THE CLASSIFICATION OF THICK SUBCATEGORIES AND BALMER'S RECONSTRUCTION THEOREM

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ABSTRACT. We classify all the localizing subcategories of the derived category $\mathbf{D}(R)$ of modules over a noetherian ring R, after developing the theory of unbounded complexes over R. Then, we use this classification to classify thick subcategories of the derived category $\mathbf{D}^b(R)_{\mathsf{proj}}$ of bounded complexes of projective modules over R, and prove Balmer's reconstruction theorem in the affine case.

INTRODUCTION

In algebraic topology and algebraic geometry, we want to find invariants for spaces, schemes, etc. that can be used to distinguished different objects. In particular, there are various triangulated categories one can associate to a given space or scheme; in topology, this can be the derived category of sheaves over the topological space, and in algebraic geometry, we can do the same for coherent sheaves or a subcategory thereof.

What is surprising about the algebro-geometric case, however, is that a certain triangulated category, namely (in good cases) the derived category of perfect complexes (together with its tensor structure), ends up being a *complete* invariant of the scheme. This is the content of

Theorem ([Bal05])**.** *For X a topologically noetherian scheme, the derived category of perfect complexes $\mathbf{D}^{\mathsf{perf}}(X)$ fully characterizes X.*

Moreover, Balmer's theorem is more explicit: it actually gives a concrete *construction* of how to construct X out of the triangulated category $\mathbf{D}^{\mathsf{perf}}(X)$, using a notion of "primes" in $\mathbf{D}^{\mathsf{perf}}(X)$ from which a spectrum is constructed, much like in commutative algebra.

Our primary aim is to prove Balmer's result in the affine case over a noetherian ring R. The motivation to do so is partially from the theory of almost modules: since there is a good notion of a derived category of almost modules over a ring R, it seems plausible that using Balmer's construction can give a geometric way of studying the category of almost modules over R. The author hopes to pursue this idea further in the future.

We pause to give a brief outline of this paper. First, in §1 we build up a theory of unbounded complexes over a ring R, using the formalism of homotopy (co)limits following [BN93]. Injective and projective resolutions for unbounded complexes over R are constructed, and are used to define tensor products and the local cohomology functor on $\mathrm{Spec}(R)$. In §2, we use this formalism to fully classify localizing subcategories of $\mathbf{D}(R) = \mathbf{D}(\mathsf{Mod}\text{-}R)$, the derived category of unbounded complexes over R. In §3, we use this classification to classify the thick subcategories of $\mathbf{D}^b(R)_{\mathsf{proj}}$, the full subcategory of $\mathbf{D}(R)$ consisting of complexes quasi-isomorphic to bounded complexes of finitely generated projective modules. We close with a proof of Balmer's theorem above in this setting.

1. Preliminaries on the (unbounded) derived category

We start with some preliminaries about the unbounded derived category of chain complexes over a ring R. The original source for this is [Spa88]. We follow instead [BN93], which gives simpler proofs at the expense of generality.

1.1. **Homotopy limits and resolutions.** Let $\mathcal{A}$ be an abelian category. Let $\mathbf{C}(\mathcal{A})$ denote the category of chain complexes of objects of $\mathcal{A}$, $\mathbf{K}(\mathcal{A})$ its corresponding homotopy category, and $\mathbf{D}(\mathcal{A}) := \mathbf{K}(\mathcal{A})/\mathbf{K}(\mathcal{A})^{\mathrm{acycl}}$ the corresponding derived category, obtained as the Verdier quotient of $\mathbf{K}(\mathcal{A})$ by the full subcategory of

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acyclic complexes. In this section, we will show that if $\mathcal{A}$ is nice enough, we can construct analogues of projective and injective resolutions in $\mathbf{D}(\mathcal{A})$.

First, suppose we have a sequence of objects and arrows in $\mathbf{C}(\mathcal{A})$:

$$\cdots \xrightarrow{f_{i+2}} X_{i+1} \xrightarrow{f_{i+1}} X_i \xrightarrow{f_i} X_{i-1} \xrightarrow{f_{i-1}} \cdots \xrightarrow{f_1} X_0 \tag{1}$$

Consider the maps $D^p \colon \prod X_i^p \to \prod X_i^p$ defined by $(x_i) \mapsto (x_i - f_{i+1}(x_{i+1}))$ and consider the bicomplex B:

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & \prod X_i^{p-1} & \longrightarrow & \prod X_i^p & \longrightarrow & \prod X_i^{p+1} & \longrightarrow \cdots \\
& \uparrow & & \uparrow & & \uparrow & \\
\cdots \longrightarrow & \prod X_i^{p-1} & \longrightarrow & \prod X_i^p & \longrightarrow & \prod X_i^{p+1} & \longrightarrow \cdots
\end{array}
\tag{2}
$$

where the horizontal maps are products $\prod d_i^p$ of the differentials $d_i^p \colon X_i^p \to X_i^{p+1}$, and the vertical maps are given by $(-1)^p D^p$.

Definition 1. The *homotopy limit* $\varprojlim X_i$ of the sequence (1) is the totalization $\mathrm{Tot}(B)$ of the bicomplex (2). Note that by construction, $\varprojlim X_i$ is functorial in the sequence X_i.

Now note that by definition,

$$\varprojlim X_i \cong \mathrm{Cone}\left(\prod X_i \xrightarrow{\prod D} \prod X_i\right)[-1], \tag{3}$$

hence $\varprojlim X_i$ fits into a distinguished triangle $\varprojlim X_i \to \prod X_i \to \prod X_i \rightsquigarrow$ in $\mathbf{K}(\mathcal{A})$. Dualizing, we also define

Definition 1*. The *homotopy colimit* $\varinjlim X_i$ of the sequence $X_0 \to X_1 \to \cdots \to X_i \to \cdots$ is the totalization $\mathrm{Tot}(B)$ of the bicomplex obtained by dualizing (2) and replacing $\prod$ with $\bigoplus$.

We now restrict to the case when $\mathcal{A}$ satisfies $\mathbf{AB}4^*$, i.e., where direct products are exact. This is satisfied, for example, when $\mathcal{A} = \mathsf{Mod}\text{-}R$ for a ring R [Wei94, Ex. A.4.5]. In this situation, we have the following lemma:

Lemma 2. *Let X be an object in $\mathbf{C}(\mathcal{A})$, where $\mathcal{A}$ satisfies $\mathbf{AB}4^*$. Let $\cdots \to X_i \to X_{i-1} \to \cdots \to X_0$ be a sequence in $\mathbf{C}(\mathcal{A})$ as in (1), together with maps $g_i \colon X \to X_i$ that are compatible with the sequence maps. If, for every n, the map $H^n(X) \to H^n(X_i)$ is an isomorphism for $i \gg 0$, then X and $\varprojlim X_i$ are quasi-isomorphic.*

Proof. The hypothesis on the maps $g_i \colon X \to X_i$ implies that the composite map $X \to \prod X_i \xrightarrow{D} \prod X_i$ is zero. We therefore have a well-defined chain map $X \to \varprojlim X_i$ by considering the diagram

$$
\begin{array}{ccc}
X^{p+1} & \xrightarrow{(\prod g_i, 0)} & \prod X_i^{p+1} \oplus \prod X_i^p \\
\uparrow & & \uparrow{\scriptstyle d+(-1)^p D} \\
X^p & \xrightarrow{(\prod g_i, 0)} & \prod X_i^p \oplus \prod X_i^{p-1}
\end{array}
$$

and noting that the $(-1)^p D$ component of the right vertical map is zero on elements in the image of X^p. Now we claim we have the short exact sequence

$$0 \longrightarrow H^n(X) \xrightarrow{\prod g_i} H^n\left(\prod X_i\right) \xrightarrow{D} H^n\left(\prod X_i\right) \longrightarrow 0$$

This is exact at $H^n(X)$ since $H^n(X) \to H^n(X_i)$ is injective for $i \gg 0$, hence $H^n(X) \to \prod H^n(X_i)$ is injective, and then since cohomology and products commute by $\mathbf{AB}4^*$. Similarly, it is exact at the rightmost term since $D \colon \prod X_i \to \prod X_i$ is surjective. Now at the middle term, since $X \to \prod X_i \to \prod X_i$ is zero, it suffices to check that $\ker D \subset \mathrm{im} \prod g_i$, but this also follows from the surjectivity of $H^n(X) \to H^n(X_i)$ for $i \gg 0$ and since the maps g_i are compatible with the sequence maps f_i, and then using $\mathbf{AB}4^*$ as before.

Finally, this implies $X \to \underleftarrow{\operatorname{holim}} X_i$ is a quasi-isomorphism since the long exact sequence on cohomology for the triangle $\underleftarrow{\operatorname{holim}} X_i \to \prod X_i \to \prod X_i \rightsquigarrow$ gives the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^n(X) & \xrightarrow{\prod g_i} & H^n\left(\prod X_i\right) & \xrightarrow{D} & H^n\left(\prod X_i\right) & \longrightarrow & 0 \\
& & \downarrow & & \| & & \| & & \\
0 & \longrightarrow & H^n(\underleftarrow{\operatorname{holim}} X_i) & \longrightarrow & H^n\left(\prod X_i\right) & \xrightarrow{D} & H^n\left(\prod X_i\right) & \longrightarrow & 0
\end{array}
$$

and then by applying the snake lemma [Wei94, Lem. 1.3.2]. $\qquad \square$

In the dual setting, we write down the following statement with a stronger conclusion:

Lemma 2*. *Let $\mathcal{A}$ be an abelian category satisfying* **AB**5*, i.e., where filtered colimits are exact. Let $X_0 \to X_1 \to \cdots \to X_i \to \cdots$ be a sequence in* $\mathbf{C}(\mathcal{A})$*. Then, there* $\underrightarrow{\operatorname{hocolim}} X_i$ *is quasi-isomorphic to* $\underrightarrow{\operatorname{colim}} X_i$*.*

Note the reason we do not prove the analogous statement for homotopy limits is that the abelian category we are interested in, $\mathsf{Mod}\text{-}R$, does not satisfy **AB**5^* [Wei94, Ex. 3.5.5].

Proof. For each n we have the short exact sequence

$$
0 \longrightarrow \bigoplus_{i=0}^{n-1} X_i \xrightarrow{1-f_i} \bigoplus_{i=0}^{n} X_i \longrightarrow X_n \longrightarrow 0
$$

where the second map comes from summing the maps $X_i \to X_n$. By **AB**5, the colimit of this short exact sequence gives rise to the sequence

$$
0 \longrightarrow \bigoplus X_i \xrightarrow{D} \bigoplus X_i \longrightarrow \underrightarrow{\operatorname{colim}} X_i \longrightarrow 0
$$

which is short exact. Note D is defined just as before. This induces a distinguished triangle $\bigoplus X_i \to \bigoplus X_i \to \underrightarrow{\operatorname{colim}} X_i \rightsquigarrow$ by [Wei94, Ex. 10.4.9], so the quasi-isomorphism follows by the dual of the isomorphism (3). $\quad \square$

We record the following lemma that will be useful in the sequel:

Lemma 3. *If I is a bounded below complex of injectives, then $I \in {}^{\perp}\mathbf{K}(\mathcal{A})^{\mathrm{acycl}}$, i.e., $\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(A, I) = 0$ for all A acyclic.*

Proof. Suppose A is acyclic and $\alpha \colon A \to I$ is a map in $\mathbf{C}(\mathcal{A})$ such that $\alpha^j = 0$ for all $j < n$ for some n.

We claim there exists a homotopy $h \colon A^{n+1} \to K^n$ such that $\alpha^n = h \circ d_A$. Note this would suffice to show the lemma since $\alpha \simeq \beta$, where

$$
\beta^j = \begin{cases} 0 & \text{if } j \leq n \\ \alpha^{n+1} - h & \text{if } j = n+1 \\ \alpha^j & \text{if } j > n+1 \end{cases}
$$

and then by applying induction.

So we want to show the claim. We have the commutative diagram

$$
\begin{array}{ccccc}
A^{n-1} & \longrightarrow & A^n & \longrightarrow & A^{n+1} \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \alpha^n} & & \downarrow{\scriptstyle \alpha^{n+1}} \\
I^{n-1} & \longrightarrow & I^n & \longrightarrow & I^{n+1}
\end{array}
$$

$\alpha^{n-1} = 0$, and so $\alpha^n(B^n) = 0$ by commutativity. Since A is acyclic, $\alpha^n(Z^n) = 0$. We therefore have the commutative diagram

where the dashed arrow exists by the injectivity of I^n. We therefore conclude that $\overline{\alpha^n}$ lifts to some $h\colon A^{n+1} \to I^n$ such that $h \circ d_A = \alpha^n$. $\qquad\square$

We can now prove our first main result.

Theorem 4. *Let $\mathcal{A}$ be an abelian category satisfying* $\mathbf{AB}4^*$ *with enough injectives. Then, every object $X \in \mathbf{C}(\mathcal{A})$ is quasi-isomorphic to an object in $\mathbf{K}(I)$, the smallest full subcategory containing the bounded below complexes of injectives that is closed under direct products and forming triangles.*

Proof. Let $\tau_{\geq n} X$ be the "good" truncation of X, that is,

$$(\tau_{\geq n} X)^j = \begin{cases} X^j & \text{if } j \geq n+1 \\ X^n/B^n & \text{if } j = n \\ 0 & \text{otherwise} \end{cases}$$

There is then a natural map $X \to \tau_{\geq n} X$ which is an isomorphism on cohomology in degree $\geq n$, and $H^j(\tau_{\geq n} X) = 0$ for all $j \leq n-1$.

Now since $\mathcal{A}$ has enough injectives, and since each $\tau_{\geq n} X$ is bounded below, we can use the Cartan-Eilenberg resolution [Wei94, §5.7] to obtain a quasi-isomorphism $q_n \colon \tau_{\geq n} X \to I_n$ where each I_n is a complex of injectives. Now we have the following diagram giving a morphism $I_{n-1} \to I_n$ in $\mathcal{D}(\mathcal{A})$:

$$\begin{array}{ccc} \tau_{\geq n-1} X & \xrightarrow{\pi} & \tau_{\geq n} X \\ {\scriptstyle q_{n-1}}\big\downarrow & & \big\downarrow{\scriptstyle q_n} \\ I_{n-1} & & I_n \end{array}$$

Now since q_n is a quasi-isomorphism, the sequence $\tau_{\geq n-1} X \to I_{n-1} \to \operatorname{Cone}(q_{n-1}) \rightsquigarrow$ is a distinguished triangle, hence applying $\operatorname{Hom}(-, I_n)$, we obtain the exact sequence

$$\operatorname{Hom}(\operatorname{Cone}(q_{n-1}), I_n) \longrightarrow \operatorname{Hom}(I_{n-1}, I_n) \longrightarrow \operatorname{Hom}(\tau_{\geq n-1} X, I_n) \longrightarrow \operatorname{Hom}(\operatorname{Cone}(q_{n-1})[-1], I_n)$$

where the Hom's are morphisms in $\mathbf{K}(\mathcal{A})$, since $\operatorname{Hom}(-, I_n)$ is a cohomological functor [Wei94, Ex. 10.2.8]. But the cone of a quasi-isomorphism is acyclic [Wei94, Cor. 1.5.4], and the two Hom groups on the side are zero by Lemma 3, hence we have an isomorphism $\operatorname{Hom}(I_{n-1}, I_n) \to \operatorname{Hom}(\tau_{\geq n-1} X, I_n)$. The map $q_n \circ \pi$ then lifts to a chain map $I_{n-1} \to I_n$.

We now have morphisms

$$X = \varprojlim(\tau_{\geq -n} X) \xrightarrow{\alpha} \operatorname{holim}(\tau_{\geq -n} X) \xrightarrow{\beta} \operatorname{holim}(I_{-n}).$$

α is a quasi-isomorphism by Lemma 2, and β is a quasi-isomorphism since it is the homotopy limit of quasi-isomorphisms. Finally, $\operatorname{holim}(I_{-n}) \in \mathbf{K}(I)$ by definition, since by (3) it was isomorphic to the cone of a morphism between two direct products of bounded below complexes of injectives. $\qquad\square$

Now Lemma 3 and Theorem 4 show that $(\mathbf{K}(\mathcal{A})^{\mathrm{acycl}}, \mathbf{K}(I))$ form an admissible pair, that is,

(1) for all $A \in \mathbf{K}(\mathcal{A})^{\mathrm{acycl}}$ and $I \in \mathbf{K}(I)$, we have $\operatorname{Hom}(A, I) = 0$;
(2) for all $X \in \mathbf{D}(\mathcal{A})$, there exists a distinguished triangle $A \to X \to I \rightsquigarrow$ with $A \in \mathbf{K}(\mathcal{A}^{\mathrm{acycl}})$ and $I \in \mathbf{K}(I)$.

By general results about admissible pairs in, e.g., [BK89, §1], we have the following

Proposition 5. *Let $\mathcal{A}$ be an abelian category with enough injectives satisfying* $\mathbf{AB}4^*$. *Then, the composition $\mathbf{K}(I) \hookrightarrow \mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ is an equivalence.*

Now let $\mathbf{K}(P)$ be the smallest full subcategory containing the bounded above complexes of projectives that is closed under direct sums and forming triangles. By dualizing the proofs and statements above, we also obtain

Proposition 5*. *Let $\mathcal{A}$ be an abelian category with enough projectives satisfying* $\mathbf{AB}4$. *Then, the composition $\mathbf{K}(P) \hookrightarrow \mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ is an equivalence.*

Using both equivalences, we can construct derived functors in the unbounded setting:

Theorem 6. *Let $\mathcal{A}$ be an abelian category with enough injectives and projectives satisfying* **AB**4 *and* **AB**4^*. *Then, there is a right derived functor* $\mathbf{R}\mathrm{Hom}\colon \mathbf{D}(\mathcal{A})^{\mathrm{op}} \times \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathsf{Ab})$ *and a left derived functor* $\otimes^{\mathbf{L}}\colon \mathbf{D}(\mathcal{A}) \times \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathsf{Ab})$ *satisfying all reasonable good properties.*

If $\mathcal{A}$ has an internal Hom *functor, then* $\mathbf{R}\mathrm{Hom}$ *can be defined to be a functor* $\mathbf{D}(\mathcal{A})^{\mathrm{op}} \times \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$, *and similarly if $\mathcal{A}$ has an internal* $\otimes$ *functor, then* $\otimes^{\mathbf{L}}$ *can be defined to be a functor* $\mathbf{D}(\mathcal{A}) \times \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$.

Proof. Both $\mathbf{R}\mathrm{Hom}$ and $\otimes^{\mathbf{L}}$ can be defined easily on $\mathbf{K}(\mathcal{A})$ as the totalizations of the Hom and tensor product bicomplexes [Wei94, 2.7.1, 2.7.4]. But now using Propositions 5 and 5*, we can define $\mathbf{R}\mathrm{Hom}$ as the functor $\mathbf{K}(P)^{\mathrm{op}} \times \mathbf{K}(I) \to \mathbf{D}(\mathsf{Ab})$ and $\otimes^{\mathbf{L}}$ as the functor $\mathbf{K}(P) \times \mathbf{K}(P) \to \mathbf{D}(\mathsf{Ab})$ by restricting the functors on $\mathbf{K}(\mathcal{A})$. The last statement follows since if Hom (resp. $\otimes$) are internal to $\mathcal{A}$, then the totalizations mentioned above land in $\mathbf{K}(\mathcal{A})$, and hence in $\mathbf{D}(\mathcal{A})$ by composing with the functor $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$. $\qquad\square$

We write down some of the "reasonable good properties" we will freely use in the sequel.

(1) $\otimes^{\mathbf{L}}$ is symmetric and associative;
(2) $\otimes^{\mathbf{L}}$ and $\mathbf{R}\mathrm{Hom}$ are bitriangulated;
(3) $\otimes^{\mathbf{L}}$ commutes with direct sums in either variable;
(4) $\mathbf{R}\mathrm{Hom}$ is sends direct sums in the first factor, and direct products in the second, to direct products.

1.2. **Local cohomology.** We provide one more application of Theorem 4, namely, the existence of a derived functor version of local cohomology on a commutative noetherian ring R. This will be useful in the next section when we prove our first main theorem.

Let R be a noetherian commutative ring, and denote the derived category of complexes of modules over R as $\mathbf{D}(R) \coloneqq \mathbf{D}(\mathsf{Mod}\text{-}R)$. What follows is a modification of [Har66, IV] to work in our setting of unbounded complexes of R-modules.

Definition 7. Let Z be a closed subset of $\operatorname{Spec} R$ and I an ideal of R defining Z. Let M be an R-module. Then, define the *module of sections of M supported on Z* to be

$$\Gamma_Z(M) = \varinjlim \mathrm{Hom}(R/I^i, M).$$

It is a fact that this functor does not depend on the choice of I [Har67, Thm. 2.8]. In the algebro-geometric context, $\Gamma_Z(M)$ corresponds to the module of sections of the associated sheaf $\tilde{M}$ with support in Z.

Definition 8. Let Z be a subset of $\operatorname{Spec}(R)$ *stable under specialization*, i.e., such that if $z_1 \in Z$, and $z_0 \in \overline{\{z_1\}}$, then $z_0 \in Z$ as well. Let $\{Z_\alpha\}$ be the collection of closed subsets of Z; note they form a poset under inclusion. We call this collection a *system of supports*. Then, define the functor

$$\Gamma_Z(M) = \varinjlim_{\alpha} \Gamma_{Z_\alpha}(M)$$

where the limit is taken over inclusion maps $Z_\alpha \hookrightarrow Z_\beta$ inducing maps $\mathrm{Hom}(R/I_\alpha^i, -) \to \mathrm{Hom}(R/I_\beta^i, -)$.

Moreover, suppose $Z' \subset Z$ is a subset also stable under specialization. Then, define

$$\Gamma_{Z/Z'}(M) = \Gamma_Z(M)/\Gamma_{Z'}(M).$$

Theorem 9. *For subsets $Z' \subset Z$ of $\operatorname{Spec}(R)$ that are stable under specialization, then there is a right-derived functor* $\mathbf{R}\Gamma_{Z/Z'}(-)\colon \mathbf{D}(R) \to \mathbf{D}(R)$ *such that the triangle* $\mathbf{R}\Gamma_{Z'}(X) \to \mathbf{R}\Gamma_Z(X) \to \mathbf{R}\Gamma_{Z/Z'}(X) \rightsquigarrow$ *is distinguished.*

Proof. As before, by Theorem 5 it suffices to define $\mathbf{R}\Gamma_Z$ and $\mathbf{R}\Gamma_{Z/Z'}(-)$ on $\mathbf{K}(I)$, that is, on the level of chain complexes.

For the second statement, let $K \in \mathbf{K}(I)$ be quasi-isomorphic to X. Then, the sequence

$$0 \longrightarrow \mathbf{R}\Gamma_{Z'}(K) \longrightarrow \mathbf{R}\Gamma_Z(K) \longrightarrow \mathbf{R}\Gamma_{Z/Z'}(K) \longrightarrow 0$$

is exact, and taking the corresponding distinguished triangle, we are done. $\qquad\square$

Remark 10. Theorem 4 was first shown by [Spa88] for sheaves of abelian groups on an arbitrary ringed space. Note this is stronger than our situation, since sheaves of abelian groups do not satisfy **AB**4^*. Until recently, there was no published proof that an analogue of Theorem 4 holds in an arbitrary Grothendieck abelian category. A proof was first published in [AJS00] using a derived analogue of the Gabriel-Popescu embedding theorem, and reducing to the case of the derived category of modules over a ring. A proof following the

classical proof due to Grothendieck [Tohoku] for injective resolutions in arbitrary Grothendieck abelian categories, is in [Ser03].

2. Localizing subcategories of $\mathbf{D}(R)$ and their classification

We now turn to the first main goal of this paper. Recall we are in the setting where R is a noetherian commutative ring, and $\mathbf{D}(R) = \mathbf{D}(\mathsf{Mod}\text{-}R)$. Our aim is to classify certain subcategories of $\mathbf{D}(R)$. We first start with the following

Definition 11. Let $\mathcal{D}$ be a triangulated category. We say a full triangulated subcategory $\mathcal{L}$ is *localizing* if it is *thick* (i.e., closed under taking direct summands), and closed under (small) direct sums.

We prove one general result about localizing categories of $\mathbf{D}(R)$.

Proposition 12. *If $\mathcal{L}$ is a localizing subcategory of $\mathbf{D}(R)$, then $\mathcal{L}$ is rigid, i.e., for every $X \in \mathcal{L}$ and $Y \in \mathbf{D}(R)$, we have that $X \otimes^{\mathbf{L}} Y \in \mathcal{L}$.*

Proof. Let $\mathcal{T}$ be the full subcategory of $\mathbf{D}(R)$ defined by

$$\mathcal{T} = \{Y \in \mathbf{D}(R) \mid X \otimes^{\mathbf{L}} Y \in \mathcal{L}, \text{ for all } X \in \mathcal{L}\}.$$

Since $\otimes^{\mathbf{L}}$ is bitriangulated and commutes with direct sums in either variable, we see that $\mathcal{T}$ is triangulated and closed under direct sums. Now $R \in \mathcal{T}$ since $X = X \otimes^{\mathbf{L}} R \in \mathcal{L}$. Now an arbitrary $Y \in \mathbf{D}(R)$ is quasi-isomorphic to a complex of projectives $J \in \mathbf{K}(P)$ by Proposition 5*. But every projective is a direct summand of a free module R^n [Wei94, Prop. 2.2.1], hence $J \in \mathcal{T}$, and therefore $Y \in \mathcal{T}$. Thus, $\mathcal{T} = \mathbf{D}(R)$. $\square$

Now we can state the theorem. Note that for every $\mathfrak{p} \in \mathrm{Spec}(R)$, we define $\kappa(\mathfrak{p})$ to be the residue field $\mathrm{Frac}(R/\mathfrak{p}) \cong R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ at $\mathfrak{p}$.

Theorem 13. *There is an inclusion-preserving bijection of sets*

$$\left\{ \begin{array}{c} \textit{Localizing subcategories} \\ \textit{of } \mathbf{D}(R) \end{array} \right\} \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} \left\{ \begin{array}{c} \textit{Subsets} \\ \textit{of } \mathrm{Spec}(R) \end{array} \right\}$$

where

$$f(\mathcal{L}) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \exists X \in \mathcal{L} \text{ with } X \otimes \kappa(\mathfrak{p}) \neq 0\},$$

$$g(P) = \textit{the smallest localizing category containing } \kappa(\mathfrak{p}), \textit{ for all } \mathfrak{p} \in P.$$

In the sequel, we will denote $\mathcal{L}_P := g(P)$, and $\mathcal{L}_{\mathfrak{p}} := g(\{\mathfrak{p}\})$.

The proof of Theorem 13 consists of a succession of relatively short lemmas. We follow [Nee92a, §2].

Lemma 14. *Let R be a commutative ring, and $\alpha \colon R \to k$ homomorphism to a field k. For any $X \in \mathbf{D}(R)$, the complex $X \otimes k$ is a direct sum of shifts of k.*

Proof. Put $X = \underrightarrow{\mathrm{hocolim}}\, X_i$, where the X_i are bounded above, using a "good" truncation as in Theorem 4. Then, $X \otimes k = \underrightarrow{\mathrm{hocolim}}(X_i \otimes k)$ since $- \otimes k$ is triangulated and commutes with direct sums. But this homotopy colimit is the same calculated in $\mathbf{D}(k)$ or $\mathbf{D}(R)$, hence is actually an object in $\mathbf{D}(k)$. Thus, $X \otimes k$ is a direct sum of shifts of k. $\square$

Now recall that the *injective hull* of an R-module M is the smallest injective containing M [Lam99, 3.31].

Lemma 15. *Let $I_{\mathfrak{p}}$ denote the injective hull of $R/\mathfrak{p}$, and let $X \in \mathbf{D}(R)$ be a complex consisting entirely of injectives, each of which is a direct sum of copies of $I_{\mathfrak{p}}$ for fixed $\mathfrak{p} \in \mathrm{Spec}(R)$. Then, $X \in \mathcal{L}_{\mathfrak{p}}$.*

Proof. First, recall that since R is noetherian, $I_{\mathfrak{p}}$ is in fact an indecomposable $R_{\mathfrak{p}}$-module [Lam99, 3.62, 3.77], and that every element of $I_{\mathfrak{p}}$ is annihilated by $\mathfrak{p}^n$ for some n [Lam99, 3.78]. Thus, the complex X has a filtration

$$0 = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X$$

where X_i is the subcomplex of X annihilated by $\mathfrak{p}^i$ for each i. Then, X_i/X_{i-1} is a complex of vector spaces over $\kappa(\mathfrak{p})$ since it is annihilated by $\mathfrak{p}$, hence the $R_{\mathfrak{p}}$-module structure descends to a $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$-module structure. Thus, X_i/X_{i-1} is quasi-isomorphic to a direct sum of shifts of $\kappa(\mathfrak{p})$, hence $X_i/X_{i-1} \in \mathcal{L}_{\mathfrak{p}}$.

Now from the short exact sequences $0 \to X_{i-1} \to X_i \to X_i/X_{i-1} \to 0$ we get distinguished triangles $X_{i-1} \to X_i \to X_i/X_{i-1} \rightsquigarrow$. By induction, this implies that $X_i \in \mathcal{L}_{\mathfrak{p}}$ for all i. But since $X = \varinjlim X_i$, and by Lemma 2*, a localizing subcategory is closed under direct limits, we see that $X \in \mathcal{L}_{\mathfrak{p}}$ as well. $\qquad\square$

Now let $X \in \mathbf{D}(R)$ be a complex. By Theorem 4, there is an equivalence of categories $\mathbf{D}(R) \to \mathbf{K}(I)$, so consider the image K of X in $\mathbf{K}(I)$ through this equivalence; recall that K will then be a complex of injective modules. Let $Y = \overline{\{\mathfrak{p}\}}$ and $Y' = Y \setminus \{\mathfrak{p}\}$, and consider the local cohomology complexes $\mathbf{R}\Gamma_{Y/Y'}(X) = \mathbf{R}\Gamma_{Y/Y'}(K)$ we constructed earlier. Since each object in K can be decomposed into injectives of the form $I_{\mathfrak{q}}$ [Lam99, 3.48, 3.62], by definition of $\mathbf{R}\Gamma_{Y/Y'}(X)$, we see that $\mathbf{R}\Gamma_{Y/Y'}(X)$ has terms that are all direct sums of injective modules of the form $I_{\mathfrak{p}}$.

Lemma 16. *Let $P \subset \mathrm{Spec}(R)$ be an arbitrary subset. Suppose $X \in \mathbf{D}(R)$ is such that for all $\mathfrak{p} \in \mathrm{Spec}(R) \setminus P$, we have*

$$\mathbf{R}\Gamma_{\overline{\{\mathfrak{p}\}}/\overline{\{\mathfrak{p}\}}\setminus\{\mathfrak{p}\}}(X) = 0.$$

Then, $X \in \mathcal{L}_P$.

Proof. Let $\mathcal{C}$ denote the collection of subsets $Y \subset \mathrm{Spec}(R)$ closed under specialization such that $\mathbf{R}\Gamma_Y(X) \in \mathcal{L}_P$. Since $\mathcal{L}_P$ is closed under direct limits by Lemma 2*, $\mathcal{C}$ is closed under increasing unions. By Zorn's lemma, there then exists a maximal element $Y \in \mathcal{C}$.

We claim $Y = \mathrm{Spec}(R)$. Suppose not; since R is noetherian, the family of closed subsets

$$\mathcal{C}' = \{\overline{\{\mathfrak{r}\}} \mid \mathfrak{r} \in \mathrm{Spec}(R) \text{ and } \overline{\{\mathfrak{r}\}} \cap (\mathrm{Spec}(R) \setminus Y)) \neq \emptyset\}$$

has a minimal element $\overline{\{\mathfrak{p}\}}$. Now if $\mathfrak{q} \in \overline{\{\mathfrak{p}\}} \cap (\mathrm{Spec}(R) \setminus Y)$, then $\overline{\{\mathfrak{q}\}} \in \mathcal{C}'$, but by minimality of $\overline{\{\mathfrak{p}\}}$, we have $\mathfrak{q} = \mathfrak{p}$, hence $Y \cup \overline{\{\mathfrak{p}\}} = Y \cup \{\mathfrak{p}\}$.

Now by definition of systems of support, we have the identification

$$\mathbf{R}\Gamma_{Y \cup \{\mathfrak{p}\}/Y}(X) = \mathbf{R}\Gamma_{\overline{\{\mathfrak{p}\}}/\overline{\{\mathfrak{p}\}}\setminus\{\mathfrak{p}\}}(X)$$

and by assumption, either $\kappa(\mathfrak{p}) \in \mathcal{L}_P$ or $\mathbf{R}\Gamma_{\overline{\{\mathfrak{p}\}}/\overline{\{\mathfrak{p}\}}\setminus\{\mathfrak{p}\}}(X) = 0$. In either case, we have

$$\mathbf{R}\Gamma_Y(X) \to \mathbf{R}\Gamma_{Y \cup \{\mathfrak{p}\}}(X) \to \mathbf{R}\Gamma_{\overline{\{\mathfrak{p}\}}/\overline{\{\mathfrak{p}\}}\setminus\{\mathfrak{p}\}}(X) \rightsquigarrow$$

by Theorem 9, which implies $\mathbf{R}\Gamma_{Y \cup \{\mathfrak{p}\}}(X) \in \mathcal{L}_P$ since $\mathbf{R}\Gamma_{\overline{\{\mathfrak{p}\}}/\overline{\{\mathfrak{p}\}}\setminus\{\mathfrak{p}\}}(X)$ consists of terms each of which is a direct sum of $I_{\mathfrak{p}}$, contradicting maximality of Y. $\qquad\square$

Lemma 17. *Let $P \subset \mathrm{Spec}(R)$, and let $X \in \mathbf{D}(R)$ be such that for all $\mathfrak{p} \in P$, $X \otimes \kappa(\mathfrak{p}) = 0$. Then, $X \otimes^{\mathbf{L}} Y = 0$ for all $Y \in \mathcal{L}_P$.*

Proof. This follows since $\otimes^{\mathbf{L}}$ is bitriangulated and commutes with direct sums in the second variable. $\qquad\square$

Lemma 18. *If $X \in \mathbf{D}(R)$ is such that for all $\mathfrak{p} \in \mathrm{Spec}(R)$, the tensor product $X \otimes \kappa(\mathfrak{p}) = 0$, then $X = 0$.*

Proof. By the previous lemma, $X \otimes^{\mathbf{L}} Y = 0$ for all $Y \in \mathcal{L}_{\mathrm{Spec}(R)}$. But $\mathcal{L}_{\mathrm{Spec}(R)} = \mathbf{D}(R)$ by Lemma 16. In particular, $X = X \otimes R = 0$. $\qquad\square$

Lemma 19. *Let $X \in \mathbf{K}(I)$ be a complex of injectives, all of which are direct sums of $I_{\mathfrak{p}}$ for a fixed point $\mathfrak{p} \in \mathrm{Spec}(R)$. Then, for all $\mathfrak{q} \neq \mathfrak{p}$, we have $X \otimes \kappa(\mathfrak{q}) = 0$.*

Proof. This is trivial since $I_{\mathfrak{p}} \otimes \kappa(\mathfrak{q}) = 0$ for any $\mathfrak{q} \neq \mathfrak{p}$. $\qquad\square$

Lemma 20. *If X is a complex of injectives as in the previous lemma, then $X \otimes \kappa(\mathfrak{p}) = 0$ if and only if $X = 0$.*

Proof. $\Leftarrow$ is trivial. For $\Rightarrow$, this is a combination of the previous two lemmas. $\qquad\square$

We are finally able to prove our theorem.

Proof of Theorem 13. Recall we have defined the maps

$$\left\{ \begin{array}{c} \text{Localizing subcategories} \\ \text{of } \mathbf{D}(R) \end{array} \right\} \underset{g}{\overset{f}{\rightleftarrows}} \left\{ \begin{array}{c} \text{Subsets} \\ \text{of } \mathrm{Spec}(R) \end{array} \right\}$$

The fact that these maps are inclusion preserving is clear by definition. We first show $f \circ g = \mathrm{id}$. Suppose $P \subset \mathrm{Spec}(R)$. Then, $g(P) = \mathcal{L}_P$, and we claim

$$f(g(P)) = f(\mathcal{L}_P) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \exists X \in \mathcal{L}_P \text{ with } X \otimes \kappa(\mathfrak{p}) \neq 0\} = P$$

where all but the last equality are true by definition. For the last equality, if $\mathfrak{p} \in P$, then $\kappa(\mathfrak{p}) \in \mathcal{L}_P$ satisfies $\kappa(\mathfrak{p}) \otimes \kappa(\mathfrak{p}) \neq 0$, hence $P \subset f(g(P))$. Conversely, suppose $\mathfrak{p} \in f(g(P))$. Then, by definition there exists $X \in \mathcal{L}_P$ such that $X \otimes \kappa(\mathfrak{p})$ such that $X \otimes \kappa(\mathfrak{p}) \neq 0$. Then, by (the contrapositive of) Lemma 17, we have $\mathfrak{p} \in P$, hence $f \circ g = \mathrm{id}$ as claimed.

Now we want to show $g \circ f = \mathrm{id}$. We first show $g(f(\mathcal{L})) = \mathcal{L}_{f(\mathcal{L})} \subset \mathcal{L}$. To do so, it suffices to show $\kappa(\mathfrak{p}) \in \mathcal{L}$ for each $\mathfrak{p} \in f(\mathcal{L})$. Now if $\mathfrak{p} \in f(\mathcal{L})$, we have that there exists in $X \in \mathcal{L}$ such that $X \otimes \kappa(\mathfrak{p}) \neq 0$. By Proposition 12, $\mathcal{L}$ is rigid so $X \otimes \kappa(\mathfrak{p}) \in \mathcal{L}$, and by Lemma 14, $X \otimes \kappa(\mathfrak{p})$ is a direct sum of shifts of $\kappa(\mathfrak{p})$, hence $\kappa(\mathfrak{p}) \in \mathcal{L}$ since $\mathcal{L}$ is thick. Thus, $g(f(\mathcal{L})) \subset \mathcal{L}$.

Conversely, we need to show $\mathcal{L} \subset g(f(\mathcal{L}))$. Suppose $X \in \mathcal{L}$. Then, $X \otimes \kappa(\mathfrak{p}) \cong \mathbf{R}\Gamma_{\overline{\{\mathfrak{p}\}}/\overline{\{\mathfrak{p}\}}\setminus\{\mathfrak{p}\}}(X) \otimes \kappa(\mathfrak{p})$, which is zero if and only if $\mathbf{R}\Gamma_{\overline{\{\mathfrak{p}\}}/\overline{\{\mathfrak{p}\}}\setminus\{\mathfrak{p}\}}(X) = 0$ by Lemma 20. Thus, by Lemma 16, we have

$$X \in g\Big(\Big\{\kappa(\mathfrak{p}), \ \mathfrak{p} \in \mathrm{Spec}(R) \mid \mathbf{R}\Gamma_{\overline{\{\mathfrak{p}\}}/\overline{\{\mathfrak{p}\}}\setminus\{\mathfrak{p}\}}(X) \neq 0\Big\}\Big) = g(\{\kappa(\mathfrak{p}), \ \mathfrak{p} \in \mathrm{Spec}(R) \mid X \otimes \kappa(\mathfrak{p}) \neq 0\}),$$

and so $X \in g(f(\mathcal{L}))$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 21. Theorem 13 has been generalized to arbitrary noetherian schemes in [AJS04], which even shows the result for formal schemes. The proofs given above incorporate some arguments from [AJS04], in which the proof follows Neeman's closely, with modifications due to working in the setting of formal schemes.

3. Thick subcategories of $\mathbf{D}^b(R)_{\mathsf{proj}}$ and their classification

We now come back to the main motivation of this paper. Recall that a subcategory of a triangulated category $\mathcal{D}$ is *thick* if it is closed under direct summands.

3.1. The derived category of perfect complexes.

Definition 22. Let $\mathbf{D}(R)$ as before. Then, we call the *derived category of perfect complexes* the full subcategory of $\mathbf{D}(R)$ consisting of complexes quasi-isomorphic to bounded complexes of finitely generated projective modules, denoted by $\mathbf{D}^b(R)_{\mathsf{proj}}$. They are exactly the compact objects in $\mathbf{D}(R)$.

Now denote

$$\mathrm{supp}(X) \coloneqq \{\mathfrak{p} \in \mathrm{Spec}(R) \mid X \otimes \kappa(\mathfrak{p}) \neq 0\}.$$

We prove the following:

Theorem 23. *There is an inclusion-preserving bijection of sets*

$$\left\{\begin{array}{c} \textit{Thick subcategories} \\ \textit{of } \mathbf{D}^b(R)_{\mathsf{proj}} \end{array}\right\} \underset{g}{\overset{f}{\rightleftarrows}} \left\{\begin{array}{c} \textit{Specialization closed} \\ \textit{subsets of } \mathrm{Spec}(R) \end{array}\right\}$$

where

$$f(\mathcal{L}) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \exists X \in \mathcal{L} \text{ with } X \otimes \kappa(\mathfrak{p}) \neq 0\},$$
$$g(P) = \{X \mid \mathrm{supp}(X) \subset P\}.$$

Note that $g(P)$ is thick follows from basic properties of $\mathrm{supp}(X)$ following from those of $\otimes^{\mathbf{L}}$. See [Tak09, Lem. 2.1] for proofs.

Remark 24. Note this result was first announced by Hopkins [Hop87], but with a slightly flawed proof. Neeman provided two proofs of this result, where the first fixes Hopkins's [Nee92a]; our proof is a simplification of Neeman's second proof, which deduces this theorem as a corollary to Theorem 13. Here, we follow the proof outline of [Iye06]. Note the theorem has been generalized to arbitrary quasi-compact quasi-separated schemes by [Tho97].

Proof. Let P be specialization closed. We want to show $f(g(P)) = P$. First, if $\mathfrak{p} \in f(g(P))$, then there exists $X \in g(P)$ such that $X \otimes \kappa(\mathfrak{p}) \neq 0$, but then $\mathfrak{p} \in P$. Conversely, suppose $\mathfrak{p} \in P$, and pick generators $\{x_1, \ldots, x_n\}$ generating $\mathfrak{p}$. Denote the Koszul complex on these x_i as K. Then, $\text{supp}(K) = V(\mathfrak{p}) \subset P$, and so $K \in g(P)$ and $\mathfrak{p} \in V(\mathfrak{p}) = \text{supp}(K) \subset f(g(P))$. Thus, $f \circ g = \text{id}$.

Now suppose $\mathcal{T}$ is a thick subcategory of $\mathbf{D}^b(R)_{\mathsf{proj}}$. We see that $\mathcal{T} \subset g(f(\mathcal{T}))$ since if $X \in \mathcal{T}$, then all $\mathfrak{p} \in \text{supp}(X)$ are contained in $f(\mathcal{T})$, hence $X \in g(f(\mathcal{T}))$. Conversely, suppose $X \in g(f(\mathcal{T}))$, and so $\text{supp}(X) \subset f(\mathcal{T})$. Then, since X is quasi-isomorphic to a bounded complex of finitely generated projective modules, $\text{supp}(X)$ is the union of supports of finitely many, finitely generated projective modules, hence is a closed subset of $\text{Spec}(R)$, and has finitely many minimal primes. Thus, there exist finitely many complexes $Y_1, \ldots, Y_s$ in $\mathcal{T}$ such that

$$\text{supp}(X) \subset \bigcup_{i=1}^{s} \text{supp}(Y_i) = \text{supp}\left(\bigoplus_{i=1}^{s} N_i\right).$$

Now by Theorem 13, the inclusion of supports above implies that X is in the localizing subcategory of $\mathbf{D}(R)$ generated by the Y_i. Now since the Y_i are compact, by [Nee92b, Lem. 2.2], we moreover have that X is in the thick subcategory of $\mathbf{D}(R)$ generated by the Y_i. $\qquad\square$

3.2. **Thick subcategories and the Balmer spectrum.** We now turn to a more detailed description of the Balmer spectrum, the construction of which is the main motivation for our paper. We start with the following definition inspired by commutative algebra.

Definition 25 ([Bal05, Def. 2.1])**.** Let $\mathcal{D}$ be a triangulated category with a tensor product $\otimes$, and $\mathcal{P}$ a full, rigid, thick, and triangulated proper subcategory of $\mathcal{D}$. We say $\mathcal{P}$ is a *prime* of $\mathcal{D}$ if

$$X \otimes Y \in \mathcal{P} \text{ implies either } X \in \mathcal{P} \text{ or } Y \in \mathcal{P}. \tag{4}$$

Then, let the *spectrum* of $\mathcal{D}$, denoted $\text{Spc}(\mathcal{D})$, be the set of all primes of $\mathcal{D}$. For any family of objects $\mathcal{S} \subset \mathcal{D}$, let $Z(\mathcal{S})$ denote the set

$$Z(\mathcal{S}) = \{\mathcal{P} \in \text{Spc}(\mathcal{D}) \mid \mathcal{S} \cap \mathcal{P} = \emptyset\}.$$

Now by definition $\cap_{j \in J} Z(\mathcal{S}_j) = Z(\cup_{j \in J} \mathcal{S}_j)$ and $Z(\mathcal{S}_1) \cup Z(\mathcal{S}_2) = Z(\mathcal{S}_1 \oplus \mathcal{S}_2)$, where $\mathcal{S}_1 \oplus \mathcal{S}_2 := \{X_1 \oplus X_2 \mid X_i \in \mathcal{S}_i \text{ for } i = 1, 2\}$. Since $Z(\mathcal{D}) = \emptyset$ and $Z(\emptyset) = \text{Spc}(\mathcal{D})$, the collection $\{Z(\mathcal{S})\}$ define the closed subsets of a topology on $\text{Spc}(\mathcal{D})$, which we call the *Zariski topology*. Finally, for any $X \in \mathcal{D}$, we denote by $\text{spp}(X)$ the following:

$$\text{spp}(X) = Z(\{X\}) = \{\mathcal{P} \in \text{Spc}(\mathcal{D}) \mid X \notin \mathcal{P}\}$$

which we call the *support* of X.

What is surprising about this construction is the following

Theorem 26 ([Bal05])**.** *There is a homeomorphism* $\varphi \colon \text{Spec}(R) \xrightarrow{\sim} \text{Spc}(\mathbf{D}^b(R)_{\mathsf{proj}})$, *with*

$$\varphi(\mathfrak{p}) = \{X \in \mathbf{D}^b(R)_{\mathsf{proj}} \mid \mathfrak{p} \notin \text{supp}(X)\}.$$

Remark 27. In fact, this can be turned into an isomorphism of locally ringed spaces; see [Bal05, Thm. 6.3]. Like most results in this paper, this result has been generalized to arbitrary quasi-compact quasi-separated schemes [BKS07].

Proof. Note that in our context, Proposition 12 and the argument using [Nee92b, Lem. 2.2] in the previous section imply that all of our thick subcategories will in fact be rigid, and so in practice in checking that a thick subcategory is prime, we only have to show the condition (4).

So, to show $\varphi(\mathfrak{p})$ is prime in $\mathbf{D}^b(R)_{\mathsf{proj}}$, we need to show that if $\mathfrak{p} \notin \text{supp}(X \otimes^{\mathbf{L}} Y)$, then $\mathfrak{p} \notin \text{supp}(X)$ or $\mathfrak{p} \notin \text{supp}(Y)$. But $\text{supp}(X \otimes^{\mathbf{L}} Y) = \text{supp}(X) \cap \text{supp}(Y)$, hence either $\mathfrak{p} \notin \text{supp}(X)$ or $\mathfrak{p} \notin \text{supp}(Y)$, i.e., either $X \in \varphi(\mathfrak{p})$ or $Y \in \varphi(\mathfrak{p})$.

Now we show φ is continuous. Let $V(\mathfrak{a})$ be closed in $\text{Spec}(R)$, where $\mathfrak{a} \subset R$ is an ideal. Then,

$$\varphi(\mathfrak{p}) \in \text{spp}(X) \iff X \notin \varphi(\mathfrak{p}) \iff \mathfrak{p} \in \text{supp}(X),$$

hence $\varphi^{-1}(\text{spp}(X)) = \text{supp}(X)$.

Finally, we want to show φ is a homeomorphism. We first prove that any closed subset $Z \subset \text{Spec}(R)$ is of the form $Z = \text{supp}(X)$ for some object $X \in \mathbf{D}(R)$. Since $\text{supp}(X_1) \cup \cdots \cup \text{supp}(X_n) = \text{supp}(X_1 \oplus \cdots \oplus X_n)$, and since $\text{Spec}(R)$ is noetherian, it suffices to prove the claim for $Z = \overline{\{\mathfrak{p}\}}$ for some $\mathfrak{p} \in \text{Spec}(R)$. But this follows since $\text{supp}(\kappa(\mathfrak{p})) = \overline{\{\mathfrak{p}\}}$.

We now show φ is injective. For $\mathfrak{p} \in \mathrm{Spec}(R)$ let $Y(\mathfrak{p}) = \{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mathfrak{q} \notin \overline{\{\mathfrak{p}\}}\}$. $Y(\mathfrak{p})$ is specialization closed by definition. We show that $\mathrm{supp}(X) \subset Y(\mathfrak{p}) \iff \mathfrak{p} \notin \mathrm{supp}(X)$. By definition, since $\mathfrak{p} \notin Y(\mathfrak{p})$, we have the implication $\implies$. Conversely, since $\mathrm{supp}(X)$ is closed, and $\mathfrak{p} \notin \mathrm{supp}(X)$, we have $\mathfrak{p} \notin \overline{\{\mathfrak{q}\}}$ for all $\mathfrak{q} \in \mathrm{supp}(X)$, giving the opposite implication. Thus,

$$g(Y(\mathfrak{p})) = \{X \in \mathbf{D}^b(R)_{\mathsf{proj}} \mid \mathrm{supp}(X) \subset Y(\mathfrak{p})\} = \{X \in \mathbf{D}^b(R)_{\mathsf{proj}} \mid \mathfrak{p} \notin \mathrm{supp}(X)\} = \varphi(\mathfrak{p}), \tag{5}$$

where g is the map from Theorem 23. In particular, if $\varphi(\mathfrak{p}_1) = \varphi(\mathfrak{p}_2)$, then $Y(\mathfrak{p}_1) = Y(\mathfrak{p}_2)$, hence $\overline{\{\mathfrak{p}_1\}} = \overline{\{\mathfrak{p}_2\}}$ which implies $\mathfrak{p}_1 = \mathfrak{p}_2$ by the uniqueness of generic points in $\mathrm{Spec}(R)$.

We now show surjectivity. Let $\mathcal{P} \subset \mathbf{D}^b(R)_{\mathsf{proj}}$ be prime. By Theorem 23, there exists a specialization closed subset P of $\mathrm{Spec}(R)$ such that $f(P) = \mathcal{P}$, where f is the map from Theorem 23. Now $\mathrm{Spec}(R) \setminus P$ is non-empty since $\mathcal{P}$ is a proper subcategory. Now let $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(R) \setminus P$. By the claim above, there exist $X, Y \in \mathbf{D}^b(R)_{\mathsf{proj}}$ such that $\overline{\{\mathfrak{p}\}} = \mathrm{supp}(X)$ and $\overline{\{\mathfrak{q}\}} = \mathrm{supp}(Y)$. Now since $\mathfrak{p}, \mathfrak{q}$ are outside P, we have that $X, Y \notin g(P) = \mathcal{P}$. Since $\mathcal{P}$ is prime, we have that $X \otimes^{\mathbf{L}} Y \notin \mathcal{P}$, i.e., there is a point $\mathfrak{r} \in \mathrm{Spec}(R) \setminus P$ such that $\mathfrak{r} \in \mathrm{supp}(X \otimes^{\mathbf{L}} Y) = \mathrm{supp}(X) \cap \mathrm{supp}(Y) = \overline{\{\mathfrak{p}\}} \cap \overline{\{\mathfrak{q}\}}$ hence $\overline{\{\mathfrak{r}\}} \subset \overline{\{\mathfrak{p}\}}$ and $\overline{\{\mathfrak{r}\}} \subset \overline{\{\mathfrak{q}\}}$. Thus, we have that the non-empty family of closed subsets

$$\mathcal{F} = \{\overline{\{\mathfrak{p}\}} \subset \mathrm{Spec}(R) \mid \mathfrak{p} \in \mathrm{Spec}(R) \setminus P\}$$

is such that any two elements have a lower bound with respect to inclusion; since $\mathrm{Spec}(R)$ is noetherian, there is a unique lower bound for $\mathcal{F}$. Thus, there exists a point $\mathfrak{p} \in \mathrm{Spec}(R) \setminus P$ such that $\mathrm{Spec}(R) \setminus P \subset \{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mathfrak{p} \in \overline{\{\mathfrak{q}\}}\}$; this is in fact an equality since $\mathfrak{p} \notin P$, which is specialization closed. So, $P = \{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mathfrak{p} \notin \overline{\{\mathfrak{q}\}}\} = Y(\mathfrak{p})$. Thus, $\mathcal{P} = g(P) = g(Y(\mathfrak{p})) = \varphi(\mathfrak{p})$ by (5), hence φ is surjective.

Finally, $\varphi^{-1}(\mathrm{spp}(X)) = \mathrm{supp}(X)$ gives that $\mathrm{spp}(X) = \varphi(\mathrm{supp}(X))$, so φ is closed since any closed subset of $\mathrm{Spec}(R)$ is of the form $\mathrm{supp}(X)$. φ is therefore a homeomorphism. $\square$

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