. *The functor $f^\star$ is left adjoint to $\int_f$.*
3. *There is a canonical morphism of functors $\int_{f!} \to \int_f$ which is an isomorphism for proper f.*
4. *If f is smooth, $f^\dagger = f^\star[2(\dim X - \dim Y)]$.*

Remark 9.18. We will be discussing constructible sheaves when we talk about the Riemann–Hilbert correspondence, so I think it's helpful to make some comments about parallels now. Recall the definition, in the setting of analytic spaces.

Definition 9.19. Let X be an analytic space. A *stratification* of X is a locally finite decomposition

$$X = \bigsqcup_{\alpha \in A} X_\alpha$$

by locally closed analytic subsets such that each X_α is smooth, and $\overline{X}_\alpha = \bigsqcup_{\beta \in B} X_\beta$ for a subset B of A. A sheaf of $\mathbf{C}_X$-modules F is called a *constructible sheaf* on X if there exists a stratification as above such that $F|_{X_\alpha}$ is a local system on X_α for each $\alpha \in A$.

If F is a sheaf on X^{an} for X an algebraic variety such that we can find a decomposition such that each X_α is an algebraic subvariety, and $F|_{X_\alpha^{\mathrm{an}}}$ is a locally constant sheaf for each $\alpha \in A$, we say that F is an *algebraically constructible sheaf*.

The reason why this is a good notion is that we can chop up X into little pieces, and try to understand global objects by glueing together sheaves recursively using diagrams of the form

$$U \xhookrightarrow[\text{open}]{j} X \xhookleftarrow[\text{closed}]{i} Z$$

where $U = X \setminus Z$. To make this glueing operation possible, there are two sequences of adjoint functors:

$$(j_!, j^! = j^*, \mathbf{R}j_*) \qquad (i^*, i_* = i_!, \mathbf{R}i^!)$$

(technically living on the derived category) that allow you to glue sheaves together. This is Grothendieck's six functor formalism (the other two are tensor product and internal hom), which you would see when looking at constructible sheaves or étale cohomology. So the analogue of these sequences of adjoint functors for holonomic D-modules is

$$\Big(\int_{j!}, j^\dagger = j^\star, \int_j\Big) \qquad \Big(i^\star, \int_i = \int_{i!}, i^\dagger\Big).$$

Proof of Theorem. For (1) and (2), we actually show stronger statements involving $\mathbf{R}\mathscr{H}\!om$. First,

$$\begin{aligned}
\mathbf{R}f_*\,\mathbf{R}\mathscr{H}\!om_{D_X}(M^\bullet, f^\dagger N^\bullet) &\simeq \mathbf{R}f_*((\omega_X \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathbf{D}_X M^\bullet) \otimes^{\mathbf{L}}_{D_X} f^\dagger N^\bullet)[-\dim X] \\
&\simeq \mathbf{R}f_*((\omega_X \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathbf{D}_X M^\bullet) \otimes^{\mathbf{L}}_{D_X} D_{X\to Y} \otimes^{\mathbf{L}}_{f^{-1}D_Y} f^{-1}N^\bullet)[-\dim Y] \\
&\simeq \mathbf{R}f_*((\omega_X \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathbf{D}_X M^\bullet) \otimes^{\mathbf{L}}_{D_X} D_{X\to Y}) \otimes^{\mathbf{L}}_{D_Y} N^\bullet)[-\dim Y] \\
&\simeq (\omega_Y \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \int_f \mathbf{D}_X M^\bullet) \otimes^{\mathbf{L}}_{D_Y} N^\bullet[-\dim Y] \\
&\simeq (\omega_Y \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathbf{D}_Y \int_{f!} M^\bullet) \otimes^{\mathbf{L}}_{D_Y} N^\bullet[-\dim Y] \\
&\simeq \mathbf{R}\mathscr{H}\!om_{D_Y}(\int_{f!} M^\bullet, N^\bullet).
\end{aligned}$$

where the fourth isomorphism is best explained as coming from the definition of derived pushforward for right D-modules. Applying duals gives the adjunction (2) in the form of

$$\mathbf{R}f_*\,\mathbf{R}\mathscr{H}\!om_{D_X}(f^\star N^\bullet, M^\bullet) \xrightarrow{\sim} \mathbf{R}\mathscr{H}\!om_{D_Y}(N^\bullet, \int_f M^\bullet).$$

For (3), we have already constructed such a morphism for proper f, in which case it was an isomorphism. So it remains to show we can define a canonical morphism in general. By Nagata's compactification theorem (or, Hironaka's resolution of singularities since all our varieties are quasi-projective), we can decompose any morphism f as

$$X \xhookrightarrow[\text{closed}]{g} X \times Y \xhookrightarrow[\text{open}]{j} \widetilde{X} \times Y \xrightarrow[\text{proper}]{p} Y$$

in which case we know $\int_{-!} \xrightarrow{\sim} \int_-$ for g and p, and so it suffices to show there exists a canonical morphism $\int_{j!} \to \int_j$. But we can observe

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{D}^b_{\mathsf{h}}(D_Y)}(\int_{j!} M^\bullet, \int_j M^\bullet) &\simeq \mathrm{Hom}_{\mathsf{D}^b_{\mathsf{h}}(D_X)}(M^\bullet, j^\dagger \int_j M^\bullet) \\
&\simeq \mathrm{Hom}\,\mathsf{D}^b_{\mathsf{h}}(D_X)(M^\bullet, M^\bullet)
\end{aligned}$$

and so we get the desired canonical morphism by pulling back the identity.

Finally, (4) is just rewriting our previous result on how $\mathbf{L}f^\circ$ commutes with taking duals. $\square$

10 November 7 and November 14: Regular holonomic D-modules (Takumi Murayama)

Recall that the (classical) Riemann–Hilbert correspondence said that the categories of local systems on X, and vector bundles with integrable connection on X, are equivalent. We also had the finiteness property from last time: if $M^\bullet \in \mathsf{D}^b_{\mathsf{h}}(D_X)$, then there is a stratification

$$X = X_0 \supset X_1 \supset \cdots \supset X_m = \emptyset$$

such that $X_r \setminus X_{r+1}$ is smooth, and $H^k(i_r^\dagger M)$ is a vector bundle with integrable connection, where $i_r \colon X_r \setminus X_{r+1} \hookrightarrow X$. It therefore looks like the Riemann–Hilbert correspondence could be "upgraded" to be an equivalence between $\mathsf{D}^b_{\mathsf{h}}(D_X)$ and the bounded derived category $\mathsf{D}^b_{\mathsf{c}}(X)$ of abelian sheaves with constructible cohomology; this, however, does not quite work, and we must pass to the subcategory of *regular holonomic* D-modules. The idea is that allowing arbitrary filtrations like that above makes singularities at the boundary of the vector bundle with integrable connection too wild.

Note there is an interpretation in terms of ODE's in [HTT08, §5.1.2]; since the speaker lacks the background needed to appreciate the notion in ODE's, we will discuss the algebraic case only. We will content ourselves with pointing out that it gives a growth condition on solutions to the differential equations [Bor+87, III, Thm. 1.3.1], in that all formal solutions are actually algebraic [Bjö93, Ch. V]. This last result is due to Kashiwara–Kawai, and the precise statement is:

Proposition 10.1 (cf. [Ber82, §5.10; HTT08, Rem. 7.3.2])**.** *If $M^\bullet \in \mathsf{D}^b(D_X)$, then $M^\bullet$ is regular holonomic if and only if*

$$\mathbf{R}\mathrm{Hom}_{D_{X,x}}(M^\bullet_x, \widehat{\mathcal{O}}_{X,x}/\mathcal{O}_{X,x}) = 0$$

for all $x \in X$.

While this seems like a nice definition, because it is difficult to prove results with, we will instead follow the approach of [HTT08, Ch. 6], which in turn is based on [Ber82, §4]. The idea is to define them for vector bundles on curves, build a definition for higher dimensions using restrictions, and then build a definition using the finite length filtrations that we know exist for holonomic D-modules.

10.1 Regular holonomic D-modules on curves [Ber82, §4.1]

Let C be a curve, and choose a smooth compactification $i \colon C \hookrightarrow \overline{C}$, where C is an open dense subset. Note this exists by, say, [Har77, Ch. I, §6]. Choose a point $p \in \overline{C} \setminus C$, which plays the role of a point at infinity of C. Let x be a local parameter at p, $\partial = \partial/\partial x$, $d = x\partial \in D_{\overline{C}}$. Denote by D^ν_C the subsheaf of subalgebras of $D_{\overline{C}}$ generated by d and $\mathcal{O}_{\overline{C}}$. Note that D^ν_C and D^ν/tD^ν do not depend on the choice of parameter x.

Definition 10.2. Let F be a $\mathcal{O}$-coherent D_C-module.

(*a*) We say F has a *regular singularity* at the point p if its direct image $\int_i F$ is a union of $\mathcal{O}$-coherent D^ν_C-submodules.

(*b*) We say F has *regular singularities* or is *RS* if it has regular singularities at all points $p \in \overline{C} \setminus C$.

Definition 10.3. Let F be a holonomic D_C-module on C. We say F is *regular (holonomic)* if its restriction to some open dense subset $U \subset C$ is a $\mathcal{O}$-coherent D_C-module with regular singularities.

By looking at how local parameters change under morphisms of curves we obtain

Lemma 10.4 [HTT08, Lem. 5.1.23]**.** *Let $f \colon C \to C'$ be a dominant morphisms of curves. Then*

(*i*) *$M \in \mathsf{Mod}_{\mathsf{h}}(D_{C'})$ is regular if and only if $f^\dagger M$ is regular; and*

(*ii*) *$N \in \mathsf{Mod}_{\mathsf{h}}(D_C)$ is regular if and only if $\int_f N$ is regular.*

Proof Sketch. By definition of regular holonomicity, we may replace C with $\overline{C}$ and C' with $\overline{C'}$. Let C_0 be an open subset such that we have the commutative diagram

$$\begin{array}{ccc} f^{-1}(C_0') =: C_0 & \xhookrightarrow{\ j\ } & C \\ {\scriptstyle \text{étale}}\big\downarrow{\scriptstyle f_0} & & {\scriptstyle f}\big\downarrow{\scriptstyle \text{proper}} \\ C_0' & \xhookrightarrow{\ j'\ } & C' \end{array}$$

such that $M|_{C_0'}$ and $N|_{C_0}$ are vector bundles with regular singularities. Note such a diagram exists by generic smoothness [Har77, Ch. III, Cor. 10.7]. Pick $p \in C \setminus C_0$ let $f(p) =: p'$. Also choose local parameters x_p and $y_{p'}$ at p, p', so $y_{p'} = x_p^{m_p}$ for some $m_p \in \mathbf{Z}_{>0}$, which implies $d_p = m_p \cdot d_{p'}$.

For (i), by definition it suffices to show $\int_{j'} M|_{C_0'}$ is a union of $\mathcal{O}$-coherent $D^{\nu}_{C'}$-submodules if and only if $\int_j (f^\dagger M)|_{C_0}$ is a union of $\mathcal{O}$-coherent D^{ν}_{C}-submodules. But this holds since

$$\int_j (f^\dagger M)|_{C_0} \simeq \int_j f_0^\dagger M|_{C_0'} \simeq f^\dagger \int_{j'} M|_{C_0'}$$

where the second isomorphism is by base change [HTT08, Thm. 1.7.3], and the fact that $\mathcal{O}$-coherence is preserved under inverse and direct images via a proper morphism. For (ii), we can repeat the same argument, except we no longer need base change, only the fact that direct image behaves well under composition. ☐

10.2 Simple holonomic modules [HTT08, §3.4]

The definition of regular holonomic D-modules will require some background material on simple holonomic modules, that is, holonomic modules that have no nontrivial D-coherent submodules or quotients. Recall Proposition 7.5(b), which said that holonomic D-modules have finite length. We want to characterize the simple holonomic D-modules that appear in the filtration.

Let $i\colon Y \to X$ be a locally closed affine embedding of smooth varieties. Then, $\mathbf{R}i_* = i_*$ and $D_{X\leftarrow Y}$ is locally free, and so we can consider $\int_i M$ and $\int_{i!} M$ as D_X-modules, which are holonomic as we have shown.

Definition 10.5. The *minimal extension* of M, denoted $L(Y, M)$, is the image of the canonical morphism $\int_{i!} M \to \int_i M$.

Theorem 10.6. *If M is simple, then $L(Y, M)$ is simple, and is the unique simple quotient of $\int_{i!} M$. Also, any simple holonomic D_X-module L is isomorphic to $L(Y, M)$ for some pair (Y, M), where Y is as above, and M is a simple vector bundle on Y.*

Proof. Let F be any simple quotient of $\int_{i!} M$. Then, since $\operatorname{Hom}(\int_{i!} M, F) \simeq \operatorname{Hom}(M, i^\dagger F) \neq 0$ and $i^\dagger F$ is simple as well as M, we see that $M \simeq i^\dagger F$, and $\int_{i!} i^\dagger F \simeq F$. Note that F must be unique, for otherwise there would be another simple quotient F', and so there is a larger quotient factoring through both F and F', a contradiction.

Now we show $L(Y, M) \simeq F$. We have

$$\operatorname{Hom}_{D_X}(\int_{i!} M, F) \simeq \operatorname{Hom}_{D_Y}(M, i^\dagger F) \simeq \operatorname{Hom}_{D_Y}(M, M) \neq 0,$$

and so we have a factorization $\int_{i!} M \twoheadrightarrow F \to \int_i M$. Since F surjects onto $L(Y, M)$ and F is simple, we have that $F \simeq L(Y, M)$.

For the second statement, let Y be an affine open dense subset of an irreducible component of $\operatorname{Supp} L$, so that $i^\dagger L$ is a vector bundle on Y; note this is possible by Proposition 7.8. Let $M = i^\dagger L$; note that this is simple by Kashiwara's equivalence. Then, we have an isomorphism

$$\operatorname{Hom}_{D_X}(\int_{i!} M, L) \simeq \operatorname{Hom}_{D_Y}(M, i^\dagger L) \simeq \operatorname{Hom}_{D_Y}(M, M) \neq 0,$$

and so there is a non-zero surjective morphism $\int_{i!} M \to L$. Thus, $L \simeq L(Y, M)$. ☐

There is also a statement where $L(Y, M)$ is the unique simple submodule of $\int_i M$; the proof is analogous.

10.3 Regular holonomic *D*-modules in general [Ber82, §§4.2–4.6]

We can now return to our discussion of regular holonomic D-modules. Let X be of arbitrary positive dimension.

Definition 10.7.

(1) Let F be an $\mathcal{O}$-coherent D_X-module. We say F has regular singularities if its restriction to any curve does.
(2) Let L be a *simple* holonomic D_X-module. We say L is regular holonomic if it is of the form $L \simeq L(Y, M)$ for M a $\mathcal{O}_Y$-coherent D_Y-module with regular singularities.
(3) A holonomic D_X-module M is regular holonomic if all its simple factors are regular holonomic. These form a category $\mathsf{Mod}_{\mathsf{rh}}(D_X)$.
(4) $M^\bullet \in \mathsf{D}^b_{\mathsf{h}}(D_X)$ is regular holonomic if its cohomology sheaves are all regular holonomic. These form a category $\mathsf{D}^b_{\mathsf{rh}}(D_X)$.

By definition, this category $\mathsf{Mod}_{\mathsf{rh}}(D_X)$ is closed under subquotients and extensions.
We state the main theorem about preservation of regular holonomicity.

Theorem 10.8. *The functors $\mathbf{D}, \int_f, f^\dagger, \int_{f!}, f^\star$ preserve regular holonomicity.*

Theorem 10.9 (Curve testing criterion)**.** *$M^\bullet \in \mathsf{D}^b_{\mathsf{rh}}(D_X)$ if and only if its restriction $i_C^\dagger M^\bullet$ to any locally closed curve C in X is regular holonomic.*

Remark 10.10. We could of course define regular holonomicity as in the curve testing criterion; however, according to [Ber82, p. 32], checking the "subquotient" properties would be much more difficult, and it is instead preferable to check the cohomological statements in the first theorem, where we actually have machinery to work with.

At least part of Theorem 10.8 is easy:

Step 1. Theorem 10.8 holds for $\mathbf{D}$.

Proof. By induction on cohomological dimension and length, it suffices to show that if $M \in \mathsf{Mod}_{\mathsf{rh}}(D_X)$ is simple, then its dual is regular holonomic. By definition, there exists $i\colon Y \hookrightarrow X$ locally closed and affine, such that $M \simeq L(Y, N)$ for some vector bundle N with regular singularities on Y. Taking duals, since $\mathbf{D}L(Y, N)$ is the image of

$$\int_{i!} \mathbf{D}N \simeq \mathbf{D}\int_i N \longrightarrow \mathbf{D}\int_{i!} N \simeq \int_i \mathbf{D}N$$

and duals of vector bundles with regular singularities have regular singularities, we are done. □

By definition of $\int_{f!}$ and $f^\star$, this means it suffices to show Theorem 10.8 for $\int_f$ and $f^\dagger$. Note, however, that the curve testing criterion implies the result for $f^\dagger$, so really, we only need to show that $\int_f$ preserves regular holonomicity and the curve testing criterion. We give a sketch of the argument, following [Ber82, §§4.4–4.6; HTT08, §6.2].

The main idea is to prove both theorems using induction on the dimension of $\operatorname{Supp} M^\bullet$, and utilize two special cases of the proposition. Note the case for curves is by definition and the Lemma from before.

10.3.1 D-modules with regular singularities along a divisor

The first special case is pushforward along an affine embedding whose complement is a simple normal crossings divisor.

Let X be an algebraic variety. Consider a smooth compactification $i\colon X \hookrightarrow \overline{X}$, such that X is open and dense, and such that $X^\nu = \overline{X} \setminus X$ has simple normal crossings. Note this exists by resolution of singularities applied to any projective compactification of X.

Denote by $\mathscr{J} \subset \mathcal{O}_{\overline{X}}$ the ideal of X^ν, by $T^\nu \subset \Theta_{\overline{X}}$ the subsheaf of vector fields preserving $\mathscr{J}$, and D^ν_X the subalgebra of $D_{\overline{X}}$ generated by T^ν and $\mathcal{O}_{X^+}$. In this situation, we have the following:

Proposition 10.11 (Deligne)**.** *Suppose F is an $\mathcal{O}$-coherent D_X-module. Then, F has regular singularities if and only if $\int_i F$ is a union of $\mathcal{O}$-coherent $D^\nu_{\overline{X}}$-modules.*

This implies the following:

Step 2. Let $f = i\colon X \to \overline{X}$ be an inclusion into a smooth compactification of X, and let M be a $\mathcal{O}$-coherent D_X-module. Then, $\int_i M$ is regular holonomic.

Proof. By Deligne's proposition, $\int_i M$ is a union of $\mathcal{O}$-coherent $D^\nu_{\overline{X}}$-modules, and so any composition factor L of $\int_i M$ does as well. If $i^\dagger L \neq 0$, then one can show (see [HTT08, Thm. 3.4.2(iii)]) that $L \simeq L(Y, N)$ for $Y \subset \overline{X}$ affine, and $N \simeq M$. So the interesting case is when when $\overline{Z} = \operatorname{Supp} L$ is an irreducible component of an intersection of some components of the divisor X^ν, in which case $L \simeq L(Z, E)$, where Z is open in $\overline{Z}$. Then, $\int_{Z \to \overline{Z}}(E)$ is a union of $\mathcal{O}$-coherent $\mathcal{O}$-coherent D^ν_Z-modules, since D^ν_Z is a quotient of the algebra D^ν_X and $\int_{Z \to \overline{Z}}(E)$ is a subquotient of $\int_i M$. □

To get to the general case from this one, you use Hironaka's resolution of singularities to construct a smooth compactification

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & \overline{X} \\ & \searrow^{f} & \downarrow{\scriptstyle \bar{f}}\ \text{proper} \\ & & Y \end{array}$$

The idea is then to decompose the proper morphism $\overline{X} \to Y$ into a closed embedding then a projection, and check regular holonomicity is preserved. Note that you have to use the curve testing criterion. If f is an affine embedding, this isn't too hard:

Step 3. If f is an affine embedding, then $\int_f$ preserces holonomicity.

Proof. If f is affine, then $\int_f$ is exact, and so by inducing on cohomological dimension and length of composition series, it suffices to show $\int_f M$ is regular holonomic when M is regular holonomic. We have the following distinguished triangle

$$\int_{f!} M \longrightarrow \int_f M \longrightarrow C_f(M) \xrightarrow{[1]}$$

and since higher cohomology for $\int_f$ and $\int_{f!}$ vanish, we know composition factors of $\int_f M$ must come from $H^* C_f(M)$ or the minimal extension $L(X, M)$. Since the latter is regular by definition, it suffices to show that $H^* C_f(M)$. Using the decomposition above, we have the analogous distinguished triangle

$$\int_{i!} M \longrightarrow \int_i M \longrightarrow C_i(M) \xrightarrow{[1]}$$

which pushes forward via $\bar{f}$ to

$$\int_{f!} M \longrightarrow \int_f M \longrightarrow \int_{\bar{f}} C_i(M) \xrightarrow{[1]}$$

We know already that $\int_{i!} M, \int_i M$ are regular holonomic by Step 2. Thus, $C_i(M)$ does also. On the other hand, $C_i(M)$ has support less than that of M, and so pushes forward to something regular holonomic. □

The general case is a bit more difficult, but amounts to showing the curve testing criterion, and using it to prove that projections also preserve regular holonomicity (since closed embeddings are already affine embeddings).

11 November 14: The Riemann–Hilbert correspondence (Takumi Murayama)

We now come to the Riemann–Hilbert correspondence for regular holonomic D-modules and constructible sheaves. We recall the notion of a constructible sheaf:

Definition 11.1. Let X be an algebraic variety. A *stratification* of X is a locally finite decomposition

$$X = \bigsqcup_{\alpha \in A} X_\alpha$$

by (Zariski-)locally closed subsets such that each X_α is smooth, and $\overline{X}_\alpha = \bigsqcup_{\beta \in B} X_\beta$ for a subset B of A. A sheaf of $\mathbf{C}_X$-modules F is called a *constructible sheaf* on X if there exists a stratification as above such that $F|_{X^{\mathrm{an}}_\alpha}$ is a local system on X^{an}_α (that is, a local system on the analytic topology) for each $\alpha \in A$.

We recall that last time, we discussed that since holonomic D-modules also have a stratification into vector bundles with integrable connection, and the Riemann–Hilbert correspondence gave a correspondence between such vector bundles with local systems, we might expect holonomic D-modules to correspond to constructible ones. This does not quite work, because arbitrary holonomic D-modules can have bad singularities. We do, however, have the following:

Theorem 11.2 (Riemann–Hilbert correspondence)**.** *For a smooth algebraic variety X, the de Rham functor*

$$\begin{aligned} \mathrm{DR}_X \colon \mathsf{D}^b_{\mathsf{rh}}(D_X) &\longrightarrow \mathsf{D}^b_c(X) \\ M^\bullet &\longrightarrow \omega_X \otimes^{\mathbf{L}}_{D_X} M^\bullet \end{aligned}$$

gives an equivalence of categories.

Note the reason why we call it the de Rham functor is because you compute it using the de Rham resolution of ω_X.

Remark 11.3. We note that this functor does not immediately make much sense: why does the image land in the constructible part of $\mathsf{D}^b(X)$? This is part of the statement of Kashiwara's constructibility theorem [HTT08, Thm. 4.6.3], but we will note this as a corollary to a fact later.

Remark 11.4. The image of $\mathsf{Mod}_{\mathsf{rh}}(D_X)$ is called the category of *perverse sheaves*. This is different from the standard definition as the objects in the heart of a t-structure on $\mathsf{D}^b_c(X)$, but the point is that DR_X preserves the t-structure. This is in the second half of [HTT08, §7.2].

There is more that can be said: six functors (but not "the" six) will commute with DR_X. We recall some notation from both sides of the Riemann–Hilbert correspondence. Let $f\colon X \to Y$ be a map of smooth algebraic varieties, and let Z be another smooth algebraic variety. Denoting by $\mathsf{D}^b_c(X)$ the category of constructible sheaves on X^{an}, we have functors

$$\begin{aligned}
\mathbf{D}_X \colon \mathsf{D}^b_{\mathsf{h}}(D_X) &\longrightarrow \mathsf{D}^b_{\mathsf{h}}(D_X)^{\mathsf{op}} & \mathbb{D}_X \colon \mathsf{D}^b_c(X) &\longrightarrow \mathsf{D}^b_c(X)^{\mathsf{op}} \\
\int_f \colon \mathsf{D}^b_{\mathsf{h}}(D_X) &\longrightarrow \mathsf{D}^b_{\mathsf{h}}(D_Y) & \mathbf{R}f_* \colon \mathsf{D}^b_c(X) &\longrightarrow \mathsf{D}^b_c(Y) \\
\int_{f!} \colon \mathsf{D}^b_{\mathsf{h}}(D_X) &\longrightarrow \mathsf{D}^b_{\mathsf{h}}(D_Y) & \mathbf{R}f_! \colon \mathsf{D}^b_c(X) &\longrightarrow \mathsf{D}^b_c(Y) \\
f^\dagger \colon \mathsf{D}^b_{\mathsf{h}}(D_Y) &\longrightarrow \mathsf{D}^b_{\mathsf{h}}(D_X) & f^! \colon \mathsf{D}^b_c(Y) &\longrightarrow \mathsf{D}^b_c(X) \\
f^\star \colon \mathsf{D}^b_{\mathsf{h}}(D_Y) &\longrightarrow \mathsf{D}^b_{\mathsf{h}}(D_X) & f^{-1} \colon \mathsf{D}^b_c(Y) &\longrightarrow \mathsf{D}^b_c(X) \\
- \boxtimes - \colon \mathsf{D}^b_{\mathsf{h}}(D_X) \times \mathsf{D}^b_{\mathsf{h}}(D_Z) &\longrightarrow \mathsf{D}^b_{\mathsf{h}}(D_{X\times Z}) & - \boxtimes - \colon \mathsf{D}^b_c(X) \times \mathsf{D}^b_c(Z) &\longrightarrow \mathsf{D}^b_c(X \times Z)
\end{aligned}$$

Theorem 11.5. DR *commutes with duals, direct image, inverse image, exceptional direct image, exceptional inverse image, and exterior products.*

We point out the constructibility statement follows from the direct image statement [Bor+87, VIII, §17].

Corollary 11.6. *If $M^\bullet \in \mathsf{D}^b_{\mathsf{rh}}(D_X)$, then $\mathrm{DR}_X M^\bullet \in \mathsf{D}^b_c(X)$.*

Proof. By induction on cohomological dimension and length of composition series, we may assume $M^\bullet \in \mathsf{Mod}_{\mathsf{rh}}(D_X)$ is simple. By induction on $\dim \operatorname{Supp} M$, we may also assume $M = \int_i L$ where $i\colon Z \hookrightarrow X$ is an affine embedding of a smooth locally closed subvariety Z of X and L is a vector bundle with regular singularities. But in this case, $\mathrm{DR}_X \int_i L \simeq \mathbf{R}i_* \mathrm{DR}_X L$ is constructible since $\mathrm{DR}_X L$ is. ☐

11.1 Proof of Riemann–Hilbert correspondence assuming Theorem 11.5

We'll take Theorem 11.5 for granted for now. We can then prove Theorem 11.2.

Proof of Theorem 11.2 (Sketch). We first show DR_X is fully faithful. In fact, we will show that we have an isomorphism of functors:

$$\mathbf{R}\mathrm{Hom}_{D_X}(M^\bullet, N^\bullet) \simeq \mathbf{R}\mathrm{Hom}_{\mathbf{C}_{X^{\mathrm{an}}}}(\mathrm{DR}_X M^\bullet, \mathrm{DR}_X N^\bullet).$$

Let $\Delta\colon X \hookrightarrow X \times X$ be the diagonal embedding, and let $p\colon X \to \mathrm{pt}$ be the projection to a point. We first have

$$\mathbf{R}\mathrm{Hom}_{D_X}(M^\bullet, N^\bullet) \simeq \int_p \Delta^\dagger(\mathbf{D}_X M^\bullet \boxtimes N^\bullet)$$

on the right hand side by Corollary 9.12, and on the left hand side, we have

$$\mathbf{R}\mathrm{Hom}_{\mathbf{C}_{X^{\mathrm{an}}}}(F^\bullet, G^\bullet) \simeq \mathbf{R}p_*\Delta^!(\mathbb{D}_X F^\bullet \boxtimes G^\bullet)$$

by applying $\mathbf{R}p_* = \mathbf{R}\Gamma$ to

$$\begin{aligned}
\Delta^!(\mathbb{D}_X F^\bullet \boxtimes G^\bullet) &\simeq \Delta^!\mathbb{D}_{X\times X}(F^\bullet \boxtimes \mathbb{D}_X G^\bullet)\\
&\simeq \mathbb{D}_X\Delta^{-1}(F^\bullet \boxtimes \mathbb{D}_X G^\bullet)\\
&\simeq \mathbb{D}_X(F^\bullet \otimes_{\mathbf{C}} \mathbb{D}_X G^\bullet)\\
&\simeq \mathbf{R}\mathscr{H}om_{\mathbf{C}}(F^\bullet \otimes_{\mathbf{C}} \mathbb{D}_X G^\bullet, \omega_X)\\
&\simeq \mathbf{R}\mathscr{H}om_{\mathbf{C}}(F^\bullet, \mathbf{R}\mathscr{H}om_{\mathbf{C}}(\mathbb{D}_X G^\bullet, \omega_X))\\
&\simeq \mathbf{R}\mathscr{H}om_{\mathbf{C}}(F^\bullet, \mathbb{D}_X^2 G^\bullet)\\
&\simeq \mathbf{R}\mathscr{H}om_{\mathbf{C}}(F^\bullet, G^\bullet)
\end{aligned}$$

The result follows by the isomorphisms

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_{\mathbf{C}_{X^{\mathrm{an}}}}(\mathrm{DR}_X M^\bullet, \mathrm{DR}_X N^\bullet) &\simeq \mathbf{R}p_*\Delta^!\big((\mathbb{D}_X \mathrm{DR}_X M^\bullet) \boxtimes \mathrm{DR}_X N^\bullet\big)\\
&\simeq \mathbf{R}p_*\Delta^!\big((\mathrm{DR}_X \mathbf{D}_X M^\bullet) \boxtimes \mathrm{DR}_X N^\bullet\big)\\
&\simeq \mathbf{R}p_*\Delta^!\big(\mathrm{DR}_{X\times X}(\mathbf{D}_X M^\bullet \boxtimes N^\bullet)\big)\\
&\simeq \mathbf{R}p_* \mathrm{DR}_X\big(\Delta^\dagger(\mathbf{D}_X M^\bullet \boxtimes N^\bullet)\big)\\
&\simeq \mathrm{DR}_{\mathrm{pt}} \int_p \Delta^\dagger(\mathbf{D}_X M^\bullet \boxtimes N^\bullet)\\
&\simeq \int_p \Delta^\dagger(\mathbf{D}_X M^\bullet \boxtimes N^\bullet)\\
&\simeq \mathbf{R}\mathrm{Hom}_{D_X}(M^\bullet, N^\bullet)
\end{aligned}$$

where in the penultimate isomorphism, we used that $\mathrm{DR}_{\mathrm{pt}} = \mathrm{id}$. Note that we did not show that DR_X actually induced this isomorphism, as pointed out in [HTT08, Rem. 7.2.3]. They refer to Saito [Sai89a, §4].

For essential surjectivity, it suffices to check that generators of $\mathsf{D}^b_c(X)$ are in the essential image of DR_X. The generators of this category are pushforwards of local systems on locally closed algebraic sets. This follows by the classical Riemann–Hilbert correspondence. □

We will now sketch some of the proof of Theorem 11.5. We won't go through all the proofs, but we would like to mention that many of them follow the same pattern as for direct images: canonical morphisms are constructed using the analytic theory of D-modules, and checking isomorphisms is reduced somehow to the classical case.

11.2 Proof of Theorem 11.5 for direct images

Step 1. There exists a canonical morphism

$$\mathrm{DR}_Y\Big(\int_f M^\bullet\Big) \longrightarrow \mathbf{R}f_* \mathrm{DR}_X(M^\bullet),$$

which is an isomorphism if f is proper.

Proof. Consider the functor

$$\int_f^{\mathrm{an}} \colon \mathsf{D}(D_X^{\mathrm{an}}) \longrightarrow \mathsf{D}(D_Y^{\mathrm{an}})$$

which is given by

$$\int_f^{\mathrm{an}} F^\bullet = \mathbf{R}f_*^{\mathrm{an}}(D_{Y\leftarrow X}^{\mathrm{an}} \otimes_{D_Y^{\mathrm{an}}} F^\bullet)$$

We claim that $\mathrm{DR} \circ \int_f^{\mathrm{an}} = \mathbf{R}f_*^{\mathrm{an}} \circ \mathrm{DR}$. Indeed,

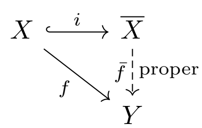
$$\begin{aligned}
\mathrm{DR}(\int_f^{\mathrm{an}} F^\bullet) &\simeq \omega_X^{\mathrm{an}} \otimes^{\mathbf{L}}_{D_Y^{\mathrm{an}}} \mathbf{R}f_*(D_{Y\leftarrow X}^{\mathrm{an}} \otimes^{\mathbf{L}}_{D_X^{\mathrm{an}}} F^\bullet) \\
&\simeq \mathbf{R}f_*(f^{-1}\omega_Y^{\mathrm{an}} \otimes^{\mathbf{L}}_{f^{-1}D_Y^{\mathrm{an}}} D_{Y\leftarrow X}^{\mathrm{an}} \otimes^{\mathbf{L}}_{D_X^{\mathrm{an}}} M^\bullet) \\
&\simeq \mathbf{R}f_*^{\mathrm{an}}(\omega_X^{\mathrm{an}} \otimes^{\mathbf{L}}_{D_X^{\mathrm{an}}} M^\bullet) \\
&\simeq \mathbf{R}f_*^{\mathrm{an}}(\mathrm{DR}_X(M^\bullet))
\end{aligned}$$

Now we use the natural morphism of functors

$$(\mathbf{R}f_*(F^\bullet))^{\mathrm{an}} \longrightarrow \mathbf{R}f_*^{\mathrm{an}}(F^{\bullet\mathrm{an}}),$$

which exists by GAGA. Note that this is not an isomorphism in general; properness is sufficient for it to be an isomorphism, however. ☐

Using the diagram



which exists by Hironaka's resolution of singularities, and since the de Rham functor commutes with pushforward via $\bar{f}$, it suffices to consider the open embedding i.

Step 2. The morphism in Step 1 is an isomorphism if $f = i\colon X \to \overline{X}$ is a smooth compactification and $M^\bullet$ is $\mathcal{O}$-coherent with regular singularities.

Proof. This is a theorem of Deligne [HTT08, Thm. 5.2.25, Prop. 5.3.6], and is part of the classical Riemann–Hilbert correspondence. ☐

Step 3. The morphism in Step 1 is an isomorphism if $f = i\colon X \to \overline{X}$ is a smooth compactification and $M^\bullet$ is regular holonomic.

Proof. By induction on cohomological dimension and length of composition series, we may assume $M^\bullet \in \mathsf{Mod}_{\mathsf{rh}}(D_X)$. By induction on $\dim \operatorname{Supp} M$, we may also assume $M = \int_j L$ where $j\colon Z \hookrightarrow X$ is an affine embedding of a smooth locally closed subvariety Z of X and L is a vector bundle with regular singularities. In this case, by using Step 2,

$$\begin{aligned}
\mathrm{DR}_Y \int_f M = \mathrm{DR}_Y \int_f \int_j L &\simeq \mathrm{DR}_Y \int_{f\circ j} L \simeq \mathbf{R}(f\circ j)_* \mathrm{DR}_Z L \\
&\simeq \mathbf{R}f_* \mathbf{R}j_* \mathrm{DR}_Z L \simeq \mathbf{R}f_* \mathrm{DR}_X \int_j L = \mathbf{R}f_* \mathrm{DR}_X M.
\end{aligned}$$ ☐

12 November 21: Hodge structures and mixed Hodge structures (Harold Blum)

12.1 Motivation

Deligne: Let X be an algebraic variety over $\mathbf{C}$. Then, $H^n(X, \mathbf{C})$ has two filtrations: the *weight filtration* W, and the *Hodge filtration* F. When X is non-singular and projective, then W is trivial, but the Hodge filtration gives the *Hodge decomposition*

$$H^i(X, \mathbf{C}) = \bigoplus_{p+q=i} H^{p,q} \quad \text{where} \quad H^{p,q} = \frac{\text{closed } (p,q) \text{ forms}}{\text{exact}}.$$

This decomposition satisfies $\overline{H^{p,q}} = H^{q,p}$, where conjugation comes from $H^n(X, \mathbf{C}) = H^n(X, \mathbf{R}) \otimes \mathbf{C}$. If we fix a polarization (X, H), we get a bilinear form

$$Q(\alpha, \beta) = (-1)^{j(j-1)/2} \int_X \alpha \wedge \beta \wedge \omega^{n-j},$$

where $\alpha, \beta \in H^j(X, \mathbf{C})$, and ω is the Kähler form coming from H. This is the usual cup product from cohomology. This bilinear form satisfies

$$Q(H^{p,q}, H^{p',q'}) = 0$$

unless $p = q'$ and $q = p'$, since $p + q = p' + q' = j$, ω is a $(1,1)$-form, and $\alpha \wedge \beta$ has to be a (j,j)-form.

12.2 Hodge structures

We next discuss abstract Hodge structures, which do not necessarily come from a variety,

Definition 12.1. A *Hodge structure (HS)* of weight n is

(1) A finitely generated abelian group $H_{\mathbf{Z}}$;

(2) A decomposition

$$H_{\mathbf{C}} \coloneqq H_{\mathbf{Z}} \otimes \mathbf{C} = \bigoplus_{p+q=n} H^{p,q}$$

such that $\overline{H^{p,q}} = H^{q,p}$.

This setup gives a filtration: if H is a Hodge structure, then we define

$$F^p H_{\mathbf{C}} \coloneqq \bigoplus_{r \geq p} H^{r,n-r}.$$

Note this is a decreasing filtration. You can then use this filtration to decompose $H_{\mathbf{C}}$:

$$H_{\mathbf{C}} = F^p H_{\mathbf{C}} \oplus \overline{F^{n-p+1} H_{\mathbf{C}}},$$

since

$$\overline{F^{n-p+1} H_{\mathbf{C}}} = \bigoplus_{r \geq n-p+1} H^{n-r,r} = \bigoplus_{r \leq p-1} H^{r,n-r}.$$

We can recover the Hodge decomposition from the filtration:

$$H^{p,q} = F^p H_{\mathbf{C}} \cap \overline{F}^q H_{\mathbf{C}},$$

where we define the notation $\overline{F}^p H_{\mathbf{C}} \coloneqq \overline{F^p H_{\mathbf{C}}}$.

Definition 12.2. A *Hodge structure (HS)* of weight n is equivalent to the following data:

(1) A finitely generated abelian group $H_{\mathbf{Z}}$;

(2) A decreasing finite filtration F^p on $H_{\mathbf{C}}$ such that $H_{\mathbf{C}} = F^p H_{\mathbf{C}} \oplus \overline{F}^{n-p+1} H_{\mathbf{C}}$ (finite here means that there exist n, m such that $F^n H_{\mathbf{C}} = 0$, $F^m H_{\mathbf{C}} = H_{\mathbf{C}}$).

To get a HS in the sense of the first definition from the second, we define

$$H^{p,q} = F^p H_{\mathbf{C}} \cap \overline{F}^q H_{\mathbf{C}}$$

when $p + q = n$. We then claim that $H^{p,q} \cap H^{p',q'} = 0$ for $p \neq p', q \neq q'$: $H^{p,q} \subseteq F^p$, and $H^{p',q'} \subseteq \overline{F}^{q'} \subseteq \overline{F}^{q+1} = \overline{F}^{n-p+1}$. Next, we need to check that $F^p H_{\mathbf{C}} = \bigoplus_{r \geq p} H^{r,n-r}$. The inclusion $\supseteq$ is obvious. In the other direction, we use the second condition to give $F^{p-1} H_{\mathbf{C}} = F^p H_{\mathbf{C}} \oplus H^{p-1,q+1}$ to get the formula we want by induction.

So we know there really is a bijection between the two objects. The reason why we need the interpretation in terms of filtrations is that these behave better under variation, and when we talk about mixed Hodge structures.

Definition 12.3. A morphism $\varphi\colon H \to H'$ of Hodge structures of weight n is a morphism of abelian groups

$$\varphi_{\mathbf{Z}}\colon H_{\mathbf{Z}} \longrightarrow H'_{\mathbf{Z}}$$

which extends to a map

$$\varphi_{\mathbf{C}}\colon H_{\mathbf{C}} \longrightarrow H'_{\mathbf{C}}$$

preserving the Hodge decomposition (so the image of $H^{p,q}$ lies in $H'^{p,q}$).

Note 12.4. If $\varphi\colon H \to H'$ is a morphism of HS's, then $\ker\varphi, \operatorname{cok}\varphi$ have HS's, induced by H and H', respectively. We can also consider

$$\operatorname{im}\varphi = \ker(H' \to \operatorname{cok}\varphi) \qquad \operatorname{coim}\varphi = \operatorname{cok}(\ker\varphi \to H).$$

We have im $\simeq$ coim.

Proposition 12.5. *Hodge structures of weight n form an abelian category.*

With respect to the filtration, we can define another notion of a morphism:

Definition 12.6. A morphism $\varphi\colon (A, F) \to (B, F)$, where A, B are R-modules, and F is a decreasing filtration, is a morphism $\varphi\colon A \to B$ such that

$$\varphi(F^n(A)) \subseteq F^n(B).$$

For such a morphism, the map $\operatorname{coim}\varphi \to \operatorname{im}\varphi$ is not always an isomorphism. To fix this problem, we define the following:

Definition 12.7. A morphism $\varphi\colon (A, F) \to (B, F)$ is *strict* if

$$\varphi(F^n(A)) = f(A) \cap F^n(B).$$

If φ is strict, then $\operatorname{coim}\varphi \xrightarrow{\sim} \operatorname{im}\varphi$.

Proposition 12.8. *If H, H' are two Hodge structures of weight n, and $\varphi_{\mathbf{Z}}\colon H_{\mathbf{Z}} \to H'_{\mathbf{Z}}$ extends to a map $\varphi_{\mathbf{C}}\colon H_{\mathbf{C}} \to H'_{\mathbf{C}}$ that is a morphism of filtered objects $(H_{\mathbf{C}}, F) \to (H'_{\mathbf{C}}, F')$, then φ gives a morphism of Hodge structures.*

Proof. Since $\varphi_{\mathbf{C}}$ respects F, F', then $\varphi_{\mathbf{C}}(F^pH_{\mathbf{C}}) \hookrightarrow \varphi_{\mathbf{C}}(F^pH'_{\mathbf{C}})$, and similarly $\varphi_{\mathbf{C}}(\overline{F}^qH_{\mathbf{C}}) \hookrightarrow \overline{F}^qH'_{\mathbf{C}}$. Thus, $\varphi_{\mathbf{C}}(H^{p,q}) = \varphi_{\mathbf{C}}(F^pH_{\mathbf{C}} \cap \overline{F}^qH_{\mathbf{C}}) \subseteq H'^{p,q}$. Note this also implies $\varphi_{\mathbf{C}}$ is strict, since

$$F^pH_{\mathbf{C}} = \bigoplus_{r \le p} H^{r,n-r} \quad \text{and} \quad F^{p'}H'_{\mathbf{C}} = \bigoplus_{r \le p} H'^{r,n-r}$$

and so $\varphi(F^pH_{\mathbf{C}}) = \varphi(H_{\mathbf{C}}) \cap \bigoplus_{r \le p} H'^{r,n-r} = \varphi(H_{\mathbf{C}}) \cap F^{p'}H'_{\mathbf{C}}$. □

We give one example:

Example 12.9 (Tate Hodge structure)**.** $H_{\mathbf{Z}} = 2\pi i\mathbf{Z} \subseteq \mathbf{C}$ and $H_{\mathbf{C}} \coloneqq H^{-1,-1}$. The weight of this Hodge structure is -2, and is denoted $\mathbf{Z}(1)$.

We also have

Definition 12.10. If H, H' are two Hodge structures, then the tensor product is defined as

$$(H \otimes H')^{p,q} = \bigoplus_{p'+p''=p} H^{p',q'} \otimes H^{p'',q''},$$

and the dual is defined as

$$H^{\vee p,q} = H^{-p,-q},$$

and with this definition (and one for hom's) we have an isomorphism

$$\operatorname{Hom}(H, H') \simeq H^{\vee} \otimes H'.$$

12.3 Mixed Hodge structures

Let $A = \mathbf{Z}, \mathbf{Q}, \mathbf{R}$ (in principle, this should be able to be any subring of $\mathbf{R}$). Then, we have

$$A \otimes \mathbf{Q} \simeq \begin{cases} \mathbf{Q} & \text{if } A = \mathbf{Z}, \mathbf{Q} \\ \mathbf{R} & \text{if } A = \mathbf{R} \end{cases}$$

Definition 12.11. An *A-mixed Hodge structure (MHS)* consists of the following data:

1. An A-module H_A of finite type,
2. (Weight filtration) A finite increasing filtration W on $H_{A\otimes\mathbf{Q}}$, and
3. (Hodge filtration) A finite decreasing filtration F on $H_{\mathbf{C}} \coloneqq H_{A\otimes\mathbf{Q}} \otimes \mathbf{C}$,

such that letting

$$\operatorname{Gr}_n^W H = W_n H_{A\otimes\mathbf{Q}} / W_{n-1} H_{A\otimes\mathbf{Q}}$$

and

$$\operatorname{Gr}_n^W H_{\mathbf{C}} \coloneqq \operatorname{Gr}_n^W H \otimes \mathbf{C},$$

then

$$F^p \operatorname{Gr}_n^W H_{\mathbf{C}} = (F^p \cap W_n + W_{n-1})/W_{n-1}$$

gives a Hodge structure of weight n.

The idea is that F induces Hodge decompositions on the associated graded pieces of $H_{A\otimes\mathbf{Q}}$ relative to the weight filtration. One way to get these is to take the direct sum of Hodge structures of different weights, and then put an appropriate weight filtration on it.

Definition 12.12. A morphism $f\colon H \to H'$ of mixed Hodge structures is a morphism $f_A\colon H_A \to H'_A$ that respects the two filtrations F, W on $H_{\mathbf{C}}$ and $H_{A\otimes\mathbf{Q}}$, respectively.

Note in particular that we do not demand strictness.

Theorem 12.13 (Deligne)**.** *The category of mixed Hodge structures is abelian.*

An important component of this is the following:

Proposition 12.14. *If $f\colon H \to H'$ is a morphism of MHS's, then f is strict with respect to both F, W.*

The idea is the following. we want a Hodge decomposition on $H_{\mathbf{C}}$ for a MHS H, and so we look at

$$0 \longrightarrow \operatorname{Gr}_{n-1}^W \longrightarrow W_n/W_{n-2} \longrightarrow \operatorname{Gr}_n^W \longrightarrow 0.$$

What we would like to say is that Hodge decompositions on either side induce one on the middle term. However, this sequence does not split with respect to mixed Hodge structures. Instead, we set

$$I^{p,q} \coloneqq (F^p \cap W_{p+q}) \cap (\overline{F}^q \cap W_{p+q} + \overline{F}^{q-1} \cap W_{p+q-2} + \overline{F}^{q-2} \cap W_{p+q-3} + \cdots)$$

in $H_{\mathbf{C}}$.

Proposition 12.15. *For $p + q = n$, the map*

$$\begin{array}{ccc} \varphi\colon W_n & \longrightarrow & \operatorname{Gr}_n^W \\ \uparrow & & \uparrow \\ I^{p,q} & \overset{\sim}{\dashrightarrow} & H^{p,q} \end{array}$$

gives an isomorphism of vector spaces, and

$$W_n = \bigoplus_{p+q\le n} I^{p,q}, \qquad F^p = \bigoplus_{p'\ge p} I^{p',q'}.$$

Remark 12.16.

- $I^{p,q} = \overline{I^{p,q}} \mod W_{p+q-2}$.
- If $\varphi\colon H \to H'$ is a morphism of mixed Hodge structures, then it sends $I^{p,q}$ to $I'^{p,q}$.

Note 12.17. If $\varphi\colon H \to H'$ is a morphism of MHS's, then it is strict with respect to F, W (by using the decompositions in the Proposition), and

$$\operatorname{coim}\varphi \xrightarrow{\sim} \operatorname{im}\varphi.$$

We should also note that $\ker\varphi$ is a MHS with filtrations induced from H.

Next time we should talk about polarizations; in this case the category becomes semisimple.

13 November 28 (Harold Blum)

We'll talk about polarized Hodge structures, and try to give examples of them and examples of Hodge structures in general.

13.1 Polarized Hodge structures

Recall that if X is compact Kähler, then we have a product

$$Q(\alpha,\beta) = (-1)^{j(j-1)/2}\int_X \alpha\wedge\beta\wedge\omega^{n-j},$$

where $[\alpha],[\beta] \in H^j(X,\mathbf{C})$. Q is a real, $(-1)^j$-symmetric, non-degenerate pairing. Moreover, $Q(H^{p,q}, H^{p',q'}) = 0$ unless $p = q'$ and $q = p'$, since you need to have an (n,n)-form in the integrand. The reason for the powers of -1 in front is that people often look at another pairing

$$h(u,v) \coloneqq Q(Cu,\overline{v}),$$

where C is the Weil operator $C|_{H^{p,q}} = i^{p-q}$. This is a hermitian form, and is positive definite.

We will now generalize this in terms of Hodge structures. Let $A = \mathbf{Q}, \mathbf{Z}, \mathbf{R}$.

Definition 13.1. A *polarization* of an A-Hodge structure $H = (H_A, H_{\mathbf{C}} = \bigoplus_{p,q=n} H^{p,q})$ of weight n is a bilinear form

$$Q\colon H_A \otimes H_A \to A$$

that is $(-1)^n$-symmetric, $H^{p,q}$ is orthogonal to $H^{p',q'}$ for $p \neq q'$ (or $q \neq p'$, but these conditions are equivalent), and the form

$$h(u,v) \coloneqq Q(Cu,\overline{v})$$

is positive definite and hermitian, and C is the Weil operator as before.

We rewrite this in a different way, using the following two constructions.

Example 13.2 (Tate Hodge structure)**.** Let $A(m)$ be our Tate Hodge structure of weight $-2m$. Define

$$A(m)_A = (2\pi i)^m \cdot A \subseteq \mathbf{C} \qquad A(m)_{\mathbf{C}} = H^{-m,-m}.$$

Example 13.3 (Tensor products)**.** Let H, H' be Hodge structures of weights n, n'. We define $H \otimes H'$, where for each $a + b = n + n'$, we have

$$(H\otimes H')^{a,b}_{\mathbf{C}} = \bigoplus_{\substack{p+p'=a\\ q+q'=b}} H^{p,q}\otimes H^{p',q'}.$$

Definition 13.4. A *polarization* of an A-Hodge structure H of weight n is a morphism of Hodge structures

$$Q\colon H\otimes H \longrightarrow A(-n)$$

which is $(-1)^n$-symmetric, such that

$$h(u,v) = (2\pi i)^n Q(Cu,\overline{v})$$

is hermitian positive definite.

Note that the $(-1)^n$-symmetry is what is necessary to make $h(u,v)$ hermitian.

Example 13.5. We will consider a polarized $\mathbf{Q}$-Hodge structure of dimension 2, and weight 1. We know $H_{\mathbf{Q}} \simeq \mathbf{Q}^2$, and asking for a skew symmetric form Q is the same as choosing a basis (e_1, e_2) for $\mathbf{Q}^2$ such that

$$Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We now examine $H_{\mathbf{C}} = H^{1,0} \oplus H^{0,1}$. Let's say $H^{1,0} = \mathbf{C} \cdot v$, where $v = (v_1, v_2)$ in real coordinates. We can choose v uniquely by dividing by v_2, so that $v = (\tau, 1)$. Then,

$$h(v,v) = iQ(v,\overline{v}) = i(\tau \cdot \overline{1} - \overline{\tau} \cdot 1) = i^2 2 \operatorname{Im}(\tau),$$

which implies that $\tau \in$ lower half plane. A different decomposition for $H_{\mathbf{C}}$ will just introduce a sign change in h.

We now move on to talk about a nice property of the category of polarized Hodge structures: the category is semisimple. In the following, we assume $A = \mathbf{Q}, \mathbf{R}$.

Proposition 13.6. *Let (H,Q) be a polarized A-Hodge structure of weight n and let $W \hookrightarrow H$ be a sub-A-Hodge structure. Then,*
(1) *Q restricts to a polarization of W;*
(2) *$W^\perp$ (with respect to h) inherits a polarized Hodge structure of weight n;*
(3) *$H = W \oplus W^\perp$ as Hodge structures.*

Proof. (1) is clear. For (2), note that

$$(W^\perp)_A = (W_A)^\perp \qquad (W^\perp)_{\mathbf{C}} = (W_{\mathbf{C}})^\perp,$$

and that $u \in (W^\perp)^{p,q}_{\mathbf{C}} \coloneqq (W^\perp_{\mathbf{C}}) \cap H^{p,q}$ if and only if $\overline{u} \in (W^\perp)^{p,q}_{\mathbf{C}}$. Finally, (3) follows by general theory about hermitian forms on vector spaces. ☐

Corollary 13.7. *The category of polarized A-Hodge structures of weight n is semisimple.*

Note that part of being semisimple is being abelian, which we checked last time for non-polarized Hodge structures; the only thing to check is that the image and coimage are the same.

13.2 Examples of the weight filtration on varieties

We now give some examples of Hodge structures on cohomology groups of varieties.

Let X be an algebraic variety over $\mathbf{C}$, and suppose $H^*(X, \mathbf{C})$ has an increasing weight filtration W. Then,
- When X is compact, $H^k(X)$ has weights $\{0, \ldots, k\}$ (i.e., $W_k = H^k(X)$ and $W_{-1} = 0$);
- When X is smooth, $H^k(X)$ has weights $\{k, \ldots, 2k\}$ (i.e., $W_{k-1} = 0$ and $W_{2k} = H^k(X)$).

Thus, when X is smooth and compact, then

$$H^k(X) \simeq \operatorname{Gr}^W_k H^k(X).$$

Example 13.8. Suppose X is a compact curve of geometric genus 1 (that is, the normalization has genus 1) with 1 node (topologically, it is a torus with two points identified). Then, $H^1(X)$ has three generators: two around the non-singular hole, and one more that passes through the node. Then, $H^1(X) = \mathbf{Q}^{\oplus 3}$, and

$$\operatorname{Gr}_j H^1(X) = \begin{cases} \mathbf{Q} & j = 0 \\ \mathbf{Q}^{\oplus 2} & j = 1 \end{cases}$$

One way to see this is as follows. Suppose $\pi \colon \widetilde{X} \to X$ is a proper birational morphism, where $\widetilde{X}$ is smooth and $\operatorname{Exc}(\pi)$ is snc. Then, we have the following:

Proposition 13.9. *We have a long exact sequence*

$$\cdots \longrightarrow H^k(X) \longrightarrow H^k(\widetilde{X}) \oplus H^k(V) \longrightarrow H^k(E) \longrightarrow H^{k+1}(X) \longrightarrow \cdots$$

where V is the discriminant, and E is the exceptional divisor. This short exact sequence is strict with respect to the weight filtration (that is, applying $\operatorname{Gr}_i^W(-)$ still gives an exact sequence).

In our example, we have

$$0 \longrightarrow \operatorname{Gr}_1^W H^1(X) \longrightarrow \operatorname{Gr}_1^W H^1(\widetilde{X}) \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Gr}_0^W H^0(X) \longrightarrow \operatorname{Gr}_0^W H^0(\widetilde{X}) \oplus \operatorname{Gr}_0^W H^0(V) \longrightarrow \operatorname{Gr}_0 H^0(E) \longrightarrow \operatorname{Gr}_0 H^1(X) \longrightarrow 0$$

The same argument will show that $\operatorname{Gr}_1 H^1(X)$ is the same as for the normalization, and $\operatorname{Gr}_0 H^1(X)$ will get contributions from the singularities.

13.3 Hodge structure on cohomology

We will now talk briefly about where the Hodge filtration comes from.

Let X be a smooth, compact variety over $\mathbf{C}$. Then, we have

$$H^i(X, \mathbf{C}) \simeq \mathbf{H}^i(X, \Omega_X^\bullet).$$

Theorem 13.10. *There is a natural filtration on the de Rham complex*

$$F^p\Omega_X^\bullet = \Omega_X^{\bullet \geq p} = \{0 \to \cdots \to 0 \to \Omega_X^p \to \Omega_X^{p+1} \to \cdots \to \Omega_X^n\}.$$

This gives rise to a Hodge structure on $H^i(X, \mathbf{C})$, where

$$F^p H^i(X, \mathbf{C}) = \operatorname{im}\bigl(\mathbf{H}^i(X, F^p\Omega_X^\bullet) \to \mathbf{H}^i(X, \Omega_X^\bullet)\bigr).$$

So now suppose that we have a variety that is not compact, i.e., U is a smooth complex variety with a compactification $U \hookrightarrow X$ such that $D \coloneqq X \smallsetminus U$ is a simple normal crossings divisor. In this case,

Theorem 13.11. *$H^k(U, \mathbf{C}) = \mathbf{H}^k\bigl(X, \Omega_X^\bullet(\log D)\bigr)$, and we get filtrations F, W on $H^k(U, \mathbf{C})$ arising from filtrations on $\Omega^\bullet(\log D)$:*

$F^p\Omega_X^\bullet(\log D) =$ *"stupid filtration" (truncation) from before*

$$W_m\Omega_X^\bullet(\log D) = \{\Omega_X^0(\log D) \to \cdots \to \Omega_X^m(\log D) \to \Omega_X^1 \wedge \Omega_X^m(\log D) \to \cdots \to \Omega_X^{n-m} \wedge \Omega_X^m(\log D)\}$$

So the forms that appear have poles of order $\leq m$ along D. These give filtrations on the hypercohomology of the de Rham complex.

Example 13.12. Let C be a compact curve, and consider $S = \{y_1, \ldots, y_m\} \subseteq C$, and let $U \coloneqq C \smallsetminus S$. Then, $\operatorname{Gr}_1^W H^1(U) \simeq H^1(C)$, and $\operatorname{Gr}_2^W H^1(U) \simeq \mathbf{C}^{m-1}$. This comes from the long exact sequence on compactly supported cohomology, and then by using Poincaré duality. This is similar to the story of the Hodge–Deligne polynomial, which is a polynomial $E(X; u, v) \in \mathbf{Z}[u, v]$ associated to a variety X such that

- If X is smooth projective, then

$$E(X) = \sum_{p,q \geq 0} (-1)^{p+1} h^{p,q}(X) u^p v^q,$$

- If $Z \hookrightarrow X \hookleftarrow U \coloneqq X \smallsetminus Z$ implies $E(X) = E(Z) + E(U)$.

Then, $E(X) \coloneqq \sum_{p,q \geq 0} \sum_{n \geq 0} (-1)^n h^{p,q}(H_c^n(X, \mathbf{Q})) u^p v^q$. But given $Z \hookrightarrow X \hookleftarrow U$, we have a long exact sequence

$$\cdots \longrightarrow \operatorname{Gr}_k H_c^n(U, \mathbf{Q}) \longrightarrow \operatorname{Gr}_k H_c^n(X, \mathbf{Q}) \longrightarrow \operatorname{Gr}_k H_c^n(Z, \mathbf{Q}) \longrightarrow \cdots$$

in which case additivity will follow.

14 December 12: Variations of Hodge structure (Takumi Murayama)

Harold discussed different kinds of Hodge structures over the last two meetings. Our goal today is to talk about how variations of Hodge structure, which originate geometrically from asking how Hodge structure vary in families. We will later present the abstract framework for a variation of Hodge structure. We mainly follow [Lit13].

14.1 Geometric variations of Hodge structure

Definition 14.1. A *family* is a smooth proper morphism $f\colon X \to S$, where X, S are both complex manifolds.

In this setting, we have the following classical result:

Theorem 14.2 (Ehresmann)**.** *The fibers X_s of a family are all diffeomorphic to each other. In particular, the Betti numbers $b^k(X_s) = h^k(X_s, \mathbf{C})$ are constant.*

In this way, you can view a family of complex manifolds as the data of a smooth manifold, plus a complex structure that varies as a function of $s \in S$.

Example 14.3. Consider the family of elliptic curves

$$\begin{array}{c} X \coloneqq \overline{\{y^2 = x(x-1)(x-\lambda)\}} \subset \mathbf{P}^2_S \\ \big\downarrow f \\ \lambda \in S \coloneqq \mathbf{C} \smallsetminus \{0,1\} \end{array}$$

Each fiber is topologically a torus. However, the complex structures differ: the map given by projection to x on a fiber X_λ is the unique 2-to-1 map of the elliptic curve to $\mathbf{P}^1$. The ramification points are $0, 1, \lambda, \infty$, which uniquely determines the analytic isomorphism class of X_λ.

We will mostly be interested when each fiber X_s is projective (in fact, compact Kähler would suffice), in which case classical Hodge theory tells us that there is a Hodge decomposition

$$H^k(X_s, \mathbf{C}) = \bigoplus_{p+q=k} H^{p,q}(X_s) \cong \bigoplus_{p+q=k} H^q(X_s, \Omega^p_{X_s}),$$

and when B is algebraic. We first note the following easy consequence of Ehresmann's theorem:

Proposition 14.4. *The Hodge numbers $h^{p,q}(X_s)$ are locally constant.*

Proof. By upper semi-continuity, the Hodge numbers $h^{p,q}(X_s) = h^q(X_s, \Omega^p_{X_s})$ are upper semi-continuous in t. But the Betti numbers are constant by Ehresmann's theorem, and

$$b^k(X_s) = \sum_{p+q=k} h^{p,q}(X_s).$$

Thus, Hodge numbers cannot jump up, because this would case the Betti number to also jump up. □

Example 14.5. The Hodge numbers for our example with elliptic curves are easy:

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$$

For a family of genus g curves, you have

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$

Variations of Hodge structure are supposed to be a relative version of Hodge structures, and so we would expect to see some decomposition theorem involving higher pushfowards. For this, we use the following:

Theorem 14.6 (Topological proper base change [Stacks, Tag 09V6])**.** *Suppose $f\colon X \to S$ is a continuous map of spaces which is universally closed and separated. Let $\mathscr{F}$ be a sheaf of abelian groups on X (or an object in $\mathsf{D}^+(X)$). Consider the cartesian diagram*

$$\begin{array}{ccc} X \times_S T & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & S \end{array}$$

Then, the canonical map

$$g^* R^i f_* \mathscr{F} \longrightarrow R^i f'_* g'^* \mathscr{F}$$

is an isomorphism. In particular, if $s \in S$ is a point, then

$$(R^i f_* \mathscr{F})_s \simeq H^i(X_s, \mathscr{F}|_{X_s}).$$

Example 14.7. Consider the case $\mathscr{F} = \mathbf{C}_X$, the constant sheaf, and $f\colon X \to S$ is a family. Then, the special case of the base change theorem says

$$(R^i f_* \mathbf{C}_X)_s \simeq H^i(X_s, \mathbf{C}).$$

The Proposition then says that $R^i f_* \mathbf{C}_X$ is a vector bundle on S, which is locally U times the vector space $H^i(X_s, \mathbf{C})$ for any $s \in U$.

Now that we have captured cohomology of a family into a single object, we want to find a version of a Hodge decomposition on $R^i f_* \mathbf{C}_X$. We could hope to exploit the Hodge decomposition in the classical setting, and extend this to a global splitting of the vector bundles $R^i f_* \mathbf{C}_X$, but this is too much to ask for, since the splittings are not holomorphic.

Example 14.8. In our family of elliptic curves, note $H^{1,0}$ varies holomorphically inside of $R^1 f_* \mathbf{C}_X$, since

$$H^0(X_s, \Omega^1_{X_s}) = \mathbf{C} \cdot \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} = \mathbf{C} \cdot \frac{dx}{y}.$$

On the other hand, since $\overline{H^{1,0}} = H^{0,1}$, we would expect $H^{0,1}$ to vary *anti*-holomorphically inside of $R^1 f_* \mathbf{C}_X$.

Instead, we recall the last description of Hodge structures given by Harold. There is a natural filtration on the de Rham complex

$$F^p \Omega^\bullet_X = \Omega^{\bullet \geq p}_X = \{0 \to \cdots \to 0 \to \Omega^p_X \to \Omega^{p+1}_X \to \cdots \to \Omega^n_X\},$$

that gives rise to a Hodge structure on $H^i(X, \mathbf{C})$, where

$$F^p H^i(X, \mathbf{C}) = \operatorname{im}\bigl(\mathbf{H}^i(X, F^p \Omega^\bullet_X) \to \mathbf{H}^i(X, \Omega^\bullet_X)\bigr).$$

The filtration exists also in the relative setting, i.e.,

$$F^p \Omega^\bullet_{X/S} = \Omega^{\bullet \geq p}_{X/S} = \{0 \to \cdots \to 0 \to \Omega^p_{X/S} \to \Omega^{p+1}_{X/S} \to \cdots \to \Omega^n_{X/S}\},$$

and what we want to do is to give a relative version of the Hodge structure.

Theorem 14.9 (Holomorphic Poincaré Lemma)**.** *Let X be a complex manifold. The canonical map*

$$\mathbf{H}^i(\Omega^\bullet_X) \longrightarrow H^*_{\mathrm{dR}}(X) \qquad (\simeq H^*(X, \mathbf{C}))$$

is an isomorphism.

Since everything is natural, combining this with the proper base change theorem implies the following:

Corollary 14.10. *Let $f\colon X \to S$ be a smooth proper map of complex manifolds. There is a canonical isomorphism*

$$R^i f_* \Omega_f^\bullet \xrightarrow{\sim} R^i f_* \mathbf{C}_X \otimes_{\mathbf{C}} \mathcal{O}_S \qquad (\simeq R^i f_* \mathbf{Z}_X \otimes_{\mathbf{Z}} \mathcal{O}_S).$$

In particular, the sheaves on the left-hand side are vector bundles.

We then get something like a Hodge structure on $R^i f_* \Omega_f^\bullet$. Note that by using the language of filtrations, we do end up getting a Hodge decomposition-like result.

Theorem 14.11. *Let $f\colon X \to S$ be as before, but let X be projective (or just compact Kähler, so there is a Hodge structure on cohomology). Then,*

- *(i)* $R^i f_* \Omega_{X/S}^{\bullet \geq p}$ *is a holomorphic vector bundle for all i and p.*
- *(ii)* $R^i f_* \Omega_{X/S}^{\bullet \geq p} \hookrightarrow R^i f_* \Omega_{X/S}^{\bullet \geq p-1}$*, and the cokernel is a vector bundle.*
- *(iii)* *We have a decreasing filtration*

$$F^p H^i(X, \mathbf{C}) = \operatorname{im}\bigl(R^i f_* \Omega_{X/S}^{\bullet \geq p} \to R^i f_* \Omega_{X/S}^\bullet\bigr)$$

such that

$$R^i f_* \Omega_{X/S}^\bullet = F^p R^i f_* \Omega_{X/S}^\bullet \oplus \overline{F^{i-p+1} R^i f_* \Omega_{X/S}^\bullet}$$

Proof. We first claim that $R^i f_* \Omega_{X/S}^k$ is a vector bundle for each i, k. But this follows from the constancy of Hodge numbers plus the base change theorem, and then by using (the analytic version of) Grauert's direct image theorem.

Now for (i), we use descending induction on p. If $p = n$, this is a special case of the above. If $p < n$, consider the short exact sequence

$$0 \longrightarrow \Omega_{X/S}^{\bullet \geq p} \longrightarrow \Omega_{X/S}^{\bullet \geq p+1} \longrightarrow \Omega_{X/S}^p \longrightarrow 0,$$

which gives a long exact sequence of higher direct images. The boundary morphisms

$$R^i f_* \Omega_{X/S}^p \longrightarrow R^{i+1} f_* \Omega_{X/S}^{\bullet \geq p}$$

are all zero by using base change and the degeneration of the Frölicher (Hodge-to-de Rham) spectral sequence, and so we have short exact sequences

$$0 \longrightarrow R^i f_* \Omega_{X/S}^{\bullet \geq p} \longrightarrow R^i f_* \Omega_{X/S}^{\bullet \geq p+1} \longrightarrow R^i f_* \Omega_{X/S}^p \longrightarrow 0.$$

This shows $R^i f_* \Omega_{X/S}^{\bullet \geq p+1}$ is the extension of vector bundles, hence is a vector bundle itself.

For (ii), the short exact sequence above has the required injective map, and the cokernel is indeed a vector bundle.

For (iii), the first part is just a reformulation of (ii), and the second part follows from the analogous decomposition on fibers. □

We note that the exact relationship between the integral structure on cohomology and the relative Hodge filtration we constructed above is subtle:

Example 14.12. Let $C \to S$ be a family of elliptic curves (like the one from before), and consider the family $X = C \times_S C \to S$ of products of elliptic curves. Then, the Picard rank

$$\rho(X_s) = \operatorname{rank}\bigl(H^2(X_s, \mathbf{Z}) \cap H^{1,1}(X_s)\bigr)$$

is not constant:

$$\rho(X_s) = \begin{cases} 3 & X_s \text{ does not have complex multiplication} \\ 4 & X_s \text{ has complex multiplication} \end{cases}$$

Note that the set of points where $\rho(X_s) = 4$ is countable, but is dense in both the Zariski and analytic topologies: they correspond to quotients $\mathbf{C}/\langle 1, \tau \rangle$ where τ is a point in the upper-half plane which is an imaginary quadratic number.

14.1.1 Griffiths transversality

Variations of Hodge structure satisfy one more property which was not part of our definition of a Hodge structure before. This property was proved by Griffiths, and basically says that the filtration in the previous Theorem plays well with the D-module structure on $R^i f_* \Omega^\bullet_{X/S}$.

First, note that

$$\mathbf{R}f_*\Omega^\bullet_{X/S} \simeq \mathbf{R}f_*(\mathrm{DR}^\bullet_f(\mathcal{O}_X))[\dim S - \dim X] \simeq \int_{X/S} \mathcal{O}_X[\dim S - \dim X]$$

as complexes of $\mathcal{O}_S$-modules. Thus, we have that

$$R^i f_* \Omega^\bullet_{X/S} \simeq \int_f^i \mathcal{O}_X[\dim S - \dim X]$$

is a vector bundle that is also a D_S-module, that is, it has an integrable connection, which we recall induces the de Rham complex below, which is a resolution of $\ker \nabla$:

$$0 \longrightarrow R^i f_* \Omega^\bullet_{X/S} \xrightarrow{\nabla} R^i f_* \Omega^\bullet_{X/S} \otimes_{\mathcal{O}_S} \Omega^1_S \xrightarrow{\nabla} R^i f_* \Omega^\bullet_{X/S} \otimes_{\mathcal{O}_S} \Omega^2_S \xrightarrow{\nabla} \cdots$$

This is called the *Gauss–Manin connection*. We can now state Griffiths' theorem.

Theorem 14.13 (Griffiths transversality)**.** *Let ∇ be the Gauss–Manin connection on $R^i f_* \Omega^\bullet_{X/S}$. Then,*

$$\nabla(F^p R^i f_* \Omega^\bullet_{X/S}) \subseteq (F^{p-1} R^i f_* \Omega^\bullet_{X/S}) \otimes \Omega^1_S$$

I think you can prove this in the following manner, using the D-module theory we have built up so far.

Proof Sketch. By the dual description of integrable connection, this is equivalent to saying

$$\nabla_\eta(F^p R^i f_* \Omega^\bullet_{X/S}) \subseteq (F^{p-1} R^i f_* \Omega^\bullet_{X/S})$$

for each operator $\nabla_\eta \in F_1 D_S = \Theta_S$. But it turns out that setting

$$F_{-p} R^i f_* \Omega^\bullet_{X/S} \coloneqq F^p R^i f_* \Omega^\bullet_{X/S},$$

this defines a filtration on $R^i f_* \Omega^\bullet_{X/S}$ by working through the definition of a D-module (or alternatively, building filtrations into our definitions of functors), which implies what we wanted. ☐

From what I can gather, this seems to be the content of [Sai88, Thm. 5.4.3].

14.2 Abstract variations of Hodge structure

We now define an abstract variation of Hodge structure, which asks for a local system to satisfy the same properties as the local system $R^i f_* \mathbf{Z}_X$.

Definition 14.14. An integral *variation of Hodge structure* of weight n on S is a $\mathbf{Z}$-local system $V_\mathbf{Z}$ on S, together with a decreasing filtration

$$F^p V_S \subseteq V_S \coloneqq V_\mathbf{Z} \otimes \mathcal{O}_S$$

by holomorphic vector bundles, satisfying the following properties:

(1) F^p induces a Hodge structure on each fiber of V_S, that is,

$$V_s = F^p V_s \oplus \overline{F^{n-p+1} V_s}.$$

(2) [Griffiths transversality] Denote $V_\mathbf{C} \coloneqq V_\mathbf{Z} \otimes \mathbf{C}$, which gives the vector bundle V_S an integrable connection ∇ by the (classical) Riemann–Hilbert correspondence. Then,

$$\nabla(F^p V_S) \subseteq F^{p-1} V_S \otimes \Omega^1_S.$$

Similarly, we can define rational or real variations of Hodge structure.

The polarized version is as follows:

Definition 14.15. An integral *polarized variation of Hodge structure* of weight n on S is a variation of Hodge structure along with a bilinear form

$$Q\colon V_{\mathbf{Z}} \otimes V_{\mathbf{Z}} \longrightarrow \mathbf{Z}$$

such that the restriction to each fiber is a polarized Hodge structure.

We also state the mixed version:

Definition 14.16. An integral *variation of mixed Hodge structure* is a $\mathbf{Z}$-local system $V_{\mathbf{Z}}$ on S, together with a decreasing filtration

$$F^p V_S \subseteq V_S \coloneqq V_{\mathbf{Z}} \otimes \mathcal{O}_S$$

by holomorphic vector bundles, and an increasing filtration

$$W_p V_{\mathbf{Q}} \coloneqq V_{\mathbf{Z}} \otimes \mathbf{Q}$$

by sublocal systems of rational vector spaces, such that

(1) F^p, W_p induce a mixed Hodge structure on each fiber of V_S.

(2) [Griffiths transversality]

Much like the smooth case, the main example is for a morphism of complex analytic varieties.

Example 14.17. $R^i f_* \mathbf{Z}_X$ is a constructible sheaf, and on each stratum you have a local system, for which filtrations can be defined locally.

One thing to mention is that there is no notion of a variation of Hodge structure on the entire constructible sheaf; I believe this is what Saito's theory of mixed Hodge module is needed for.

14.3 One application: Hodge loci

We end with an application of variations of Hodge structure from [Voi03, §5.3.1].

Let $U \subset S$ be a connected open set and let λ be a section of $V_{\mathbf{Z}}$ on U. Let $\lambda_u \in V_{\mathbf{Z},u}$ denote the value of λ at u, which can then be viewed as a holomorphic section of V_S via the inclusion $V_{\mathbf{Z}} \subset V_S$.

Definition 14.18. The *Hodge locus* U_λ^p defined by λ is the set

$$U_\lambda^p \coloneqq \{u \in U \mid \lambda_u \in F^p V_{S,u}\}.$$

Example 14.19. Let $n = 2p$ and let $\lambda \in R^i f_* \mathbf{Z}_X$. Then, the Hodge locus U_λ^p is the set of points u where the class λ is a Hodge class, that is, an element of $H^{p,p}(X_u) \cap H^p(X_u, \mathbf{Z})$. The Hodge conjecture states that the Hodge locus is the set of points for which a multiple of λ is the class of an algebraic cycle. So studying Hodge loci is a first step in attacking the Hodge conjecture.

Lemma 14.20. *Every U_λ^p is a complex analytic subset of U, with a natural scheme structure.*

Here by "scheme structure" we just mean non-reduced structure as a complex analytic space.

Proof. U_λ^p is the zero locus of $\overline{\lambda} \in V_S/F^p V_S$, which is holomorphic. □

Remark 14.21. Cattani, Deligne, and Kaplan showed that if $V_{\mathbf{Z}} = R^i f_* \mathbf{Z}_X$, then the Hodge loci are *algebraic*. This is considered evidence in favor of the Hodge conjecture.

Note that the proof of the Lemma above implies that U_λ^p can be defined locally by $N = \operatorname{rank} V_s/F^p V_S$ equations. Griffiths transversality can be used to show the following:

Proposition 14.22. *The analytic set U_λ^p can be defined locally by*

$$h^{p-1} \coloneqq \dim(F^{p-1}V_S/F^pV_S)$$

equations.

Proof. Let $u \in U_\lambda^p$. In a neighborhood of u, choose a decomposition

$$V_S/F^pV_S = F^{p-1}V_S/F^pV_S \oplus F$$

as holomorphic vector bundles, so that

$$\overline{\lambda} = \overline{\lambda}_{p-1} + \overline{\lambda}_F.$$

Then, $U_\lambda^p \subseteq V_\lambda^p \subseteq U$ where V_λ^p is defined by $\overline{\lambda}_{p-1}$. We want to show $U_\lambda^p = V_\lambda^p$.

Assume for simplicity that V_λ^p is smooth, and replace U by V_λ^p. It suffices to show that if $U_\ell = \operatorname{Spec}(\mathcal{O}_{U,u}/\mathfrak{m}_u^{\ell+1})$ is the ℓth infinitesimal neighborhood of U, then $U_\lambda^p \cap U_\ell = U_\ell$. We prove this by induction on ℓ.

The base case $\ell = -1$ is trivial. For the inductive case, suppose by inductive hypothesis that $U_\lambda^p \cap U_\ell = U_\ell$, so that the section λ of V_S lies in F^pV_S up to order ℓ, i.e.,

$$\lambda = \mu + \sum_i \alpha_i\sigma_i + \sum_j \beta_j\tau_j$$

for some $\mu \in F^pV_S$, $\alpha_i, \beta_j \in \mathfrak{m}_u^{\ell+1}$, and $\sigma_i \in F^{p-1}V_S$, $\tau_j \in F$. Since $\overline{\lambda}_{p-1} = 0$ on U, we see that $\lambda - \mu$ maps to zero in $(\mathfrak{m}_u^{\ell+1}/\mathfrak{m}_u^{\ell+2})F^{p-1}V_S/F^pV_S$, and so after modifying μ, we may assume that $\alpha_i = 0 \bmod \mathfrak{m}_u^{\ell+2}$.

Now applying ∇ to this equation, we obtain

$$-\nabla\mu = \sum_i d\alpha_i \otimes \sigma_i + \sum_j d\beta_j \otimes \tau_j \in F^{p-1}V_S \otimes \Omega_U^1$$

since λ is flat, and by transversality. Looking at this modulo F^pV_S, we see that we may assume $\beta_j = 0$. Looking at this modulo $\mathfrak{m}_u^{\ell+1}$, we have that $d\alpha_i = 0$, and so $\alpha_i \in \mathfrak{m}_u^{\ell+2}$. Thus,

$$\lambda \in F^pV_S \mod \mathfrak{m}_u^{\ell+2}V_S,$$

i.e., $U_\lambda^p \cap U_{\ell+1} = U_{\ell+1}$. □

15 January 23: *V*-filtrations and vanishing cycles (Harold Blum)

Vanishing and nearby cycles are things from topology, which can be studied in terms of perverse sheaves. V-filtrations are the analogue on the D-module side.

15.1 Example

We start with a simple example:

Example 15.1. Let X be a family of elliptic curves $\{y^2 = x(x-1)(x-t)\} \subseteq \mathbf{P}^2 \times \Delta$ over the open unit disc Δ that degenerate to a nodal curve at $t = 0$. The first homology is $H_1(X_t, \mathbf{Z}) = \mathbf{Z} \cdot \langle \alpha_t, \beta_t \rangle$. We want to study what happens when $t \to 0$. Heuristically, it looks like $\alpha_t \mapsto 0$ (a "vanishing cycle"), and $\beta_t \mapsto \beta_0$ (a "nearby cycle"). Now let $T \in \pi_1(\Delta \smallsetminus \{0\}, t)$. Then, we can think about what the action of T is on the generator of homology:

$$T(\alpha_t) = \alpha_t \qquad T(\beta_t) = \beta_t \pm \delta\alpha_t.$$

The key word here is "Picard–Lefschetz theory."

15.2 Vanishing cycles

Let X be a complex manifold, and let $f\colon X \to \Delta$ be a map that is submersive away from 0. Then, let $e\colon \mathbf{H} \to \Delta \smallsetminus \{0\}$ be the map $z \mapsto e^{2\pi i z}$, that is, the universal cover of $\Delta \smallsetminus \{0\}$. We can then construct the following diagram:

$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{k} & X & \xleftarrow{i} & X_0 \\ \big\downarrow & \lrcorner & \big\downarrow f & & \big\downarrow \\ \mathbf{H} & \xrightarrow{e} & \Delta & \longleftarrow & \{0\} \end{array}$$

Now we can construct vanishing and nearby cycles using the language of constructible sheaves.

Let K be a complex of constructible sheaves on X (that is, a complex of $\mathbf{C}$-vector spaces with constructible cohomology).

Definition 15.2 (*Nearby cycles*)**.** $\psi_f K = i^{-1}\mathbf{R}k_*k^{-1}K$.

By pulling back to $\widetilde{X}$, we are "gaining information" about the behavior of fibers away from 0. By adjunction, we have a map $K \to \mathbf{R}k_*k^{-1}K$, and so there is a map $i^{-1}K \to \psi_f K$.

Definition 15.3 (*Vanishing cycles*)**.** $\phi_f K = \mathrm{Cone}(i^{-1}K \to \psi_f K)$. (Is this clearly functorial? Recall that cones are unique, but only up to non-unique isomorphism. This could work out if we took the Cone in terms of dg categories.)

Theorem 15.4 (Gabber)**.** *If K is perverse, then ${}^p\psi_f K \coloneqq \psi_f K[-1]$ and ${}^p\phi_f K \coloneqq \phi_f[-1]$ are perverse.*

Since the category of perverse sheaves sheaves is abelian, we can try decomposing them.

We can define the action of T on ${}^p\psi_f K, {}^p\phi_f K$ by $z \mapsto z+1$ on $\mathbf{H}$. More precisely, this map induces an isomorphism j of $\widetilde{X}$ via base change, such that $k \circ j = k$; via adjunction, we have a map

$$\mathbf{R}k_*k^{-1}K \longrightarrow \mathbf{R}k_*j^{-1}k^{-1}K \simeq \mathbf{R}k_*k^{-1}K,$$

and then shifting $[-1]$. (Using j^{-1} or $\mathbf{R}j_*$ corresponds to a sign change $z \mapsto z \pm 1$.)

We can then look at the *generalized eigenspaces* by looking at

$${}^p\psi_{f,\lambda} = \ker(T - \lambda \cdot \mathrm{id})^m \qquad {}^p\phi_{f,\lambda} = \ker(T - \lambda \cdot \mathrm{id})^m$$

for $m \gg 0$, where the kernel is taken in the category of perverse sheaves. (If K is a local system, then this should be compatible with what happens on stalks, i.e., you should get back the classical description using Milnor fibers.)

We have the decomposition

$${}^p\psi_f = \bigoplus_{\lambda \in \mathbf{C}^*} {}^p\psi_{f,\lambda} \qquad {}^p\phi_f = \bigoplus_{\lambda \in \mathbf{C}^*} {}^p\phi_{f,\lambda}.$$

Claim 15.5. *${}^p\psi_f \simeq {}^p\phi_f$ for $\lambda \neq 1$.*

The reason is that the eigenvalues on $i^{-1}K$ are 1 (they correspond to the identity).

15.3 *V*-filtrations

Let $f\colon X \to \mathbf{C}$, where X is a variety over $\mathbf{C}$. We want an analogue of the above story on the D-module side; to do this, we will construct a filtration on $M \in \mathsf{Mod}_{\mathsf{c}}(D_X)$ with respect to $f^{-1}(0)$.

Case 1. f is smooth.

In local coordinates, $x_1, \ldots, x_n, t$, $\{t = 0\} \leftrightarrow f^{-1}(0)$. Then, we can define

$$\begin{aligned} V_0 D_X &\coloneqq \{P \in D_X \mid P \cdot (t)^i \subseteq (t)^i \text{ for } i \geq 0\} \\ &= \left\{ P \in D_X \;\middle|\; P = \sum_{\beta - \gamma \geq 0} h_{\alpha,\beta,\gamma}(x)\partial_x^\alpha t^\beta \partial_t^\gamma \right\} \\ V_j D_X &\coloneqq \{P \in D_X \mid P \cdot (t)^i \subseteq (t)^{i+j} \text{ for } i \geq 0\} \end{aligned}$$

Definition 15.6. A *V-filtration* on $M \in \mathsf{Mod_c}(D_X)$ is a decreasing filtration by coherent $V^0 D_X$-modules such that

1. $\{V^\alpha\}_\alpha$ is indexed by $\mathbf{Q}$, discretely ($\mathrm{Gr}_V^\alpha = V^\alpha/V^{>\alpha} \neq 0$ for a discrete set of α (do the denominators have to be the same?)), and left continuously ($V^\alpha = \bigcap_{\beta<\alpha} V^\beta$);
2. $tV^\alpha \subseteq V^{\alpha+1}$, $\partial_t V^\alpha \subseteq V^{\alpha-1}$, and $tV^\alpha = V^{\alpha+1}$ for $\alpha \gg 0$ (> 0?);
3. $\partial_t t - \alpha$ is nilpotent on $\mathrm{Gr}_V^\alpha = V^\alpha/V^{>\alpha}$.

Examples 15.7. Here, $V^\alpha = V^{\lceil\alpha\rceil}$ if $\alpha \in \mathbf{Q}$ (this is the opposite continuity as for multiplier ideals).

1. $\mathcal{O}_X$ has a V-filtration, where $V^m\mathcal{O}_X = (t)^{m-1}$, since $\partial t \cdot t^{m-1} = mt^{m-1}$.
2. If $(\mathscr{E}, \nabla)$ is a vector bundle with flat connection, then letting $V^m\mathscr{E} := t^{m-1}\mathscr{E}$ gives a V-filtration, since if $s \in \mathscr{E}$, then
$$\partial_t t t^{m-1} s = \partial_t(t^m)s + t^m\partial s = mt^{m-1}s + t^m(\partial s).$$
3. Suppose $M \in \mathsf{Mod_c}(D_X)$ is supported on $(t = 0)$. In this case, we saw the V-filtration already when we talked about Kashiwara's equivalence: Let
$$M^j = \{s \in M \mid \partial_t t \cdot s = j \cdot s\},$$
in which case $M = \bigoplus_{j=0}^\infty M^{-j}$. We set
$$V^m M = \bigoplus_{j=m}^{\infty} M^j.$$
Alternatively, $M \simeq N \otimes_{\mathbf{C}} \mathbf{C}[\partial]$, in which case $M^j \simeq N \otimes_{\mathbf{C}} \partial^j$.

Theorem 15.8 (Kashiwara–Malgrange)**.** *For $M \in \mathsf{Mod_c}(D_X)$, there exists a V-filtration along $(t = 0)$ if M is regular holonomic, with quasi-unipotent monodromy (i.e., T acts on the perverse sheaf* $\mathrm{DR}(M)$ *with eigenvalues that are roots of unity).*

If f is a family of varieties and $M = \mathbf{C}_X$, then the quasi-unipotence of the monodromy action is a result from Hodge theory. Note that the quasi-unipotence condition is what corresponds to the indexing by $\mathbf{Q}$ in the definition of a V-filtration; when indexed by $\mathbf{C}$, the quasi-unipotence condition in the Theorem goes away.

One way to think about this is in terms of the Bernstein polynomial, which is the minimal polynomial of the action of $\partial_t\cdot$. The roots of this polynomial correspond to eigenvalues.

Proposition 15.9. *V filtrations are unique.*

Proof. Given two V-filtrations $U^\bullet, V^\bullet$ on $M \in \mathsf{Mod_c}(D_X)$, our goal is to show that $U^\alpha \subseteq V^\alpha$ for all α; by symmetry, this shows that they are in fact equal.

We first claim that for $\alpha \neq \beta$,
$$U^\alpha \cap V^\beta = U^{>\alpha} \cap V^\beta + U^\alpha \cap V^{>\beta}. \tag{15.1}$$
"$\supseteq$" is clear; for "$\subseteq$," consider the quotient
$$U^\alpha \cap V^\beta/(U^{>\alpha} \cap V^\beta + U^\alpha \cap V^{>\beta}).$$
Then, $(\partial t - \alpha)$ and $(\partial t - \beta)$ are both nilpotent, and so the quotient is zero.

Next, we claim that
$$U^\alpha \subseteq U^{>\alpha} + V^\alpha. \tag{15.2}$$
Fix $w \in U^\alpha$; then, there exists β such that $w \in V^\beta$. Then, write $w = w_1 + w_2$, where $w_1 \in U^{>\alpha}$ and $w_2 \in V^{>\beta}$. We may then replace w by w_2 (since $\beta \geq \alpha$ would imply we are done), and so $w \in U^\alpha$ and $w \in V^{\beta_1}$, where $\beta_1 > \beta$. Repeating this argument, since the V-filtration jumps on a discrete set, we have that $w \in V^\alpha$.

Next, we claim that for $\beta \gg 0$, we have
$$U^\alpha \subseteq U^\beta + V^\alpha \tag{15.3}$$

Also, we will show that $U^\alpha \subseteq t^\ell U^\gamma + V^\alpha$ for $\ell \gg 0$ and $\gamma \gg 0$. Using the previous step twice, we have

$$\begin{aligned} U^\alpha &\subseteq U^{>\alpha} + V^\alpha \\ U^{>\alpha} = U^{\alpha_1} &\subseteq U^{>\alpha_1} + V^{\alpha_1} \end{aligned}$$

Combining these, we have

$$U^\alpha \subseteq U^{>\alpha_1} + V^{\alpha_1} + V^\alpha \subseteq U^{>\alpha_1} + V^\alpha.$$

Now the discreteness argument from before gives the claim.

We now want to show that for $\ell \gg 0$,

$$t^\ell U^\gamma \subseteq V^\alpha. \tag{15.4}$$

First, write $U^\gamma = \sum_i V^0 D_X \cdot u_i$, and so $U^\gamma \subseteq V^\delta$ for $\delta \ll 0$. $t^\ell U^\delta \subseteq t^\ell V^\delta \subseteq V^{\delta+\ell} \subseteq V^\alpha$ for $\partial + \ell \geq \alpha$. □

Case 2. $f\colon X \to \mathbf{C}$, where $f^{-1}(0)$ is not necessarily smooth.

We reduce to the smooth case by using the graph map

$$(\mathrm{id}, f)\colon X \longrightarrow X \times \mathbf{C}.$$

Let t correspond to the coordinate for $\mathbf{C}$. Then, if $M \in \mathsf{Mod_c}(D_X)$, we want to reduce to Case 1 where the divisor is smooth, and so we consider the direct image module $M_f \coloneqq M \otimes_{\mathbf{C}} \mathbf{C}[\partial_t]$. Then, if there exists a V-filtration on M_f along $X \times \{0\}$, we set

$$V^\alpha M \coloneqq V^\alpha M_f \cap M \otimes 1.$$

It is more useful to study the V-filtration on $X \times \mathbf{C}$, though:

Theorem 15.10. *If M is regular holonomic, then*

$$\mathrm{DR}(\mathrm{Gr}_V^\alpha M_f) \simeq \begin{cases} {}^p\phi_{f,\lambda} & \text{if } \alpha \in [0,1) \\ {}^p\psi_{f,\lambda} & \text{if } \alpha \in (0,1] \end{cases}$$

where $\lambda = e^{-2\pi i\alpha}$.

Note that there is an assertion that $\mathrm{Gr}_V^\alpha M_f$ is regular holonomic. A theorem of Budur–Saito is that if $M = \mathcal{O}_X$, you get the multiplier ideal.

16 January 30: Pure Hodge modules (Takumi Murayama)

16.1 A functorial definition for vanishing cycles

We first answer a question we had last time: *How can we define vanishing cycles functorially?*

You can apparently make our definition using cones functorial by using dg-enhancements of all the functors in sight. Instead, we give a different construction. Recall the setup:

$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{k} & X & \xleftarrow{i} & X_0 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f} & & \downarrow \\ \mathbf{H} & \xrightarrow{e} & \Delta & \longleftarrow & \{0\} \end{array}$$

where $f\colon X \to \Delta$ is a map that is submersive away from 0.

Proposition 16.1 [KS94, §8.6]**.** *$\phi_f K \simeq i^{-1}\mathbf{R}\mathscr{H}om(f^{-1}C, K)[1]$, where*

$$C \coloneqq \Big\{ 0 \longrightarrow e_!\underline{\mathbf{C}}_{\mathbf{H}} \xrightarrow{\mathrm{tr}} \underset{\deg 0}{\underline{\mathbf{C}}_\Delta} \longrightarrow 0 \Big\}$$

and the map tr *is the adjunction morphism $e_!\underline{\mathbf{C}}_{\mathbf{H}} \cong e_!e^*\underline{\mathbf{C}}_\Delta \to \underline{\mathbf{C}}_\Delta$.*

Proof. C fits into the exact sequence

$$0 \longrightarrow \underline{\mathbf{C}}_\Delta \longrightarrow C \longrightarrow e_! \underline{\mathbf{C}}_{\mathbf{H}}[1] \longrightarrow 0,$$

which gives the distinguished triangles

$$\begin{array}{ccccccc} \underline{\mathbf{C}}_\Delta & \longrightarrow & C & \longrightarrow & e_!\underline{\mathbf{C}}_{\mathbf{H}}[1] & \rightsquigarrow \\ \underline{\mathbf{C}}_X & \longrightarrow & f^{-1}C & \longrightarrow & f^{-1}e_!\underline{\mathbf{C}}_{\mathbf{H}}[1] & \rightsquigarrow \\ & & & & \wr\| & \\ & & & & k_!\underline{\mathbf{C}}_{\widetilde{X}}[1] & \end{array}$$

where we applied f^{-1} in the second line, and the isomorphism is by base change. Now applying the functor $i^{-1}\,\mathbf{R}\mathscr{H}om(-,K)[1]$, we obtain

$$i^{-1}\,\mathbf{R}\mathscr{H}om(k_!\underline{\mathbf{C}}_{\widetilde{X}}, K) \longrightarrow \phi_f K \longrightarrow i^{-1}K[1] \rightsquigarrow$$

We can identify the leftmost entry as follows:

$$i^{-1}\,\mathbf{R}\mathscr{H}om(k_!\underline{\mathbf{C}}_{\widetilde{X}}, K) \cong i^{-1}\mathbf{R}k_*\,\mathbf{R}\mathscr{H}om(\underline{\mathbf{C}}_{\widetilde{X}}, k^{-1}K) \cong i^{-1}\mathbf{R}k_*k^{-1}K =: \psi_f K$$

by Verdier duality. □

Another question we had is why we have direct sum decompositions

$$^p\psi_f = \bigoplus_{\lambda\in\mathbf{C}^*} {}^p\psi_{f,\lambda} \qquad {}^p\phi_f = \bigoplus_{\lambda\in\mathbf{C}^*} {}^p\phi_{f,\lambda}$$

into generalized eigenspaces for the monodromy action T; this is [Rei10, Lem. 4.2].

16.2 Introduction to Hodge modules

Remark 16.2. We work with the sheaf D_X of rings of holomorphic differentials on a complex variety X of dimension n, and with *right* D-modules, following [Sai88; Sai89b; Sch14a]. Using right D-modules makes the direct image and duality functors much easier to describe, which are important since duality functors are used to define polarizations, and direct image functors are necessary to define mixed Hodge modules on singular spaces, via Kashiwara's equivalence. Moreover, we will always work with Hodge modules with $\mathbf{Q}$-structure.

16.2.1 Motivation [Sch14a, §4]

We first give some motivation. Let X be a smooth complex variety. Recall that the Riemann–Hilbert correspondence gives an equivalence between the category of regular holonomic D-modules and the category of perverse sheaves:

$$\begin{aligned} \mathrm{DR}_X\colon \mathsf{Mod}_{\mathsf{rh}}(D_X) &\xrightarrow{\sim} \mathsf{Perv}(X) \\ \mathcal{M} &\longmapsto \mathcal{M} \otimes^{\mathbf{L}}_{D_X} \mathcal{O}_X \end{aligned}$$

where we note the definition of DR_X differs from the version we saw before since we are now using right D-modules (explicitly, this can be calculated using the Spencer resolution (5.7)). The category $\mathsf{Loc}(X)$ of local systems lives in $\mathsf{Perv}(X)$, and we saw that local systems often carry a (mixed) variation of Hodge structures, which consisted of the following data:

1. A $\mathbf{Q}$-local system V;
2. A Hodge filtration $F^\bullet$ on the vector bundle $V \otimes_{\mathbf{Q}} \mathcal{O}_X$ satisfying Griffiths transversality;
3. A weight filtration $W_\bullet$ on the local system V (for mixed Hodge structures).

We also have a notion of polarization for Hodge structures.

Goal 16.3. Extend the notion of polarizable mixed Hodge structures to D-modules.

A pure Hodge module $(\mathcal{M}, F_\bullet, K)$ consists of the following data:

1. A perverse sheaf K with coefficients in $\mathbf{Q}$;
2. A regular holonomic D-module $\mathcal{M}$ such that $\mathrm{DR}(\mathcal{M}) \simeq \mathbf{C} \otimes_{\mathbf{Q}} K$;
3. A good filtration $F_\bullet$ on $\mathcal{M}$.

although there will be additional properties that we will require (in particular, conditions about weights). The exact relationship to Hodge structures is that the category of Hodge modules on a point is equivalent to the category of Hodge structures.

16.2.2 An example: the canonical bundle and vanishing theorems [Pop16, §§5, 9; Sch14a, §5]

Before we get to definitions, we give a motivating example of where mixed Hodge modules can be used in algebraic geometry.

Example 16.4. Consider the canonical bundle ω_X on a complex variety X. Recall that ω_X is a right D_X-module via the Lie derivative (Claim 1.15). Consider the filtration given by

$$F_{-n-1}\omega_X = 0 \qquad \text{and} \qquad F_{-n}\omega_X = \omega_X,$$

and consider the perverse sheaf $\mathbf{Q}_X[n]$. Then, $\mathbf{Q}_X^H[n] := (\omega_X, F_\bullet, \mathbf{Q}_X[n])$ is the "trivial" Hodge module. Here,

$$\mathrm{DR}_X(\omega_X) = \Bigl\{ \mathcal{O}_X \longrightarrow \Omega_X^1 \longrightarrow \cdots \longrightarrow \underset{\deg 0}{\Omega_X^n} \Bigr\}. \tag{16.1}$$

By the holomorphic Poincaré lemma, this is quasi-isomorphic to the complex $\mathbf{C}_X[n]$, i.e., the complex with one nonzero entry at degree $-n$. Since $\mathbf{C} \otimes_{\mathbf{Q}} \mathbf{Q}_X[n] \simeq \mathbf{C}_X[n]$, this satisfies the properties given above.

In complex algebraic geometry, some of the most useful results are vanishing theorems, which involve the canonical bundle in some way. We recall a few:

Theorem 16.5.

(i) [Kodaira–Akizuki–Nakano] *If X is a smooth projective variety, then for all ample line bundles L on X,*

$$H^q(X, \Omega_X^p \otimes L) = 0 \quad \text{for all } p + q > n.$$

(ii) [Kollár] *If $f\colon X \to Y$ is a morphism of projective varieties, where X is smooth, then for all ample line bundles L on Y,*

$$H^q(Y, R^j f_*(\omega_X) \otimes L) = 0 \quad \text{for all } q > 0,\ j \in \mathbf{Z}.$$

Saito proved the following theorem about mixed Hodge modules with implies both these results. Note that we are not being too careful about whether sheaves live on X or X^{an}.

Theorem 16.6 (Saito's vanishing theorem [Sch14a, Thm. 24.1])**.** *Let $(\mathcal{M}, F_\bullet, K)$ be a mixed Hodge module on a projective variety X. Then,*

$$H^q\bigl(X, \mathrm{gr}_p^F \mathrm{DR}(\mathcal{M}) \otimes L\bigr) = 0$$

for all $q > 0$, $p \in \mathbf{Z}$, and ample L.

Proof of Theorem 16.5 assuming Theorem 16.6. For (*i*), we let $M = \omega_X$ as in Example 16.4. Then, (16.1) implies

$$\mathrm{gr}_{-p}^F \mathrm{DR}(\omega_X) = \Omega_X^p[n-p],$$

where we recall the order of a k-form is $-k$ in the proof of Lemma 5.19. Then,

$$H^q(X, \Omega_X^p \otimes L) = H^q(X, \Omega_X^p \otimes L) = H^{p+q-n}(X, \mathrm{gr}_{-p}^F \mathrm{DR}(\omega_X) \otimes L) = 0$$

by Theorem 16.6.

For (*ii*), let $f\colon X \to Y$ be a morphism to a projective variety Y. There is a way to define the direct image $f_*(\mathcal{M}, F_\bullet, K)$ of a mixed Hodge module, such that its action on $\mathcal{M}$ and K are as we would expect, and such that the filtration behaves well with respect to the Riemann–Hilbert correspondence:

$$R^j f_* \,\mathrm{gr}_p^F \mathrm{DR}_X(\mathcal{M}) \simeq \mathrm{gr}_p^F \mathrm{DR}_Y\bigl(\textstyle\int_f^j \mathcal{M}\bigr).$$

This is known as "Saito's formula" [Pop16, p. 9]. For $p = n$ and $\mathcal{M} = \omega_X$, this yields

$$R^j f_* \omega_X \simeq \operatorname{gr}_n^F \mathrm{DR}_Y(\textstyle\int_f^j \omega_X).$$

Thus, applying Theorem 16.6 to $\mathcal{M} = \int_f^j \omega_X$, we obtain

$$H^q(Y, R^j f_*(\omega_X) \otimes L) = H^q\big(Y, \operatorname{gr}_n^F \mathrm{DR}_Y(\textstyle\int_f^j \omega_X) \otimes L\big) = 0. \qquad \square$$

Of course, the proof of Theorem 16.6 is difficult, but it illustrates the power of Saito's theory.

16.3 Pure Hodge modules [Sai89b, §3; Sch14a, Pt. B]

We now start giving definitions. Recall that D_X is the sheaf of rings of holomorphic differentials.

16.3.1 Filtered D-modules with $\mathbf{Q}$-structure [Sch14a, §7]

We first construct a larger category of filtered D-modules, together with perverse sheaves giving their rational structure, which will contain the category of pure Hodge modules.

Definition 16.7. Let $\mathsf{MF}_{\mathsf{rh}}(D_X)$ be the category of filtered D_X-modules $(\mathcal{M}, F_\bullet)$ such that $\mathcal{M}$ is regular holonomic, and $F_\bullet$ is a good filtration; morphisms are given by morphisms of D-modules which respect the filtration. Consider the commutative diagram of functors

$$\begin{array}{ccccc}
\mathsf{MF}_{\mathsf{rh}}(D_X, \mathbf{Q}) & & \xrightarrow{\quad\text{rat}\quad} & & \mathsf{Perv}_{\mathbf{Q}}(X) \\
\big\downarrow \lrcorner & & & & \big\downarrow {\scriptstyle \mathbf{C} \otimes_{\mathbf{Q}} -} \\
\mathsf{MF}_{\mathsf{rh}}(D_X) & \xrightarrow[(\mathcal{M}, F_\bullet) \mapsto \mathcal{M}]{\text{Forget}} & \mathsf{Mod}_{\mathsf{rh}}(D_X) & \xrightarrow[\sim]{\mathrm{DR}_X} & \mathsf{Perv}(X)
\end{array}$$

The fiber product $\mathsf{MF}_{\mathsf{rh}}(D_X, \mathbf{Q})$ is the *category of filtered regular holonomic D-modules with $\mathbf{Q}$-structure*. Explicitly, objects of $\mathsf{MF}_{\mathsf{rh}}(D_X, \mathbf{Q})$ are triples $M = (\mathcal{M}, F_\bullet, K)$ where

1. K is a perverse sheaf with coefficients in $\mathbf{Q}$;
2. $\mathcal{M}$ is a regular holonomic right D_X-module, together with an isomorphism
$$\alpha\colon\ \mathrm{DR}_X(\mathcal{M}) \xrightarrow{\sim} \mathbf{C} \otimes_{\mathbf{Q}} K;$$
3. $F_\bullet$ is a good filtration of $\mathcal{M}$, which recall is an increasing filtration such that $F_p\mathcal{M} \cdot F_k D_X \subseteq F_{p+k}\mathcal{M}$, and such that $F_p\mathcal{M}$ is $\mathcal{O}_X$-coherent for all p (by Proposition 3.2, this is equivalent to $\operatorname{gr}_\bullet^F \mathcal{M}$ being coherent over $\operatorname{gr}_\bullet^F D_X$).

Morphisms are given by pairs of morphisms for $(\mathcal{M}, F_\bullet)$ and K which respect the isomorphism α.

Pure and mixed Hodge modules will form a subcategory of $\mathsf{MF}_{\mathsf{rh}}$. Before we define them, we give some examples.

Example 16.8. Let V be a ($\mathbf{Q}$-)variation of Hodge structures of weight w. We can associate to it the following element of $\mathsf{MF}_{\mathsf{rh}}(D_X, \mathbf{Q})$:

1. $K = V[n]$;
2. $\mathcal{M} = \omega_X \otimes_{\mathcal{O}_X} V_X$, where $V_X \coloneqq \mathcal{O}_X \otimes_{\mathbf{Q}} V$;
3. $F_p\mathcal{M} = \omega_X \otimes_{\mathcal{O}_X} F^{-p-n} V_X$.

We will see that this gives a pure Hodge module of weight $w + n$. We have

$$\mathrm{DR}_X(\mathcal{M}) = \Big\{ V_X \longrightarrow \Omega_X^1 \otimes V_X \longrightarrow \cdots \longrightarrow \underset{\deg 0}{\Omega_X^n \otimes V_X} \Big\} \simeq \mathbf{C} \otimes_{\mathbf{Q}} V[n] = \mathbf{C} \otimes_{\mathbf{Q}} K$$

by the holomorphic Poincaré lemma. Note that the example $\mathbf{Q}_X^H[n]$ in Example 16.4 is a special case, where $V = \mathbf{Q}_X$ is the constant variation of Hodge structure of weight 0.

Example 16.9. The *Tate twist* of $(\mathcal{M}, F_\bullet, K)$ by an integer $k \in \mathbf{Z}$ is the new triple

$$M(k) \coloneqq \big(\mathcal{M}, F_{\bullet-k}, K \otimes_{\mathbf{Q}} \mathbf{Q}(k)\big),$$

where $\mathbf{Q}(k) = (2\pi i)^k\mathbf{Q} \subseteq \mathbf{C}$.

Example 16.10. Recall that if $f\colon X \to Y$ is an arbitrary morphism of complex varieties, then $R^j f_*\mathbf{Q}_X$ is a constructible sheaf, but is *not* a local system in general (for example, if f is a closed embedding). But it still defines a filtered D-module with $\mathbf{Q}$-structure, which fixes our complaint before that variations of Hodge structure don't push forward (Example 14.17).

It turns out that the category $\mathsf{MF}_{\mathsf{rh}}(D_X, \mathbf{Q})$ is too large for the purposes of Hodge theory (why?). We want to find a subcategory that is small enough to be manageable, but large enough to contain all polarizable variations of Hodge structure. In particular, the category of pure Hodge modules will be semisimple, with components coming from polarizable variations of Hodge structure on subvarieties.

16.3.2 Nearby and vanishing cycles for filtered D-modules [Sai88, §§3.4, 5.1; Sch14a, §9]

To construct the subcategory of pure Hodge modules in $\mathsf{MF}_{\mathsf{rh}}(D_X, \mathbf{Q})$, we will use nearby and vanishing cycles. These will be different from those we defined in Definition 15.6 since we are now using right D-modules.

Let $M = (\mathcal{M}, F_\bullet, K) \in \mathsf{MF}_{\mathsf{rh}}(D_X, \mathbf{Q})$, and let $f\colon X \to \mathbf{C}$ be a non-constant holomorphic function.

Goal 16.11. Define nearby cycles $\psi_f M$ and vanishing cycles $\phi_f M$ in the category $\mathsf{MF}_{\mathsf{rh}}(D_X, \mathbf{Q})$.

As we did for left D_X-modules in §15.3, consider the graph embedding

$$(\mathrm{id}, f)\colon X \hookrightarrow X \times \mathbf{C} \eqqcolon Y.$$

Let t be the coordinate on $\mathbf{C}$, and consider the direct image $\mathcal{M}_f$ of the D-module given by

$$\mathcal{M}_f = \int_{(\mathrm{id},f)} \mathcal{M} = \mathcal{M}[\partial_t]$$
$$F_\bullet\mathcal{M}_f = F_\bullet \int_{(\mathrm{id},f)} \mathcal{M} = \bigoplus_{i=0}^{\infty} F_{\bullet-i}\mathcal{M} \otimes \partial_t^i$$

where $\int_g \coloneqq \mathbf{R}g_*(- \otimes^{\mathbf{L}}_{D_X} D_{X\to Y})$ is the direct image for right D-modules; note that by Proposition 5.22, this functor is the same as the naïve direct image functor from §4.2.1.

Now recall that $V_0 D_Y = \{P \in D_Y \mid P \cdot (t) \subseteq (t)\}$.

Definition 16.12. A *V-filtration* on $\mathcal{M}_f$ is an exhaustive *increasing* filtration by coherent V_0D_Y-modules such that

(1) $\{V_\alpha\}_\alpha$ is indexed by $\mathbf{Q}$, discretely, and right continuously ($V_\alpha = \bigcap_{\beta>\alpha} V_\beta$);
(2) $V_\alpha \cdot t \subseteq V_{\alpha-1}$, $V_\alpha \cdot \partial_t \subseteq V_{\alpha+1}$, and $V_\alpha \cdot t = V_{\alpha-1}$ for $\alpha < 0$;
(3) $t\partial_t - \alpha$ is nilpotent on $\mathrm{gr}^V_\alpha = V_\alpha/V_{<\alpha}$.

Recall that by Theorem 15.8 and Proposition 15.9, the V-filtration exists and is unique *as long as the eigenvalues of the monodromy operator T on ${}^p\psi_f K$ are roots of unity.*

To define nearby and vanishing cycles for filtered D-modules, we recall the following:

Theorem 15.10 (cf. [Sai88, Prop. 3.4.12])**.** *If M is regular holonomic, then*

$$\mathrm{DR}(\mathrm{gr}^V_\alpha \mathcal{M}_f) \simeq \begin{cases} {}^p\psi_{f,\lambda}(\mathrm{DR}(\mathcal{M})) & \text{if } -1 \le \alpha < 0 \\ {}^p\phi_{f,\lambda}(\mathrm{DR}(\mathcal{M})) & \text{if } -1 < \alpha \le 0 \end{cases}$$

where $\lambda = e^{2\pi i\alpha}$.

So we know what the D-module part of $\psi_f M$ should be; we still need to determine what filtration to put on it. By the decomposition of nearby and vanishing cycles by generalized eigenspaces

$$ {}^p\psi_f = \bigoplus_{\lambda \in \mathbf{C}^*} {}^p\psi_{f,\lambda} \qquad {}^p\phi_f = \bigoplus_{\lambda \in \mathbf{C}^*} {}^p\phi_{f,\lambda} $$

we have

$$ \begin{aligned} \mathrm{DR}\Biggl(\bigoplus_{-1 \le \alpha < 0} \operatorname{gr}_\alpha^V \mathcal{M}_f\Biggr) &\simeq \mathbf{C} \otimes_{\mathbf{Q}} {}^p\psi_f K \\ \mathrm{DR}\Biggl(\bigoplus_{-1 < \alpha \le 0} \operatorname{gr}_\alpha^V \mathcal{M}_f\Biggr) &\simeq \mathbf{C} \otimes_{\mathbf{Q}} {}^p\phi_f K \end{aligned} $$

The two D-modules on the left-hand side are regular holonomic; we now want to put a filtration on them. The "correct" thing to do is to ensure the V-filtration is compatible with the filtration $F_\bullet$:

$$ F_p \operatorname{gr}_\alpha^V \mathcal{M}_f = \frac{F_p\mathcal{M}_f \cap V_\alpha \mathcal{M}_f}{F_p\mathcal{M}_f \cap V_{<\alpha}\mathcal{M}_f}, \tag{16.2} $$

in which case we make the following definition:

Definition 16.13. Let $f\colon X \to \mathbf{C}$ be a non-constant holomorphic function, and let $M = (\mathcal{M}, F_\bullet, K) \in \mathsf{MF}_{\mathsf{rh}}(D_X, \mathbf{Q})$. Then, the *nearby and vanishing cycles* are

$$ \begin{aligned} \psi_f M &\coloneqq \bigoplus_{-1 \le \alpha < 0} \bigl(\operatorname{gr}_\alpha^V \mathcal{M}_f, F_{\bullet-1}, {}^p\psi_{f,e^{2\pi i\alpha}} K\bigr) \\ \psi_{f,1} M &\coloneqq \bigl(\operatorname{gr}_{-1}^V \mathcal{M}_f, F_{\bullet-1}, {}^p\psi_{f,1} K\bigr) \\ \phi_{f,1} M &\coloneqq \bigl(\operatorname{gr}_0^V \mathcal{M}_f, F_\bullet, {}^p\phi_{f,1} K\bigr) \end{aligned} $$

As long as the filtration defined in (16.2) is good for all $-1 \le \alpha \le 0$, these are objects in $\mathsf{MF}_{\mathsf{rh}}(D_X, \mathbf{Q})$.

Just as for perverse sheaves, we also define two morphisms:

$$ \begin{array}{ccc} \psi_{f,1} M & \xrightarrow{\quad \mathrm{can} \quad} & \phi_{f,1} M \\ \bigl(\operatorname{gr}_{-1}^V \mathcal{M}_f, F_{\bullet-1}\bigr) & \xrightarrow{\cdot \partial_t} & \bigl(\operatorname{gr}_0^V \mathcal{M}_f, F_\bullet\bigr) \\ {}^p\psi_{f,1} K & \xrightarrow{\quad \mathrm{can} \quad} & {}^p\phi_{f,1} K \end{array} \qquad \begin{array}{ccc} \phi_{g,1} M & \xrightarrow{\quad \mathrm{Var} \quad} & \psi_{g,1} M(-1) \\ \bigl(\operatorname{gr}_0^V \mathcal{M}_f, F_\bullet\bigr) & \xrightarrow{\cdot t} & \bigl(\operatorname{gr}_{-1}^V \mathcal{M}_f, F_\bullet\bigr) \\ {}^p\phi_{f,1} K & \xrightarrow{\quad \mathrm{Var} \quad} & {}^p\psi_{f,1} K(-1) \end{array} $$

One has to check that the morphisms match via the Riemann–Hilbert correspondence [Sai88, Lem. 3.4.11].

Remark 16.14. Except for $\lambda = 1$, the individual perverse sheaves ${}^p\psi_{f,\lambda}$ are not guaranteed to be defined over $\mathbf{Q}$, and so they must be packaged together in the definition of $\psi_f M$. I don't know why we don't similarly define $\phi_f M$.

16.3.3 A preliminary definition [Sai89b, n° 3.2]

We can now give a definition for pure Hodge modules, which admittedly is not very enlightening.

Definition 16.15. The category $\mathsf{HM}(X, w)$ of *Hodge modules of weight w* on X is the largest full subcategory of $\mathsf{MF}_{\mathsf{rh}}(X, \mathbf{Q})$ satisfying the following properties:

1. $\mathsf{HM}(\mathrm{pt}, w)$ is the category of $\mathbf{Q}$-Hodge structures of weight w;
2. If $\operatorname{Supp} \mathcal{M} = \{x\}$ for $M = (\mathcal{M}, F_\bullet, K) \in \mathsf{HM}(X, w)$, then there exists $M' \in \mathsf{HM}(\mathrm{pt}, w)$ such that $i_* M' = M$, where $i\colon \{x\} \hookrightarrow X$ is the inclusion map;
3. If $M \in \mathsf{HM}(X, w)$, then for every non-constant holomorphic function $f\colon U \to \mathbf{C}$ defined on an open subset $U \subseteq X$,
   (a) M has quasi-unipotent monodromy along g;

(b) $\mathrm{gr}^W_{i-w+1}\,\psi_f M, \mathrm{gr}^W_{i-w}\,\phi_{f,1} M \in \mathsf{HM}(U,i)$ for all i, where W is the monodromy filtration;
(c) $\phi_{f,1} = \mathrm{im}\,\mathrm{can} \oplus \ker \mathrm{Var}$.

Here, the monodromy filtration is the unique filtration induced by the nilpotent endomorphism

$$N = \frac{1}{2\pi i}\log T_u \colon \psi_f M \longrightarrow \psi_f M(-1)$$

satisfying $NW_i \subset W_{i-2}$ and $N^r\colon\ \mathrm{gr}^W_r \cong \mathrm{gr}^W_{-r}$.

17 February 13 (Harold Blum)

Today's material is a bit technical, so we start by motivating it with the decomposition theorem.

17.1 The decomposition theorem

Definition 17.1. Let X be a complex manifold, and let $Z \subseteq X$ be an irreducible closed subvariety. A Hodge module M has *strict support* Z if every subobject and every quotient object of M has support equal to Z.

Theorem 17.2. *Let X be a complex manifold, and let $Z \subseteq X$ be an irreducible closed subvariety. Then,*
(1) *Every polarized variation of $\mathbf{Q}$-Hodge structures of weight $w - \dim Z$ on $U \subseteq Z$ extends to an object of $\mathsf{HM}^p_Z(X,w)$, the category of polarizable Hodge modules of weight w with strict support Z.*
(2) *Every object of $\mathsf{HM}^p_Z(X,w)$ is obtained in this way.*

The first statement is much harder to prove, but the second is not as bad. It comes from the conditions we put on what it means to be a pure Hodge module.

- Our definition from last time in fact requires that every $M \in \mathsf{HM}(X,w)$ to have a decomposition

$$M = \bigoplus_{Z \subseteq X} M_Z.$$

- If $(\mathcal{M}, F_\bullet\mathcal{M}, K) \in \mathsf{HM}^p_Z(X,w)$, then there exists $U \subseteq Z$ such that $K|_U$ is a local system. Moreover, $(\mathcal{M}, F_\bullet\mathcal{M}, K)$ is determined by the restriction. The filtration is the hard part to check: K and $\mathcal{M}$ are already determined by the minimal extension functor from the theory of perverse sheaves.

17.2 Decomposition by strict support

The first thing we will discuss is why Hodge modules actually do have this decomposition by strict support, that is, we want to show that if $M \in \mathsf{HM}^p(X,w)$, then $M = \bigoplus_{Z\subseteq X} M_Z$, where M_Z have strict support Z.

Proposition 17.3. *Let $f\colon X \to \mathbf{C}$ be a holomorphic function, and let $\mathcal{M}$ be a regular holonomic D-module on X. Then, $\mathcal{M}$ has a nonzero subobject (resp. quotient object) with support in $f^{-1}(0)$ if and only if $t\colon\ \mathrm{gr}^V_0\,\mathcal{M}_f \to \mathrm{gr}^V_{-1}\,\mathcal{M}_f$ is injective (resp. $\partial_t\colon\ \mathrm{gr}^V_{-1}\,\mathcal{M}_f \to \mathrm{gr}^V_0\,\mathcal{M}_f$ is surjective).*

Proof. $\Rightarrow$. If $\mathcal{M}$ has a subobject of the required form, then $\mathcal{M}[\partial] \simeq \mathcal{M}_f$ has a subobject $\mathcal{N}$ by adjoining ∂ with $\mathrm{Supp}\,\mathcal{N} \subseteq X \times \{0\}$. By the proof of Kashiwara's equivalence, we know that

$$\mathcal{N} \simeq \bigoplus_{i=0}^{\infty} \mathcal{N}_0 \otimes \partial^i$$

and we also know what the V-filtration looks like:

$$V_\alpha = \bigoplus_{i=0}^{\lceil\alpha\rceil} \mathcal{N}_0 \otimes \partial^i$$

if $\alpha \geq 0$, and $V_\alpha = 0$ if $\alpha < 0$. We then have the commutative diagram

$$\begin{array}{ccc} \mathrm{gr}^V_0\,\mathcal{M}_f & \longrightarrow & \mathrm{gr}^V_{-1}\,\mathcal{M}_f \\ \cup\!\mid & & \cup\!\mid \\ \mathrm{gr}^V_0\,\mathcal{N} & \longrightarrow & \mathrm{gr}^V_{-1}\,\mathcal{N} \end{array}$$

since the V-filtration is unique. But the map on the bottom is the zero map by looking at the V-filtration, hence the top map must not be injective. The same argument works for quotient objects.

$\Leftarrow$. Suppose $t\colon \operatorname{gr}_0^V \mathcal{M}_f \to \operatorname{gr}_{-1}^V \mathcal{M}_f$ is not injective, so the map $V_0 \to V_{-1}/V_{<-1}$ is not injective. This implies$V_0 \to V_{-1}$ is not injective, since we saw before that $t\colon V_{<0} \to V_{<-1}$ is an isomorphism. Now consider the sub-D-module $\mathcal{N}$ of $\mathcal{M}_f$ generated by $\ker(t\colon V_0 \to V_{-1})$. We claim

$$\mathcal{N} = \sum_{i \geq 0} \ker(t\colon V_0 \to V_{-1}) \cdot \partial^i.$$

Suppose $s \in \ker(t\colon V_0 \to V_{-1})$; what we need to show is that $s \cdot \partial^i$ has support on $X \times \{0\}$. If $i = 1$, then

$$(s \cdot \partial) \cdot t^2 = s \cdot \partial t \cdot t = s(t\partial + 1) \cdot t = 0,$$

and a similar argument works for $i > 1$.

For the quotient version, you have to look at the cokernel of $\partial\colon \mathcal{M}_f \to \mathcal{M}_f$. □

The next Proposition tells you when a decomposition exists into subobjects with strict support.

Proposition 17.4. *The decomposition*

$$\operatorname{gr}_0^V \mathcal{M}_f = \ker(t\colon \operatorname{gr}_0^V \mathcal{M}_f \to \operatorname{gr}_{-1}^V \mathcal{M}_f) \oplus \operatorname{im}(\partial_t\colon \operatorname{gr}_{-1}^V \mathcal{M}_f \to \operatorname{gr}_0^V \mathcal{M}_f) \tag{17.1}$$

holds if and only if $\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}''$ *where* $\operatorname{Supp} \mathcal{M} \subseteq f^{-1}(0)$ *and* $\mathcal{M}''$ *has no nonzero subobjects or quotient objects supported in* $f^{-1}(0)$.

Proof. $\Leftarrow$. Assume there exists a decomposition $\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}''$ as stated. Then, $\operatorname{gr}_0^V \mathcal{M}_f \simeq \operatorname{gr}_0^V \mathcal{M}'_f \oplus \operatorname{gr}_0^V \mathcal{M}''_f$, since the V-filtration on the summands is induced from $\mathcal{M}$. Now for $\mathcal{M}''$, the map t is injective, and the map ∂ is surjective by the previous Proposition, and for $\mathcal{M}'$, ∂ acts by zero since it is supported on $X \times \{0\}$. Thus, $\ker(t) = \operatorname{gr}_0^V(\mathcal{M}')$, and $\operatorname{im}(\partial_t) = \operatorname{gr}_0^V(\mathcal{M}'')$. □

We note that the decomposition is unique: $\mathcal{M}'_1 \oplus \mathcal{M}''_1 \xrightarrow{\sim} \mathcal{M}'_2 \oplus \mathcal{M}''_2$ implies $\mathcal{M}'_1 \xrightarrow{\sim} \mathcal{M}'_2$ and $\mathcal{M}''_1 \xrightarrow{\sim} \mathcal{M}''_2$ by the requirement on supports.

Proposition 17.5. *If $\mathcal{M}$ is a regular holonomic D-module and* (17.1) *holds for all functions $f\colon U \to \mathbf{C}$, where the open sets U may vary, then $\mathcal{M}$ decomposes as a direct sum of D-modules with strict support.*

Note that the condition in (17.1) is the condition $\phi_{f,1} = \operatorname{im} \operatorname{can} \oplus \ker \operatorname{Var}$.

17.3 Compatibility with filtrations

Given a triple $(\mathcal{M}, F_\bullet \mathcal{M}, K)$ (not necessarily a Hodge module) such that $\mathcal{M}$ has strict support on $Z \subseteq X$, there exists a Zariski-open set $U \subseteq X$ such that $\mathcal{M}|_U$ is a vector bundle with integrable connection.

Goal 17.6. Recover $F_\bullet \mathcal{M}$ from $F_\bullet \mathcal{M}|_U$.

This is what we need to put filtrations on extensions of perverse sheaves and D-modules. Conditions on the V-filtration will ensure that we can indeed recover the filtration in this way.

Suppose $f\colon X \to \mathbf{C}$ is a holomorphic function such that $f^{-1}(0) \supsetneq Z$. We know that $t\colon \operatorname{gr}_0^V \mathcal{M}_f \to \operatorname{gr}_{-1}^V \mathcal{M}_f$ is injective from the previous Proposition, since there are no subobjects with support in $f^{-1}(0)$, and similarly $\partial_t\colon \operatorname{gr}_{-1}^V \mathcal{M}_f \to \operatorname{gr}_0^V \mathcal{M}_f$ is surjective. We therefore get

$$\mathcal{M}_f = \sum_{i \geq 0} V_{<0} \mathcal{M}_f \cdot \partial_t^i,$$

since $\partial\colon \operatorname{gr}_\alpha^V \mathcal{M}_f \to \operatorname{gr}_{\alpha+1}^V \mathcal{M}_f$ is surjective for $\alpha \geq -1$. Now, if $V_{-1}\mathcal{M}_f \xrightarrow{\partial} V_0\mathcal{M}_f/V_{<0}\mathcal{M}_f$ is surjective, then $V_0\mathcal{M}_f$ is in the sum; using the same argument for other parts of the V-filtation, we can generate all of $\mathcal{M}_f$ by the components on the right-hand side.

Claim 17.7. *If $t\colon F_pV_\alpha \to F_pV_{\alpha-1}$ is surjective for all $\alpha < 0$, then*

$$F_pV_{<0}\mathcal{M} = V_{<0}\mathcal{M} \cap j_*j^*F_p\mathcal{M},$$

where $j\colon X \smallsetminus f^{-1}(0) \hookrightarrow X$ is the open inclusion.

Proof. $\subseteq$ is trivial. For $\supseteq$, we replace X with its image under the graph $\Gamma_f\colon X \hookrightarrow X \times \mathbf{C}$. Suppose that $s \in V_{<0}\mathcal{M}_f \cap j_*j^*F_p\mathcal{M}_f$. Then, $s \in V_{<0}\mathcal{M}_f$, and there exists m such that $t^m s \in F_p\mathcal{M}_f$. We then claim that $s \in F_p\mathcal{M}_f$. Then, $t^m s \in V_\alpha$ for some $\alpha < -m$, and since $t\colon V_\alpha\mathcal{M}_f \overset{\sim}{\to} V_{\alpha-1}\mathcal{M}_f$ for $\alpha < 0$, we see that $s \in V_0\mathcal{M}_f$. ☐

Claim 17.8. *Fix p. Suppose $\partial_t\colon F_p\operatorname{gr}^V_\alpha \to F_p\operatorname{gr}^V_{\alpha+1}$ is surjective for $\alpha \geq -1$. Then,*

$$F_pM = \sum_{i=0}^{\infty} F_{p-i}V_{<0}\cdot\partial^i.$$

Proof. $\supseteq$ is trivial since F is a good filtration, so ∂^i maps F_{p-i} to F_p. For the other direction, the idea is to work inductively with the base case when $\alpha = -1$. ☐

The point of these two claims is that replacing $F_{p-i}V_{<0}$ with the formula in the previous claim, this determines the behavior of the filtration on $\mathcal{M}_f$ from the locus away from the fiber $f^{-1}(0)$.

Definition 17.9. We say $(\mathcal{M}', F_\bullet\mathcal{M})$ is *quasi-unipotent along* $\{f = 0\}$, where $f\colon X \to \mathbf{C}$ is a holomorphic function, if

- all eigenvalues of ${}^p\psi_f K$ are roots of unity;
- $t\colon F_pV_\alpha \to F_pV_{\alpha-1}$ is surjective for all $\alpha < 0$ and all p;
- $\partial_t\colon F_p\operatorname{gr}^V_\alpha \to F_{p+1}\operatorname{gr}^V_{\alpha+1}$ is surjective for all $\alpha > -1$ and all p.

The last two conditions are natural from the point of view of the previous claims.

We say $(\mathcal{M}, F_\bullet\mathcal{M})$ is *regular along* $\{f = 0\}$ if $F_\bullet\operatorname{gr}^V_\alpha$ is a good filtration for $-1 \leq \alpha \leq 0$.

18 February 20 and March 6: Functors on Hodge modules and an application (Takumi Murayama)

Today, we will introduce functors for Hodge modules in order to prove Popa and Schnell's theorem on zeros of one-forms on varieties of general type [PS14]. See [Sch14b] for an introductory account of the proof.

18.1 Statement of Popa and Schnell's theorem on zeros of one-forms [PS14]

Today's goal will be to introduce mixed Hodge modules, to the extent that we will need to prove the following:

Conjecture 18.1 [HK05, Conj. 1.1; LZ05, Conj. 1]**.** *If X is a smooth complex projective variety of general type, then the zero locus*

$$Z(\omega) \coloneqq \bigl\{x \in X \bigm| \omega(T_xX) = 0\bigr\}$$

of every global holomorphic one-form $\omega \in H^0(X, \Omega_X)$ is nonempty.

Note this is trivial if X is a curve, since then, $\Omega_X = \omega_X$ is the inverse of an ample line bundle; the case for surfaces was proved by Carrell [Car74]. To motivate Popa and Schnell's approach to this problem [PS14], we recall the following.

Recall 18.2 (see, e.g., [Huy05, §3.3])**.** Let X be a compact Kähler manifold of dimension n. Then, the *Albanese variety* is

$$\operatorname{Alb}(X) \coloneqq \frac{H^0(X, \Omega_X)^\vee}{H_1(X, \mathbf{Z})}$$

where we identify $H_1(X, \mathbf{Z})$ with its image via the map

$$\begin{aligned} H_1(X, \mathbf{Z}) &\longrightarrow H^0(X, \Omega_X)^\vee \\ [\gamma] &\longmapsto \Big(\alpha \mapsto \int_\gamma \alpha\Big) \end{aligned}$$

For a fixed base point $x_0 \in X$, the *Albanese morphism* is

$$\begin{aligned} \operatorname{alb}\colon X &\longrightarrow \operatorname{Alb}(X) \\ x &\longmapsto \Big(\alpha \mapsto \int_{x_0}^{x} \alpha\Big) \end{aligned}$$

Note that the integral here depends on the choice of path connecting x_0 and x, but their difference is an integral over a closed path γ, hence defines a well-defined point in $\operatorname{alb}(x) \in \operatorname{Alb}(X)$. Moreover, different choices of x_0 correspond to a translation on $\operatorname{Alb}(X)$. By choosing a basis for $H^0(X, \Omega_X)$ in coordinates, the albanese map induces an isomorphism

$$\operatorname{alb}^*\colon H^0\big(\operatorname{Alb}(X), \Omega_{\operatorname{Alb}(X)}\big) \xrightarrow{\sim} H^0(X, \Omega_X). \tag{18.1}$$

This suggests that we can use the geometry of the abelian variety $\operatorname{Alb}(X)$ to study the zeros of one-forms on X. More precisely, Popa and Schnell consider all morphisms $X \to A$, where A is an abelian variety, and show the following:

Theorem 18.3 [PS14, Thm. 2.1]**.** *Let X be a smooth complex projective variety, and let*

$$f\colon X \longrightarrow A$$

be a morphism to an abelian variety. Suppose for some $d \geq 1$ and some ample line bundle L, we have

$$H^0\big(X, \omega_X^{\otimes d} \otimes f^*L^{-1}\big) \neq 0.$$

Then, for every $\omega \in H^0(A, \Omega_A^1)$, the zero locus $Z(f^*\omega)$ is nonempty.

Proof that Theorem 18.3 implies Conjecture 18.1 [PS14, n° 4]. Let $\operatorname{alb}\colon X \to \operatorname{Alb}(X)$ be the Albanese morphism. Then, since X is of general type, the canonical bundle ω_X is big, and so for any ample line bundle L on $\operatorname{Alb}(X)$,

$$H^0\big(X, \omega_X^{\otimes d} \otimes f^*L^{-1}\big) \neq 0$$

for $d \gg 0$, since the divisor associated to $\omega_X^{\otimes d} \otimes f^*L^{-1}$ lies in the big cone of divisors. Conjecture 18.1 then follows from Theorem 18.3 by the isomorphism (18.1). □

A similar argument [PS14, n° 4] shows that for varieties not necessarily of general type, there can be non-vanishing one-forms, but they form a subspace in $H^0(X, \Omega_X)$ of dimension $\dim X - \kappa(X)$.

18.2 Strategy of proof [PS14, n° 10]

We now give a sketch of the key ideas of the proof. Denote $V \coloneqq H^0(A, \Omega_A^1)$. To study the vanishing of $f^*\omega$, we consider the following incidence variety:

$$\begin{array}{ccc} & Z_f \coloneqq \big\{(X, \omega) \in X \times V \bigm| x \in Z(f^*\omega)\big\} & \\ \swarrow & & \searrow \\ X & & V \end{array}$$

With this notation, it suffices to show that $Z_f \to V$ is surjective. Since our goal was to involve the geometry of the abelian variety A somehow, it ends up being better to study the push-forward of this incidence variety to $A \times V$:

$$\begin{array}{ccc} & S_f \coloneqq (f \times \mathrm{id}_V)(Z_f) = \big\{(a, \omega) \in A \times V \bigm| \exists\, x \in f^{-1}(a) \text{ such that } x \in Z(f^*\omega)\big\} & \\ \swarrow & & \searrow \\ X & & V \end{array}$$

Since the condition in brackets holds if and only if ω kills $\operatorname{im}(T_xX \to T_aA)$, we see that S_f measures the singularities of the map $f\colon X \to A$; for example, f is smooth if and only if $S_f = A \times \{0\}$.

Now to show $Z_f \to V$ is surjective, it suffices to show $S_f \to V$ is surjective since $S_f = (f \times \mathrm{id}_V)(Z_f)$ by definition. The proof of this follows in two steps:

Proposition 18.4. *There exists a Hodge module $(\mathcal{M}, F)$ on A, and a graded $\operatorname{Sym}(\Theta_A)$-submodule $\mathscr{F}_\bullet \subseteq \operatorname{gr}^F_\bullet \mathcal{M}$ corresponding to coherent sheaves $\mathscr{F}$ and $\mathscr{G} := \operatorname{gr}^F \mathcal{M}$ on $T^*A = A \times V$, such that*

(1) *For some $k \in \mathbf{Z}$, the sheaf $\mathscr{F}_k$ is isomorphic to $L \otimes f_*\mathcal{O}_X$, where L is an ample line bundle on A;*
(2) $\operatorname{Supp} \mathscr{F} \subseteq S_f$.

Proposition 18.5. *If the objects in Proposition 18.4 hold, then*

$$p_{2*}(\operatorname{gr}^F \mathcal{M} \otimes p_1^*\alpha)$$

is locally free on V for generic $\alpha \in \operatorname{Pic}^0(A)$, where

$$\begin{array}{ccc} & A \times V & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ A & & V \end{array} \tag{18.2}$$

Proof of Theorem 18.3 assuming Proposition 18.5 [PS14, Prop. 10.2]*.* We have an inclusion

$$p_{2*}(\mathscr{F} \otimes p_1^*\alpha) \subseteq p_{2*}(\mathscr{G} \otimes p_1^*\alpha).$$

We first show the left-hand side is nonzero. It is the sheaf on V corresponding to the $\operatorname{Sym}(V^*)$-module $\bigoplus_k H^0(A, \mathscr{F}_k \otimes \alpha)$, one of whose summands is $H^0(A, L \otimes f_*\mathcal{O}_X \otimes \alpha)$. But this is nonzero since otherwise, $f(X)$ is contained in every general translate of the ample divisor L, which cannot occur. Note also that its support is contained in $p_2(S_f)$.

Now the right-hand side is locally free by Proposition 18.5, hence the left-hand side cannot be torsion. This implies $p_2(S_f) = V$. □

18.3 Technical preliminaries [Sch14a; Pop17, §2]

Before we begin the proofs of Propositions 18.4 and 18.5, we need to develop some more functorial formalism for Hodge modules. Like for D-modules, there are several important functors: we will focus on the duality, direct image, and inverse image functors. Since we are working with right D-modules, the direct image functor is the easiest to discuss, and will be the most important. We will also discuss polarizations.

18.3.1 Strictness

Definition 18.6. A morphism $f\colon (\mathcal{M}, F) \to (\mathcal{N}, F)$ of filtered D-modules is *strict* if

$$f(F_k\mathcal{M}) = F_k\mathcal{N} \cap f(\mathcal{M})$$

for all k. A complex $(\mathcal{M}^\bullet, F_\bullet)$ of filtered D-modules is strict if all of its differentials are strict.

Lemma 18.7. *If a complex $(\mathcal{M}^\bullet, F_\bullet)$ of filtered D-modules is strict, then $\mathcal{H}^i\mathcal{M}^\bullet$ is a filtered D-module with filtration*

$$F_k(\mathcal{H}^i\mathcal{M}^\bullet) := \operatorname{im}\bigl(\mathcal{H}^i(F_k\mathcal{M}^\bullet) \longrightarrow \mathcal{H}^i\mathcal{M}^\bullet\bigr).$$

Proof idea. Strictness means that subquotients have well-defined filtrations, similar to what we saw for strict morphisms of Hodge structures. □

18.3.2 The duality functor and polarizations [Sai88, §2.4; Sch14a, §§13,29]

We start with the duality functor, which is surprisingly subtle.

Definition 18.8. Given two filtered D-modules $(\mathcal{M}, F)$ and $(\mathcal{N}, F)$, we define

$$F_k \mathscr{H}om_D^F\big((\mathcal{M}, F), (\mathcal{N}, F)\big) \coloneqq \big\{\phi \in \mathscr{H}om_D(\mathcal{M}, \mathcal{N}) \mathrel{\big|} \phi(F_i\mathcal{M}) \subset F_{p+i}\mathcal{N} \text{ for all } i\big\}.$$

Given a filtered complex of D-modules $(\mathcal{M}^\bullet, F)$ on an n-dimensional variety, the dual is given by

$$\mathbf{D}\mathcal{M}^\bullet \coloneqq \mathbf{R}\mathscr{H}om_{D_X}(\mathcal{M}^\bullet, \omega_X \otimes^{\mathbf{L}}_{\mathcal{O}_X} D_X)[n],$$

where ω_X is given by the filtration induced from the de Rham complex, $\omega_X \otimes^{\mathbf{L}}_{\mathcal{O}_X} D_X$ has the tensor product filtration, and the $\mathbf{R}\mathscr{H}om$ has a filtration as described above.

Strictness is not preserved for arbitrary filtered D-modules, but luckily, they are for Hodge modules:

Theorem 18.9 [Sai88, Lem. 5.1.13; Sch14a, Thm. 29.3]**.** *If $M = (\mathcal{M}, F_\bullet) \in \mathsf{HM}(X, w)$, then the dual complex is strict and underlies a Hodge module $\mathbf{D}M \in \mathsf{HM}(X, -w)$.*

Since our definitions are recursive, we might expect to (and indeed, Saito does) prove this by induction on $\dim \operatorname{Supp} M$. The key fact used is that $\mathbf{D}$ is compatible with nearby and vanishing cycles.

With this in mind, we may define polarizations:

Definition 18.10 (Polarizations)**.** Let $M = (\mathcal{M}, F, K) \in \mathsf{HM}(X, w)$. Then, a *polarization* on M is an isomorphism $K(w) \to \mathbf{D}K$ such that

(1) It is nondegenerate and compatible with the filtration, i.e., it extends to an isomorphism $M(w) \simeq \mathbf{D}M$ of Hodge modules;

(2) For each summand M_Z in the decomposition of M by strict support, and for every locally defined holomorphic function $f\colon U \to \mathbf{C}$ that is not identically zero on $U \cap Z$, the induced morphism

$$^p\psi_f K_Z(w) \longrightarrow \mathbf{D}(^p\psi_f K_Z)$$

is a "polarization of Hodge–Lefschetz type";

(3) If $\dim \operatorname{Supp} M_Z = 0$, then $K_Z(w) \to \mathbf{D}K_Z$ is induced by a polarization of Hodge structures.

The subcategory of *polarizable Hodge modules* of weight w is denoted

$$\mathsf{HM}^p(X, w) \subseteq \mathsf{HM}(X, w).$$

18.3.3 Direct images of Hodge modules [Sch14a, §27]

Let $f\colon X \to Y$ be a morphism. Recall that on the level of D-modules, we already have an exact functor

$$\begin{aligned} \int_f\colon \mathsf{D}^b(D_X) &\longrightarrow \mathsf{D}^b(D_Y) \\ \mathcal{M}^\bullet &\longmapsto \mathbf{R}f_*(\mathcal{M}^\bullet \otimes^{\mathbf{L}}_{D_X} D_{X\to Y}) \end{aligned}$$

When applying the same definition to our category of triples, this functor is automatically compatible with the $\mathbf{Q}$-perverse sheaf by the Riemann–Hilbert correspondence. The issue is that we have not defined a filtration on $\int_f \mathcal{M}^\bullet$.

Recall the graph factorization:

$$\begin{array}{ccc} X & \hookrightarrow & X \times Y \\ & \overset{f}{\searrow} & \downarrow{\scriptstyle p_2} \\ & & Y \end{array}$$

where $X \hookrightarrow X \times Y$ is a closed embedding. We therefore only need to define how the filtration works for closed embeddings and projections.

Definition 18.11 (Filtration for closed immersions)**.** Let $i\colon X \hookrightarrow Y$ be a closed embedding. Then,

$$(D_{X\to Y}, F_\bullet D_{X\to Y}) \coloneqq \mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}(D_Y, F_\bullet D_Y).$$

The direct image $\int_i \mathcal{M}^\bullet$ of $\mathcal{M}^\bullet \in \mathsf{D}^b(D_X)$ has the tensor product filtration; locally, if $\mathcal{M}^\bullet = \mathcal{M}$ is an actual D-module, then

$$F_k\Big(\int_i \mathcal{M}\Big) = \sum_{\alpha\in\mathbf{N}^r} F_{k-|\alpha|}\mathcal{M} \otimes \partial^\alpha,$$

where $X = (t_1 = \cdots = t_r = 0)$, and ∂_i are the corresponding differentials. This preserves strictness of complexes.

For projections, things are a bit more difficult.

Definition 18.12 (Filtration for projections)**.** Let $p_2\colon X \times Y \to Y$ be the projection map. Recall (Proposition 5.21) that the direct image of $\mathcal{M}^\bullet \in \mathsf{D}^b(D_{X\times Y})$ is given by

$$\int_{p_2} \mathcal{M}^\bullet \simeq \mathbf{R}p_{2*}\big(\mathrm{DR}_{X\times Y/Y}(\mathcal{M}^\bullet)\big).$$

The filtration on $\int_{p_2} \mathcal{M}^\bullet$ is given by

$$F_k\Big(\int_{p_2} \mathcal{M}^\bullet\Big) \coloneqq \mathbf{R}p_{2*} F_k\, \mathrm{DR}_{X\times Y/Y}(\mathcal{M}),$$

where $\mathrm{DR}_{X\times Y/Y}(\mathcal{M})$ is given the filtration induced by the tensor product of the filtrations on Ω_X^p and on $\mathcal{M}$, and $\mathbf{R}p_{2*}$ is computed using the canonical Godement resolution by flasque sheaves [God73, §4.3].

There is an issue here, in that it is no longer obvious that the direct image should preserve strictness! This becomes important in examples, since, for example, we deduced Kollár's vanishing theorem Theorem 16.5(*ii*) by using the Hodge module $\int_f^j \omega_X$, which would not necessarily even be a filtered D-module unless we could show that $\int_f \omega_X$ is strict. This is one of the main results of [Sai88].

Theorem 18.13 (Direct Image Theorem [Sch14a, Thm. 16.1])**.** *Let $f\colon X \to Y$ be a projective morphism between two complex manifolds, and let $M \in \mathsf{HM}^p(X, w)$. Then, the complex $\int_f(\mathcal{M}, F_\bullet\mathcal{M})$ is strict, and $\mathcal{H}^i f_* M \in \mathsf{HM}^p(Y, w+i)$.*

Proof Sketch. The proof is quite involved; for details, see [Sai88, §5.3; Sch14a, §17].

Since Hodge modules were defined recursively, it is natural to split M up into components with strict support Z, and then prove Theorem 18.13 by induction on $\dim Z$. The argument is in three parts:

(1) Prove the theorem for $\dim X = 1$;
(2) Prove the theorem in the case $\dim f(Z) \geq 1$;
(3) Prove the theorem in the case $\dim f(Z) = 0$.

For (1), it is in fact true that $X = \mathbf{P}^1$ is the only necessary case, in which cause $Z = \mathbf{P}^1$, or is a disjoint union of points. The latter case is trivial; in the former case, the argument is due to Zucker.

For (2), the key point is to use nearby and vanishing cycles on Y to reduce the dimension on Z, and use the inductive hypothesis. To show that $\mathcal{H}^i f_* M \in \mathsf{HM}(Y, w+i)$, consider an arbitrary locally defined holomorphic function $g\colon U \to \mathbf{C}$. First suppose $f(Z) \not\subseteq g^{-1}(0)$, and consider the composition $h = g \circ f$. Then, we have the spectral sequence from a filtration:

$$E_1^{p,q} = \mathcal{H}^{p+q} f_*(\mathrm{gr}_{-p}^W \psi_h M) \Longrightarrow \mathcal{H}^{p+q} f_* \psi_h M,$$

where W denotes the monodromy weight filtration form Definition 16.15.

- By induction $\mathrm{gr}_{-p}^W \psi_h M \in \mathsf{HM}^p(X, w-1-p)$, hence $E_1^{p,q} \in \mathsf{HM}^p(X, w-1+q)$.
- Compatibility of direct image with nearby and vanishing cycles (part of the Riemann–Hilbert correspondence) implies

$$\psi_g \mathcal{H}^i f_* M \simeq \mathcal{H}^i f_* \psi_h M \qquad \text{and} \qquad \phi_{g,1} \mathcal{H}^i f_* M \simeq \mathcal{H}^i f_* \phi_{h,1} M,$$

so the objects on the left-hand side are Hodge modules on Y by induction.

- Strictness is obtained by checking locally in small analytic neighborhoods of $g^{-1}(0)$.

Other properties follow more or less from the spectral sequence above. A subtle part of the argument is that decomposition by strict support uses the polarization in an essential way.

Finally, for (3), we may assume X is projective space and Y is a point (by using the graph factorization). The idea is to use a pencil of hyperplane sections, and use the inductive hypothesis. Let $\pi\colon \widetilde{X} \to X$ be the blowup of X at a general linear subspace of codimension 2; we obtain a diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\pi} & X \\ {\scriptstyle p}\big\downarrow & \overset{\tilde{f}}{\searrow} & \big\downarrow{\scriptstyle f} \\ \mathbf{P}^1 & \longrightarrow & \{*\} \end{array}$$

To apply the inductive hypothesis, we use the structure theorem to induce a Hodge module $\widetilde{M}$ on the strict transform of Z in $\widetilde{X}$. Now the claim for f follows by combining the inductive hypotheses for π, p, and $\mathbf{P}^1 \to \{*\}$. □

18.3.4 Inverse images [Sch14a, §30]

We will only define these for smooth morphisms, since the general definition is quite difficult.

Definition 18.14. Let $f\colon Y \to X$ be a smooth morphism, and let $M = (\mathcal{M}, F, K) \in \mathsf{HM}^p(X, w)$. Then, letting $r = \dim Y - \dim X$, define

$$\begin{aligned} \widetilde{M} &= (\widetilde{\mathcal{M}}, F, \widetilde{K}) \\ \widetilde{\mathcal{M}} &= \omega_{Y/X} \otimes_{\mathcal{O}_Y} f^*\mathcal{M} \\ F_p\widetilde{\mathcal{M}} &= \omega_{Y/X} \otimes_{\mathcal{O}_Y} f^*F_{p+r}\mathcal{M} \\ \widetilde{K} &= f^{-1}K(-r) \end{aligned}$$

The factor $\omega_{Y/X}$ is crucial to make sure $\mathrm{DR}(\widetilde{\mathcal{M}}) \simeq \mathbf{C} \otimes_{\mathbf{Q}} \widetilde{K}$.

Theorem 18.15 [Sch14a, Thm. 30.1]**.** *If $M \in \mathsf{HM}^p(X, w)$, then $\widetilde{M} \in \mathsf{HM}^p(Y, w + r)$.*

Proof Sketch. You can't prove this directly; instead, you must use the structure theorem. Suppose M has strict support Z. Then, its pullback is a generically defined polarizable variation of Hodge structure on $f^{-1}(Z)$ of the same weight, hence extends to an object of $\mathsf{HM}^p_{f^{-1}(Z)}(Y, w + r)$. You then check that this extension is isomorphic to $\widetilde{M}$. □

Example 18.16. We can finally show $\mathbf{Q}^H_X[n]$ from Example 16.4 is a Hodge module for smooth X! The constant Hodge structure $(\mathcal{O}_{\mathrm{pt}}, F, \mathbf{Q})$ on a point is a Hodge module by definition. Its pullback then gives the Hodge module $\mathbf{Q}^H_X[n]$, where we note that this is why we get ω_X to be the underlying D-module, with the "trivial" filtration, for the "trivial" Hodge module $\mathbf{Q}^H_X[n]$.

18.4 Proof of Proposition 18.5, assuming Proposition 18.4 [PS14, nº 10]

Step 1 [PS13, Lem. 2.5] (use Saito Vanishing)**.** *If A is an abelian variety, and $(\mathcal{M}, F)$ is a Hodge module on A, then $H^i(A, \mathrm{gr}^F_k \mathcal{M} \otimes L) = 0$ for all $i > 0$ and $k \in \mathbf{Z}$.*

Proof. For each $k \in \mathbf{Z}$, consider the complex of coherent sheaves

$$\mathrm{gr}^F_k \mathrm{DR}_A(\mathcal{M}) = \Big\{ \mathrm{gr}^F_k \mathcal{M} \longrightarrow \Omega^1_A \otimes \mathrm{gr}^F_{k+1}\mathcal{M} \longrightarrow \cdots \longrightarrow \underbrace{\Omega^g_A \otimes \mathrm{gr}^F_{k+g}\mathcal{M}}_{\deg 0} \Big\}.$$

where $g = \dim A$. Recall that Saito's vanishing theorem 16.6 says that for all $i > 0$,

$$\mathbf{H}^i\big(A, \mathrm{gr}^F_k \mathrm{DR}_A(\mathcal{M}) \otimes L\big) = 0.$$

Let p such that $F_{p-1}\mathcal{M} = 0$ but $F_p\mathcal{M} \neq 0$. We proceed by induction on $k \geq p$. If $k = p$, then

$$\operatorname{gr}^F_{p-g} \operatorname{DR}_A(\mathcal{M}) = \Big\{0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow \underset{\deg 0}{\operatorname{gr}^F_p \mathcal{M}}\Big\}$$

using that $\Omega^1_A \simeq \mathcal{O}_A^{\oplus g}$, and so $H^i(A, \operatorname{gr}^F_g \mathcal{M} \otimes L) = 0$ for all $i > 0$.

If $k > p$, then letting

$$E_k \coloneqq \Big\{\operatorname{gr}^F_k \mathcal{M} \longrightarrow \Omega^1_A \otimes \operatorname{gr}^F_{k+1} \mathcal{M} \longrightarrow \cdots \longrightarrow \Omega^{g-1}_A \otimes \operatorname{gr}^F_{k+g-1} \mathcal{M} \longrightarrow \underset{\deg 0}{0}\Big\},$$

we have the distinguished triangle

$$E_k \longrightarrow \operatorname{gr}^F_k \operatorname{DR}_A(\mathcal{M}) \longrightarrow \operatorname{gr}^F_{k+g} \mathcal{M} \xrightarrow{[1]}$$

Note E_k satisfies $\mathbf{H}^i(A, E_k \otimes L) = 0$ for all $i > 0$ by inductive hypothesis and using the hypercohomology spectral sequence. Since the middle complex also has vanishing higher hypercohomology, we have $H^i(A, \operatorname{gr}^F_k \mathcal{M} \otimes L) = 0$ for all $i > 0$ as desired. □

Step 2 [PS13, Thm. 1.1] (Vanishing ⇒ Generic vanishing)**.** *If $A, (\mathcal{M}, F)$ are as in Step 1, and if $\alpha \in \operatorname{Pic}^0(A)$ is generic, then $H^i(A, \operatorname{gr}^F_k \mathcal{M} \otimes \alpha) = 0$ for all $i > 0$ and $k \in \mathbf{Z}$.*

Proof. A result of Hacon [Hac04] (as cited in [Sch13, Thm. 25.5]) says that it suffices to show that for every finite étale morphism $\varphi\colon A' \to A$ between abelian varieties, and for all L' ample on A', we have

$$H^i\big(A', \varphi^* \operatorname{gr}^F_k \mathcal{M} \otimes L'\big) = 0$$

for all $i > 0$.

Let $(\mathcal{N}, F) = \varphi^*(\mathcal{M}, F)$ be the pullback of the Hodge module $(\mathcal{M}, F)$ to A'. Then, $F_k\mathcal{N} = \varphi^* F_k\mathcal{M}$ since φ is étale (Definition 18.14). Then,

$$H^i\big(A', \varphi^* \operatorname{gr}^F_k \mathcal{M} \otimes L'\big) = H^i\big(A', \operatorname{gr}^F_k \mathcal{N} \otimes L'\big) = 0$$

by Step 1. □

Step 3. *With notation as in (18.2), the complex $\mathbf{R}p_{2*}(\operatorname{gr}^F \mathcal{M} \otimes p_1^*\alpha) \in \mathsf{D}^b(\mathsf{Coh}\, V)$ is concentrated in degree 0 if $\alpha \in \operatorname{Pic}^0(A)$ is generic.*

Proof. We have that the sheaf $R^i p_{2*}(\operatorname{gr}^F \mathcal{M} \otimes p_1^*\alpha)$ corresponds to the module

$$H^i\big(A \times V, \operatorname{gr}^F \mathcal{M} \otimes p_1^*\alpha\big)$$

over $V \simeq \mathbf{A}^g$ by [Har77, Prop. 8.5]. Now by the equivalence of modules on T^*A and graded modules over $\operatorname{Sym}(\Theta_A)$ given by p_{1*}, we have

$$H^i\big(A \times V, \operatorname{gr}^F \mathcal{M} \otimes p_1^*\alpha\big) \cong H^i\big(A, p_{1*}(\operatorname{gr}^F \mathcal{M} \otimes p_1^*\alpha)\big) \cong \bigoplus_{k \in \mathbf{Z}} H^i\big(A, \operatorname{gr}^F_k \mathcal{M} \otimes \alpha\big) = 0$$

by using the projection formula and by Step 2. □

Step 4. *$p_{w*}(\operatorname{gr}^F \mathcal{M} \otimes p_q^*\alpha)$ is locally free on V for generic $\alpha \in \operatorname{Pic}^0(A)$.*

Proof. $p_{w*}(\operatorname{gr}^F \mathcal{M} \otimes p_1^*\alpha)$ is locally free if and only if $\mathbf{R}\mathscr{H}\!om(p_{2*}(\operatorname{gr}^F \mathcal{M} \otimes p_1^*\alpha), \mathcal{O}_V)$ is concentrated in degree zero (you can see this by considering locally free sheaves as sheaves whose stalks are all projective modules, in which case vanishing of Ext groups is exactly the condition for projectivity). But

$$\begin{aligned}
\mathbf{R}\mathscr{H}\!om(p_{2*}(\operatorname{gr}^f \mathcal{M} \otimes p_1^*\alpha), \mathcal{O}_V) &= \mathbf{R}\mathscr{H}\!om(\mathbf{R}p_{2*}(\operatorname{gr}^F \mathcal{M} \otimes p_1^*\alpha), \mathcal{O}_V) \\
&\underset{\mathrm{GD}}{\cong} \mathbf{R}p_{2*}\big(\mathbf{R}\mathscr{H}\!om\big((\operatorname{gr}^F \mathcal{M} \otimes p_1^*\alpha), \mathcal{O}_{A \times V}[n]\big)\big) \\
&= \mathbf{R}p_{2*}\big(p_1^*\alpha^{-1} \otimes \mathbf{R}\mathscr{H}\!om(\operatorname{gr}^F \mathcal{M}, \mathcal{O}_{A \times V}[n])\big)
\end{aligned}$$

Now $\mathbf{R}\mathscr{H}\!om(\operatorname{gr}^F \mathcal{M}, \mathcal{O}_{A \times V}[n])$ is of the form $\operatorname{gr}^F \mathcal{N}$, where $(\mathcal{N}, F)$ is the dual Hodge module to $(\mathcal{M}, F)$ (Definition 18.8). Thus, the complex above is concentrated in degree zero by Step 3. □

18.5 Construction of objects as in Proposition 18.4 [PS14, n$^{\text{o}}$s 11–17]

Recall our main goal:

Theorem 18.3. *Let X be a smooth complex projective variety, and let*

$$f\colon X \longrightarrow A$$

be a morphism to an abelian variety. Suppose for some $d \geq 1$ and some ample line bundle L, we have

$$H^0\big(X, \omega_X^{\otimes d} \otimes f^*L^{-1}\big) \neq 0. \tag{18.3}$$

Then, for every $\omega \in H^0(A, \Omega_A^1)$, the zero locus $Z(f^*\omega)$ is nonempty.

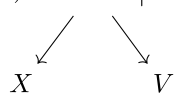
We restrict to the case where $\kappa(X) \geq 0$, for otherwise the statement is vacuous.

We still need to show the following:

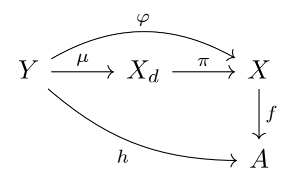
Proposition 18.4. *There exists a Hodge module $(\mathcal{M}, F)$ on A, and a graded $\mathrm{Sym}(\Theta_A)$-submodule $\mathscr{F}_\bullet \subseteq \mathrm{gr}_\bullet^F \mathcal{M}$ corresponding to coherent sheaves $\mathscr{F}$ and $\mathscr{G} := \mathrm{gr}^F \mathcal{M}$ on $T^*A = A \times V$, such that*

(1) *For some $k \in \mathbf{Z}$, the sheaf $\mathscr{F}_k$ is isomorphic to $L \otimes f_*\mathcal{O}_X$, where L is an ample line bundle on A;*
(2) $\mathrm{Supp}\,\mathscr{F} \subseteq S_f$, *where*

$$S_f := (f \times \mathrm{id}_V)(Z_f) = \big\{(a,\omega) \in A \times V \bigm| \exists\, x \in f^{-1}(a) \text{ such that } x \in Z(f^*\omega)\big\}$$



The Hodge module $(\mathcal{M}, F)$ on A will be of the form $\mathcal{H}^0 \int_h (\omega_Y, F)$ for some morphism $h\colon Y \to A$ from a smooth projective variety. The map h will fit into the following diagram:



where $\pi\colon X_d \to X$ is a cyclic branched cover of degree d, and $\mu\colon Y \to X_d$ is a resolution of singularities. We will need to change our setting somewhat to be able to take this cyclic cover.

18.5.1 Preliminaries on cyclic covers [EV92, §3]

We start with some background on cyclic covers. This is also treated in [KM98, Def. 2.50; Laz04, Prop. 4.1.6].

Let B be a line bundle on a variety X, and suppose

$$0 \neq s \in H^0(X, B^{\otimes d}).$$

This determines a morphism of line bundles

$$s\colon \mathcal{O}_X \longrightarrow B^{\otimes d},$$

which gives the sheaf

$$\mathcal{A}' = \bigoplus_{i=0}^{d-1} B^{\otimes -i}$$

the structure of a graded $\mathcal{O}_X$-algebra, via the multiplication

$$B^{\otimes -i} \otimes B^{\otimes -j} \xrightarrow{\ \sim\ } B^{\otimes -(i+j)} \otimes \mathcal{O}_X \xrightarrow{\mathrm{id} \otimes s} B^{\otimes d-(i+j)}$$

if $i + j \geq d - 1$. The associated affine morphism

$$\pi\colon X_d \coloneqq \mathbf{Spec}_X \mathcal{A}' \longrightarrow X$$

is a branched covering of degree d, with branch locus $Z(s)$, and such that s acquires a dth root. One way to see this is as follows. By our construction of X_d, we have an inclusion

$$X_d \subseteq \mathbf{V}(B^{-1})$$

where notation is as in [Har77, Ch. II, Exc. 5.18(a)]. Denoting $p\colon \mathbf{V}(B^{-1}) \to X$, the pullback p^*B has a tautological section

$$t\colon \mathcal{O}_{\mathbf{V}(B^{-1})} \longrightarrow p^*B$$

corresponding to the map

$$\operatorname{Sym}_{\mathcal{O}_X}(B^{-1}) \longrightarrow B \otimes \operatorname{Sym}_{\mathcal{O}_X}(B^{-1})$$

of graded quasi-coherent sheaves on X, which is the canonical inclusion of the sheaf on the left as a direct summand. Then, X_d is given by the vanishing of $t^m - p^*s \in H^0\big(\mathbf{V}(B^{-1}), p^*B^{\otimes d}\big)$. In particular, t restricts to a section

$$0 \neq t|_{X_d} \in H^0(X_d, \pi^*B).$$

18.5.2 The construction

To construct the cyclic cover, we need to be in a setting where the hypothesis in (18.3) has $L \simeq L'^{\otimes d}$ for some ample line bundle L'.

Lemma 18.17 [PS14, Lem. 11.1]**.** *After finite étale base change, we may assume without loss of generality that* $H^0\big(X, (\omega_X \otimes f^*L^{-1})^{\otimes d}\big) \neq 0$.

Proof. Consider the diagram

$$\begin{CD} X' @>>> X \\ @V{f'}VV @VV{f}V \\ A @>{[2d]}>> A \end{CD}$$

where $[2d]$ is the multiplication-by-$2d$ morphism on A. Vanishing of one-forms on X can be detected on X' since f' is finite étale. Finally, by [Mum08, §6, Cor. 3], we have that

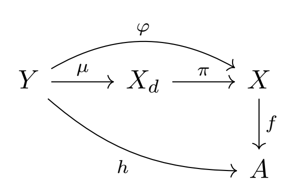
$$[2d]^*L \simeq L^{\otimes d(d+1)} \otimes [-1]^*L^{\otimes d(d-1)} \simeq (L^{\otimes(d+1)} \otimes [-1]^*L^{\otimes(d-1)})^{\otimes d}$$

is the dth power of an ample line bundle. Thus, we may replace X with X' and L with $[2d]^*L$ to conclude. □

From now on, we will take d to be the *smallest* power of $B \coloneqq \omega_X \otimes f^*L^{-1}$ which has a section, and consider the dth cylic cover

$$\pi\colon X_d \longrightarrow X$$

associated to a nonzero section $s \in H^0(X, B^{\otimes d})$. Since d is minimal, X_d is irreducible (by the same local description as before). Let $\mu\colon Y \to X_d$ be a resolution of singularities, which is an isomorphism over the complement of $Z(s)$, and consider the diagram



Remark 18.18. [Sch14b, p. 5] notes that the original construction due to [VZ01] is a bit finicky: the resolution μ and the section s have to be chosen very carefully to avoid singularities. This is where Hodge modules show their utility: they can be pulled back and pushed forward by arbitrary morphisms.

Let $S = \operatorname{Sym} H^0(A, \Omega^1_A)^\vee$. Consider the complex of graded $\mathcal{O}_X \otimes S$-modules

$$C_{X,\bullet} := \Big\{ \mathcal{O}_X \otimes S_{\bullet-g} \longrightarrow \Omega^1_X \otimes S_{\bullet-g+1} \longrightarrow \cdots \longrightarrow \underbrace{\Omega^n_X \otimes S_{\bullet-g+n}}_{\deg 0} \Big\}$$

where $g = \dim A$ and $n = \dim X$, and the differential is induced by the evaluation morphism $V \otimes \mathcal{O}_X \to \Omega^1_X$: letting $\omega_i \in V$ be a basis and $s_i \in S_1$ be the dual basis, the differential is

$$\begin{aligned} d\colon \Omega^p_X \otimes S_{\bullet-g+p} &\longrightarrow \Omega^{p+1}_X \otimes S_{\bullet-g+p+1} \\ \theta \otimes s &\longmapsto \sum_{i=1}^{g} \big(\theta \wedge f^*\omega_i\big) \otimes s_i s \end{aligned}$$

We can similarly define $C_{Y,\bullet}$.

Proposition 18.19 [PS14, Prop. 17.1]**.** *There is a morphism of complexes of graded $\mathcal{O}_A \otimes S$-modules*

$$\mathbf{R}f_*(B^{-1} \otimes C_{X,\bullet}) \longrightarrow \mathbf{R}h_* C_{Y,\bullet}. \tag{18.4}$$

Moreover, letting

$$\mathscr{F}_\bullet := \operatorname{im}\big(R^0 f_*(B^{-1} \otimes C_{X,\bullet}) \longrightarrow R^0 h_* C_{Y,\bullet}\big) \qquad (\mathcal{M}, F) = \mathcal{H}^0 \textstyle\int_h (\omega_Y, F),$$

we have that $\mathscr{F}_\bullet \subseteq \operatorname{gr}^F_\bullet \mathcal{M}$, and

1. *$\mathscr{F}_{g-n} \simeq L \otimes f_*\mathcal{O}_X$, where L is an ample line bundle on A;*
2. *$\operatorname{Supp} \mathscr{F} \subseteq S_f$, where*

$$S_f := (f \times \mathrm{id}_V)(Z_f) = \big\{ (a, \omega) \in A \times V \bigm| \exists\, x \in f^{-1}(a) \text{ such that } x \in Z(f^*\omega) \big\}$$

$$\swarrow \qquad \searrow$$

$$X \qquad\qquad V$$

Note that $\mathcal{H}^0 \int_h (\omega_Y, F)$ is indeed a Hodge module since h is projective, and then by the Direct Image Theorem 18.13.

Step 1 ([PS14, Lem. 13.1; PS16, Prop. 2.8])**.** *Construction of the morphism* (18.4).

Construction. By adjunction, it suffices to construct a morphism of complexes

$$\varphi^*(B^{-1} \otimes C_{X,\bullet}) \longrightarrow C_{Y,\bullet}.$$

Since $H^0(X_d, \pi^*B) \neq 0$ by construction of the cyclic cover, we have a nontrivial injective morphism

$$\pi^* B^{-1} \longrightarrow \mathcal{O}_{X_d}.$$

Applying μ^* gives a nontrivial morphism

$$\varphi^* B^{-1} \longrightarrow \mathcal{O}_Y,$$

and tensoring this with $\varphi^*\Omega^k_X$ gives a morphism

$$\varphi^*(B^{-1} \otimes \Omega^k_X) \longrightarrow \varphi^*\Omega^k_X \longrightarrow \Omega^k_Y, \tag{18.5}$$

where the last map is induced by pullback of forms. Tensoring this with $S_{\bullet-g+p}$ gives the desired morphism, where we note that this definition commutes with the differentials since $\varphi^*(f^*\omega) = h^*\omega$ for every $\omega \in V$. ☐

Remark 18.20. We will need later that the morphism (18.5) is injective.

Step 2 [PS14, Lem. 14.1]**.** *The support of C_X, the complex of coherent sheaves on $X \times V$ associated to $C_{X,\bullet}$, has support equal to $Z_f \subseteq X \times V$, where*

$$Z_f := \big\{ (X, \omega) \in X \times V \bigm| x \in Z(f^*\omega) \big\}$$

$$\swarrow \qquad \searrow$$

$$X \qquad\qquad V$$

Proof. If $p_1 \colon X \times V \to X$ denotes the first projection, then

$$C_X = \Big\{ p_1^*\mathcal{O}_X \longrightarrow p_1^*\Omega_X^1 \longrightarrow \cdots \longrightarrow \underset{\deg 0}{p_1^*\Omega_X^n} \Big\}$$

is the pullback of the Koszul resolution of the structure sheaf of the zero section of T^*X via the morphism

$$\begin{aligned} df \colon X \times V &\longrightarrow T^*X \\ (x, \omega) &\longmapsto f^*\omega(x) \end{aligned}$$

Thus, $\operatorname{Supp} C_X = df^{-1}(0) = Z_f$. □

Step 3. $\operatorname{Supp} \mathscr{F} \subseteq S_f$.

Proof. First, by definition, we note that $\mathscr{F}$, by definition, is a quotient of the sheaf

$$R^0(f \times \mathrm{id})_*(p_1^*B^{-1} \otimes C_X).$$

Now

$$\operatorname{Supp}(p_1^*B^{-1} \otimes C_X) = \operatorname{Supp} C_X = Z_f,$$

so $\operatorname{Supp} \mathscr{F} \subseteq (f \times \mathrm{id})(Z_f) =: S_f$. □

Step 4 [PS13, Prop. 2.11; PS14, Lem. 15.1]**.** $\mathscr{F}_\bullet \subseteq \mathrm{gr}_\bullet^F \mathcal{M}$.

Proof. By tracking the filtrations, a theorem of Laumon [PS13, Thm. 2.4] says

$$\mathrm{gr}_\bullet^F \big(\textstyle\int_h (\omega_Y, F) \big) \simeq \mathbf{R} h_* C_{Y,\bullet}$$

as gradeed modules over $\mathcal{O}_A \otimes S$. By the Direct Image Theorem 18.13, taking graded pieces commutes with taking cohomology, so we have

$$\mathrm{gr}_\bullet^F \big(\mathcal{H}^0 \textstyle\int_h (\omega_Y, F) \big) \simeq R^0 h_* C_{Y,\bullet}.$$

Since $\mathscr{F}_\bullet$ was defined as the image of the morphism

$$R^0 f_*(B^{-1} \otimes C_{X,\bullet}) \longrightarrow R^0 h_* C_{Y,\bullet}$$

induced by (18.4), we are done. □

Step 5. $\mathscr{F}_{g-n} \simeq L \otimes f_*\mathcal{O}_X$.

Proof. By definition, we have that $C_{X,g-n} = \omega_X$ and $C_{Y,g-n} = \omega_Y$. Now the morphism

$$f^*L = B^{-1} \otimes \omega_X \longrightarrow \varphi_*\omega_Y$$

is injective since the morphism (18.5) was injective, and injectivity is preserved by adjunction since $\mathcal{O}_Y \hookrightarrow \varphi_*\mathcal{O}_Z$. After pushing forward to A, we have that the resulting morphism

$$L \otimes f_*\mathcal{O}_X \simeq f_*f^*L \longrightarrow h_*\omega_Y$$

is still injective. But $\mathrm{gr}_{g-n}^F \mathcal{M} \simeq h_*\omega_Y$, and so $\mathscr{F}_{g-n} \simeq L \otimes f_*\mathcal{O}_X$. □

19 March 27: Mixed Hodge modules (Takumi Murayama)

Today, I will introduce mixed Hodge modules on complex manifolds and algebraic mixed Hodge modules on complex algebraic varieties. They are the Hodge module analogue of variations of mixed Hodge structures, and will become important in applications. One reason for this is that in Hodge theory, cohomology groups of smooth varieties have pure Hodge structures, but as we saw before, cohomology groups of singular and/or non-complete varieties give examples of mixed Hodge structures (§13.2).

19.1 Weakly mixed Hodge modules [Sch14a, §19]

Recall 19.1. In the definition of pure Hodge modules, we had the following category of filtered regular holonomic D-modules with $\mathbf{Q}$-structure:

$$\mathsf{MF}_{\mathsf{rh}}(D_X, \mathbf{Q}) = \left\{ (\mathcal{M}, F_\bullet, K) \;\middle|\; \begin{array}{l} 1.\ K \in \mathsf{Perv}_{\mathbf{Q}}(X) \\ 2.\ \mathcal{M} \in \mathsf{Mod}_{\mathsf{rh}}(D_X) \text{ with an isomorphism} \\ \quad \alpha\colon \operatorname{DR}_X(\mathcal{M}) \xrightarrow{\sim} \mathbf{C} \otimes_{\mathbf{Q}} K \\ 3.\ F_\bullet \text{ is a good filtration of } \mathcal{M} \end{array} \right\}$$

We then defined the category $\mathsf{HM}(X, w)$ of Hodge modules of weight w as a certain subcategory, which we defined recursively.

Weakly mixed Hodge modules will be the ambient category in which mixed Hodge modules will form a subcategory. The point is that they add in the information of a weight filtration $W_\bullet$ that was not in the definition of $\mathsf{MF}_{\mathsf{rh}}(D_X, \mathbf{Q})$.

Definition 19.2. The category $\mathsf{MHW}(X)$ of *weakly mixed Hodge modules* has objects $(M, W_\bullet)$, where $M = (\mathcal{M}, F_\bullet, K) \in \mathsf{MF}_{\mathsf{rh}}(D_X, \mathbf{Q})$, together with a finite increasing filtration by objects in $\mathsf{MF}_{\mathsf{rh}}(D_X, \mathbf{Q})$, compatible with α, and with the property that

$$\operatorname{gr}_\ell^W M \in \mathsf{HM}(X, \ell).$$

Morphisms are *strict* morphisms in $\mathsf{MF}_{\mathsf{rh}}(D_X, \mathbf{Q})$ that *strictly* respect the filtration $W_\bullet$. A weakly mixed Hodge module is *graded-polarizable* if each $\operatorname{gr}_\ell^W M$ is polarizable; these form the category $\mathsf{MHW}^p(X)$.

Remark 19.3. [Sch14a, p. 30] says that strictness of morphisms is automatic.

This category $\mathsf{MHW}(X)$ is abelian [Sai88, Prop. 5.1.14]. Some results from the pure case carry over, via abstract nonsense about filtered objects in an abelian category:

Proposition 19.4 [Sai90, Prop. 2.15]**.** *Let $(M, W_\bullet) \in \mathsf{MHW}^p(X)$, and consider a projective morphism $f\colon X \to Y$. Then, the spectral sequence*

$$E_1^{p,q} = \mathcal{H}^{p+q} f_*\big(\operatorname{gr}_{-p}^W M\big) \Rightarrow \mathcal{H}^{p+q} f_* M$$

associated to the filtration W degenerates at E_2, and each $\mathcal{H}^i f_* M$, together with the filtration induced by the spectral sequence, lies in $\mathsf{MHW}^p(Y)$.

Proof Sketch. All morphisms on the E_2 page are zero by the Direct Image Theorem 18.13 for pure Hodge modules (which says the complex $f_*\big(\operatorname{gr}_{-p}^W M\big)$ is strict of weight $-p+i$), and the fact that morphisms in $\mathsf{MHW}^p(X)$ must respect the weight filtration. □

The condition on the gr^W basically says that any object of $\mathsf{MHW}^p(X)$ is a repeated extension of objects in $\mathsf{HM}^p(X, \ell)$. The issue, though, is that allowing arbitrary extensions of Hodge modules is not a good idea. This issue already appears for mixed Hodge structures:

Example 19.5 [Sch14a, Ex. 19.1]**.** Consider a mixed variation of Hodge structure on Δ^* that is an extension of $\mathbf{Z}(0)$ and $\mathbf{Z}(1)$:

$$0 \longrightarrow \mathbf{Z}(0) \longrightarrow \mathcal{V} \longrightarrow \mathbf{Z}(1) \longrightarrow 0.$$

Since $\operatorname{Ext}^1_{\mathsf{MHS}}(\mathbf{Z}(0), \mathbf{Z}(1)) \simeq \mathbf{C}^\times$, the data of $\mathcal{V}$ is equivalent to the data of a holomorphic function $f\colon \Delta^* \to \mathbf{C}^\times$, which may have an essential singularity at the origin.

Since we are trying to stay in the realm of regular holonomic D-modules, it seems reasonable to restrict to variations of Hodge structure that have controlled singularities at the boundary. For mixed Hodge structures, these form the class of *admissible* mixed Hodge structures; we will define a Hodge module analogue of this notion.

19.2 Definition and properties of mixed Hodge modules [Sch14a, §§20–21]

Here we give a simpler definition than [Sai90, §2.d], following [Sch14a, §20]. Again, the idea is to work recursively using nearby and vanishing cycles, but we have to also ensure that there is some analogue of the admissibility condition for mixed Hodge structures.

Definition 19.6. Let $(M, W_\bullet) \in \mathsf{MHW}^p(X)$, and fix $f\colon X \to \mathbf{C}$ a non-constant holomorphic function on X. We say that $(M, W_\bullet)$ is *admissible along* $\{f = 0\}$ if the following hold:

1. $(\mathcal{M}, F_\bullet)$ is quasi-unipotent and regular along $\{f = 0\}$;
2. The three filtrations $F_\bullet\mathcal{M}_f$, $V_\bullet\mathcal{M}_f$, and $W_\bullet\mathcal{M}_f$ are compatible;
3. Consider the naïve limit filtrations

$$L_i(\psi_f M) = \psi_f(W_{i+1}M) \qquad L_i(\phi_{f,1}M) = \phi_{f,1}(W_iM),$$

   which are preserved by the nilpotent endomorphism $N = (2\pi i)^{-1}\log T_u$. Then, the *relative monodromy filtrations*

$$W_\bullet(\psi_f M) = W_\bullet\big(N, L_\bullet(\psi_f M)\big) \qquad W_\bullet(\phi_{f,1}M) = W_\bullet\big(N, L_\bullet(\phi_{f,1}M)\big)$$

   for the action of N exist. See [Sai90, eq. 1.1.3–4]

Remark 19.7. Condition (1) is automatic by induction on the length of $W_\bullet$, together with the same property for pure Hodge modules. Condition (3) did not appear in the pure case, since the existence of the monodromy filtration was automatic absent the extra filtration $L_\bullet$.

Remark 19.8. The condition of admissibility along $\{f = 0\}$ can be thought of as saying that the restriction of $(M, W_\bullet M)$ to the open subset $X \smallsetminus f^{-1}(0)$ is admissible relative to X, that is, it puts conditions on the singularities of M at the boundary.

Now we can define the category of mixed Hodge modules; they are defined recursively, just as in the pure case.

Definition 19.9. Consider a weakly mixed Hodge module $(M, W_\bullet) \in \mathsf{MHW}(X)$, and let $f\colon U \to \mathbf{C}$ be a locally defined holomorphic function. Then, we say that $(M, W_\bullet)$ is a *mixed Hodge module* if

1. The pair $(M, W_\bullet)$ is admissible along $\{f = 0\}$;
2. Both $(\psi_f M, W_\bullet)$ and $(\phi_f M, W_\bullet)$ are mixed Hodge modules, whenever $f^{-1}(0)$ does not contain any irreducible components of $U \cap \operatorname{Supp}\mathcal{M}$ (this makes sense since nearby and vanishing cycles are supported on a subset of strictly smaller dimension).

We denote by

$$\mathsf{MHM}(X) \subseteq \mathsf{MHW}(X)$$
$$\mathsf{MHM}^p(X) \subseteq \mathsf{MHM}(X) \cap \mathsf{MHW}^p(X)$$

the full subcategories of all mixed Hodge modules and graded-polarizable mixed Hodge modules, respectively. Morphisms are given by morphisms in $\mathsf{MF}_{\mathsf{rh}}(D_X, \mathbf{Q})$ that are strictly compatible with the weight filtrations.

Properties 19.10.

1. $\mathsf{MHM}(X)$ and $\mathsf{MHM}^p(X)$ are abelian (since subquotients in $\mathsf{MHW}(X)$ for a mixed Hodge module is still a mixed Hodge module);
2. $\mathsf{MHM}(X)$ and $\mathsf{MHM}^p(X)$ are stable under application of nearby and vanishing cycles (by definition);
3. If $f\colon X \to Y$ is projective, then there are cohomological direct image functors

$$\mathcal{H}^i f_*\colon \mathsf{MHM}^p(X) \longrightarrow \mathsf{MHM}^p(Y)$$

   (apply Proposition 19.4);
4. If $f\colon Y \to X$ is an arbitrary morphism, then there are cohomological inverse image functors

$$\mathcal{H}^i f^*\colon \mathsf{MHM}^p(X) \longrightarrow \mathsf{MHM}^p(Y)$$
$$\mathcal{H}^i f^!\colon \mathsf{MHM}^p(X) \longrightarrow \mathsf{MHM}^p(Y)$$

   (use the $\widetilde{\mathcal{M}}$ construction in §18.3.4 together with the definition on [Sch14a, p. 41]);

(5) There is a duality functor

$$\mathbf{D}\colon \mathsf{MHM}^p(X) \longrightarrow \mathsf{MHM}^p(X)^{\mathsf{op}}$$

compatible with Verdier duality for perverse sheaves (extend the results in §18.3.2).

We also state a result relating variations of mixed Hodge structure and mixed Hodge modules.

Theorem 19.11 [Sch14a, Thm. 21.1]**.** *Let X be a complex manifold, and $Z \subseteq X$ an irreducible closed analytic subvariety of X. Then, a graded-polarizable variation of mixed Hodge structure on a Zariski-open subset of Z can be extended to $\mathsf{MHM}(X)$ if and only if it is admissible relative to Z.*

Remark 19.12. In particular, this says why we cannot hope to have functors $j_*, j_!$ in general: for an open embedding $j\colon U \to X$, an object of $\mathsf{MHM}^p(U)$ does not carry an admissibility condition for the boundary $X \smallsetminus U$.

There is also a notion of glueing, which we will not discuss; see [Sch14a, pp. 30–31].

19.3 Algebraic mixed Hodge modules [Sch14a, §22]

Since we are ultimately interested in algebraic varieties, we want to restrict ourselves to algebraic objects. The way to do so is to recall that for complete varieties, Serre's GAGA theorem says that algebraic coherent sheaves are the same as analytic ones. While GAGA does not a priori hold for filtered D-modules, we can still use this idea to make the following definition:

Definition 19.13. Let X be a complex algebraic variety, and let $\overline{X}$ be a compaxtification. The category $\mathsf{MHM}_{\mathrm{alg}}(X)$ of *algebraic mixed Hodge modules* is the image of the restriction functor $\mathsf{MHM}^p(\overline{X}^{\mathrm{an}}) \to \mathsf{MHM}^p(X^{\mathrm{an}})$.

One can show this definition is independent of the choice of $\overline{X}$ (the idea is to use that any two complete algebraic varieties containing X as a dense Zariski-open subset are birationally equivalent). In this case, the perverse sheaf $\operatorname{rat} M$ is constructible with respect to an algebraic stratification, and the coherent sheaves $F_i\mathcal{M}$ are algebraic. There is also an intrinsic definition not mentioning $\overline{X}$; see [Sch14a, Thm. 22.3].

We note that by definition, this means that there is always a functor $j_*\colon \mathsf{MHM}^p(U) \to \mathsf{MHM}^p(X)$ for an open embedding, so using the usual trick of decomposing an arbitrary morphism between quasi-projective varieties into an open embedding followed by a projective morphism, we can define pushforwards for arbitrary morphisms.

19.4 Derived categories [Sch14a, §23]

Since $\mathsf{MHM}_{\mathrm{alg}}(X)$ is abelian, we can define a derived category:

Definition 19.14. The *bounded derived category of algebraic mixed Hodge modules* is denoted by $\mathsf{D}^b\,\mathsf{MHM}_{\mathrm{alg}}(X)$. By construction, there is an exact functor

$$\operatorname{rat}\colon \mathsf{D}^b\,\mathsf{MHM}_{\mathrm{alg}}(X) \longrightarrow \mathsf{D}^b_c(\mathbf{Q}_X)$$

to the bounded derived category of algebraically constructible complexes.

A similar definition of course makes sense in the analytic setting. However, the advantage here is that the six-functor formalism works in this setting:

1. The duality functor $\mathbf{D}$ exists since $\mathbf{D}\colon \mathsf{MHM}_{\mathrm{alg}}(X) \to \mathsf{MHM}_{\mathrm{alg}}(X)^{\mathsf{op}}$ is an exact functor.
2. The inverse image functors $\mathcal{H}^j f^*$, $\mathcal{H}^j f^!$ exist for arbitrary morphisms $f\colon X \to Y$, and the functors f^*, $f^!$ on the derived category are defined by working with Čech complexes for suitable affine open coverings.
3. The direct image functors $\mathcal{H}^j f_*$ and $\mathcal{H}^j f_!$ exist for *projective* morphisms, and in general one can use that pushforwards via open embeddings are well-defined for algebraic mixed Hodge modules. Again, one needs to use Čech resolutions via affine open coverings to make sense of the derived versions [Sai90, Thm. 4.3].
4. You then have to check they satisfy the standard compatibility and adjunction relations.

Example 19.15 [Sch14a, Ex. 23.1]**.** Let $f\colon X \to \mathrm{pt}$. Then, one has $\mathbf{Q}_X^H[n] \simeq f^*\mathbf{Q}(0)[n] \in \mathsf{D}^b\,\mathsf{MHM}_{\mathrm{alg}}(X)$, where $n = \dim X$. We then have that

$$H^i(X, \mathbf{Q}_X^H) = H^i f_* \mathbf{Q}_X^H \qquad H_c^i(X, \mathbf{Q}_X^H) = H^i f_! \mathbf{Q}_X^H$$

are graded-polarizable mixed Hodge structures. It is a difficult theorem that these mixed Hodge structures are the same as the ones defined by Deligne.

19.5 Weights [Sch14a, §23]

We finally mention how the yoga of weights works for mixed Hodge modules; it works in the same way as for mixed complexes in the theory of perverse sheaves.

Definition 19.16. We say that a complex $M \in \mathsf{D}^b\,\mathsf{MHM}_{\mathrm{alg}}(X)$ is
(a) *mixed of weight* $\le w$ if $\mathrm{gr}_i^W \mathcal{H}^j(M) = 0$ for all $i > j + w$;
(b) *mixed of weight* $\ge w$ if $\mathrm{gr}_i^W \mathcal{H}^j(M) = 0$ for all $i < j + w$;
(c) *pure of weight* w if $\mathrm{gr}_i^W \mathcal{H}^j(M) = 0$ for $i \ne j + w$.

Saito showed the following:

$$\begin{aligned} M \text{ mixed of weight} \le w &\implies f_!M, f^*M \text{ mixed of weight} \le w\\ M \text{ mixed of weight} \ge w &\implies f_*M, f^!M \text{ mixed of weight} \ge w\\ M \text{ mixed of weight} \le w &\iff \mathbf{D}M \text{ mixed of weight} \ge -w \end{aligned}$$

In particular, pure complexes are stable under direct images by proper morphisms and under the duality functor. He also shows that

$$\left.\begin{matrix} M_1 \text{ mixed of weight} \le w_1\\ M_2 \text{ mixed of weight} \ge w_2 \end{matrix}\right\} \implies \mathrm{Ext}^i(M_1, M_2) = 0 \text{ for all } i > w_1 - w_2.$$

This formally implies that every pure complex splits into the direct sum of its cohomology sheaves, that is,

$$M \simeq \bigoplus_{j \in \mathbf{Z}} \mathcal{H}^j(M)[-j] \in \mathsf{D}^b\,\mathsf{MHM}_{\mathrm{alg}}(X).$$

20 April 2: Algebraic mixed Hodge modules (Takumi Murayama)

Last time, we discussed algebraic mixed Hodge modules briefly. Today, we will present Saito's new definition for algebraic mixed Hodge modules, following [Sai13]. We will discuss some material necessary to talk about direct images.

20.1 Definitions and statement of results

Let X be a smooth complex algebraic variety. Recall we had the category $\mathsf{MHW}^p(X)$ of *weakly mixed Hodge modules*, whose objects were pairs $(M, W_\bullet)$ with $M = (\mathcal{M}, F_\bullet, K) \in \mathsf{MF}_{\mathsf{rh}}(D_X, \mathbf{Q})$ and $W_\bullet$ was a finite increasing filtration on M such that $\mathrm{gr}_\ell^W \in \mathsf{HM}^p(X, \ell)$.

The definition of an algebraic mixed Hodge module as follows:

Definition 20.1. Let $(M, W_\bullet) \in \mathsf{MHW}^p(X)$. Then, the category of *(algebraic) mixed Hodge modules* $\mathsf{MHM}_{\mathrm{alg}}(X)$ is the abelian full subcategory of $\mathsf{MHW}^p(X)$ defined by increasing induction on the dimension of support as follows.

Let $(M, W_\bullet) \in \mathsf{MHW}^p(X)$ with $\operatorname{Supp} \mathcal{M} = X$. Then, $(M, W_\bullet) \in \mathsf{MHM}_{\mathrm{alg}}(X)$ if and only if, for every $x \in X$, there is a Zariski-open neighborhood U_x of x in X and a regular function $g\colon U_x \to \mathbf{C}$ such that $U_x' \coloneqq U_x \smallsetminus g^{-1}(0)$ is smooth and dense in U_x, and the following two conditions are satisfied:
(C1) The restriction $M' \coloneqq M|_{U_x'}$ is an admissible variation of mixed Hodge structure;
(C2) The nearby and vanishing cycle functors along $\{g = 0\}$ are *well-defined* for $M|_{U_x}$, and $\phi_{g,1} M|_{U_x} \in \mathsf{MHM}_{\mathrm{alg}}\big(g^{-1}(0)\big)$.

We say that the nearby and vanishing cycles functor along $\{g = 0\}$ are *well-defined* if the following two conditions are satisfied:

(W1) The three filtrations $F_\bullet, W_\bullet, V_\bullet$ on $\mathcal{M}_g$ are compatible, where $\mathcal{M}_g$ is the direct image of the underlying filtered D-module via the graph embedding $i_g \colon U \to U \times \mathbf{C}$;

(W2) There is a relative monodromy filtration $W_\bullet$ for the action of the nilpotent part N of the monodromy on $\psi_g K[-1]$ and $\phi_{g,1}K[-1]$ with respect to $L_\bullet \coloneqq \psi_g W_\bullet[-1]$ and $\phi_{g,1}W_\bullet$. Explicitly, defining the filtrations

$$L_i(\psi_f M) = \psi_f(W_{i+1}M) \qquad\qquad L_i(\phi_{f,1}M) = \phi_{f,1}(W_i M),$$

which are preserved by the nilpotent endomorphism $N = (2\pi i)^{-1}\log T_u$, the *relative monodromy filtrations*

$$W_\bullet(\psi_f M) = W_\bullet\bigl(N, L_\bullet(\psi_f M)\bigr) \qquad W_\bullet(\phi_{f,1}M) = W_\bullet\bigl(N, L_\bullet(\phi_{f,1}M)\bigr)$$

are given inductively by

$$\begin{aligned} W_{-i+k}L_k M &= W_{-i+k}L_{k-1} + N^i\bigl(S^i W_{i+k}L_k M\bigr) && \text{for } i > 0\\ W_{i+k}L_k M &= \ker\bigl(N^{i+1} \colon L_k M \to S^{-i-1}(L_k M/W_{-i-2+k}L_k M)\bigr) && \text{for } i \geq 0 \end{aligned}$$

This is the admissibility condition (Definition 19.6) we had before.

Remark 20.2. You can define mixed Hodge modules on singular varieties by using local embeddings into smooth varieties and using Kashiwara's equivalence; see [Sai13, nº 1.2].

The main results in [Sai13] are as follows:

Theorem 20.3 [Sai13, Thm. 1]**.** *Conditions* (C1) *and* (C2) *are independent of the choice of U_x, g. More precisely, suppose* (C1) *and* (C2) *are satisfied for some choice of U_x, g. Then,* (C2) *is satisfied for every choice of U_x, g, and* (C1) *is satisfied if the underlying perverse sheaf of $\mathcal{M}'$ is a local system.*

Theorem 20.4 [Sai13, Thm. 2]**.** *The categories $\mathsf{MHM}_{\mathrm{alg}}(X)$ for complex algebraic varieties X are stable under canonically defined functors*

$$\mathcal{H}^j f_*,\ \mathcal{H}^j f_!,\ \mathcal{H}^j f^*,\ \mathcal{H}^j f^!,\ \psi_g,\ \phi_{g,1},\ \boxtimes.$$

These functors are compatible with the corresponding functors for the underlying perverse sheaves.

Combining the latter theorem with results from [Sai90] will lead to the following:

Corollary 20.5 [Sai13, Cor. 1]**.** *There are canonically defined functors*

$$f_*,\ f_!,\ f^*,\ f^!,\ \psi_g,\ \phi_{g,1},\ \boxtimes,\ \otimes,\ \mathscr{H}\!om$$

between the bounded derived categories $\mathsf{D}^b\,\mathsf{MHM}_{\mathrm{alg}}(X)$ for complex algebraic varieties X, so that we have canonical isomorphisms $H^j f_* \cong \mathcal{H}^j f_*$. These functors are compatible with the corresponding functors for the underlying complexes of sheaves with constructible cohomology.

20.2 Open direct images [Sai13, nº 2.3]

We want to understand how f_* works for open immersions, since that is the key to defining f_* and $f_!$.

Let X be a smooth complex algebraic variety as before, and let D be a Cartier divisor (in the singular case, you use locally principal divisors).

Definition 20.6. Let $M' \in \mathsf{MHW}(X \smallsetminus D)$. We say that the open direct images $j_!M', j_*M'$ are *well-defined* if there are $M'_!, M'_* \in \mathsf{MHW}(X)$ whose underlying perverse sheaves are respectively isomorphic to $j_!K', j_*K'$, where K' is the underlying perverse sheaf of M', and the following condition is satisfied:

(W3) For any locally defined regular function g such that $g^{-1}(0)_{\mathrm{red}} = D_{\mathrm{red}}$, the nearby and vanishing cycle functors $\psi_g, \phi_{g,1}$ along $\{g = 0\}$ are well-defined for $M'_!, M'_*$.

Theorem 20.7 (cf. [Sai90, Thm. 3.27])**.** *The direct image j_*M' exists.*

The proof requires some work. We will assume that $D = \{x_1 \cdots x_r = 0,\ r \le n\}$, $D_i = \{x_i = 0\}$ is normal crossing divisor. The general case follows by an application of Hironaka's resolution of singularities; see the end of the proof of [Sai90, Thm. 3.27] and the citations therein.

In the following, $U \coloneqq X \setminus D$. We will also specialize to the case where the perverse sheaf underlying $\mathcal{M}'$ is actually a local system.

We will first show the existence of $(M, W_\bullet) \in \mathsf{MHW}(X)^p$ such that the underlying perverse sheaf is isomorphic to j_*K'. We already know that the underlying perverse sheaf should be j_*K', and so we define the underlying D-module to be the following:

Definition 20.8 [Sai90, n$^{\text{o}}$ 3.10]**.** We define

$$j_*^{\text{reg}}\mathcal{M}' \coloneqq \mathrm{DR}^{-1}\, j_*\, \mathrm{DR}\, \mathcal{M}'$$
$$j_!^{\text{reg}}\mathcal{M}' \coloneqq \mathrm{DR}^{-1}\, j_!\, \mathrm{DR}\, \mathcal{M}'$$

To define the filtration takes a lot of work. What Saito does is to define the filtration for left D-modules. Before we do so, we need to mention a construction we skipped before, which comes up in the study of Deligne's Riemann–Hilbert correspondence.

Theorem 20.9 (see [HTT08, Thms. 5.2.17, 5.2.20])**.** *There exists a canonical extension $\widetilde{\mathcal{L}'}$ of $\mathcal{L}'$ on X, together with lattices $\widetilde{\mathcal{L}'}^{\ge\alpha}$ and $\widetilde{\mathcal{L}'}^{>\alpha}$, such that the eigenvalues of* $\operatorname{res}\nabla$ *along each D_i is contained in* $[\alpha, \alpha+1)$ *(resp. $(\alpha, \alpha+1)$).*

We can then define a filtration F on $\widetilde{\mathcal{L}'}^{\ge\alpha}$ and $\widetilde{\mathcal{L}'}^{>\alpha}$:

$$F_p\widetilde{\mathcal{L}'}^{\ge\alpha} = \widetilde{\mathcal{L}'}^{\ge\alpha} \cap j_*F_p\mathcal{L}'$$
$$F_p\widetilde{\mathcal{L}'}^{>\alpha} = \widetilde{\mathcal{L}'}^{>\alpha} \cap j_*F_p\mathcal{L}'$$

The filtrations on $j_*^{\text{reg}}\mathcal{M}', j_!^{\text{reg}}\mathcal{M}'$ can then be defined as

$$F_pj_*^{\text{reg}}\mathcal{M}' = \sum \bigl(\omega_X \otimes F_i\widetilde{\mathcal{L}'}^{\ge -1}\bigr)F_{p-i}D_X$$
$$F_pj_!^{\text{reg}}\mathcal{M}' = \sum \bigl(\omega_X \otimes F_i\widetilde{\mathcal{L}'}^{> -1}\bigr)F_{p-i}D_X$$

The weight filtration is harder to write down; see [Sai90, Prop. 2.8].

20.3 Passing to the derived category [Beĭ87, §3]

Now to pass to the derived category, we proceed as follows. If $f\colon X \to Y$, then we can define f_* and $f_!$ to be the right-derived functors of $\mathcal{H}^0f_*$ and $\mathcal{H}^0f_!$, respectively. Otherwise, we fix compatible Čech covers on X and Y, and simultaneously resolve everything term in the resolutions by acyclics for $\mathcal{H}^0f_*$ respectively $\mathcal{H}^0f_!$; this is possible by [Beĭ87, Lem. 3.3]. One then applies the functor $\mathcal{H}^0f_*$ respectively $\mathcal{H}^0f_!$, and then totalizes this complex.

21 April 3 Kodaira–Saito vanishing (Harold Blum)

We first state Kodaira vanishing:

Theorem 21.1. *Let X be a smooth projective variety over* $\mathbf{C}$*, and let L be a ample line bundle on X. Then, we have $H^i(X, L^{-1}) = 0$ for $i < \dim X$.*

We give a very quick sketch of the proof, from [Laz04]. The proof of the Kodaira–Saito vanishing theorem will be similar.

Proof Sketch. Let m such that L^m is very ample, and write $L^m = \mathcal{O}(D)$ for some smooth hyperplane section $D \in |L^m|$. We then take a m-cyclic cover $\widetilde{X} \to X$ corresponding to this divisor D. Then, there exists a divisor $D' \in |\pi^* L|$ such that D' is smooth. There is a short exact sequence

$$0 \longrightarrow \mathcal{O}_{X'}(-D') \longrightarrow \mathcal{O}_{X'} \longrightarrow \mathcal{O}_{D'} \longrightarrow 0$$

on X'. Using the Lefschetz hyperplane theorem, we obtain that $H^i(\mathcal{O}_{X'}(-D')) = 0$ for $i < \dim X'$, and we also have $H^i(\mathcal{O}_{X'}(-D')) \simeq H^i(X, \pi_*\mathcal{O}_{X'}(-D')) \simeq H^i(X, \pi_*\mathcal{O}_{X'} \otimes L^\vee)$. By the cyclic cover construction, $\pi_*\mathcal{O}_{X'} \simeq \mathcal{O}_X \oplus L^{-1} \oplus \cdots \oplus L^{-(m-1)}$. □

We will do a similar proof with topological input for the Kodaira–Saito Vanishing theorem. Here, we used:

Theorem 21.2 (Lefschetz hyperplane)**.** *If $D \subseteq X$, then $H^i(X, \mathbf{Z}) \to H^i(D, \mathbf{Z})$ is an isomorphism if $i < \dim X - 1$, and is injective if $i = \dim X - 1$. This map respects the Hodge decomposition, and so the same statement holds for $H^i(X, \mathcal{O}_X) \to H^i(D, \mathcal{O}_D)$.*

Note that the point of the m-cyclic cover was to reduce to the case where D is a smooth hyperplane section. For Kodaira–Saito vanishing, m-cyclic covers will be used in a different way.

Theorem 21.3 (Saito)**.** *Let X be a projective algebraic variety over $\mathbf{C}$, and let L be an ample line bundle. Let $(\mathcal{M}, F_\bullet\mathcal{M})$ be a filtered regular holonomic D-module that underlies an algebraic mixed Hodge module on X. Then,*

$$H^i\big(X, \operatorname{gr}_k^F \operatorname{DR}(\mathcal{M}) \otimes L\big) = 0 \qquad \text{for } i > 0$$
$$H^i\big(X, \operatorname{gr}_k^F \operatorname{DR}(\mathcal{M}) \otimes L^{-1}\big) = 0 \qquad \text{for } i < 0$$

Note here that X is not necessarily smooth, but we will assume this for the proof.

21.1 Initial reductions

It is enough to consider the case when $(\mathcal{M}, F)$ underlies a pure, polarized Hodge module with strict support X. This is because:

- $W_\bullet$ is a finite filtration, and we use short exact sequences

$$0 \longrightarrow W_i \longrightarrow W_{i+1} \longrightarrow \operatorname{gr}^W_{i+1} \longrightarrow 0.$$

  We also need that $\operatorname{gr}_k^F \circ \operatorname{DR}(-)$ is exact.
- Also need the fact that algebraic mixed Hodge modules are polarizable.

For the condition on supports, you need to consider the case the support of $(\mathcal{M}, F)$ has strict support. This means that the proof might go out of the setting where the variety X is smooth.

21.2 Key tools

Theorem 21.4 (Artin–Grothendieck vanishing)**.** *Let V be an affine variety over $\mathbf{C}$, and let $\mathscr{F}$ be a constructible sheaf on V. Then, $H^i(V, \mathscr{F}) = 0$ for all $i > \dim V$.*

21.2.1 Non-characteristic pullbacks

Let $f\colon X \to Y$ be a morphism of smooth complex manifolds, and let $(\mathcal{M}, F)$ be a filtered D_Y-module.

Definition 21.5. We say that f is *non-characteristic* with respect to $(\mathcal{M}, F)$ if

- $H^i(X, f^{-1}\operatorname{gr}^F \mathcal{M} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_X} \mathcal{O}_X) = 0$ for $i > 0$, where H^i is hypercohomology.
- The map $df^*\colon p_2^{-1}\big(\operatorname{Char}(\mathcal{M})\big) \to T^*X$ is finite, where $p_2\colon X \times_Y T^*Y \to T^*Y$, and

$$\begin{aligned} df\colon X \times_Y T^*Y &\longrightarrow T^*X \\ (x, \omega) &\longmapsto df^*\omega(x) \end{aligned}$$

This is automatic if f is smooth [HTT08].

Proposition 21.6 (Saito)**.** *If f is non-characteristic with respect to $(\mathcal{M}, F)$, then*

(1) *There exists a filtered pullback*
$$f^*(\mathcal{M}, F) = (\widetilde{\mathcal{M}}, F)[-d],$$
where $d = \dim X - \dim Y$, and
$$\widetilde{\mathcal{M}} = f^{-1}\mathcal{M} \otimes_{f^{-1}\mathcal{O}_X} \omega_{X/Y}$$
$$F_p\widetilde{\mathcal{M}} = f^{-1}F_{p+d}\mathcal{M} \otimes_{f^{-1}\mathcal{O}_X} \omega_{X/Y}$$

(2) *If $(\mathcal{M}, F)$ underlies a pure Hodge module, so does $f^*(\mathcal{M}, F)$.*

Remark 21.7. If $(\mathcal{M}, F)$ is a pure Hodge module, and f is smooth along $\operatorname{Sing}(\mathcal{M}, F)$, then f is non-characteristic with respect to $(\mathcal{M}, F)$.

22 April 10 (Harold Blum)

We will finish Kodaira–Saito vanishing. Today, we will show the following:

Theorem 22.1. *Let X be a smooth complex projective variety, let L be an ample line bundle on X, and let $(\mathcal{M}, F)$ be a filtered D-module that underlies a pure, polarizable Hodge module M with strict support X, i.e., $M \in \mathsf{HM}^p_X(X, d)$. Then,*
$$H^i(X, \operatorname{gr}^F_k \operatorname{DR}(M) \otimes L^{-1}) = 0$$
for $i < 0$.

Proof. Let $m > 1$ such that $L^{\otimes m}$ is very ample, and let $s \in H^0(X, L^{\otimes m})$ be a general section with vanishing locus $Z(s) = D$ with complement $U = X \smallsetminus D$. Denote
$$D = V(s) \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U = X \smallsetminus D.$$

Saito shows there exists a short exact sequence
$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}(*D) \longrightarrow \mathcal{H}^1 i^! \mathcal{M} \longrightarrow 0,$$
which implies there exists an exact sequence
$$\operatorname{gr}^F_k \operatorname{DR}(\mathcal{M}) \otimes L^{-1} \longrightarrow \operatorname{gr}^F_k \operatorname{DR}(*D) \otimes L^{-1} \longrightarrow \operatorname{gr}^F_k \operatorname{DR}(\mathcal{H}^1 i^! \mathcal{M}) \otimes L^{-1} \longrightarrow \operatorname{gr}^F_k \operatorname{DR}(\mathcal{M}) \otimes L^{-1} \otimes L^{-1}[1].$$
By induction on dimension,
$$H^i(\operatorname{gr}_k \operatorname{DR}(\mathcal{H}^1 i^! \mathcal{M}) \otimes L^{-1}) = 0$$
for $i < 0$. We want to show that the same thing holds for the $\mathcal{M}(*D)$ term, that is, $H^i(\operatorname{gr}_k \operatorname{DR}(\mathcal{M}(*D)) \otimes L^{-1}) = 0$ for $i < 0$.

Goal 22.2. Realize $\mathcal{M}(*D) \otimes L^{-1}$ as coming from a pure Hodge module.

Set $f \colon \widetilde{X} \to X$ be the m-cyclic cover along D, and let $f^*D = mE$. Set
$$\widetilde{L} \coloneqq \operatorname{cok}(\mathcal{O}_X \to f_*\mathcal{O}_{\widetilde{X}}),$$
where we recall $f_*\mathcal{O}_{\widetilde{X}} \simeq \mathcal{O}_X \oplus L^{-1} \oplus \cdots \oplus L^{-(m-1)}$.

Claim 22.3. *There is a short exact sequence*
$$0 \longrightarrow M \longrightarrow f_*f^*M \longrightarrow \widetilde{M} \longrightarrow 0$$
in $\mathsf{HM}^p(X, d)$ such that
$$(\widetilde{\mathcal{M}}, F) \simeq \bigl((\mathcal{M}(*D), F) \otimes_{\mathcal{O}_X} \widetilde{L}\bigr) \simeq j_*\bigl((\mathcal{M}|_U, F) \otimes_{\mathcal{O}_U} \widetilde{L}|_U\bigr).$$

Taking the claim for granted for now, we prove that

$$H^i(\operatorname{gr}_k \operatorname{DR}(\mathcal{M}(*D)) \otimes L^{-1}) = 0$$

for $i < 0$. Let $\widetilde{P}$ be the $\mathbf{Q}$-perverse sheaf associated to $\widetilde{M}$, and let $\widetilde{P}_{\mathbf{C}} = \widetilde{P} \otimes_{\mathbf{Q}} \mathbf{C}$. We have $j_* j^{-1} \widetilde{P}_{\mathbf{C}} = \widetilde{P}_{\mathbf{C}}$, so that

$$H^i(X, \widetilde{P}_{\mathbf{C}}) = H^i(U, j^{-1} \widetilde{P}_{\mathbf{C}}).$$

The cohomology on the right-hand side can be computed by the spectral sequence

$$H^p(U, \mathcal{H}^q(j^{-1} \widetilde{P}_{\mathbf{C}})) \Rightarrow H^{p+q}(U, j^{-1}(\widetilde{P}_{\mathbf{C}})).$$

Then, $\dim \operatorname{Supp} \mathcal{H}^q(j^{-1} \widetilde{P}_{\mathbf{C}}) \le -q$, hence the left-hand side vanishes when $p > -q$ by Artin–Grothendieck vanishing (Theorem 21.4). The right-hand side is therefore zero for $p + q > 0$. Thus, $H^i(X, \widetilde{P}_{\mathbf{C}}) = 0$ for $i > 0$, and Verdier duality says that

$$H^{-1}(X, \widetilde{P}_{\mathbf{C}}) \simeq H^i(X, \mathbf{D} \widetilde{P}_{\mathbf{C}})^\vee.$$

Since the Hodge module is polarizable, however, we have an isomorphism $\mathbf{D} \widetilde{P}_{\mathbf{C}} \simeq \widetilde{P}_{\mathbf{C}}(d)$. Thus, $H^i(X, \widetilde{P}_{\mathbf{C}}) = 0$ for $i < 0$, and $H^i(X, \operatorname{DR}(\widetilde{\mathcal{M}})) = 0$ for $i \neq 0$.

To get vanishing for the graded pieces, we use the spectral sequence for the filtration F:

$$E_1^{p,q} = H^{p+q}(\operatorname{gr}_{-k} \operatorname{DR}(\widetilde{\mathcal{M}})) \Rightarrow H^{p+q}(\operatorname{DR}(\mathcal{M})).$$

This degenerates at E_1 by the direct image theorem for $X \to \mathrm{pt}$. This shows the vanishing we want, since Claim 22.3 implies $\mathcal{M}(*D) \otimes L^{-1}$ is a direct summand of $\widetilde{\mathcal{M}} \otimes L^{-1}$.

We now return to Claim 22.3. Recall that

$$f^* \mathcal{M} \simeq f^{-1} \mathcal{M} \otimes_{f^{-1} \mathcal{O}_X} \mathcal{O}_{\widetilde{X}} \otimes \omega_{\widetilde{X}/X} \simeq f^{-1} \mathcal{M} \otimes_{f^{-1} \mathcal{O}_X} \mathcal{O}_{\widetilde{X}} \otimes \mathcal{O}_{\widetilde{X}}\big((m-1)E\big).$$

When f is non-characteristic, this D-module underlies a pure Hodge module; since we chose $s \in H^0(L^{\otimes m})$ to be general, f is non-characteristic for M (this is similar to Noether normalization). Then,

$$f_* f^* \mathcal{M} \simeq f_*\Big(f^{-1} \mathcal{M} \otimes_{f^{-1} \mathcal{O}_X} \mathcal{O}_{\widetilde{X}}\big((m-1)E\big) \otimes_{D_{\widetilde{X}}} D_{\widetilde{X} \to X}\Big).$$

Now recall $D_{\widetilde{X} \to X} \coloneqq \mathcal{O}_{\widetilde{X}} \otimes_{f^{-1} \mathcal{O}_X} D_X$. Choose $x_1, \ldots, x_n$ locally on X such that $\{x_1 = 0\} = D$, and let $A = \mathcal{O}_{X,x}$. We then have

$$\frac{A[y]}{y^m - x_1} \simeq \mathcal{O}_{\widetilde{X}, f^{-1}(x)}$$

with local coordinates $y, x_2, \ldots, x_n$. For example, if we have a map $\mathbf{A}^1_s \to \mathbf{A}^1_t$ mapping $s \mapsto s^m$, then the transfer D-module is given by

$$k[s] \otimes_{k[t]} k[t, \partial_t] = k[s, \partial_t]$$

where $\partial_s = m \cdot s^{m-1} \partial_t$ by the description of inverse images as in §4.1. ☐

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