Math 732: Topics in Algebraic Geometry II Rationality of Algebraic Varieties

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Course Description

A fundamental problem in algebraic geometry is to determine which varieties are rational, that is, birational to the projective space. Several important developments in the field have been motivated by this question. The main goal of the course is to describe two recent directions of study in this area. One approach goes back to Iskovskikh and Manin, who proved that smooth, 3-dimensional quartic hypersurfaces are not rational. This relies on ideas and methods from higher-dimensional birational geometry. The second approach, based on recent work of Claire Voisin and many other people, relies on a systematic use of decomposition of the diagonal and invariants such as Chow groups, Brauer groups, etc. to prove irrationality. Both directions will give us motivation to introduce and discuss some important concepts and results in algebraic geometry.

The following is a rough outline of the course: Introduction

1. Rationality in dimension 2 (Castelnuovo's criterion).

Part I

2. Introduction to Chow groups and intersection theory.

- 3. Decomposition of the diagonal and non-stable-rationality.
- 4. Examples of classes on non-stably-rational varieties.

Part II

- 5. Introduction to singularities of pairs and vanishing theorems.
- 6. Birational rigidity of Fano hypersurfaces of index 1.

Mircea Mustață's own lecture notes are available at:

http://www-personal.umich.edu/~mmustata/lectures_rationality.html

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1 January 5

The theory of rationality centers around the question: Which varieties are rational, i.e., birational to the projective space \mathbf{P}^n ? This theory is very interesting because it uses vastly different methods. On one hand, showing something is rational usually requires a lot of explicit geometric constructions that are classical in flavor. On the other hand, showing something is *not* rational requires more complicated invariants like Chow groups, etc., many of them arising from birational geometry.

We first give a brief outline of the course:

Introduction: Some definitions and easy properties; examples

In particular, we will study surfaces (Castelnuovo's rationality criterion), cubic three-folds and quartic three-folds, and the unirationality of hypersurfaces of small degree. Note that for these last examples, it is already hard to show they are not rational. These examples will build intuition and help you develop tools to study more complicated cases.

Part I: Stable rationality

We will review some things about Chow groups and intersection theory, and then study the theory of "decomposition of the diagonal" and how it relates to stable rationality, following ideas of Voisin and others.

Part II: Birational rigidity of certain Fano varieties

A key example here is that of hypersurfaces of degree n in \mathbf{P}^n , which are "extremal" Fano varieties. Birationally rigid varieties are those for which birational self-maps can all be extended to automorphisms; for the aforementioned hypersurfaces, the group of birational self-maps will be finite, and so they cannot be rational. The main ideas in this theory come from Mori theory, and so we will need to talk about invariants of singularities and vanishing theorems.

Time and interest permitting, we may also discuss rationally connected varieties, and other methods for showing non-rationality via intermediate Jacobians.

1.1 The Lüroth problem and its solution for curves

Today we work over an arbitrary field k. Note that for most of the time in this course, we will assume k to be algebraically closed; sometimes we will need to discuss the non-closed case, and often we will need to restrict to the case when char(k) = 0. Also, while we will state results in the greatest generality they are known, we will often only prove things under some assumptions.

Let X be a variety over k, i.e., an integral scheme that is separated and of finite type over k. Let n denote the dimension of X.

Definition 1.1. X is rational if it is birational to \mathbf{P}_k^n over k. Equivalently, X is rational if its function field K(X) is isomorphic to $k(x_0, \ldots, x_n)$ as an extension of k.

Remark 1.2. When studying rationality, we may always replace X by any Y birational to X. For example, we can assume that X is affine, but it is more useful to assume that X is projective, so that geometric methods are available.

If char k = 0, then by Hironaka's resolution of singularities, we may also assume that X is smooth. This will be useful in Part II, when we will need to consider singular varieties, in which case it will be important to understand what their resolutions of singularities look like.

Definition 1.3. X is unirational if there is a dominant rational map $\mathbf{P}^N \dashrightarrow X$.

Remarks 1.4.

- (1) Clear: If X and Y are birational, then X is unirational if and only if Y is unirational.
- (2) If X is unirational, then there is a rational dominant map $\mathbf{P}^n \dashrightarrow X$, where $n \coloneqq \dim X$.

Proof when k *is infinite.* The idea is to keep replacing \mathbf{P}^N by a general hyperplane. If $\mathbf{P}^N \dashrightarrow X$ is a rational dominant map, then there exists a commutative diagram



where $N \ge n$. If N > n, then consider a general fiber $f^{-1}(x)$ of f, which is of dimension N - n. Now choose a general hyperplane $H \subseteq \mathbf{P}_k^N$, which satisfies $H \cap U \ne \emptyset$ and $\dim(H \cap f^{-1}(x)) < \dim f^{-1}(x)$. The map $H \dashrightarrow X$ is still rational dominant, for otherwise all fibers would have dimension $\ge (N-1) - (n-1)$, which is not the case for $H \cap f^{-1}(x)$. Now repeat.

This remark is still true if k is finite.

(3) Clear: rationality implies unirationality.

This last remark prompts the following question about its converse:

Classical Question 1.5 (Lüroth's problem). Does unirationality imply rationality? Equivalently, given field extensions $k \subseteq K \subseteq k(x_1, \ldots, x_N)$, is K purely transcendental over k?

Note that the latter question is equivalent to the former since K is a finitely generated field extension of k (it is a subfield of $k(x_1, \ldots, x_N)$), and so the existence of the second inclusion $K \subseteq k(x_1, \ldots, x_N)$ implies there is a rational dominant map from \mathbf{P}^n to X, where X is some variety with function field K. Note that the finite generation statement fails pretty badly for rings, but is okay for fields!

Theorem 1.6 (Lüroth). The answer is "yes" in dimension one: a unirational curve is automatically rational.

We will give two arguments. The first is a geometric argument that works over an algebraically closed field of characteristic zero. We will also give an algebraic argument without these restrictions.

Argument 1 (geometric) [Har77, Ch. IV, Ex. 2.5.5]. Assume that $k = \overline{k}$, and char(k) = 0. Suppose that X/k is a unirational curve; we may assume that X is a smooth projective curve (by completion and normalization). Then, by assumption, there is a rational dominant map $\mathbf{P}^1 \to X$, which in fact extends to a morphism $\mathbf{P}^1 \to X$, which is separable since char(k) = 0. By the Riemann-Hurwitz formula [Har77, Ch. IV, Prop. 2.3],

$$K_{\mathbf{P}^1} \sim f^*(K_X) + \underbrace{\operatorname{Ram}(f)}_{\text{effective}}.$$

Taking degrees, we obtain

$$-2 = \deg f \cdot (2g_X - 2) + \deg(\operatorname{Ram}(f)) \ge \deg f \cdot (2g_X - 2)$$

This implies that $g_X = 0$, and so $X \simeq \mathbf{P}^1$.

It is also instructive to see an elementary argument, in the way they would've done it in the 19th century. This works over an arbitrary ground field k.

Argument 2 (algebraic). We show that if $k \subseteq K \subseteq k(t)$ is a sequence of field extensions such that $\operatorname{trdeg}_k K = 1$, then $K \simeq k(X)$. Since $\operatorname{trdeg}_k K = 1$, it is enough to find some $a \in K$ such that K = k(a). First, the second extension $K \subseteq k(t)$ must be algebraic, so t is algebraic over K. Let

The second extension $\mathbf{M} \subseteq \kappa(t)$ must be algebraic, so t is algebraic over \mathbf{M} .

$$f(X) = X^n + a_1 X^{n-1} + \dots + a_n \in K[X]$$

be the minimal polynomial of t over K. Since t is transcendental over k, we cannot have all $a_i \in k$, and so there is some i such that $a_i \in K \setminus k$. We will show that in this case, $K = k(a_i)$. We know that the $a_i \in K \subseteq k(t)$, and so we may write

$$a_i = \frac{u(t)}{v(t)},$$

where $u, v \in k[t]$ are relatively prime, and where at least one of them of positive degree. Now consider the following polynomial:

$$F(X) = u(X) - a_i v(X) \in K[X].$$

Since F(t) = 0, we have that $f(X) \mid F(X)$ in K[X] by minimality of f(X), and so

$$u(X) - a_i v(X) = f(X)g(X).$$
(1.1)

where $g \in K[X]$. We then claim the following:

Claim 1.7. $g \in K$.

Showing the Claim would conclude the proof, for then we have a sequence of extensions

$$k \hookrightarrow k(a_i) \hookrightarrow K \hookrightarrow k(t)$$

where $[k(t): K] = [k(t): k(a_i)]$ by the Claim, which implies $K = k(a_i)$.

To prove the Claim, the first step is to get rid of all denominators: by multiplying (1.1) by a suitable nonzero element of k[t], we get a relation

$$c(t)(u(X)v(t) - v(X)u(t)) = f_1(t, X)g_1(t, X),$$

where $f_1(t, X), g_1(t, X) \in k[t, X]$ are obtained from f resp. g by multiplication by an element in k[t]. Note that k[t, X] is a UFD, and so we can get rid of c(t) by successively dividing by prime factors of c(t) to get a relation

$$u(X)v(t) - v(X)u(t) = f_2(t, X)g_2(t, X),$$
(1.2)

where now $f_2(t, X), g_2(t, X) \in k[t, X]$ are obtained from f, g by multiplication by a nonzero element in k(t). The trick is now to look at the degrees in t on both sides. First,

$$\deg_t \left(u(X)v(t) - v(X)u(t) \right) \le \max\{\deg u(t), \deg v(t)\},\$$

Letting $f_2(t, X) = \gamma_0(t)X^n + \cdots + \gamma_n(t)$, we see that

$$rac{\gamma_i(t)}{\gamma_0(t)} = a_i(t) = rac{u(t)}{v(t)},$$

where u(t), v(t) were relatively prime. This implies that in fact,

$$\deg_t(f_2(t,X)) \ge \max\{\deg u(t), \deg v(t)\}.$$

By looking at degrees in t on both sides of the relation (1.2), we have that $\deg_t(g_2(t, X)) = 0$, hence $g_2 \in k[X]$.

Now we claim that $g_2 \in k$. For sake of contradiction, suppose that $g_2 \notin k$. Then, there is a root $\gamma \in k$ such that $g_2(\gamma) = 0$, which implies that $u(\gamma)v(t) = v(\gamma)u(t)$. But since u(t) and v(t) are relatively prime, and are not both constants, we must have $u(\gamma) = 0 = v(\gamma)$. This contradicts that u, v are relatively prime, and so $g_2 \in k$. Finally, since $g \in K[X]$ and $g = g_2 \cdot (\text{element of } k(t))$, we have that $g \in K$.

One interesting thing to point out is that Colliot-Thélène and others do a lot of concrete computations like this one to prove (uni)rationality.

This settles the Lüroth problem in dimension one. Already, for surfaces the situation is a bit more complicated, and here the results are only one-hundred years old. We'll see that the answer to the Lüroth problem in dimension two is still "yes" in characteristic zero, but "no" in positive characteristic (because of inseparable phenomena). The answer is also "no" in dimensions ≥ 3 . We will come to the higher-dimensional case later; for now, we want to understand surfaces.

1.2 Birational invariance of plurigenera

What we want to do now is to discuss the first invariants that allow one to show a variety is not rational. These invariants are elementary, and are constructed in terms of the canonical divisor. This will allow us to see explicit examples of non-rational varieties.

General Goal 1.8. Find invariants that agree on varieties that are birational to each other, but take on different values for a given example when compared to \mathbf{P}^n .

Definition 1.9. Let X be a smooth complete variety over k. The plurigenera of X are

$$p_m(X) = \dim_k H^0(X, \omega_X^{\otimes m})$$

for each positive integer m.

Proposition 1.10 [Har77, Ch. II, Ex. 8.8]. The plurigenera $p_m(X)$ are birational invariants for smooth complete varieties, for all $m \ge 1$.

The argument is exactly the same as the case when m = 1, which is [Har77, Ch. II, Thm. 8.19]. We give an argument here, since we will need a variation of the argument later.

Proof. Suppose that there is a map $X \to Y$ that is a birational map of smooth complete varieties. Since X is normal and Y is complete, this extends to a morphism $f: U \to Y$ for $U \subseteq X$ open, such that $\operatorname{codim}(X \setminus U, X) \ge 2$. Now we just have to compare forms on X to those on U: there is a canonical morphism $f^*\Omega^1_Y \to \Omega^1_U$ given by pulling back forms, which is an isomorphism over an open subset of U (e.g., where f is an isomorphism). Taking top exterior powers and mth tensor powers, we obtain a morphism

$$f^*(\Omega^n_Y)^{\otimes m} \longrightarrow (\Omega^n_U)^{\otimes m},$$

which is still generically an isomorphism. Since both sheaves are locally free, they are both torsion-free, and so the morphism is actually an injective map of sheaves. Thus, we have the following commutative diagram:

$$\begin{array}{ccc} H^0(U, f^*(\Omega^n_Y)^{\otimes m}) & \longrightarrow & H^0(U, (\Omega^n_U)^{\otimes m}) \\ & & \uparrow & & \\ & & & f^{\wr} \\ H^0(Y, (\Omega^n_Y)^{\otimes m}) & & & H^0(X, (\Omega^n_X)^{\otimes m}) \end{array}$$

where the left vertical arrow is injective since f is dominant, and the right vertical arrow is an isomorphism since $\operatorname{codim}(X \setminus U, X) \ge 2$, $(\Omega_X^n)^{\otimes m}$ locally free, and X normal (S_2) . This shows that $p_m(Y) \le p_m(X)$ for all $m \ge 1$, hence by symmetry, you get equality.

Remark 1.11.

- (1) The same argument also works to show that the dimensions $h^0(X, (\Omega_X^j)^{\otimes m})$ are birational invariants for smooth complete varieties for all $j \ge 0, m \ge 1$. These invariants for m = 1 are called *Hodge numbers*.
- (2) The same argument implies that if $f: X \to Y$ is a dominant, separable (i.e., generically étale, which breaks down in positive characteristic) rational map between smooth complete varieties of the same dimension, then $p_m(Y) \leq p_m(X)$.
- (3) $\dim_k H^i(X, \mathcal{O}_X)$ are birational invariants for smooth complete varieties.

Proof. If char(k) = 0, then you can use Hodge symmetry, which says

$$h^{i}(X,\Omega_{X}^{j}) = h^{j}(X,\Omega_{X}^{i})$$

$$(1.3)$$

for all i, j (this fails in positive characteristic in general). Thus, $h^i(X, \mathcal{O}_X) = h^0(X, \Omega_X^i)$, which is a birational invariant by (1). In positive characteristic, the argument due to Chatzistamatiou–Rülling [CR11] is much more involved, and very recent.

Corollary 1.12. If X is a rational smooth complete variety, then $p_m(X) = 0$ for all $m \ge 1$, and $h^i(X, \mathcal{O}_X) = 0$ for all $i \ge 1$.

Proof. This is the case for \mathbf{P}^n : the statement for plurigenera follows since ω_X is negative, and $h^i(X, \mathcal{O}_X) = 0$ by the cohomology of projective space [Har77, Thm. 5.1].

This gives us first examples of non-rational varieties.

Example 1.13. Let $X \subseteq \mathbf{P}^n$ be a smooth hypersurface of degree $d \ge n+1$. Then, X is not rational. More generally, if $X \subseteq \mathbf{P}^n$ is a smooth complete intersection of type (d_1, \ldots, d_r) with $d_1 + \cdots + d_r \ge n+1$, then X is not rational.

Proof. By the adjunction formula, you can write down what the canonical bundle is explicitly:

$$\omega_X \simeq \mathcal{O}_X(d_1 + \dots + d_r - n - 1).$$

Then, $h^0(X, \omega_X) \neq 0$ under our assumption, hence X is not rational.

These are cheap examples of non-rational varieties; varieties in the other range are more interesting.

Remark 1.14. It is important that we assume that X is smooth.

Example 1.15. Suppose that $X \subseteq \mathbf{P}_k^n$ is a hypersurface of degree d, and suppose that $p \in X(k)$ is a rational point. Then, $\operatorname{mult}_p X \leq d$, and equality holds if and only if X is a cone with vertex p over a hypersurface in \mathbf{P}^{n-1} .

If $\operatorname{mult}_p X = d - 1$, then X is rational. The idea is simple: you parametrize \mathbf{P}^{n-1} by the lines of \mathbf{P}^n that pass through p, which gives a generically one-to-one map since lines through p intersect X at at most one more point. We will write this out carefully next time.

If d = 2, then the condition $\operatorname{mult}_p X = d - 1 = 1$ says that the point p is smooth. The upshot is that if X is a smooth quadric hypersurface such that $X(k) \neq \emptyset$, then X is rational. Note that this is a statement we will use even when k is not algebraically closed.

Next time, we will discuss related rationality properties, specifically that of stable rationality and uniruledness. We will then start discussing cubic surfaces. In particular, we will prove that all cubic surfaces are rational, which is not proved in [Har77, Ch. V, §4]. This uses Castelnuovo's criterion for rationality, and the proof will be useful for arguments later.

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2.1 Birational invariance of plurigenera (continued)

Last time, we showed in Proposition 1.10 that if $X \rightarrow Y$ is a rational separable dominant map, and if X, Y are smooth projective varieties of the same dimension, then

$$h^0(X, (\Omega^q_X)^{\otimes m}) \ge h^0(Y, (\Omega^q_Y)^{\otimes m}).$$

In particular, these are birational invariants. This is most used when $q = \dim X = \dim Y$, in which case we have $p_m(X) = h^0(X, \omega_X^{\otimes m})$, and in characteristic zero, $h^0(X, \Omega_X^q) = h^q(X, \mathcal{O}_X)$ (Remark 1.11).

Remark 2.1. The quantities $h^0(X, (\omega_X^{-1})^{\otimes m})$ are not birational invariants. For example, if $f: Y = \operatorname{Bl}_p X \to X$ is the blowup of a smooth projective surface X at a point p with exceptional divisor E, then $\omega_Y \simeq f^* \omega_X(E)$, and global sections do not change. However,

$$H^0(Y, \omega_Y^{-1}) \simeq H^0(X, \omega_X^{-1} \otimes \mathscr{I}_p) \subsetneq H^0(X, \omega_X^{-1})$$

as long as p is not in the base locus of the complete linear system associated to ω_X^{-1} , even for $X = \mathbf{P}^2$.

Knowing these are birational invariants, we can use them to show that certain spaces are not rational, since we saw that for rational varieties, we have $p_m(X) = 0$ for all $m \ge 1$, and $h^i(X, \mathcal{O}_X) = 0$ for all $i \ge 1$, since these hold for projective space (Corollary 1.12).

Exercise 2.2. Show that if X is a smooth projective variety that is rational, then

$$h^0(X, (\Omega^q_X)^{\otimes m}) = 0$$

for all m, q > 0.

Proof by notetaker. By Remark 1.11(1), it suffices to show that $h^0(\mathbf{P}^n, (\Omega_{\mathbf{P}^n}^q)^{\otimes m}) = 0$. Fix q > 0; we will induce on m > 0. First suppose that m = 0. Consider the Euler exact sequence [Har77, Ch. II, Thm. 8.13]

$$0 \longrightarrow \Omega_{\mathbf{P}^n} \longrightarrow \mathcal{O}_{\mathbf{P}^n}(-1)^{\oplus (n+1)} \longrightarrow \mathcal{O}_{\mathbf{P}^n} \longrightarrow 0$$

By taking the qth exterior power as in [Hir95, Thm. $4.1.3^*$], we obtain the short exact sequence

$$0 \longrightarrow \Omega_{\mathbf{P}^n}^q \longrightarrow \mathcal{O}_{\mathbf{P}^n}^{\oplus \binom{n+1}{q}}(-q) \longrightarrow \Omega_{\mathbf{P}^n}^{q-1} \longrightarrow 0.$$
(2.1)

Taking global sections, we then obtain

$$H^0(\mathbf{P}^n, \Omega^q_{\mathbf{P}^n}) \subseteq H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(-q))^{\oplus \binom{n+1}{q}} = 0$$

for all q > 0 by the cohomology of projective space [Har77, Thm. 5.1(a)].

It now remains to consider the case when m > 0. By tensoring the exact sequence (2.1) by $(\Omega_{\mathbf{P}^n}^q)^{\otimes (m-1)}$, and then by taking global sections, we obtain the injection

$$H^{0}(\mathbf{P}^{n}, (\Omega_{X}^{q})^{\otimes m}) \subseteq H^{0}(\mathbf{P}^{n}, (\Omega_{X}^{q})^{\otimes (m-1)}(-q))^{\oplus \binom{n+1}{q}} \subseteq H^{0}(\mathbf{P}^{n}, (\Omega_{X}^{q})^{\otimes (m-1)})^{\oplus \binom{n+1}{q}} = 0$$

by inductive hypothesis on m.

Remark 2.3. In characteristic zero, the same invariants vanish even if X is just *unirational*, since the dominant rational map from projective space is separable, and then by applying Remark 1.11(2).

There were two questions from last time: first, whether the characteristic zero assumption is necessary in this Remark, and second, to what degree smoothness is necessary in Proposition 1.10.

For the first question, recall that we saw in Example 1.13 that if $X \subset \mathbf{P}^n$ is a smooth projective hypersurface of degree $d \ge n + 1$, then X is not rational or even unirational in characteristic zero. On the other hand, in characteristic p, we will see unirational surfaces in \mathbf{P}^3 of large degree can be unirational [Shi74, Prop. 1]. For the second question, we make the following:

Remark 2.4. The invariant $h^0(X, \omega_X)$ is not a birational invariant in general if X is not smooth, since a resolution $f: Y \to X$ will have a canonical divisor of the form $f^*K_X + \sum a_i E_i$, where the a_i could be negative. Demanding that $a_i \ge 0$ is exactly the condition that says X has canonical singularities. Moreover, a high degree hypersurface that is singular can still be rational, but have sections, as we will see below.

Example 2.5. Let $X \subset \mathbf{P}_k^n$ be a hypersurface of degree d, and let $p \in X(k)$. After changing coordinates, we may assume that

$$p = (1, 0, 0, \dots, 0) \in U = (x_0 \neq 0) \simeq \mathbf{A}_k^n.$$

If X = (f = 0), then $X \cap U = (g = 0) \subseteq U$, where $g = f(1, x_1, \dots, x_n)$. Now write

$$g = \sum_{i \leq d} g_i$$
 where g_i are homogeneous of degree i ,

and where $\operatorname{mult}_p X = \min\{i \mid g_i \neq 0\} \leq d$. Then, the $\operatorname{mult}_p X = d$ if and only if $f \in k[x_1, \ldots, x_n]$, i.e., X is a cone with vertex p over a hypersurface in \mathbf{P}^{n-1} .

Now we claim that if X is a variety such that $\operatorname{mult}_p X = d - 1$, then X is rational. The geometric idea is that lines through the point p are parametrized by \mathbf{P}^{n-1} , and intersect X at at most one more point. We define a birational map

$$\mathbf{P}^{n-1} = \{ \text{lines in } \mathbf{P}^n \text{ through } p \} \dashrightarrow X$$

as follows. Given $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in k^n$, we want to look at $X \cap \{(t\lambda_1, t\lambda_2, \ldots, t\lambda_n) \mid t \in k\}$, which is given by

$$g(t\lambda_1, t\lambda_2, \dots, t\lambda_n) = t^{d-1}(g_{d-1}(\lambda) + tg_d(\lambda)) = 0.$$

By assumption, $g_{d-1} \neq 0$ (by multiplicity condition), and $g_d \neq 0$ (otherwise, $f = x_0 g$ is not irreducible). Define a map

$$\mathbf{P}^{n-1} \smallsetminus \{g_d = 0\} \xrightarrow{\varphi} X$$
$$(\lambda_1, \lambda_2, \dots, \lambda_n) \longmapsto \left(1, -\frac{g_{d-1}(\lambda)}{g_d(\lambda)}\lambda_1, \dots, -\frac{g_{d-1}(\lambda)}{g_d(\lambda)}\lambda_n\right)$$

and in the other direction,

$$(X \cap U) \smallsetminus \{p\} \xrightarrow{\psi} \mathbf{P}^{n-1}$$
$$(1, \lambda_1, \dots, \lambda_n) \longmapsto (\lambda_1, \dots, \lambda_n)$$

You can check that φ, ψ are inverse to each other.

An important special case is that if d = 2, i.e., $X \subset \mathbf{P}_k^n$ is an integral quadric such that $(X_{sm})(k) \neq \emptyset$, then X is rational. We will use this even if k is not algebraically closed, but in the algebraically closed case, this says that all integral quadrics are rational.

Remark 2.6. If k is infinite, and X is unirational, then $X(k) \neq \emptyset$, since any open subsets in \mathbf{P}^n have rational points.

Remark 2.7. This is okay even if k is finite and X is complete by Nishimura's Lemma, which says that if $Y \dashrightarrow X$ is a rational map, where X is complete, and Y is smooth with $Y(k) \neq \emptyset$, then $X(k) \neq \emptyset$. See [KSC04, Sol. to Exc. 1.12] for a proof.

2.2 Other properties close to rationality

Before moving on, we will discuss two more classes of varieties which are close to being rational. One will come up often, and the other encompass the varieties we just saw as examples.

2.2.1 Stably rational varieties

Definition 2.8. A variety X/k is stably rational if $X \times \mathbf{P}_k^n$ is rational for some *n*.

- It is clear that if X and Y are birational, then X is stably rational if and only if Y is stably rational.
- Rational \implies stably rational \implies unirational. For the last implication, we have



This breaks up the Lüroth problem into two parts, which are both known to not be equivalences:

- Artin–Mumford examples [AM72] are unirational but not stably rational.
- Beauville, Colliot-Thélène, Sansuc, and Swinnerton-Dyer [Bea+85] found a complex threefold X such that $X \times \mathbf{P}^3$ is rational, but X is not rational.

It is hard to show varieties are not stably rational. There is no good way to show that a variety is not unirational.

2.2.2 Uniruled varieties

These show up in birational geometry; we may discuss them when we discuss rationally connected varieties.

Definition 2.9. A variety X over k is uniruled if there is a variety Y of dimension dim X - 1 and a dominant rational map $Y \times \mathbf{P}^1 \dashrightarrow X$.

- It is clear that this is a birational property.
- A point is *not* uniruled, but in higher dimensions, unirational \implies uniruled:



• Uniruledness is much weaker than unirationality, and is much better behaved. It is well-understood due to Mori's theory about deformations of rational curves.

Remark 2.10. Uniruled varieties are far from being unirational, e.g., $C \times \mathbf{P}^1$, where C is an elliptic curve, is uniruled but not unirational.

Remark 2.11. All examples we saw of non-(uni)rational varieties are not uniruled because of the following:

Proposition 2.12. Suppose X is a smooth projective variety in characteristic zero. If X is uniruled, then $p_m(X) = 0$ for all $m \ge 1$.

Proof. We have by definition a map $\varphi: Y^{n-1} \times \mathbf{P}^1 \longrightarrow X^n$ that is dominant. Since we are in characteristic zero, we may assume that Y is also smooth and projective by Hironaka's resolution of singularities, and that the map is separable. Then, we saw last time that $p_m(X) \leq p_m(Y \times \mathbf{P}^1)$. Now consider the two projection maps



Since the tangent bundle of the product is the direct sum of those of the two factors, we have $\omega_{Y \times \mathbf{P}^1} \cong p^* \omega_Y \otimes q^* \mathcal{O}(-2)$. Thus, by the Künneth formula, $H^0(\omega_{Y \times \mathbf{P}^1}^{\otimes m}) \simeq H^0(Y, \omega_Y^{\otimes m}) \otimes H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-2m)) = 0$. \Box

The reason why this is actually interesting is that we expect the converse to be true:

Conjecture 2.13. If char(k) = 0, and X/k is a smooth projective variety, and if $p_m(X) = 0$ for all $m \ge 1$, then X is uniruled. (This is known if dim $X \le 3$.)

This gives a purely numerical criterion for uniruledness. While people have looked for a similar purely numerical criterion for rationality for a long time, such a criterion probably does not exist.

2.3 Cubic surfaces

For the next few weeks, we will discuss examples of unirational and stably rational varieties. The first thing to discuss is cubic surfaces.

Goal 2.14. Every smooth cubic surfaces in \mathbf{P}_k^3 where $k = \overline{k}$ is rational.

In Hartshorne's book, he shows that general cubics satisfy this by showing that a general smooth cubic surface is isomorphic to the blowup of \mathbf{P}^2 at six points. This is indicative of this field in general, where it is usually much easier to show that a property holds for objects that are general in moduli. Showing a property holds for *all* objects, however, tends to be a bit tricker.

We will show the following:

Theorem 2.15. Every hypersurface $X \subset \mathbf{P}^3$ of degree 3 contains a line.

{

Note 2.16. There are such X that contain infinitely many lines: nonreduced ones, or cones over elliptic curves in \mathbf{P}^2 .

Theorem 2.17. If X is smooth, then

- (1) X contains precisely 27 lines;
- (2) X contains two disjoint lines.

The idea is to setup an incidence correspondence that parametrizes lines that lie in X.

Proof of Theorem 2.15. Let **P** be the projective space parametrizing cubic hypersurfaces of degree 3 in \mathbf{P}^3 , which is isomorphic to $\mathbf{P}^{\binom{3+3}{3}-1} = \mathbf{P}^{19}$. Let G be the Grassmannian parametrizing lines in \mathbf{P}^3 , which is isomorphic to G(2,4), which is a smooth projective variety of dimension 4. Let

Claim 2.18. M is closed in $\mathbf{P} \times G$.

Proof of Claim. Consider the open subset $V \subset G$ which consists of those lines L which are spanned by the rows of the matrix

$$\begin{pmatrix} 1 & 0 & a_1 & a_2 \\ 0 & 1 & b_1 & b_2 \end{pmatrix}$$

Suppose X is a hypersurface corresponding to $f_c = \sum c_{\alpha} x^{\alpha}$, where $c = (c_{\alpha})$, $\alpha = (\alpha_0, \ldots, \alpha_3)$, and $\alpha_0 + \cdots + \alpha_3 = 3$. If L is the line corresponding to the matrix above, then $L \subset X$ if and only if $f_c(s(1, 0, a_1, a_2) + t(0, 1, b_1, b_2)) = 0$ for all s, t. Write

$$f_c(s, t, sa_1 + tb_1, sa_2 + tb_2) = \sum_{i=0}^{3} \underbrace{F_i(a, b, c)}_{\text{linear in } c} s^i t^{3-i}.$$

Hence, $M \cap (\mathbf{P} \times V) \subset \mathbf{P} \times V$ is cut out by (F_0, \ldots, F_3) , and so $M \cap (\mathbf{P} \times V)$ is closed. Finally, since open subsets of the form V cover G, we see that M is closed in $\mathbf{P} \times G$.

The next step is to understand the projection map q. The incidence correspondence will allow us to deduce that p is then surjective.

Claim 2.19. Locally on G, the projection map q is trivial with fiber \mathbf{P}^{15} .

Proof of Claim. We can check this on V (other charts in G are of a similar form). M is cut out by four equations F_0, \ldots, F_3 , and since the equations are linear in c, it is enough to check that for all $L \in G$, the fiber $q^{-1}(L) \subseteq \mathbf{P}$ has codimension 4. Choose coordinates such that $L = (x_2 = x_3 = 0)$. Then, $L \subset V(f_c)$ if and only if the coefficients of $x_0^3, x_0^2 x_1, x_0^1 x_1^2, x_1^3$ in f_c are zero.

This implies that M is smooth, irreducible, and of dimension 15 + 4 = 19.

Now consider $p: M \to \mathbf{P}$, which is a morphism between projective varieties of dimension 19. For Theorem 2.15, we need to show that p is surjective. Since p is proper (hence closed), we only need p to be dominant. So it is enough to find a zero-dimensional fiber by the theorem on dimension of fibers. We check this explicitly for our favorite cubic:

Example 2.20 (char(k) \neq 3). Consider the Fermat cubic $Q: x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$. This is smooth (e.g., by the Jacobian criterion), and by symmetry, we can consider lines L of the form $x_0 = \alpha x_2 + \beta x_3$ and $x_1 = \gamma x_2 + \delta x_3$. Then, $L \subset Q$ if and only if

$$(\alpha x_2 + \beta x_3)^3 + (\gamma x_2 + \delta x_3)^3 + x_2^3 + x_3^3 = 0$$

in $k[x_2, x_3]$. This gives the system of equations

$$\alpha^3 + \gamma^3 + 1 = 0, \tag{2.2}$$

$$\alpha^2 \beta + \gamma^2 \delta = 0, \tag{2.3}$$

$$\alpha\beta^2 + \gamma\delta^2 = 0, \tag{2.4}$$

$$\beta^3 + \delta^3 + 1 = 0. \tag{2.5}$$

If $\alpha, \beta, \gamma, \delta \neq 0$, then (2.3) implies

$$\delta = -\frac{\alpha^2 \beta}{\delta^2}.$$

Combined with (2.4), we have that

 $\alpha^3 + \gamma^3 = 0,$

which contradicts (2.2). Thus, one of $\alpha, \beta, \gamma, \delta$ must be zero.

If $\alpha = 0$, then $\gamma \delta = 0$, but $\gamma^3 = -1$ and so we must have $\delta = 0$ and $\beta^3 = -1$. Thus, L is given by $x_0 = \beta x_3$ and $x_1 = \gamma x_2$. There are nine such lines corresponding to different choices of roots of -1.

By symmetry, there are two more sets of nine lines, corresponding to the two other choices of two variables out of x_0, x_1, x_2, x_3 .

We note for later that our explicit description above says that there are pairs of disjoint lines. \Box

Exercise 2.21. Show that for every line $L \subset Q$, there are precisely ten lines that meet it.

There are two ways to study lines on a cubic surface. If you know the description of a cubic surface as a blowup of \mathbf{P}^2 at six points, then it is relatively easy to show Theorems 2.15 and 2.17; see [Har77, Ch. V, §4]. On the other hand, our proof of Theorem 2.15 just required understanding the Fermat cubic well. Theorem 2.17 says that any smooth cubic surface has exactly 27 lines, which corresponds to the fact that the projection map $p: M \to \mathbf{P}$ is étale. We also need to show that there are two disjoint lines. The ultimate goal is to use these lines to get a birational map to projective space.

3 January 12

Three administrative comments:

- 1. Since this is a topics course, there will be no exam at the end.
- 2. There will not be homework to be turned in. However, there will be exercises and facts to check given in class, which you are encouraged to do. Mircea will fill in proofs of these statements in his lecture notes.
- 3. Office hours will be by appointment.

3.1 Cubic surfaces (continued)

We return to what we were discussing last time. Recall that we work over an algebraically closed filed $k = \overline{k}$. So far, we have shown the following:

Theorem 2.15. Every hypersurface $X \subset \mathbf{P}^3$ of degree 3 contains a line.

We remind ourselves of the setup, since we need to use it for some proofs today.

Recall that $\mathbf{P} = \mathbf{P}^{19}$ parametrizes all hypersurfaces of degree 3 in \mathbf{P}^3 , and that the Grassmannian $G = G(\mathbf{P}^1, \mathbf{P}^3)$ is a four-dimensional, smooth projective variety that parametrizes lines in \mathbf{P}^3 . We then considered the incidence correspondence



We showed that q is a projective bundle of relative dimension 15, and so M is smooth and irreducible of dimension 19. In Example 2.20, we gave an explicit example (when $\operatorname{char}(k) \neq 3$) of a smooth surface containing precisely 27 lines, some of which are not disjoint. Since the first projection p above therefore has one fiber which is finite, we see that that p is surjective by the theorem on dimension of fibers, i.e., for every cubic hypersurface $X \in \mathbf{P}$, we have that $p^{-1}(X) \neq \emptyset$.

We now want to show the following:

Theorem 2.17. If X is smooth, then

- (1) X contains precisely 27 lines;
- (2) X contains two disjoint lines.

The basic idea is that we want to prove that the first projection p is étale after restricting to the locus of $U \subseteq \mathbf{P}$ of smooth hypersurfaces:

Proposition 3.1. The induced map $p^{-1}(U) \to U$ is étale and finite.

Proof of Theorem 2.17(1) assuming Proposition 3.1. Since $p^{-1}(U) \to U$ is finite, we see that any smooth cubic hypersurface in \mathbf{P}^3 has only finitely many lines. Moreover, since U is connected, we have that all fibers of p over U have the same number of points, that is, every smooth cubic hypersurface in \mathbf{P}^3 contains the same number of lines. Since we already gave a point in U where the fiber of p had degree 27 in Example 2.20, we conclude that all smooth cubic surfaces in \mathbf{P}^3 have 27 lines.

Proof of Proposition 3.1. We first recall how we wrote down equations for the lines that lie on a given surface X. Recall that we covered G by affine opens V, which consisted of lines that are given as the span of the rows of

$$\begin{pmatrix} 1 & 0 & a_1 & a_2 \\ 0 & 1 & b_1 & b_2 \end{pmatrix}$$

The condition that the lines lie on the cubic surface give equations F_0, \ldots, F_3 for $M \cap q^{-1}(V)$, where

$$f_c(s, t, a_1s + b_1t, a_2s + b_2t) = \sum_{i=0}^{3} F_i(a, b, c)s^i t^{3-i},$$
(3.1)

and $X = (f_c = 0)$.

Since $p^{-1}(U) \to U$ is proper, we only need to check it is étale, since a proper étale map is finite. To check that $p^{-1}(U) \to U$ is étale at some point $(x, L) \in U \times V$, it is equivalent to show that the Jacobian matrix

$$A = \frac{\partial(F_0, F_1, F_2, F_3)}{\partial(a_1, a_2, b_1, b_2)}(x, L)$$

has rank 4 by the Jacobian criterion. We need to differentiate (3.1) with respect to different variables. First, we may assume (after changing coordinates) that L corresponds to $(a_1, a_2, b_1, b_2) = 0$. Then, we apply the partial derivative $\frac{\partial}{\partial a_1}\Big|_{(c,0)}$ to (3.1) to obtain

$$s\frac{\partial f_c}{\partial x_2}(s,t,0,0) = \sum_{i=0}^3 \frac{\partial F_i}{\partial a_1}(0,0,c)s^it^{3-i}.$$

The first column of A is then given by the coefficients of the left-hand side. Similarly, the other columns of A correspond to the coefficients of the partial derivatives

$$s rac{\partial f_c}{\partial x_3}(s,t,0,0) \qquad t rac{\partial f_c}{\partial x_2}(s,t,0,0) \qquad t rac{\partial f}{\partial x_3}(s,t,0,0).$$

Suppose the matrix A has rank < 4, so there is a non-trivial linear combination of the columns that is zero, i.e., there exist $\lambda_1, \lambda_2, \mu_1, \mu_2 \in k$, not all zero, such that

$$\frac{\partial f_c}{\partial x_2}(s,t,0,0)(\lambda_1 s + \mu_1 t) + \frac{\partial f_c}{\partial x_3}(s,t,0,0)(\lambda_2 s + \mu_2 t) = 0.$$

$$(3.2)$$

Since f_c has degree 3, its partial derivatives are homogeneous of degree 2 in s, t. Moreover, since we are over an algebraically closed field, these partial derivatives are products of linear factors. The equality (3.2) then says that $\frac{\partial f_c}{\partial x_2}(s, t, 0, 0)$ and $\frac{\partial f_c}{\partial x_3}(s, t, 0, 0)$ must have a common factor. In other words, there is a common root $(s_0, t_0) \neq (0, 0)$, such that

$$\frac{\partial f_c}{\partial x_2}(s_0, t_0, 0, 0) = 0 = \frac{\partial f_c}{\partial x_3}(s_0, t_0, 0, 0).$$

Finally, we claim that the other partial derivatives also vanish automatically at (s_0, t_0) . By assumption, the line L is contained in $X = (f_c = 0)$, so we have that $f_c(s, t, 0, 0) = 0$. Thus,

$$\frac{\partial f_c}{\partial x_0}(s_0, t_0, 0, 0) = 0 = \frac{\partial f_c}{\partial x_1}(s_0, t_0, 0, 0),$$

and the point $(s_0, t_0, 0, 0) \in L \subset X$ is in fact a singular point of X, which is a contradiction.

We are now ready to prove Theorem 2.17(2). The idea is to imitate what we did so far. We know one smooth cubic surface has disjoint lines, so we want to show all of them have disjoint lines, using incidence correspondences.

Proof of Theorem 2.17(2). Consider the following:

$$p^{-1}(U) \times_U p^{-1}(U) = \left\{ (X, L_1, L_2) \middle| \begin{array}{c} X \text{ smooth} \\ L_1, L_2 \subseteq X \end{array} \right\}$$

$$\downarrow^{\text{étale, finite}} \\ U$$

Let W be the set of all triples that correspond to lines in X that are disjoint:

$$W = \left\{ (X, L_1, L_2) \middle| \begin{array}{c} X \text{ smooth} \\ L_1, L_2 \subseteq X \\ L_1 \cap L_2 = \emptyset \end{array} \right\}.$$

We want to show that W is a union of connected components of $p^{-1}(U) \times_U p^{-1}(U)$. This would imply all fibers of $W \to U$ have the same number of elements, and so since Example 2.20 is an example of a smooth cubic surface with two disjoint lines, the assertion of Theorem 2.17(2) will follow.

Our method will be to understand the locus corresponding to lines that *do* intersect, that is, we want to understand the incidence correspondence

Note that the condition $L_1 \cap L_2 \neq \emptyset$ is a closed condition, so R is a closed subscheme of $\mathbf{P} \times G \times G$. We first show that Z is irreducible of dimension 7. We will need the following:

Parenthesis 3.2 [Sha13, Thm. 1.26]. If $f: A \to B$ is a morphism of schemes over k such that

(1) f is proper,

(2) B is irreducible, and

(3) all fibers of f are irreducible of the same dimension,

then A is irreducible.

Now note that Z is an incidence correspondence with two projection maps:



where $f^{-1}(L)$ are lines that intersect L. The fibers

$$f^{-1}(L) = \{ L' \in G \mid L \cap L' \neq \emptyset \}$$

fit into another incidence correspondence



where the map to $f^{-1}(L)$ is surjective. The fibers of the map to L are isomorphic to \mathbf{P}^2 , which implies that T is irreducible of dimension 3. The map to $f^{-1}(L)$ has general fiber equal to a single point, which therefore

implies that $f^{-1}(L)$ is irreducible of dimension 3. We therefore conclude that Z is irreducible of dimension 7, since G has dimension 4.

Finally, to get information about R, we have to understand the fibers of β . Let $(L_1, L_2) \in Z$, and consider $\beta^{-1}(L_1, L_2)$, where we may assume that $L_1 = (x_0 = x_1 = 0)$ and $L_2 = (x_0 = x_2 = 0)$, after a change of coordinates. Then, $L_1, L_2 \subset (f = 0)$, so we must have that the coefficients of $x_2^3, x_2^2x_3, x_2x_3^2, x_3^3$ and $x_1^3, x_1^2x_3, x_1x_3^2$ in f are all zero. Thus, the fiber $\beta^{-1}(L_1, L_2) \subset \mathbf{P}^{19}$ is a linear subspace of codimension 7. But Z has dimension 7, so we see that R is irreducible, and dim $R = \dim \mathbf{P} = 19$. Note that we have an embedding

$$R \hookrightarrow M \times_{\mathbf{P}} M,$$

where over U, the right-hand side is of dimension 19. Thus, the subscheme of R lying over U is equal to a connected component of $p^{-1}(U)$, and so the component W we are interested in is a union of the other connected components.

What's important for us is that any smooth cubic surface over $k = \overline{k}$ contains two disjoint lines.

3.2 Some rational cubic hypersurfaces

We will now prove a rationality criterion for higher dimensional cubic hypsurfaces in projective space, that works for even-dimensional cubics of a certain form.

Theorem 3.3. If $X \subseteq \mathbf{P}^{2m+1}$ is a smooth cubic hypersurface that contains two disjoint m-dimensional linear subspaces (over the ground field), then X is rational.

The condition on linear subspaces is restrictive: it is a closed condition, and for example for fourfolds, it is a codimension 2 condition. We will discuss more about what this locus looks like later.

Corollary 3.4. Every smooth cubic surface in \mathbf{P}^3 over $k = \overline{k}$ is rational.

Let $\Lambda_1, \Lambda_2 \subseteq X$ be the two disjoint linear subspaces. Before proving Theorem 3.3, we show the following: Claim 3.5. For all $P \in \mathbf{P}^{2m+1} \setminus (\Lambda_1 \cup \Lambda_2)$, there is a line L_P containing P, such that L_P intersects both Λ_1 and Λ_2 . Moreover, the assignment

$$\mathbf{P}^{2m+1} \smallsetminus (\Lambda_1 \cup \Lambda_2) \longrightarrow \Lambda_1 \times \Lambda_2 \\
P \longmapsto (L_P \cap \Lambda_1, L_P \cap \Lambda_2)$$
(3.3)

is a morphism.

This is straightforward if we write things down in coordinates.

Proof of Claim 3.5. Choose coordinates such that

$$\Lambda_1 = (x_0 = \dots = x_m = 0),$$

$$\Lambda_2 = (x_{m+1} = \dots = x_{2m+1} = 0).$$

and choose points

$$Q_1 = (0, \dots, 0, a_0, \dots, a_m) \in \Lambda_1, Q_2 = (b_0, \dots, b_m, 0, \dots, 0) \in \Lambda_2.$$

Then,

$$\overline{Q_1Q_2} = \left\{ sQ_1 + tQ_2 = (tb_0, \dots, tb_m, sa_0, \dots, sa_m) \mid (s,t) \in \mathbf{P}^1 \right\}.$$

Suppose $P \notin \Lambda_1 \cup \Lambda_2$, with coordinates $P = (\lambda_0, \ldots, \lambda_{2m+1})$. Then, $P \in \overline{Q_1 Q_2}$ if and only if there exist $s, t \neq 0$ such that $(\lambda_0, \ldots, \lambda_m) = t(b_0, \ldots, b_m)$, and $(\lambda_{m+1}, \ldots, \lambda_{2m+1}) = s(a_0, \ldots, a_m)$. This implies the existence and uniqueness of L_P , and the map is given by

$$(\lambda_0, \dots, \lambda_{2m+1}) \longmapsto ((\lambda_0, \dots, \lambda_m), (\lambda_{m+1}, \dots, \lambda_{2m+1})) \in \Lambda_1 \times \Lambda_2.$$

We can now prove Theorem 3.3.

Proof of Theorem 3.3. Restricting the map (3.3) to our cubic hypersurface X, we get a map

$$X \smallsetminus (\Lambda_1 \cup \Lambda_2) \xrightarrow{\varphi} \Lambda_1 \times \Lambda_2 \simeq \mathbf{P}^m \times \mathbf{P}^m.$$

The target space is rational. Now for a general point $(P_1, P_2) \in \Lambda_1 \times \Lambda_2$, we see that $\overline{P_1P_2} \not\subseteq X$; otherwise, X would be the whole projective space, a contradiction. Thus, for such P_1, P_2 , the line $\overline{P_1P_2}$ intersects X in a scheme of length 3. By assumption, this scheme contains P_1, P_2 , and so there is at most one other point, that is, every fiber of φ contains at most one other point. If $\operatorname{char}(k) = 0$, this shows that φ is birational. \Box

Exercise 3.6. Write down an explicit formula for φ^{-1} to check birationality also in characteristic p.

The moral of the story is that over $k = \overline{k}$, any smooth cubic surface is rational.

Remark 3.7. If X is a smooth cubic surface in \mathbf{P}^3 , choosing two disjoint lines L_1, L_2 in X gives a rational map

$$X \smallsetminus (L_1 \cup L_2) \longrightarrow \mathbf{P}^1 \times \mathbf{P}^1.$$

This map in fact extends to X, and exhibits X as the blowup of $\mathbf{P}^1 \times \mathbf{P}^1$ at five points.

Exercise 3.8. Show that the blowup $W \to \mathbf{P}^1 \times \mathbf{P}^1$ at one point is isomorphic to the blowup $\operatorname{Bl}_{\{P,Q\}} \mathbf{P}^2$, and that under this isomorphism the exceptional divisor corresponds to the strict transform of \overline{PQ} .

This implies that any smooth cubic surface in \mathbf{P}^3 is isomorphic to the blowup of \mathbf{P}^2 at six points. The six points must be general (not on a conic, no three on a line): If we denote $f: X \to \mathbf{P}^2$ to be the blowup map, then $\omega_X \simeq f^* \mathcal{O}(-3) \otimes \mathcal{O}(-E_1 - \cdots - E_6)$. But X is a cubic surface in \mathbf{P}^3 , so $\omega_X^{-1} = \mathcal{O}_X(1)$ is (very) ample. If the blown up points were not general, then the strict transform of a special curve would contradict the positivity of ω_X^{-1} .

Fact 3.9. If Y is the blowup of Y as six general points, then ω_Y^{-1} is very ample, and embeds Y as a cubic surface in \mathbf{P}^3 [Har77, IV, Cor. 4.7].

This concludes what we wanted to say for cubic surface.

We next want to understand the condition that a cubic hypersurface of dimension 2m in \mathbf{P}^{2m+1} contains two disjoint linear subspaces of dimension m. The idea is to look at an appropriate incidence correspondence, although the fibers of the projections will be difficult to understand.

Proposition 3.10. Let $X \subset \mathbf{P}^n$ be a hypersurface of degree ≥ 2 .

- (1) If $\Lambda \subset X$ is a linear subspace of dimension r, then $r \leq \frac{n-1}{2}$.
- (2) If $r = \frac{n-1}{2}$, and X is smooth, then X contains only finitely many such subspaces.

The first is simple, but the second is more subtle: we have to use the fact that linear subspaces are parametrized by the Hilbert scheme. We will then discuss the proof that *all* cubic hypersurfaces are unirational, and then discuss unirationality for higher degree hypersurfaces.

4 January 17

4.1 Some rational cubic hypersurfaces (continued)

Last time, we saw that if $X \subseteq \mathbf{P}^{2m+1}$ is a smooth cubic hypersurface containing two disjoint *m*-planes, then, X is rational.

Example 4.1. If the ground field $k = \overline{k}$, and m = 1, then X is always rational.

Question 4.2. How many such X do we have for $m \ge 2$?

As before, the idea is to setup the correct incidence correspondence, and compute dimensions. First, however, we will prove the following result about what linear subspaces could possibly live in X:

Proposition 4.3. Let L be an r-dimensional plane contained in the smooth locus of X, where $X \subseteq \mathbf{P}^n$ is a hypersurface of degree $d \ge 2$.

(1)
$$r \leq \frac{n-1}{2};$$

(2) If $r = \frac{n-1}{2}$ and $d \ge 3$, then there are only finitely many such r-planes in $X_{\rm sm}$.

In the second part, we will need some facts about Hilbert schemes (Facts 4.4 and 4.5) that we will only show next week (strictly speaking, we will only show it for linear subspaces in a hypersurface, using incidence correspondences).

Proof. For (1), suppose L is cut out by $(\ell_1, \ldots, \ell_{n-r} = 0)$. Let X = (f = 0). Since $L \subseteq X$, we have that

$$f = \sum_{i=1}^{n-r} \ell_i f_i, \tag{4.1}$$

where $f_i \in \Gamma(\mathbf{P}^n, \mathcal{O}(d-1))$. Let $g_i = f_i|_L \in \Gamma(L, \mathcal{O}_L(d-1))$.

We first claim that for all $P \in L$, there is some f_i that does not vanish at P. If not, then (4.1) gives that $\operatorname{mult}_P X \ge 2$, contradicting that $L \subseteq X_{\operatorname{sm}}$.

To finish (1), suppose that $r > \frac{n-1}{2}$, which is equivalent to $2r \ge n$. Then, we have that dim $L = r \ge n-r$, and so by the intersection theory on $L \cong \mathbf{P}^r$, we have that the polynomials g_1, \ldots, g_{n-r} of degree $d-1 \ge 1$ have a common solution in L. This contradicts that the f_i do not have a common zero in L, which we saw in the previous paragraph. Thus, $r \le \frac{n-1}{2}$, giving (1).

For (2), suppose that n = 2r + 1 above. Note that $g_1, \ldots, g_{n-r} \in S$, the homogeneous coordinate ring of L. These form a regular sequence: in Spec(S), their zero locus is just the origin, since they have no common solution on L. We then need the following general fact about parameter spaces of subschemes:

Fact 4.4. If $X \subseteq \mathbf{P}^n$ is a scheme, then the subschemes of X with fixed Hilbert polynomial are parametrized by a projective scheme Hilb_P.

In particular, taking P to be the Hilbert polynomial of an r-plane, that is,

$$P(T) = \binom{T+r}{r},$$

then Hilb_P parametrizes r-dimensional linear subspaces of X. We also need:

Fact 4.5. If X, Y are smooth, then

$$T_{[Y]}$$
 Hilb_P = $H^0(Y, N_{Y/X}),$

where $N_{Y/X}$ is the normal bundle.

It is therefore enough to show that for all linear subspaces $L \subseteq X_{sm}$, where dim L = r, and n = 2r + 1, we have that $H^0(L, N_{L/X}) = 0$. This would show that the corresponding point $[L] \in \text{Hilb}_P$ is an isolated, reduced point.

Since L, \mathbf{P}^n are smooth, and $L \subset X_{sm}$, we have the short exact sequence of normal bundles

$$0 \longrightarrow N_{L/X} \longrightarrow N_{L/\mathbf{P}^n} \longrightarrow N_{X/\mathbf{P}^n} |_L \longrightarrow 0.$$

Note that $N_{X/\mathbf{P}^n} = \mathcal{O}_X(d)$. Parenthesis 4.6.

(1) Let \mathscr{E} be locally free on Y smooth, rank m, and let $s \in \Gamma(Y, \mathscr{E})$ such that the zero locus $Z(s) \subseteq Y$ of s has pure codimension m. Locally, if $\mathscr{E} \simeq \mathcal{O}_Y^{\oplus m}$, then s corresponds to an m-tuple (g_1, \ldots, g_m) such that g_1, \ldots, g_m form a regular sequence. The section $s \colon \mathcal{O}_Y \to \mathscr{E}$ gives a map $\mathscr{E}^{\vee} \to \mathcal{O}_Y$ by dualizing, and we have the corresponding Koszul complex:

$$0 \longrightarrow \wedge^{m} \mathscr{E}^{\vee} \longrightarrow \cdots \longrightarrow \wedge^{2} \mathscr{E}^{\vee} \longrightarrow \mathscr{E}^{\vee} \longrightarrow \mathscr{O}_{Y} \longrightarrow 0$$

Since s is locally defined by a regular sequence, the above complex gives a resolution of $\mathscr{I}_{Z(s)}$:

$$\wedge^2 \mathscr{E}^{\vee} \longrightarrow \mathscr{E}^{\vee} \longrightarrow \mathscr{I}_{Z(s)} \longrightarrow 0$$

Tensoring by $\mathcal{O}_{Z(s)}$, we obtain

$$\wedge^2 \mathscr{E}^{\vee}|_{Z(s)} \xrightarrow{0} \mathscr{E}^{\vee}|_{Z(s)} \longrightarrow \mathscr{I}_{Z(s)}|_{Z(s)} \longrightarrow 0$$

so that $N_{Z(s)/Y} \simeq \mathscr{E}|_{Z(s)}$. (2) If $\mathbf{P}^n = \mathbf{P}(V)$,¹ then a linear subspace is given by a quotient $V \to V/W$, which induces an inclusion $\mathbf{P}(V/W) \hookrightarrow \mathbf{P}(V)$. On V, there is a canonical map

$$W \otimes \mathcal{O}_{\mathbf{P}(V)} \hookrightarrow V \otimes \mathcal{O}_{\mathbf{P}(V)} \twoheadrightarrow \mathcal{O}_{\mathbf{P}(V)}(1),$$

which defines a section of $s \in \Gamma(\mathbf{P}(V), W^* \otimes \mathcal{O}_{\mathbf{P}(V)}(1))$, such that $Z(s) = \mathbf{P}(V/W)$. This satisfies the condition in (1), and so it follows that $N_{\mathbf{P}(V/W)/\mathbf{P}(V)} \simeq W^* \otimes \mathcal{O}_{\mathbf{P}(V/W)}(1)$.

In our case, $N_{L/\mathbf{P}^n} \simeq \bigoplus_{i=1}^{n-r} \mathcal{O}_L(1)$. You can then check that the map

$$\bigoplus_{i=1}^{n-r} \mathcal{O}_L(1) \longrightarrow \mathcal{O}_L(d)$$

is defined by g_1, \ldots, g_{n-r} . We then see

$$H^{0}(L, N_{L/X}) = \ker \left(H^{0}(L, \mathcal{O}_{L}(1))^{\oplus (n-r)} \xrightarrow{(g_{1}, \dots, g_{n-r})} H^{0}(L, \mathcal{O}_{L}(d)) \right).$$

$$(4.2)$$

Since the g_i 's form a regular sequence, the Koszul complex on g_1, \ldots, g_{n-r} is exact, and so the relations on the g_i are generated in degrees $\geq d - 1 = 2$. This implies that $H^0(L, N_{L/X}) = 0$, since the kernel of the map in (4.2) is zero.

This calculation allows us to show the following:

Proposition 4.7. Let **P** be the projective space parametrizing cubic hypersurfaces in \mathbf{P}^{2m+1} . Then, the subset of \mathbf{P} corresponding to smooth cubic hypersurfaces containing two disjoint m-planes is constructible, irreducible, and of dimension $(m+1)^2(m+4) - 1$.

Proof. The idea is to setup the correct incidence correspondence. Let

G := G(m+1, 2m+2) =Grassmannian of *m*-linear subspaces in \mathbf{P}^{2m+1} ,

which is smooth, projective, and of dimension $(m+1)^2$. Let

$$U := \{ (L_1, L_2) \in G \times G \mid L_1 \cap L_2 = \emptyset \} \subseteq_{\text{open}} G \times G,$$

which is an open subset of $G \times G$ since the condition on linear subspaces is given by the non-vanishing of certain minors.

Now let $V \subseteq \mathbf{P}$ be the open subset of the $\binom{2m+4}{3}$ -dimensional space \mathbf{P} corresponding to smooth cubic hypersurfaces. Now consider the incidence correspondence below:



¹Note that we use the Grothendieck convention that $\mathbf{P}(-)$ denotes the projective space/bundle of hyperplanes.

We want to calculate the fibers of q. Let $(L_1, L_2) \in U$, and suppose after change of coordinates that

$$L_1 = (x_0 = \dots = x_m = 0)$$
 $L_2 = (x_1 = \dots = x_{2m+1} = 0)$

Then, $L_1, L_2 \subseteq V(f)$ if and only if

$$f \in (x_0, \dots, x_m) \cap (x_{m+1}, \dots, x_{2m+1}) = (x_i x_j \mid 0 \le i \le m, \ m+1 \le j \le 2m+1),$$

and therefore $q^{-1}(L_1, L_2)$ is a projective space of dimension $(m+1)^2(m+2) - 1$. By Parenthesis 3.2, we see that M is irreducible, and that

$$\dim M = 2(m+1)^2 + (m+1)^2(m+2) - 1 = (m+1)^2(m+4) - 1,$$

since $G \times G$ is $2(m+1)^2$ -dimensional, and $U \subseteq G \times G$ is open.

It remains to show that M_V is nonempty, for if this were the case, then M_V would be dense and therefore have the same dimension as M. It suffices to exhibit one smooth cubic hypersurface in \mathbf{P}^{2m+1} that contains two disjoint *m*-planes:

Example 4.8. If $\operatorname{char}(k) \neq 3$, then the hypersurface $X_0^3 + \cdots + X_{2m+1}^3 = 0$ contains two disjoint *m*-linear subspaces, using a similar argument to Example 2.20. Hence, $M_V \neq \emptyset$ in $\operatorname{char} \neq 3$.

Finally, the set $p(M_V)$ of smooth hypersurfaces containing two disjoint *m*-linear subsapces is constructible by Chevalley's theorem [Har77, Ch. II, Exc. 3.19], and is irreducible since it is the image of the irreducible set *M*. This set is also of dimension = dim *M* since by Proposition 4.3(2), the map $M_V \to \mathbf{P}$ has finite fibers. \Box

So the story for even-dimensional cubic hypersurfaces containing two disjoint linear subspaces is pretty clear. We describe the story for cubic fourfolds:

Example 4.9. Let m = 2, so we are considering cubic hypersurfaces in \mathbf{P}^5 . Then,

$$\dim \mathbf{P} = \frac{2m+4}{3} - 1 = 55$$

The space of (smooth) cubics containing two disjoint 2-planes is 53-dimensional. This has codimension 2 in \mathbf{P} . It is easy to check that the set

$$D = \left\{ \begin{array}{c} \text{smooth cubics in} \\ \mathbf{P}^5 \text{ containing 2-planes} \end{array} \right\}$$

is a divisor in V. Hassett (in his thesis [Has99]) gave an infinite family of divisors in D such that for each of them, the general element gives a rational cubic fourfold. This description uses Hodge theory and facts about derived categories of cubic fourfolds. It is conjectured that there are countably many divisors in V whose general elements correspond to rational cubic fourfolds, and that the complement of these divisors corresponds to non-rational cubic fourfolds.

Remark 4.10. In contrast to the even-dimensional case, there are no known examples of smooth odddimensional cubic hypersurfaces that are rational.

4.2 Unirationality of cubic hypersurfaces

We now move on to unirationality results for cubic hypersurfaces, namely the following:

Goal 4.11. If $X \subseteq \mathbf{P}^n$ is a smooth cubic hypersurface containing a line, then X is unirational (this is always the case if $k = \overline{k}$).

The exact geometric condition is necessary to make inductive arguments work. The idea is to project from the given line in X.

4.2.1 Review of projections

Let $\mathbf{P} = \mathbf{P}(V)$ be a projective space over a field k, and fix a linear subspace $\mathbf{P}(V/W) \hookrightarrow \mathbf{P}(V)$ corresponding to the short exact sequence

$$0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0.$$

The inclusion $W \hookrightarrow V$ defines a rational map

$$\varphi \colon \mathbf{P}(V) \smallsetminus \mathbf{P}(V/W) \longrightarrow \mathbf{P}(W),$$

which is called the projection with center $\mathbf{P}(V/W)$.

To describe this map more geometrically, choose a splitting of $W \hookrightarrow V$, which gives an embedding $\mathbf{P}(W) \hookrightarrow \mathbf{P}(V)$ such that $\mathbf{P}(W) \cap \mathbf{P}(V/W) = \emptyset$. Then, φ is defined by

$$\varphi(P) = \left\{ \begin{array}{l} \text{linear span of} \\ \mathbf{P}(V/W) \text{ and } P \end{array} \right\} \cap \mathbf{P}(W).$$

If we choose coordinates such that $\mathbf{P}(W) = (x_{r+1} = \cdots = x_n = 0)$ and $\mathbf{P}(V/W) = (x_0 = \cdots = x_r = 0)$, then $\varphi(x_0, \ldots, x_n) = (x_0, \ldots, x_r)$.

Goal 4.12. If

$$Bl_{\mathbf{P}(V/W)} \mathbf{P}(V)$$

$$\downarrow$$

$$\mathbf{P}(V) \xrightarrow{\varphi} \mathbf{P}(W)$$

there is a lift $\operatorname{Bl}_{\mathbf{P}(V/W)} \mathbf{P}(V) \to \mathbf{P}(W)$ of φ that realizes $\operatorname{Bl}_{\mathbf{P}(V/W)} \mathbf{P}(V)$ as a projective bundle over $\mathbf{P}(W)$.

We'll start by describing the projective bundle. Fix a splitting of $W \hookrightarrow V$, so that there is an isomorphism $V \simeq W \oplus V/W$. On $\mathbf{P}(W)$, consider $\mathscr{E} = \mathcal{O}_{\mathbf{P}(W)}(1) \oplus (V/W \otimes \mathcal{O}_{\mathbf{P}(W)})$. Consider the projective bundle associated to \mathscr{E}

$$B \coloneqq \mathbf{P}(\mathscr{E})$$

$$\downarrow^{g}$$

$$\mathbf{P}(W)$$

and let $\mathscr{L} = \mathcal{O}_B(1)$. Note that \mathscr{E} is globally generated since $\mathcal{O}_{\mathbf{P}(W)}(1)$ is, and since the other summands are trivial. Thus, we have a surjection

$$\begin{array}{c} H^{0}(\mathbf{P}(W), \mathscr{E}) \otimes \mathcal{O}_{\mathbf{P}(W)} & \longrightarrow \mathscr{E} \\ \| \\ (H^{0}(\mathcal{O}_{\mathbf{P}(W)}(1)) \oplus V/W) \otimes \mathcal{O}_{\mathbf{P}(W)} \\ \| \\ (W \oplus V/W) \otimes \mathcal{O}_{\mathbf{P}(W)} \\ \| \\ V \oplus \mathcal{O}_{\mathbf{P}(W)} \end{array}$$

We therefore get a corresponding closed embedding



By composing j with the first projection, we get a map $h: \mathbf{P}(\mathscr{E}) \to \mathbf{P}(V)$, which we will show next time is actually the blowup map $\mathrm{Bl}_{\mathbf{P}(V/W)} \mathbf{P}(V) \to \mathbf{P}(V)$.

Remark 4.13. You can check that

- $h^*\mathcal{O}_{\mathbf{P}(V)}(1) \simeq \mathscr{L};$
- h is defined by the map $H^0(B, \mathscr{L}) = H^0(\mathbf{P}(W), \mathscr{E}) \cong V$; and
- there is a surjection $H^0(B, \mathscr{L}) \otimes \mathcal{O}_B \twoheadrightarrow \mathscr{L}$.

What we need to do first next time is to understand the exceptional divisor of the map h. We will then prove a unirationality criterion for quadric bundles, which will be the key ingredient necessary to achieve Goal 4.11.

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5.1 Unitationality of cubic hypersurfaces (continued)

The goal of today is to prove the following result about unirationality:

Theorem 5.1. If $X \subset \mathbf{P}^n$ is a smooth cubic hypersurface over k where $n \ge 3$ and $\operatorname{char}(k) = 0$, and X contains a line, then X is unirational.

We will use the smoothness and characteristic assumption when we use generic smoothness. As a direct corollary, we show the following:

Corollary 5.2. If $k = \overline{k}$ and char(k) = 0, and $X \subset \mathbf{P}^n$, $n \ge 3$ is a smooth cubic hypersurface, then X is unirational.

Proof of Corollary 5.2. If $\Lambda \subseteq \mathbf{P}^n$ is a three-dimensional plane not contained in X, then the intersection $\Lambda \cap X \subseteq \Lambda \simeq \mathbf{P}^3$ is a cubic surface. Thus, $\Lambda \cap X$ contains a line by Theorem 2.15, and X therefore contains a line, hence X is rational by Theorem 5.1.

We start with some preparations.

5.1.1 Projections

Suppose V is a vector space over k, and $W \subsetneq V$ is a proper subspace inducing an inclusion $\mathbf{P}(V/W) \hookrightarrow \mathbf{P}(V)$ and the projection $\mathbf{P}(V) \smallsetminus \mathbf{P}(V/W) \xrightarrow{\varphi} \mathbf{P}(W)$. Recall that in this setting, we wanted to show the following:

Goal 5.3. φ gives a morphism $\operatorname{Bl}_{\mathbf{P}(V/W)}\mathbf{P}(V) \to \mathbf{P}(W)$, which realizes $\operatorname{Bl}_{\mathbf{P}(V/W)}\mathbf{P}(V)$ as a projective bundle over $\mathbf{P}(W)$.

This can be proved in local coordinates, but then it is complicated to identify what vector bundle gives rise to the projective bundle structure on $\operatorname{Bl}_{\mathbf{P}(V/W)}\mathbf{P}(V)$. We will therefore give a more formal argument.

Choose a splitting of $W \hookrightarrow V$, so that there is a canonical isomorphism $V \cong W \oplus V/W$. Then, on $\mathbf{P}(W)$, consider the vector bundle $\mathscr{E} = \mathcal{O}_{\mathbf{P}(W)}(1) \oplus (V/W \otimes \mathcal{O}_{\mathbf{P}(W)})$. Let

$$B := \mathbf{P}(\mathscr{E})$$

$$\downarrow^{g}$$

$$\mathbf{P}(W)$$

and $\mathscr{L} = \mathcal{O}_B(1)$. Then, \mathscr{E} is globally generated, so there is a canonical surjection

$$\underbrace{H^0(\mathbf{P}(W),\mathscr{E})}_{=V}\otimes\mathcal{O}_{\mathbf{P}(W)}\twoheadrightarrow\mathscr{E}$$

so $h: \mathbf{P}(\mathscr{E}) \to \mathbf{P}(V)$ realizes $\mathbf{P}(\mathscr{E})$ as a projective subbundle of the projective bundle $\mathbf{P}(V) \times \mathbf{P}(W) \to \mathbf{P}(V)$:



We denote g to be the projection from $\mathbf{P}(\mathscr{E})$ onto the second component of the larger projective bundle $\mathbf{P}(V) \times \mathbf{P}(W)$. Last time, we wanted to show the following:

Claim 5.4. h is the morphism defined by $\mathscr{L} = \mathcal{O}_B(1)$ and the surjection

$$\underbrace{H^0(B,\mathscr{L})}_{=V=H^0(\mathbf{P}(W),\mathscr{E})} \otimes \mathcal{O}_B \longrightarrow \mathscr{L}$$

Proof. First, on $\mathbf{P}(W)$ the surjection

$$\mathscr{E} \longrightarrow V/W \otimes \mathcal{O}_{\mathbf{P}(W)} \tag{5.1}$$

gives a diagram



so $\mathbf{P}(V/W) \times \mathbf{P}(W)$ is a projective subbundle of $\mathbf{P}(\mathscr{E})$ over $\mathbf{P}(W)$. We then claim that $E = \mathbf{P}(V/W) \times \mathbf{P}(W)$ is the exceptional divisor of a blowup. Note that E is a smooth irreducible divisor on B; to see that it is a divisor, note that the surjection (5.1) realizes $V/W \otimes \mathcal{O}_{\mathbf{P}(W)}$ as a codimension 1 quotient of \mathscr{E} .

First, we have an isomorphism $\mathscr{L}|_E \simeq \operatorname{pr}_1^* \mathcal{O}_{\mathbf{P}(V/W)}(1)$, which implies that we have a cartesian square



We then claim that $h^{-1}(\mathbf{P}(V/W)) = E$. To do so, we use the following general fact: Note 5.5. If $\varphi \colon \mathscr{E}_1 \twoheadrightarrow \mathscr{E}_2$ is a surjective morphism of locally free sheaves on Y, then there is an embedding $\mathbf{P}(\mathscr{E}_2) \hookrightarrow \mathbf{P}(\mathscr{E}_1)$. The subscheme is the zero locus of the morphism of vector bundles

$$\pi^*(\ker\varphi) \hookrightarrow \pi^*\mathscr{E}_1 \longrightarrow \mathcal{O}(1),$$

where $\pi \colon \mathbf{P}(\mathscr{E}_1) \to Y$.

In particular, we can apply this Note to say that $\mathbf{P}(V/W)$ is the zero locus of

$$W \otimes \mathcal{O}_{\mathbf{P}(V)} \hookrightarrow V \otimes \mathcal{O}_{\mathbf{P}(V)} \twoheadrightarrow \mathcal{O}_{\mathbf{P}(V)}(1),$$

and so $h^{-1}(\mathbf{P}(V/W))$ is the zero locus of the pullback of this morphism, i.e., the zero locus of the composition

$$W \otimes \mathcal{O}_B \hookrightarrow V \otimes \mathcal{O}_B \twoheadrightarrow h^* \mathcal{O}_{\mathbf{P}(V)}(1) = \mathscr{L}.$$

Similarly, E is the zero locus of

$$g^*\mathcal{O}_{\mathbf{P}(W)}(1) \hookrightarrow g^*\mathscr{E} \longrightarrow \mathscr{L}.$$

These two morphisms are not exactly the same, but there is a commutative diagram

$$\begin{array}{cccc} W\otimes \mathcal{O}_B & \longrightarrow V\otimes \mathcal{O}_B & \longrightarrow \mathscr{L} \\ & & & & & \\ & & & & \\ g^*\mathcal{O}_{\mathbf{P}(W)}(1) & \longrightarrow g^*\mathscr{E} & \longrightarrow \mathscr{L} \end{array}$$

Since the left vertical map is surjective, we see that the two horizontal compositions have the same zero loci, which implies Claim 5.4.

Proposition 5.6. *h* is the blowup of $\mathbf{P}(V)$ along the subspace $\mathbf{P}(V/W)$ with exceptional divisor *E*. Moreover, $g \circ h^{-1}$ is the projection map $\varphi : \mathbf{P}(V) \setminus \mathbf{P}(V/W) \to \mathbf{P}(W)$.

Proof. We have the commutative diagram



A point $x \in B$ is given by the following data:

- a point y = g(x) in $\mathbf{P}(W)$, i.e., a nonzero map $y: W \to k$; and
- a nonzero map

The point h(x) corresponds to a nonzero morphism

$$W \oplus V/W = V \longrightarrow k$$

proj
 $M/\ker(y) \oplus V/W$

Exercise 5.7. This implies that as a subset of $\mathbf{P}(V) \times \mathbf{P}(W)$, *B* is the graph of the rational map φ . (To check this, recall that φ takes a point $V \to k$ to the composition $W \to V \to k$.)

In particular, the map h is an isomorphism over $\mathbf{P}(V) \setminus \mathbf{P}(V/W)$. Thus, by the universal property of the blowup (see Remark 5.8 below), we must have that h factors through $\operatorname{Bl}_{\mathbf{P}(V/W)} \mathbf{P}(V)$, i.e., we have

$$\begin{array}{ccc} B & \stackrel{\alpha}{\longrightarrow} & \operatorname{Bl}_{\mathbf{P}(V/W)} \mathbf{P}(V) & \longrightarrow & \mathbf{P}(V) \\ & & & \cup \\ & & & \\ F = \text{exceptional divisor} \end{array}$$

Then, Claim 5.4 implies that $\alpha^* F = E$, and that α is an isomorphism over the complement of F. This implies that α has no exceptional divisor, and so α is an isomorphism.

Remark 5.8. Recall that the universal property of the blowup [Har77, Ch. II, Prop. 7.14] says the following: If $Z \subset X$ is a closed subscheme of a noetherian scheme X and $\beta: Y \to X$ is any morphism such that $\beta^{-1}(Z)$ is an effective Cartier divisor, then there is a unique morphism $Y \to \text{Bl}_Z X$ making the diagram



commute.

Remark 5.9. Let a splitting $V \cong W \oplus V/W$ be given as before, and let $Q \in \mathbf{P}(W)$. Then, $g^{-1}(Q) \subseteq \mathbf{P}(V)$ is the linear span of $\mathbf{P}(V/W)$ and Q by the geometric description of the projection map φ , and consider $E \cap g^{-1}(Q)$. This closed subscheme is identified with $\mathbf{P}(V/W) \times \{Q\}$ under the identification $\mathbf{P}(V/W) \times \mathbf{P}(W)$. We can fit these objects into the diagram below:



Remark 5.10. Suppose $\Lambda_1, \Lambda_2 \subseteq \mathbf{P}(V)$ are two linear subspaces such that

$$\Lambda_1 \cap \Lambda_2 = \emptyset$$
 and $\dim \Lambda_1 + \dim \Lambda_2 = n - 1$

Then, denoting $\Lambda_1 = \mathbf{P}(V/W_1)$ and $\Lambda_2 = \mathbf{P}(V/W_2)$, we have

$$W_1 + W_2 = V$$
 and $\dim W_1 + \dim W_2 = \dim V$.

This implies that $V = W_1 \oplus W_2$, and so we can apply the previous discussion.

5.1.2 Quadric bundles

We also need some facts about quadric bundles before proving Theorem 5.1.

Definition 5.11. A morphism of varieties $f: X \to S$ is a quadric bundle if there exists a factorization



where $n \ge 2$, such that $f^{-1}(s) \hookrightarrow \mathbf{P}_{k(s)}^n$ is a quadric that is smooth for general s. A quadric bundle is a conic bundle if n = 2. A rational quadric bundle is a rational map $\varphi \colon X \dashrightarrow S$ such that there exist open subsets $U \subseteq X, V \subseteq S$ such that φ is represented by a quadric bundle $U \to V$.

Remark 5.12. Given a morphism

$$\begin{array}{ccc} X & \longrightarrow & \mathbf{P}^n \times S \\ f \\ & \swarrow & & & \\ S \end{array}$$

we have that f is a rational quadric bundle if and only if the generic fiber



is a smooth quadric. Moreover, smoothness is automatic if X itself is smooth and char(k) = 0.

Proposition 5.13. Let $f: X \dashrightarrow S$ be a rational quadric bundle and $Y \hookrightarrow X$ a subvariety. Then,

- (1) If f induces a birational map $Y \dashrightarrow S$ and S is rational, then X is rational.
- (2) If $Y \rightarrow S$ is dominant, and Y is unirational, then X is unirational.

The hypothesis in (1) says that there is a rational section; the hypothesis in (2) (if $\dim Y = \dim S$) says that there is a multisection.

Proof. By replacing X, S with open subsets, we may assume that f is a quadric bundle.

For (1), we have the cartesian square below, where X_{η} is a smooth quadric in $\mathbf{P}_{k(S)}^{n}$:



The birational map $Y \to S$ induces the isomorphism on spectra of their generic points, and so by using the universal property of the fiber product, we obtain a k(S)-rational point on X_{η} . Thus, X_{η} is rational over k(S), i.e., birational to some $\mathbf{P}_{k(S)}^{m}$. We can therefore find some open subset $U \subseteq S$ such that $f^{-1}(U)$ is birational to $U \times \mathbf{P}_{k}^{m}$. But U is rational by assumption, since it is an open subset of S, and so X is rational. For (2), we know that Y is unirational, so there exists a dominant morphism $g: Z \to Y$, where Z is

rational. Now consider the diagram below:



The generic fiber of h is the base change of X_{η} via $k(S) \hookrightarrow k(Z)$, and so $(X \times_S Z)_{\eta}$ is a smooth quadric over k(Z). Thus, after replacing Z by an open subset, we may assume that $X \times_S Z$ is a variety. The map $i \circ g$ gives a section of h, by the same argument as in (1), and since Z is rational, (1) implies that $X \times_S Z$ is rational. Since $Z \to S$ is dominant, we have that $X \times_S Z \to X$ is dominant, and so X is unirational. \Box

5.1.3 Proof of Theorem 5.1

We can now prove our main goal from today.

Proof of Theorem 5.1. Let $L \subset X$ be the given line in X, where X is a smooth cubic hypersurface in \mathbf{P}^n . Choose $\Lambda \subset \mathbf{P}^n$ which is a linear subspace of dimension n-2 such that $\Lambda \cap L = \emptyset$, and consider the projection

$$f: \mathbf{P}^n \smallsetminus L \longrightarrow \Lambda,$$

which induces a morphism $g: X \setminus L \to \Lambda$. As before, denote $B = \operatorname{Bl}_L \mathbf{P}^n$; we have an induced morphism

$$\widetilde{f} \colon B \longrightarrow \Lambda$$

We also denote $\widetilde{X} \simeq \operatorname{Bl}_L X$ to be the strict transform of X in B, in which case we have an induced morphism

$$\widetilde{g} \colon X \longrightarrow \Lambda.$$

These morphisms fit into the following commutative diagram:



Since $Y = E \cap \widetilde{X}$ is a projective bundle over L, we know that Y is smooth and rational.

Next time, we will show that $\widetilde{g} \colon \widetilde{X} \to \Lambda$ is a rational conic bundle, and that we have a diagram



and so Proposition 5.13(2) implies \widetilde{X} is unirational, hence X is also unirational.

Next, we will consider when hypersurfaces contain projective linear subspaces of larger dimension. We will then prove Castelnuovo's criterion, and then introduce more sophisticated tools.

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6.1 Unitationality of cubic hypersurfaces (continued)

6.1.1 Proof of Theorem 5.1 (continued)

We want to finish the proof from last time.

Recall our setup: $\operatorname{char}(k) = 0$, and $X \subset \mathbf{P}^n$ is a smooth cubic hypersurface containing a line $\Lambda \subset X$. We wanted to show that X is unirational. This implies over any algebraically closed field, a smooth cubic hypersurface is unirational; for induction reasons, however, we will need this more general formulation.

Proof of Theorem 5.1. Recall $\mathbf{P}(W) \subset \mathbf{P}^n$ is a (n-2)-dimensional projective linear subspace such that $\Lambda \cap \mathbf{P}(W) = \emptyset$. We considered the projection with center Λ :

$$\varphi \colon \mathbf{P}^n \smallsetminus \Lambda \longrightarrow \mathbf{P}(W).$$

We saw last time that φ lifts to a morphism $f: \widetilde{\mathbf{P}^n} \to \mathbf{P}(W)$, where $\pi: \widetilde{\mathbf{P}^n} \to \mathbf{P}^n$ is the blowup along Λ , and so we have the following commutative diagram:



We also showed (in Remarks 5.9 and 5.10) that $i = (f, \pi) : \widetilde{\mathbf{P}^n} \hookrightarrow \mathbf{P}^n \times \mathbf{P}(W)$ realizes $\widetilde{\mathbf{P}^n}$ as a sub-projective bundle of $\mathbf{P}^n \times \mathbf{P}(W)$ over $\mathbf{P}(W)$. Thus, for $Q \in \mathbf{P}(W)$, we have the diagram



Let \widetilde{X} be the strict transform of X on $\widetilde{\mathbf{P}^n}$, let $Y = \widetilde{X} \cap E$, and consider the inclusion

$$Y \longleftrightarrow \widetilde{X} \simeq \operatorname{Bl}_{\Lambda} X$$

which identifies Y as the exceptional divisor of the blowup of X along Λ . Then, both \widetilde{X} and Y are smooth and irreducible, and since Y is a projective bundle over Λ , it is rational.

Let $g = f|_{\widetilde{X}} \colon \widetilde{X} \to \mathbf{P}(W)$. We claim that this is a rational conic bundle. To prove this, it suffices to check that it becomes a rational conic bundle after replacing k with \overline{k} . Since we are in characteristic zero, we have that for general $Q \in \mathbf{P}(W)$, the fiber $g^{-1}(Q)$ is smooth, and is contained in $f^{-1}(Q) \simeq \mathbf{P}^2$. Thus, $g^{-1}(Q)$ is connected (it is a smooth plane curve), and we claim that it is a conic. Note that for general Q, the intersection $X \cap \langle \Lambda, Q \rangle$ of X and the linear span of Λ and Q is not completely contained in Λ , which implies

$$g^{-1}(Q) = \overline{g^{-1}(Q) \smallsetminus E}.$$

Now $X \cap \langle \Lambda, Q \rangle$ is a curve of degree 3 containing the line Λ that is also embedded in $\langle \Lambda, Q \rangle \simeq \mathbf{P}^2$, and so $g^{-1}(Q) \subseteq \langle \Lambda, Q \rangle$ is a smooth conic. This shows $g \colon \widetilde{X} \to \mathbf{P}(W)$ is a rational conic bundle. Now we have a commutative diagram



where $g|_Y$ is a 2-to-1 finite map: for $Q \in \mathbf{P}(W)$ general,

$$g^{-1}(Q) \cap Y \cong \Lambda \cap (\langle \Lambda, Q \rangle \cap X) =$$
two points.

Since Y is rational, Proposition 5.13(2) implies \widetilde{X} is unirational, so X is unirational.

This finishes the story about cubic hypersurfaces: all cubic hypersurfaces (over an algebraically closed field) are unirational, and some are rational in even dimension. It is also known that general cubic threefolds are not rational; this relies on Hodge theory.

We next want to discuss what happens for higher degree hypersurfaces in projective space. For example, rationality of quartic threefolds in \mathbf{P}^4 is completely open. We will see, however, that if the degree is fixed and the dimension of the ambient space is large enough, then similar arguments from before will still allow us to deduce (uni)rationality of these hypersurfaces.

Before this, however, we need to talk more about incidence correspondences; in particular, we want to prove the promised results about tangent spaces. This will allow us to answer the following:

Question 6.1. What is the necessary and sufficient condition for any hypersurface to contain an *r*-dimensional linear projective space?

6.2 More on incidence correspondences

Let \mathbf{P}^n be fixed, and let r < n. Let G be the Grassmannian parametrizing r-dimensional linear subspaces of \mathbf{P}^n . This is a smooth, irreducible variety of dimension (r+1)(n-r). Let $d \ge 2$, and let \mathbf{P} be the projective

space parametrizing hypersurfaces of degree d in $\mathbf{P}^n = \mathbf{P}(V)$. If $V = H^0(\mathbf{P}^n, \mathcal{O}(1))$, then $\mathbf{P}^n = \mathbf{P}(\text{Sym}^d(V)^*)$ has dimension $\binom{n+d}{n} - 1$. Consider the following incidence correspondence:



Recall that we may find equations for I as a subscheme of $\mathbf{P} \times G$ as follows. Let $V \subseteq G$ be the open affine chart consisting of linear subspaces generated by the rows of the matrix

 $\begin{pmatrix} 1 & 0 & \cdots & 0 & a_{0,r+1} & \cdots & a_{0n} \\ 0 & 1 & \cdots & 0 & a_{1,r+1} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a_{r,r+1} & \cdots & a_{rn} \end{pmatrix}$

If X is defined by $f_c = \sum c_{\alpha} x^{\alpha} = 0$, where $c = (c_{\alpha})$ and $\alpha = (\alpha_0, \ldots, \alpha_n)$ satisfies $\sum \alpha_i = d$, then

$$\Lambda \subseteq (f_c = 0) \iff f_c(s_0 L_0 + \dots + s_r L_r) = 0 \text{ in } k[s_0, \dots, s_r].$$

Explicitly, we may write the equation on the right-hand side as

$$f_c(s_0L_0 + \dots + s_rL_r) = f_c\Big(s_0, \dots, s_r, \sum_{i=0}^r a_{i,r+1}s_i, \dots, \sum_{i=0}^r a_{i,n}s_i\Big) = \sum_{\beta} F_{\beta}(c, a)s^{\beta}$$

where $\beta = (\beta_0, \dots, \beta_r)$ and $\sum \beta_i = d$. Then, the ideal $(F_\beta(c, a))_\beta$ defines the subscheme $I \cap (\mathbf{P} \times V)$ in $\mathbf{P} \times V$.

Proposition 6.2. The second projection $q: I \to G$ realizes I as a projective subbundle of $\mathbf{P} \times G$ of codimension $\binom{r+d}{r}$ over G. In particular, I is smooth and irreducible, and

dim
$$I = (r+1)(n-r) + {\binom{n+d}{n}} - {\binom{r+d}{r}} - 1.$$

Proof. Since all coordinate charts of G look like that of the open subset $V \subseteq G$ (after change of coordinates), it suffices to check that over V, we have that $I \hookrightarrow \mathbf{P} \times V$ is cut out by $\binom{d+r}{d}$ equations, linear in the cvariables. Moreover, for each $\Lambda \in G$, we must show that the fiber $q^{-1}(\Lambda)$ is a projective space in \mathbf{P} of the right codimension. To see this, choose coordinates such that $\Lambda = (x_{r+1} = \cdots = x_n = 0)$, so that $\Lambda \subset V(\sum c_{\alpha} x^{\alpha})$ if and only if the coefficients in $f_c = \sum c_{\alpha} x^{\alpha}$ of the monomials in $k[x_0, \ldots, x_r]$ of degree d are all zero. \Box

Now we want to study what happens on the level of tangent spaces.

Definition 6.3. If $X \subseteq \mathbf{P}^n$ is a hypersurface of degree d, then the Fano scheme of r-dimensional linear subspaces on X is $p^{-1}([X]) \hookrightarrow G$, where p^{-1} denotes the scheme-theoretic fiber. We denote the Fano scheme by $F_r(X)$.

If K/k is a field extension, then the K-valued points of $F_r(X)$ are in bijection with the K-linear subspaces on X_K . We may also think of $F_r(X)$ as the Hilbert scheme parametrizing r-dimensional linear subspaces on X, which we note have the same Hilbert polynomial. Note that in principle, the Fano scheme might have non-reduced structure.

Goal 6.4.

- (1) Describe $T_{[\Lambda]}F_r(X)$.
- (2) Give a sufficient condition for p to be smooth at $([X], [\Lambda])$, so that in particular, $F_r(X)$ is smooth at $[\Lambda]$.

Both can be deduced from properties of Hilbert schemes, but we want to give a direct proof.

Fix $(X, \Lambda) \in I$. Choose linear generators $\ell_1, \ldots, \ell_{n-r}$ for the ideal of Λ in \mathbf{P}^n , and choose an equation f for X. Since $\Lambda \subset X$, we can write $f = \sum_{i=1}^{n-r} \ell_i f_i$. Recall $\mathbf{P}^n = \mathbf{P}(V)$ and $f \in \text{Sym}^d(V)$, so that $f_i \in \text{Sym}^{d-1}(V) = H^0(\mathbf{P}^n, \mathcal{O}(d-1))$. Let $g_i = f_i|_{\Lambda} \in H^0(\Lambda, \mathcal{O}_{\Lambda}(d-1))$. Note that the g_i 's do not depend on the choice of f_i 's, since if $\sum \ell_i f_i = \sum \ell_i f'_i$, then $f_i - f'_i \in (\ell_1, \ldots, \ell_{n-r})$ because $\ell_1, \ldots, \ell_{n-r}$ form a regular sequence.

Now consider the map

$$\Phi = \Phi_{X,\Lambda} \colon H^0(\Lambda, \mathcal{O}_{\Lambda}(1)) \otimes k^{n-r} \longrightarrow H^0(\Lambda, \mathcal{O}_{\Lambda}(d))$$
$$u \otimes e_j \longmapsto ug_j$$

where e_1, \ldots, e_{n-r} is a basis for k^{n-r} .

Exercise 6.5. Show that if we replace k^{n-r} by $H^0(\mathbf{P}^n, \mathscr{I}_{\Lambda/\mathbf{P}^n}(1))^*$ where $e_j \mapsto \ell_j^*$, then the resulting map only depends on the choice of f.

Theorem 6.6.

- (i) $T_{\Lambda}F_r(X) \simeq \ker(\Phi)$.
- (ii) If Φ is surjective, then $p: I \to \mathbf{P}$ is smooth at (X, Λ) . In particular, $F_r(X)$ is smooth at Λ .

Proof. For (i), we have

$$f_c\left(x_1, \dots, x_r, \sum_{i=0}^r a_{i,r+1}x_i, \dots, \sum_{i=0}^r a_{i,n}x_i\right) = \sum_{\beta = (\beta_0, \dots, \beta_r)} F_\beta(c, a) x^\beta.$$
(6.1)

Choose coordinates such that $\Lambda = (a_{ij} = 0)$. We will need to understand the partial derivatives of F_{β} in variables $a_{i,j}$. Differentiating (6.1) with respect to $a_{i,j}$ and comparing at (c, 0) gives

$$\frac{\partial F_{\beta}}{\partial a_{i,j}}(c,0) = \text{coefficient of } x_i \frac{\partial f_c}{\partial x_j}(x_0,\ldots,x_r,0,\ldots,0) \text{ in } x^{\beta}.$$

Choosing coordinates such that $\Lambda = (x_{r+1} = \cdots = x_r = 0)$, and $f = \sum_{i=1}^{n-r} x_{r+i} f_i$, we have

$$x_i \frac{\partial f_c}{\partial x_j}(x_0, \dots, x_r, 0, \dots, 0) = x_i f_{j-r}(x_0, \dots, x_r, 0, \dots, 0) = x_i g_{j-r}(x_0, \dots, x_r).$$

Recall that

$$T_{\Lambda}F_{r}(X) = \left\{ (u_{i,j})_{\substack{0 \le i \le r \\ r+1 \le j \le n}} \middle| \sum_{i,j} \frac{\partial F_{\beta}}{\partial a_{i,j}}(c,0) \cdot u_{i,j} = 0 \ \forall \beta \right\}.$$

This condition is equivalent to

$$\sum_{i,j} u_{i,j} x_i g_{j-r}(x_1,\ldots,x_r) = 0.$$

Now if $\alpha_j = \sum_i u_{i,j} x_j$, then

$$(u_{i,j}) \in T_{\Lambda}F_r(X) \iff \sum_j \alpha_j g_{j-r} = 0 \iff (\alpha_{r+1}, \dots, \alpha_n) \in \ker(\Phi).$$

For (*ii*), we have that P is smooth at (X, Λ) if and only if the map $dp_{X,\Lambda}$ in the diagram



is surjective. We will need the following linear algebraic lemma:

Lemma 6.7. A linear map $\varphi \colon k^m \oplus k^n \to k^m$ given by a matrix $A \in M_{m,m+n}(k)$ induces a surjective map



if and only if the first m columns of A are in the linear span of the last n columns of A. In particular, $\ker(\varphi) \to k^m$ is surjective if the last n columns of A span k^m .

Exercise 6.8. Prove this.

Lemma 6.7 implies dp is surjective at P if $\sum_{\beta} \frac{\partial F_{\beta}}{\partial a_{ij}}(c,0)x^{\beta}$ is spanned by the space of degree d homogeneous polynomials in $k[x_0,\ldots,x_r]$. This latter condition is equivalent to $x^i g_{j-r}(x_0,\ldots,x_r)$ spanning the space $H^0(\Lambda, \mathcal{O}_{\Lambda}(d))$. This holds if and only if Φ is surjective. \Box

We now connect this to our previous description using normal bundles:

Recall 6.9. If $\Lambda = (x_{r+1} = \cdots = x_n = 0)$ and $f = \sum x_i f_i$, $g_i = f_i|_{\Lambda}$, then we have an exact sequence

$$0 \longrightarrow N_{\Lambda/X} \longrightarrow N_{\Lambda/\mathbf{P}^n} \longrightarrow N_{X/\mathbf{P}^n} |_X \longrightarrow 0,$$

where we assume $\Lambda \subset X_{\rm sm}$. The surjection here is the map

$$\bigoplus_{i=r+1}^{n} \mathcal{O}_{\Lambda}(1) \xrightarrow{(g_{r+1},\ldots,g_n)} \mathcal{O}_{X}(d).$$

Taking global sections, we obtain the map

$$H^0(N_{\Lambda/\mathbf{P}^n}) \longrightarrow H^0(N_{X/\mathbf{P}^n}|_X),$$

which is the map Φ from before. Then, ker $\Phi = H^0(N_{\Lambda/X})$ and coker $\Phi = H^1(N_{\Lambda/X})$ since $H^1(\Lambda, \mathcal{O}_{\Lambda}(1)) = 0$.

Next time, we will use Theorem 6.6 to determine when p is surjective. There is a naïve guess using a diumension count; this condition will end up being sufficient, but the proof is a bit subtle. We will then talk about low degree hypersurfaces in projective space.

7 January 26

7.1 Comments on Theorem 5.1

In our discussion of the projection away from a line on a cubic hypersurface, we needed to check that the residual intersection of a plane passing through the given line was a conic, resulting in a conic bundle over what we called $\mathbf{P}(W)$ (see §6.1.1). However, a priori we must check that the intersection of the plane and the cubic is not of multiplicity > 1 along the line. This can be done in local coordinates, and we will do the calculation next time when we discuss unirationality of general hypersurfaces in \mathbf{P}^n of degree $d \ll n$.

Moreover, our proof in $\S6.1.1$ actually tells us the degree of the dominant rational map from projective space:

Exercise 7.1. Let X be a cubic hypersuface in \mathbf{P}^n which contains a line. Show that our proof from last time that X is unirational in fact shows that there exists a degree 2 rational dominant map $\mathbf{P}^{n-1} \dashrightarrow X$.

7.2Varieties containing linear subspaces

Recall 7.2. We had a smoothness criterion for the projection morphism from an incidence correspondence. Suppose $X \subset \mathbf{P}^n$ is a hypersurface of degree $d, \Lambda \subset \mathbf{P}^n$ is an r-dimensional linear subspace, and choose generators $\ell_1, \ldots, \ell_{n-r}$ for $H^0(\mathscr{I}_{\Lambda}(1))$. If f is the equation of X, then we can write $f = \sum_{i=1}^{n-r} \ell_i f_i$ where deg $f_i = d - 1$. Writing $g_i = f_i|_{\Lambda} \in H^0(\Lambda, \mathcal{O}_{\Lambda}(d-1))$, the linear span of the g_i is independent of choices. Let

$$\Phi: H^0(\Lambda, \mathcal{O}_{\Lambda}(1)) \otimes k^{n-r} \longrightarrow H^0(\Lambda, \mathcal{O}_{\Lambda}(d))$$
$$u \otimes e_j \longmapsto ug_j$$

We showed the following:

Theorem 6.6. Consider the incidence correspondence



where **P** is the projective space parametrizing degree d hypersurfaces in \mathbf{P}^n , and G is the Grassmannian of r-dimensional linear subspaces of \mathbf{P}^n . Then;

- (i) $\ker(\Phi) \simeq T_{([X],[\Lambda])}p^{-1}([X]).$ (ii) If Φ is surjective, then p is smooth at $([X], [\Lambda]).$

Note 7.3. We want to determine when p is dominant, that is, under what numerical conditions a general degree d hypersurface always contain a linear subspace. It is tricky to determine the image of p in general.

Remark 7.4. The incidence correspondence I can be defined over any field; this definition is also compatible with ground field extension. In particular, the fact that $I \to \mathbf{P}$ is a projective bundle, as well as the smoothness criterion in Theorem 6.6 apply over any ground field. We will need to use this later when we discuss unirationality again later.

One restriction for p to be surjective is that dim $I \ge \dim \mathbf{P}$:

Recall 7.5 (Proposition 6.2). We have dim $I = \dim \mathbf{P} + (r+1)(n-r) - {d+r \choose d}$, so dim $I \ge \dim \mathbf{P}$ if and only $if (r+1)(n-r) \ge \binom{d+r}{d}.$

This condition suffices for p to be surjective.

Theorem 7.6. Given $r \le n$ and $d \ge 2$, the morphism p is surjective if and only if (1) $(r+1)(n-r) \ge \binom{d+r}{d}$ if $d \ge 3$; and (2) $2r+1 \le n$ if d = 2.

Proof. By the dimension count in Proposition 6.2, the condition in (1) holds if p is surjective. On the other hand, if X is a smooth hypersurface of degree $d \ge 2$ that contains an r-dimensional subspace, then $2r + 1 \le n$, and so the condition in (2) holds if p is surjective as well.

For the converse, we may assume that k is algebraically closed by replacing k with \overline{k} : fibers over a point cannot become nonempty after base change. Moreover, it suffices to show p is dominant, since p is proper.

We first consider (2). We restrict to the case when $\operatorname{char}(k) \neq 2$, and leave the $\operatorname{char}(k) = 2$ case as an exercise. We want to show that every smooth quadric contains an r-dimensional plane if $r \leq \frac{n-r}{2}$. By the classification of quadratic forms (or completing the square), we can write $f = \sum x_i^2$. By changing coordinates again, we may assume that f is one of the following:

Case 1.
$$f = \sum_{i=0}^{k} x_{2i} x_{2i+1}$$
 if $n = 2k + 1$.
Case 2. $f = x_{2k}^2 + \sum_{i=0}^{k-1} x_{2i} x_{2i+1}$ if $n = 2k$.

Now consider

$$L = (x_0 = x_2 = \dots = x_{2k} = 0) \subseteq (f = 0).$$

Then, dim L = n - (k + 1), and you can check that $n - (k + 1) \ge r$ since $r \le \frac{n-1}{2}$. Now consider when $d \ge 3$, and suppose $(r+1)(n-r) \ge \binom{d+r}{d}$. It suffices to find a pair $([X], [\Lambda]) \in I$ such that p is smooth at this point, since this implies the map p is dominant. By Theorem 6.6(ii), it moreover suffices to show that the map Φ is surjective. Note that for any subspace of $H^0(\Lambda, \mathcal{O}_{\Lambda}(d-1))$ generated by g_1, \ldots, g_{n-r} , we have the commutative diagram

and so by commutativity, it suffices to show that there is some choice of g_i that makes the map ψ surjective, since lifting the g_i to $f_1, \ldots, f_{n-r} \in H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d-1))$ would define a hypersurface $X = (f = \sum \ell_i f_i = 0)$ such that the map Φ is surjective.

We translate this into the language of linear algebra. Let $W = H^0(\Lambda, \mathcal{O}_{\Lambda}(1))$. We want to show that there exists a subspace $U \subseteq S^{d-1}(W)$ generated by n-r elements such that the multiplication map

$$W \otimes U \longrightarrow S^d(W)$$

surjective. This is clearly okay if $n - r \ge \dim S^{d-1}(W)$; otherwise, we are done by Lemma 7.7, which we prove below. \square

Lemma 7.7. Let $k = \overline{k}$, and let W be a k-vector space of dimension r + 1. If $d \ge 3$ and $\ell \le \dim S^{d-1}(W)$ are such that $\ell(r+1) \ge \binom{r+d}{d}$, then there exists $U \subseteq S^{d-1}(W)$ of dimension ℓ such that the map

$$W \otimes U \longrightarrow S^d(W) \tag{7.1}$$

is surjective.

Proof. Note first that given $U \subseteq S^{d-1}(W)$, the surjectivity of (7.1) fails if and only if there exists $\varphi: S^d(W) \to S^{d-1}(W)$ k that vanishes on the image of $W \otimes U$.

Suppose now that φ is fixed. This map φ gives a map

$$W \otimes S^{d-1}(W) \longrightarrow S^d(W) \xrightarrow{\varphi} k$$

by composition. Giving such a morphism is equivalent to giving a morphism

$$W \xrightarrow{\alpha_{\varphi}} (S^{d-1}(W))^*.$$

Given $U \subseteq S^{d-1}(W)$, the map φ vanishes on the image of $W \otimes U$ if and only if the composition

$$\begin{array}{c} W \xrightarrow{\alpha_{\varphi}} (S^{d-1}(W))^* \\ \downarrow \\ U^* \end{array}$$

is zero, which is equivalent to saying that U^* is a rank ℓ quotient of $\operatorname{coker}(\alpha_{\varphi})$. Now consider $W_{\varphi} = \ker(\alpha_{\varphi})$. Then, by definition, φ vanishes on the image of $W_{\varphi} \otimes S^{d-1}(W)$. The exact sequence

$$W_{\varphi} \otimes S^{d-1}(W) \longrightarrow S^d(W) \longrightarrow S^d(W/W_{\varphi}) \longrightarrow 0,$$

implies that φ induces a nonzero map $S^d(W/W_{\varphi}) \to k$.

We will translate this into the langauge of algebraic geometry: the set of subspaces U that do not satisfy the condition in the theorem is the union of images of algebraic varieties Z_m , where each Z_m corresponds to those φ such that $\dim_k W_{\varphi} = m$.

Fix m such that $0 \le m \le r + 1 = \dim W$. Let $A = A_m$ be the Grasmannian variety parametrizing m-dimensional subspaces of W. On A, we have the short exact sequence

$$0 \longrightarrow S \longrightarrow W \otimes \mathcal{O}_A \longrightarrow Q \longrightarrow 0,$$

where S is the tautological subbundle. On A, consider $B = B_m = \mathbf{P}_A(S^d(Q))$. A point in B is given by a subspace $W' \hookrightarrow W$ and a nonzero map $\overline{\varphi} \colon S^d(W/W') \to k$ that factors φ :



On B, we have the morphism of locally free sheaves

$$\alpha \colon W \otimes \mathcal{O}_B \longrightarrow (S^{d-1}(W))^* \otimes \mathcal{O}_B,$$

which over a point $(W', \overline{\varphi})$ is given by α_{φ} . Inside B, let B' be the open subset where α has rank (r+1) - m. Suppose $B' \neq \emptyset$; when it is empty, we are okay. Otherwise, on B', ker (α) is the pullback of S from the Grassmannian, and coker (α) is locally free of rank $\binom{r+d-1}{d-1} - (r+1-m)$.

Let $C = C_m$ be the Grassmannian bundle on B' parametrizing rank ℓ quotients of coker(α), and let G be the Grassmannian variety parametrizing ℓ -dimensional subspaces of $S^{d-1}(W)$, which are the same thing as ℓ -dimensional quotients of $(S^{d-1}(W))^*$. There is a morphism

$$C_m \longrightarrow G$$
$$\left(W', \overline{\varphi}, \operatorname{coker}(\alpha) \xrightarrow{\rho} R\right) \longmapsto \left(S^{d-1}(W)^* \to \operatorname{coker}(\alpha)_{(W', \overline{\varphi})} \xrightarrow{\rho} R_{(W', \overline{\varphi})}\right)$$

where R is a rank ℓ quotient of coker(α), and the subscripts on the right-hand side denote fibers of the vector bundles over the point corresponding to $(W', \overline{\varphi})$ in B'.

By the discussion at the beginning of the proof, we want to show that the complement of the union of the images of the C_m is nonempty. Hence it is enough to show that $\dim(C_m) < \dim(G)$ for all m. Note

$$\dim(G) = \ell\left(\binom{r+d-1}{d-1} - \ell\right) \\ \dim(C_m) = m(r+1-m) + \binom{d+r-m}{d} - 1 + \ell\left(\binom{r+d-1}{d-1} - (r+1-m) - \ell\right)$$

Comparing the two, we see that $\dim(C_m) < \dim(G)$ for all $0 \le m \le r+1$ if and only if

$$\binom{d+r-m}{d} \le (\ell-m)(r+1-m) \tag{7.2}$$

for $0 \le m \le r+1$. Note that for m = r+1, both sides are zero, and the inequality for m = 0 is exactly the inequality in the hypothesis of the Lemma.

Exercise 7.8. If $d \ge 3$, then the condition for m = 0 implies the condition for m = 1.

By replacing m by r - m in the statement of Exercise 7.8, we see that the inequality (7.2) for m implies the same inequality (7.2) for m + 1. Thus, (7.2) holds for all $0 \le m \le r + 1$.

We note that this an interesting example of an algebro-geometric proof of a purely linear algebraic statement. We also note that the assumption that k is algebraically closed is used to show that the complement of a proper closed subvariety contains a k-valued point; this only requires that k is infinite.

7.3 Unirationality of hypersurfaces of higher degree

We now want to show the statement for unirationality of higher degree hypersurfaces that we promised before.

Goal 7.9. Given any $d \ge 4$, if $n \gg d$, a general hypersurface of degree d on \mathbf{P}^n is unirational.

The idea is to use the same approach as for the case of cubics. Given a hypersurface $X \subseteq \mathbf{P}^n$ containing a suitable linear subspace Λ , projecting away from Λ gives a rational map $X \dashrightarrow \mathbf{P}(W)$. As before, we blowup X along Λ to get an actual morphism $\tilde{X} = \operatorname{Bl}_{\Lambda} X \xrightarrow{g} \mathbf{P}(W)$, where the exceptional divisor of $\tilde{X} \to X$ is Y. Now an easy computation shows that the generic fiber of g is a hypersurface of degree d-1 in $\mathbf{P}^n_{k(\mathbf{P}(W))}$. By induction, we would then be done, as long as we can find a linear subspace in the generic fiber of g. This will be possible if we are in a situation where the general fibers of $g|_Y$ are hypersurfaces of degree d-1 in Λ . This will allow us to get a linear subspace on the generic fiber of Y, after a possible (controllable) field extension.

Suppose $X \subset \mathbf{P}_k^n$ is a hypersurface of degree d, and let $\Lambda \subset X$ be an r-dimensional linear subspace, which is defined by $(\ell_1, \ldots, \ell_{n-r})$. Then, we can write X = (f = 0), where $f = \sum \ell_i f_i$ and $g_i = f_i|_{\Lambda} \in H^0(\Lambda, \mathcal{O}_{\Lambda}(d-1))$. We will say that (X, Λ) satisfies condition (*) if g_1, \ldots, g_{n-r} span $H^0(\Lambda, \mathcal{O}_{\Lambda}(d-1))$. This is a very strong condition: it implies smoothness of the projection p at that point $([X], [\Lambda])$ by Theorem 6.6(*ii*). In particular, condition (*) requires n - r to be very large:

$$\binom{r+1}{d-1} \le n-r.$$

Definition 7.10. For $d \ge 3$, we define n_d recursively such that

$$n_3 = 3$$
 $n_d = n_{d-1} + \binom{n_{d-1} + 1}{d-1}.$

Theorem 7.11. If $k = \overline{k}$ and $X \subseteq \mathbf{P}^n$ is a general hypersurface of degree d in an n-dimensional projective space where $n \ge n_d$, then X is unirational.

This will finish the study of unirationality; we will start Castelnuovo's criterion afterward.

8 January 31

There is no class next week.

8.1 General statements about (uni)rationality

We start with some general remarks that we probably should have stated when we defined (uni)rationality. *Remarks* 8.1.

(i) If X/k is a rational variety, and K/k is a field extension, then $X_K = X \times_{\text{Spec } k} \text{Spec } K$ is irreducible and rational (with the reduced structure).

Proof. There is an open subset $U \subseteq X$ such that there is an open immersion $U \hookrightarrow \mathbf{A}_k^n$ for some n. Then, $U_K \hookrightarrow \mathbf{A}_K^n$ is also an open immersion, and since U_K is irreducible and $U_K \hookrightarrow X_K$ is dominant, we see that X_K is irreducible and rational.

(ii) If X/k is a unirational variety, then X_K is irreducible and unirational (with the reduced structure).

Proof. If there exists a dominant map $Y \to X$, where Y is rational over k, then its base change $Y_K \to X_K$ is dominant. This scheme Y_K is irreducible and rational by (i), so X_K is irreducible and unirational.

We also pick out the following statement from our previous statements about quadric bundles:

Proposition 8.2. Let $f: X \to Y$ be a dominant morphism of varieties over k.

- (1) If Y is rational and the generic fiber X_{η} is rational over k(Y), then Y is X is rational.
- (2) If Y is unirational and the generic fiber X_{η} is unirational over k(Y), then X is unirational.
Thus, to get information about the variety Y, you just need information about the total space of a fibration (e.g., a quadric bundle) and the generic fiber of the fibration. Note here that the generic fiber is always reduced, but when we ask for it to be rational, we are also demanding that it is irreducible.

Proof. Consider the following cartesian diagram:



 X_{η} is rational, so X_{η} is birational to $\mathbf{A}_{k(\eta)}^{n}$, for some *n*. Thus, there exists $V \subseteq Y$ open such that $f^{-1}(V)$ is birational to $V \times \mathbf{A}^{n}$. But since V is birational to some \mathbf{A}^{m} , we see that $f^{-1}(V)$ (and therefore X) is birational to \mathbf{A}^{m+n} .

Now suppose Y is unirational, so there exists a dominant morphism $Z \to Y$ where Z is a rational variety. Consider the following commutative diagram with cartesian squares:



Now X_{η} is unirational, and so the extension $(X_{\eta})_{k(Z)}$ is irreducible and unirational by Remark 8.1(*ii*). Let W be the closure of the image of $(X_{\eta})_{k(Z)}$ in $X \times_Y Z$. Then, $(X_{\eta})_{k(Z)}$ is the generic fiber of $W \to Z$. Since it is unirational, there exists a dominant morphism $V \to (X_{\eta})_{k(Z)}$, where V is rational over k(Z). Thus, there exists $Z_0 \subseteq Z$ open and a dominant rational map $Z_0 \times \mathbf{A}^n \to X \times_Y Z_0$ such that $Z_0 \times \mathbf{A}^n$ dominates the irreducible component W of $X \times_Y Z$, which dominates X. Rationality of Z_0 implies we get a dominant rational map $\mathbf{A}^{n+m} \dashrightarrow X$.

8.2 Unirationality of hypersurfaces of higher degree (continued)

We are now ready to prove unirationality of hypersurfaces of degree d in n-dimensional projective space, where $n \gg d$.

Recall 8.3. Suppose $X \subseteq \mathbf{P}^n$ is a hypersurface, and suppose $\Lambda \subset X$ is a linear subspace. We want a condition that makes an inductive argument work.

Let $\Lambda = (\ell_1 = \cdots = \ell_{n-r} = 0)$, X = (f = 0) where $f = \sum_{i=1}^{n-r} \ell_i f_i$, and $g_i = f_i|_{\Lambda} \in H^0(\Lambda, \mathcal{O}_{\Lambda}(d-1))$. We will say that (X, Λ) satisfies (*) if g_1, \ldots, g_{n-r} span $H^0(\Lambda, \mathcal{O}_{\Lambda}(d-1))$.

Remarks 8.4.

- (1) There exists $U \subseteq I$, where I is the incidence correspondence, such that for all K/k, and a K-valued point (X_K, Λ_K) of I, we have that $(X_K, \Lambda_K) \in U$ if and only if (X_K, Λ_K) satisfies (*), since the condition is open. This set U is nonempty if and only if $n r \ge \binom{r+d-1}{d-1}$.
- (2) Suppose that $L \subset \mathbf{P}^n$ is another linear subspace such that
 - $\Lambda \subset L;$
 - $L \not\subseteq X$.

Put $Y = X \cap L$. Then, (X, Λ) satisfies (*) if and only if (Y, Λ) satisfies (*). This follows since $Y \hookrightarrow L$ is also cut out by f, and so the linear spans of the g_i 's for the two pairs are the same.

Definition 8.5. We define inductively the numbers n(d) for $d \ge 3$:

$$n(3) = 3$$
 $n(d) = n(d-1) + \binom{n_{d-1}+1}{d-1}.$

Convention: n(2) = 1.

Theorem 8.6 (Morin [Mor42]). Suppose our ground field k is of characteristic 0. If $d \ge 3$ and $X \hookrightarrow \mathbf{P}^n$ is a hypersurface of degree d containing Λ , which is a linear subspace of dimension n(d-1) such that (X, Λ) satisfies condition (*) and $n \ge n(d)$, then X is unirational.

Corollary 8.7. If $k = \overline{k}$ and char(k) = 0, then a general hypersurface of degree d in \mathbf{P}^n with $n \ge n(d)$ is unirational.

Proof of Corollary 8.7. Let U be the open subset of the incidence correspondence from Remark 8.4(1) with r = n(d-1). Then, U is nonempty since $n \ge n(d)$, and moreover, the projection $p: I \to \mathbf{P}$ is smooth at all points of U (since (*) implies the hypothesis for Theorem 6.6(*ii*)). In particular, the map p is dominant. Thus, a general hypersurface in \mathbf{P}^n contains a linear subspace Λ as in Theorem 8.6, hence is unirational. \Box

Note the bound for d = 4 is $n(4) = 3 + {6 \choose 3} = 23$. It is expected that a general quartic hypersurface in \mathbf{P}^4 should not be unirational, but it is unclear where they start being unirational.

Remark 8.8. Harris, Mazur, and Pandharipande [HMP98] showed that if $n \gg d$, then every smooth hypersurface of degree d in \mathbf{P}^n is unirational. This is closer to the statement we had for smooth cubics.

Proof of Theorem 8.6, following [PS92]. The idea is to project away from Λ and look at the residual intersection with X. We will do so by induction on $d \geq 3$. If d = 3, every cubic hypersurface in \mathbf{P}^n for $n \geq 3$ is unirational if it contains a line by Theorem 5.1.

Now suppose d > 3. We proceed as in the case of cubic hypersurfaces. Let $\Lambda \subset X$, and choose $\mathbf{P}(W) \subset \mathbf{P}^n$ such that $\Lambda \cap \mathbf{P}(W) = \emptyset$ and dim $\Lambda + \dim \mathbf{P}(W) = n - 1$. We then get a projection $\mathbf{P}^n \dashrightarrow \mathbf{P}(W)$, which becomes a morphism on the blowup $\operatorname{Bl}_{\Lambda} \mathbf{P}^n = \widetilde{\mathbf{P}^n}$ along Λ . Consider the strict transform \widetilde{X} of X, and let $E \subset \widetilde{\mathbf{P}^n}$ be the exceptional divisor. Let $Y = \widetilde{X} \cap E$, so $\widetilde{X} = \operatorname{Bl}_{\Lambda} X$ with exceptional divisor Y, and in particular, both \widetilde{X} and Y are smooth. These spaces fit into the following diagram:



Claim 8.9. There is an open subset \mathscr{U} of $\mathbf{P}(W)$ such that for all field extensions K/k and for all K-valued points y: Spec $K \to \mathscr{U}$ such that $g^{-1}(y) \subseteq \langle \Lambda, y \rangle \simeq \mathbf{P}_{K}^{n(d-1)+1}$ and $h^{-1}(y) \subseteq \Lambda_{K}$ are hypersurfaces of degree d-1, with $g^{-1}(y)$ smooth.

Proof of Claim. Given any point $y: \operatorname{Spec} K \to \mathbf{P}(W)$, we first describe $X_K \cap \langle \Lambda_K, y \rangle$. Denote r = n(d-1), and choose coordinates on \mathbf{P}^n such that

$$\Lambda = (x_{r+1} = \dots = x_n = 0)$$
$$\mathbf{P}(W) = (x_0 = \dots = x_r = 0)$$

Then, y corresponds to a point $(\lambda_{r+1}, \ldots, \lambda_n)$, where $\lambda_i \in K$. On $\langle \Lambda_K, y \rangle$, we have coordinates $a_0, \ldots, a_r, a_{r+1}$ such that $\Lambda = (a_{r+1} = 0)$ and

$$X_K \cap \langle \Lambda_K, y \rangle = (f(a_0, \dots, a_r, a_{r+1}\lambda_{r+1}, \dots, a_{r+1}\lambda_n) = 0).$$

Let $f = \sum_{i=r+1}^{n} x_i f_i$. Then,

$$f(a_0, \dots, a_r, a_{r+1}\lambda_{r+1}, \dots, a_{r+1}\lambda_n) = a_{r+1} \cdot \sum_{\substack{i=r+1\\ i=r+1}}^n \lambda_i f_i(\cdots)$$
restriction to Λ is given by
$$\sum_{\substack{i=r+1\\ \sum_{i=r+1}^n \lambda_i g_i=0}}^{n} \lambda_i g_i = 0$$

We conclude that

$$\mathscr{U} = \left\{ (\lambda_{r+1}, \dots, \lambda_n) \mid \sum_{i=r+1}^n \lambda_i g_i \neq 0 \right\} \subseteq \mathbf{P}(W)$$

is open, and if y is a K-valued point of \mathscr{U} , then $X_K \cap \langle \Lambda_K, y \rangle = \Lambda_K + Z$, where $Z|_{\Lambda_K}$ is defined by $\sum \lambda_i g_i$. Since \widetilde{X} is smooth and char(k) = 0, we can use generic smoothness so that after replacing \mathscr{U} by a smaller subset, each fiber $g^{-1}(y)$ is smooth for all K-valued points y of \mathscr{U} . Since $g^{-1}(y)$ is a hypersurface in \mathbf{P}^n , smooth implies irreducible, and so the hypersurface

$$g^{-1}(y) = \overline{g^{-1}(y) \setminus E} = \overline{X_K \cap \langle \Lambda_K, y \rangle \setminus \Lambda}$$

is of degree d-1 in $\mathbf{P}^{n(d-1)+1}$. Also, $h^{-1}(y) \subset \Lambda$ is defined by $\sum \lambda_i g_i = 0$, hence is also of degree d-1. \Box

The idea now is to use Proposition 8.2: it suffices to show that the generic fiber of $g: \widetilde{X} \to \mathbf{P}(W)$ is unirational. To do so, we need to find a linear subspace of the generic fiber of the fibration g to apply the inductive hypothesis. Such a linear subspace may not exist; we need to enlarge the ground field so it does.

Let \mathbf{P}^s be the projective space parametrizing hypersurfaces of degree d-1 in Λ . Letting \mathscr{U} be as in Claim 8.9, we have a morphism $q: \mathscr{U} \to \mathbf{P}^s$ and a cartesian diagram



where \mathscr{H} is the universal family of hypersurfaces.

We now look at the incidence correspondence \mathscr{V} , which consists of those degree d-1 hypersurfaces parametrized by \mathscr{U} that contain a linear space of dimension n(d-2); this is the pullback of the incidence correspondence I from before to \mathscr{U} . We therefore have the following commutative diagram:



Since I is a projective bundle over G rational, I is rational. The map q is a linear projection map $(\lambda_{r+1}, \ldots, \lambda_n) \mapsto \sum \lambda_i g_i$ restricted to an open set, and the assumption (*) implies q is dominant. Since q is a projection, \mathscr{V} is irreducible, reduced, and birational to $I \times$ (affine space), and is therefore rational. Note also that $\mathscr{V} \to \mathscr{U}$ is dominant since $I \to \mathbf{P}^s$ is dominant.

We now will base change to $k(\mathscr{V})$, and apply induction. Note that a K-valued point of \mathscr{V} is equivalent to giving a K-valued point z: Spec $K \to \mathscr{U}$ and a linear subspace of $h^{-1}(z)$. We then have a commutative diagram with cartesian squares



Let $L = k(\mathcal{V})$; then, $X' \hookrightarrow \mathbf{P}_L^n$ is a smooth hypersurface of degree d-1, and $Y' \hookrightarrow \Lambda_L$ is a hypersurface of degree d-1. Thus, Y' contains an n(d-2) linear subspace corresponding to the canonical map $\operatorname{Spec} k(\mathcal{V}) \to \mathcal{V}$. Moreover, the pair (Y', Λ') satisfies condition (*) since the map $\operatorname{Spec} L \to \mathcal{V} \to I$ is dominant, hence the generic point of \mathcal{V} maps to the generic point of I, and the image of the closed point lies in the open subset U

in Remark 8.4(1). Remark 8.4(2) implies (X', Λ') satisfies (*). By induction, we conclude that $X' \hookrightarrow \mathbf{P}_L^n$ is unirational. If W is the image of $X' \to \mathscr{V} \times_{\mathscr{U}} g^{-1}(\mathscr{U})$, we still have that X' is the generic fiber of the map

 $\begin{array}{c}
W \\
\downarrow \\
\mathscr{V} = \text{rational}
\end{array}$

which implies W is unirational by Proposition 8.2(2). Then, since W dominates X, we see X is unirational. \Box

A similar argument works for complete intersections, but is messier to write down.

This completes Part I of the course. We will next prove Castelnuovo's rationality criterion, and then present Artin and Mumford's example of a unirational threefold that is not rational. But first, we will spend some time with preliminaries on singular cohomology of algebraic varieties and Brauer groups.

9 February 2

As a reminder, there is no class next week.

9.1 Overview of topology of algebraic varieties

We will give a brief review of aspects of algebraic topology that we will use to study algebraic varieties.

9.1.1 Singular (co)homology

We first review some basic things about singular cohomology.

Let A be an abelian group. One can define covariant functors

$$H^i(-,A)\colon \mathsf{Top}\longrightarrow \mathsf{Ab}$$

called homology, and contravariant functors

$$H^i(-,A)\colon \mathsf{Top}\longrightarrow \mathsf{Ab}$$

called cohomology. Thus, if $f: X \to Y$ is a continuous map, then there are maps

$$f_* \colon H_i(X, A) \longrightarrow H_i(Y, A)$$
$$f^* \colon H^i(Y, A) \longrightarrow H^i(X, A)$$

More generally, one can define covariant functors

$$H_i(-,-,A): \left\{ \begin{array}{c} \text{pairs of} \\ \text{topological spaces} \end{array} \right\} \longrightarrow \mathsf{Ab}$$

where a pair of topological spaces is a pair (X, Y) where $Y \subset X$ is a subspace. One can also define contravariant functors

$$H^{i}(-,-,A) \colon \left\{ \begin{array}{c} \text{pairs of} \\ \text{topological spaces} \end{array} \right\} \longrightarrow \mathsf{Ab}$$

The homology and cohomology functors satisfy the following properties: (1) If $X = \{*\}$, then

$$H_0(X, A) \simeq A \simeq H^0(X, A),$$

$$H_p(X, A) = 0 = H^p(X, A) \quad \text{for } p > 0.$$

(2) If $X = \prod_{i \in I} X_i$, then

$$H_p(X, A) \simeq \bigoplus_{i \in I} H_p(X_i, A),$$
$$H^p(X, A) \simeq \prod_{i \in I} H^p(X_i, A).$$

- (3) (Homotopy invariance) If $f: X \to Y$ and $g: X \to Y$ are homotopic, then $f_* = g_*$ and $f^* = g^*$.
- (4) Given a pair (X, Y), there are long exact sequences

$$\cdots \longrightarrow H_i(Y, A) \longrightarrow H_i(X, A) \longrightarrow H_i(X, Y, A) \longrightarrow H_{i-1}(Y, A) \longrightarrow \cdots$$
$$\cdots \longrightarrow H^i(X, Y, A) \longrightarrow H^i(X, A) \longrightarrow H^i(Y, A) \longrightarrow H^{i+1}(X, Y, A) \longrightarrow \cdots$$

(5) (Excision) If (X, Y) is a pair and $U \subset X$ is such that $\overline{U} \subseteq Int(Y)$, then the inclusion map induces isomorphisms

$$\begin{array}{c} H_i(X \smallsetminus U, Y \smallsetminus U, A) \xrightarrow{\sim} H_i(X, Y, A) \\ H^i(X, Y, A) \xrightarrow{\sim} H^i(X \smallsetminus U, Y \smallsetminus U, A) \end{array}$$

$$(9.1)$$

Note that for the spaces we are interested in, we will be able to describe singular cohomology in terms of sheaf cohomology; see Remark 9.12. However, it is difficult to discuss cohomology of pairs using sheaf cohomology, although it is possible [Bre97, Ch. II, §12].

Easy Properties 9.1. $H_0(X, A) \simeq A^{\oplus |\pi_0(X)|}$, and $H^0(X, A) \simeq A^{\prod |\pi_0(X)|}$, where $\pi_0(X)$ denotes the set of path-connected components of X. In particular, there is a degree map

deg:
$$H_0(X, A) \longrightarrow A$$

which is an isomorphism if X is path-connected.

We will now state some theorems we will use relating (co)homology groups with coefficients other than \mathbf{Z} to the integral (co)homology groups. In the sequel, we restrict our coefficient abelian group A to one of $\mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C}$.

Universal Coefficient Theorem 9.2. For every X, we have functorial short exact sequences:

$$0 \longrightarrow H_p(X, \mathbf{Z}) \otimes_{\mathbf{Z}} A \longrightarrow H_p(X, A) \longrightarrow \operatorname{Tor}_1^{\mathbf{Z}} (H_{p-1}(X, \mathbf{Z}), A) \longrightarrow 0$$
$$0 \longrightarrow \operatorname{Ext}_{\mathbf{Z}}^1 (H_{p-1}(X, \mathbf{Z}), A) \longrightarrow H^p(X, A) \longrightarrow \operatorname{Hom}_{\mathbf{Z}} (H_p(X, \mathbf{Z}), A) \longrightarrow 0$$

which are non-canonically split.

Consequences 9.3.

• Let $A = \mathbf{Z}$ in the second short exact sequence. Since $\operatorname{Ext}_{\mathbf{Z}}^{1}(B, \mathbf{Z})$ is torsion for any abelian group B, and $\operatorname{Hom}_{\mathbf{Z}}(B, \mathbf{Z})$ is torsion-free for any abelian group B we have

$$H^{p}(X, \mathbf{Z})_{\text{tors}} \cong \text{Ext}_{\mathbf{Z}}^{1} (H_{p-1}(X, \mathbf{Z}), \mathbf{Z})$$
$$H^{p}(X, \mathbf{Z}) /_{\text{tors}} \cong \text{Hom}_{\mathbf{Z}} (H_{p}(X, \mathbf{Z}), \mathbf{Z})$$

The case when p = 1 is especially important to show that H^3_{tors} is a birational invariant: Let $f: X \to Y$ be a blowup of a subvariety Z. Then, $H^1(Z, \mathbb{Z})$ is torsion-free by the discussion above, since $H_0(Z, \mathbb{Z})$ is free. On the other hand, $H^3(X, \mathbb{Z})$ has contributions from $H^3(Y, \mathbb{Z})$ and also from $H^1(Z, \mathbb{Z})$, and so the torsion part does not change.

• If K is a field containing \mathbf{Q} , we have

$$H_p(X,K) = H_p(X,\mathbf{Z}) \otimes_{\mathbf{Z}} K$$
$$H^p(X,K) = H_p(X,K)^*$$

Now suppose R is a ring. Then, the abelian group

$$H^*(X,R) \coloneqq \bigoplus_{p \ge 0} H^p(X,R)$$

has a multiplicative structure defined by the *cup product*

$$H^{p}(X,R) \times H^{q}(X,R) \longrightarrow H^{p+q}(X,R)$$
$$(\alpha,\beta) \longmapsto \alpha \cup \beta$$

This product is graded and graded-commutative. Also, we have the cap product

$$H^{p}(X,R) \times H_{q}(X,R) \longrightarrow H_{q-p}(X,R)$$
$$(\alpha,\beta) \longmapsto \alpha \cap \beta$$

which makes $H_*(X, R) := \bigoplus_{p \ge 0} H_p(X, R)$ a module over $H^*(X, R)$. Compatibility between the two is given by the following:

Projection Formula 9.4. If $f: X \to Y$ is a continuous map of topological spaces, then

$$f_*(f^*(\alpha) \cap \beta) = \alpha \cap f_*(\beta)$$

for all $\alpha \in H^*(Y)$ and $\beta \in H_*(X)$.

We now state the Künneth theorem, which describes the (co)homology of a product space in terms of the (co)homology of its factors:

Künneth formula 9.5. If X, Y are topological spaces, then we have a functorial short exact sequence

$$0 \longrightarrow \bigoplus_{p+q=m} \left(H_p(X, \mathbf{Z}) \otimes H_q(Y, \mathbf{Z}) \right) \longrightarrow H_m(X \times Y, \mathbf{Z}) \longrightarrow \bigoplus_{p+q=m-1} \operatorname{Tor}_1^{\mathbf{Z}} \left(H_p(X, \mathbf{Z}), H_q(Y, \mathbf{Z}) \right) \longrightarrow 0,$$

which is non-canonically split.

If all $H_p(X, \mathbf{Z})$ are finitely generated abelian groups, then we have a short exact sequence

$$0 \longrightarrow \bigoplus_{p+q=m} \left(H^p(X, \mathbf{Z}) \otimes H^q(Y, \mathbf{Z}) \right) \xrightarrow{\alpha} H^m(X \times Y, \mathbf{Z}) \longrightarrow \bigoplus_{p+q=m-1} \operatorname{Tor}_1^{\mathbf{Z}} \left(H^p(X, \mathbf{Z}), H^q(X, \mathbf{Z}) \right) \longrightarrow 0,$$

which is non-canonically split if also $H_i(Y, \mathbf{Z})$ is finitely generated for all *i*.

Note 9.6. In the second exact sequence, $\alpha(u \otimes v) = \operatorname{pr}_1^*(u) \cup \operatorname{pr}_2^*(v)$.

The restriction on homology groups in the cohomological Künneth formula are fulfilled in the setting we will usually be in:

Fact 9.7. If X is a compact manifold, all homology groups $H_i(X, \mathbf{Z})$ are finitely generated.

Poincaré Duality 9.8. Suppose X is a real compact manifold of dimension n. Then, if X has an orientation, there is a fundamental class $\mu_X \in H_n(X, \mathbf{Z})$, such that the map

$$H^{i}(X, \mathbf{Z}) \longrightarrow H_{n-i}(X, \mathbf{Z})$$
$$\alpha \longmapsto \alpha \cap \mu_{X}$$

is an isomorphism.

Note that there is no duality statement here!

Now we note that the Poincaré duality isomorphism is compatible with the cup and cap products, that is, the following diagram commutes:

$$(\alpha, \beta) \longmapsto \operatorname{deg}((\alpha \cup \beta) \cap \mu_X)$$

$$H^i(X, \mathbf{Z}) \times H^{n-i}(X, \mathbf{Z}) \longrightarrow \mathbf{Z}$$

$$\| \qquad \wr \downarrow_{\mathrm{PD}} \qquad \|$$

$$H^i(X, \mathbf{Z}) \times H_i(X, \mathbf{Z}) \longrightarrow \operatorname{deg}(\alpha \cap \beta)$$

The Universal Coefficient Theorem 9.2 implies that after killing torsion, we get a perfect pairing

$$H^{i}(X, \mathbf{Z}) /_{\text{tors}} \times H^{n-i}(X, \mathbf{Z}) /_{\text{tors}} \longrightarrow \mathbf{Z}.$$

In particular, applying $-\otimes \mathbf{Q}$ gives a perfect pairing

$$H^i(X, \mathbf{Q}) \times H^{n-i}(X, \mathbf{Q}) \longrightarrow \mathbf{Q}.$$

9.1.2 Complex algebraic varieties

We now specialize to the case of complex algebraic varieties, which is the setting we are interested in.

Let X be a variety (not necessarily irreducible) over **C**. We have a topological space X^{an} , which has the same underlying set as X, but has the "classical" topology. If X is affine, then we have a closed embedding $X \hookrightarrow \mathbf{A}^n_{\mathbf{C}} = \mathbf{C}^n$, and so we can define

 $X^{\mathrm{an}} \coloneqq (X, \mathrm{subspace topology coming from } \mathbf{C}^n \text{ with the Euclidean topology}).$

This is independent of the embedding.

If X is arbitrary and $X = \bigcup_{i=1}^{r} U_i$ is an affine cover, then we can glue U_i^{an} to get X^{an} . We therefore obtain a functor

$$\left. \begin{array}{c} \text{complex} \\ \text{algebraic varieties} \end{array} \right\} \longrightarrow \mathsf{Top}.$$

When we talk about the singular (co)homology of X, we really mean that of X^{an} .

Properties 9.9.

- (1) If X is irreducible (more generally, connected), then X^{an} is connected [SGA1, Exp. XII, Prop. 2.4].
- (2) Since X is separated, X^{an} is Hausdorff [GAGA, Prop. 2].
- (3) If X is complete, then X^{an} is compact [GAGA, Prop. 6]. This holds more generally for morphisms $f: X \to Y$: if f is proper, then f^{an} is proper in the topological sense [SGA1, Prop. 3.2(v)].
- (4) If X is smooth of dimension n, then X^{an} is a complex manifold of dimension n (e.g., by the Jacobian criterion). In particular, X^{an} is a real manifold, and comes with a canonical orientation. [SGA1, Exp. XII, Prop. 2.1(iv)]

In fact, X^{an} also carries a sheaf $\mathcal{O}_{X^{an}}$ of holomorphic functions on X.

Definition 9.10. If X is affine, and $X \hookrightarrow \mathbf{A}^n_{\mathbf{C}} = \mathbf{C}^n$ is a closed embedding, a function on an open subset of X is holomorphic if locally, it is the restriction of a holomorphic function on an open subset of \mathbf{C}^n .

• The definition is independent of the embedding.

• The definition of $\mathcal{O}_{X^{\mathrm{an}}}$ is given by gluing the corresponding sheaves on affine open subsets. This makes $(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$ a locally ringed space, and we have a morphism of locally ringed spaces

$$(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \xrightarrow{\iota} (X, \mathcal{O}_X).$$

The map on topological spaces is just the identity, which is continuous since the topology on X^{an} is finer than that on X, and is a map on structure sheaves since algebraic functions are analytic by definition. In particular, for all $x \in X$, there is a morphism

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X^{\mathrm{an}},x}$$

which induces an isomorphism after completion along their maximal ideals. For example, if $x \in X$ is smooth of dimension n, then $\mathcal{O}_{X,x}$ is hard to describe, but $\mathcal{O}_{X^{\mathrm{an}},x} \simeq \mathbb{C}\{z_1,\ldots,z_n\}$ is the ring of convergent power series; after completion, they are both isomorphic to the ring $\mathbb{C}[\![z_1,\ldots,z_n]\!]$ of formal power series.

Theorem 9.11 [GAGA, n° 12; SGA1, §4]. If $\mathscr{F} \in \mathsf{Coh}(X)$, then $\mathscr{F}^{\mathrm{an}} := \iota^* \mathscr{F}$ is a coherent sheaf of $\mathcal{O}_{X^{\mathrm{an}}}$ -modules. If X is projective (or even complete), this gives an equivalence of categories

$$\left\{ \begin{array}{c} coherent \ sheaves \ of \\ \mathcal{O}_X \text{-}modules \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} coherent \ sheaves \ of \\ \mathcal{O}_{X^{\mathrm{an}}} \text{-}modules \end{array} \right\}$$

which preserves the subcategories of locally free sheaves. In particular, the canonical morphism $\operatorname{Pic}(X) \to \operatorname{Pic}(X^{\operatorname{an}}) \simeq H^1(X^{\operatorname{an}}, \mathcal{O}^*_{X^{\operatorname{an}}})$ is an isomorphism. Moreover, if $\mathscr{F} \in \operatorname{Coh}(X)$, then we have a canonical isomorphism

$$H^i(X,\mathscr{F}) \xrightarrow{\sim} H^i(X^{\mathrm{an}},\mathscr{F}^{\mathrm{an}}).$$

One more remark about singular cohomology:

Remark 9.12. Since X^{an} is a nice topological space, for all abelian groups A, we have a canonical isomorphism

$$H^{i}(X^{\mathrm{an}}, A) \simeq H^{i}(X^{\mathrm{an}}, \underline{A}),$$

where \underline{A} denotes the constant sheaf with values in A, and the right-hand side denotes sheaf cohomology.

Exponential Sequence If X is any complex algebraic variety, we have a short exact sequence

$$0 \longrightarrow \underline{\mathbf{Z}} \longrightarrow \mathcal{O}_{X^{\mathrm{an}}} \longrightarrow \mathcal{O}_{X^{\mathrm{an}}}^* \longrightarrow 0$$
$$\varphi \longmapsto \exp(2\pi i \varphi)$$

of sheaves on X^{an} . Note the surjectivity of $\exp(2\pi i -)$ comes from the existence of logarithms (locally). The long exact sequence on cohomology contains the sequence

where $c^{1}(-)$ is the first Chern class map for line bundles.

Fundamental Class If X is a complete irreducible complex algebraic variety of dimension n, then we have a class $\mu_X \in H_{2n}(X, \mathbb{Z})$. If X is smooth, then X is a real manifold of dimension 2n with canonical orientation, and in this case μ_X is the one from before.

If X is not smooth, then by Hironaka's resolution of singularities [Hir64], we have a proper birational morphism $f: Y \to X$ such that Y is smooth. Then, we can define $\mu_X := f_* \mu_Y$.

- This is independent of choice of f.
- If $g: Z \to X$ is a dominant, generically finite morphism of complete varieties, then $g_*\mu_Z = \deg(f) \cdot \mu_X$.
- There is a symmetric multilinear map on X, defined by

$$\underbrace{H^2(X, \mathbf{Z}) \times \cdots \times H^2(X, \mathbf{Z})}_{n \text{ times}} \longrightarrow \mathbf{Z}$$

$$(\alpha_1, \dots, \alpha_n) \longmapsto \alpha_1 \cdots \alpha_n$$

$$\exists i$$

$$\deg ((\alpha_1 \cup \cdots \cup \alpha_n) \cap \mu_X)$$

This is compatible with intersection numbers of line bundles: if $L_1, \ldots, L_n \in \operatorname{Pic}(X)$, then

$$(L_1 \cdots L_n) = c^1(L_1) \cdots c^1(L_n) \in \mathbf{Z}.$$

• If X is a complete complex algebraic variety, then X^{an} admits a finite triangulation; moreover, if $Y \hookrightarrow X$ is a closed subset, then there are compatible triangulations of X and Y [Hir75]. This is useful to show that the finiteness of Betti numbers for any complex algebraic variety Z. Recall that the *i*th Betti number $b_i(Z)$ is the rank of $H_i(Z, \mathbf{Z})$. By Nagata compactification, there is an

Recall that the *i*th Betti number $b_i(Z)$ is the rank of $H_i(Z, \mathbf{Z})$. By Nagata compactification, there is an open embedding $Z \hookrightarrow X$ where X is complete. Then, both X and $X \smallsetminus Z$ have compatible triangulations, and a suitable long exact sequence can be used to show $H_i(Z, \mathbf{Z})$ is of finite rank for all *i*.

Next time, we will talk about the Hodge decomposition. We can then discuss Castelnuovo's criterion for rationality; we will also talk about a result of Noether and Enriques on when a fibration over a curve is a (birationally) ruled surface. We will also do some calculations on the cohomology of a blowup.

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10.1 Overview of topology of algebraic varieties (continued)

Last time we discussed background on the topology of algebraic varieties. We will need two more things:

Hodge decomposition If X is a smooth projective variety over \mathbf{C} , then

$$H^{i}(X, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C} = H^{i}(X, \mathbf{C}) = \bigoplus_{p+q=i} H^{p,q} \text{ where } H^{p,q} \simeq H^{q}(X, \Omega_{X}^{p}).$$
 (10.1)

Here, $H^q(X, \Omega_X^p)$ can be either the algebraic or analytic sheaf cohomology by GAGA (Theorem 9.11). Moreover, the conjugation on **C** gives an automorphism on the left-hand side, which induces an automorphism on the right-hand side mapping $H^{p,q}$ to $H^{q,p}$, i.e., there is a symmetry when you interchange p and q.

Noether's formula If X is a smooth projective complex surface, then

$$\chi(\mathcal{O}_X) = \frac{1}{12} \big((K_X^2) + \chi_{\rm top}(X) \big), \tag{10.2}$$

where $\chi(\mathcal{O}_X)$ is the sheaf-cohomological Euler characteristic, and where

$$\chi_{\text{top}} \coloneqq b_0 - b_1 + b_2 - b_3 + b_4$$

= 2 - 2b_1 + b_2
= 2 - 4q + b_2

is the topological Euler characteristic. Note the first equality is by Poincaré duality 9.8, and the second is by the Hodge decomposition (10.1). Noether's formula (10.2) comes from Hirzebruch–Riemann–Roch [Ful98, Cor. 15.2.1], which a priori gives the formula

$$\chi(\mathcal{O}_X) = \frac{1}{12} \big((K_X^2) + \deg c^2(T_X) \big),$$

and we can replace deg $c^2(T_X)$ with $\chi_{top}(X)$; see [Ful98, Ex. 15.2.2]. There is also version in positive characteristic using ℓ -adic cohomology to define the topological Euler characteristic.

10.2 Rationality and unirationality for surfaces

We will now spend today and next time discussing (uni)rationality for surfaces. This has a different flavor from previous material, using tools such as the adjunction formula and Riemann–Roch. The goal is the following: **Theorem 10.1** (Castelnuovo). If X is a smooth projective surface over $k = \overline{k}$, then X is rational if and only if $p_2(X) = 0 = q(X)$, where $p_2(X) = h^0(X, \omega_X^{\otimes 2})$, and the irregularity is $q(X) = h^1(X, \mathcal{O}_X)$.

We will only prove this in characteristic zero; in positive characteristic, see [Băd01, Ch. 13; Mil80, Thm. 3.30]. Note the proof in positive characteristic is more complicated than just replacing singular cohomology with étale cohomology.

The point of Theorem 10.1 is that for surfaces, there is a nice cohomological criterion for rationality, and so rationality is fairly well understood, in contrast to the higher-dimensional case.

Remark 10.2. Theorem 10.1 gives a positive answer to Lüroth's problem for surfaces in characteristic zero: If X is a smooth projective unirational surface in characteristic zero, then $p_m(X) = 0$ for all $m \ge 1$ by Proposition 1.10, and $h^i(X, \mathcal{O}_X) = 0$ for all $i \ge 1$ by Remark 1.11(3). Theorem 10.1 then implies that X is rational.

However, we will show next time that the answer to Lüroth's problem is "no" in positive characteristic: Shioda produced examples of unirational but non-rational surfaces in positive characteristic [Shi74].

Remark 10.3. Since $p_2(X) = 0$ implies $p_1(X) = 0$, it is tempting to ask if $p_1(X) = 0 = q(X)$ implies rationality. This is false: Enriques surfaces and Godeaux surfaces (the latter of which are of general type) give counterexamples.

Remark 10.4. Let X be a Del Pezzo surface, i.e., a smooth projective surface such that $-K_X$ is ample (in higher dimensions, these are called Fano varieties). Then, $p_2(X) = 0$, and in characteristic zero, $h^1(X, \mathcal{O}_X) = h^1(X, \omega_X \otimes \omega_X^{-1}) = 0$ by Kodaira vanishing, since ω_X^{-1} is ample and then by applying Kodaira vanishing. By Theorem 10.1, we deduce that X is rational.

In fact, you do not need characteristic zero here: it is a classical result that if X is a Del Pezzo surface, then X is either $\mathbf{P}^1 \times \mathbf{P}^1$ or the blowup of \mathbf{P}^2 at ≤ 8 points [Kol96, §III.3]. At least for the cubic surface, however, you can compute cohomology directly since it a cubic surface is a divisor in \mathbf{P}^3 , so you do not need Kodaira vanishing. Thus, Theorem 10.1 gives another proof of the fact that a smooth cubic surface in \mathbf{P}^3 is rational.

Today, we will prove some preliminary results toward Castelnuovo's theorem. We first recall the following basic facts about geometry on surfaces:

Riemann–Roch on surfaces [Har77, Ch. V, Thm. 1.6] For any divisor D,

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{2}((D^2) - (D \cdot K_X)).$$
(10.3)

Adjunction formula [Har77, Ch. V, Exc. 1.3] If D is an irreducible effective divisor on X, then

$$2p_a(D) - 2 = (D^2) + (D \cdot K_X).$$
(10.4)

The idea for the proof of Theorem 10.1 is that we will write down a map to \mathbf{P}^1 whose general fiber is \mathbf{P}^1 , and then show it is a \mathbf{P}^1 -bundle using the following:

Theorem 10.5 (Noether–Enriques). Let X be a smooth projective surface, and let C be a smooth projective curve. If $f: X \to C$ is a morphism such that $f^{-1}(x) \simeq \mathbf{P}^1$ for some $x \in C$, then there is an open neighborhood U of x such that $f^{-1}(U) \simeq U \times \mathbf{P}^1$ over U.

This means that X is ruled: X is birational to $U \times \mathbf{P}^1$.

Proof. The proof proceeds in three steps.

Step 1. $h^2(X, \mathcal{O}_X) = 0$. Let $F = f^{-1}(x) \simeq \mathbf{P}^1$. Then, since F is a fiber of f, we have that $F^2 = 0$. By the adjunction formula (10.4),

$$-2 = \underbrace{2p_a(F)}_{0} - 2 = \underbrace{(F^2)}_{0} + (F \cdot K_X) = F \cdot K_X.$$

Thus, by Serre duality, $h^2(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X(K_X))$. If this is not zero, then there exists some divisor $D \ge 0$ such that $D \sim K_X$. Then, D satisfies $D \cdot F \ge 0$ since D is effective, and F is an irreducible divisor such that $F^2 \ge 0$. This contradicts that $(D \cdot F) = -2$.

Step 2. There exists a divisor H on X such that $(H \cdot F) = 1$.

The idea is that this divisor H will correspond to the Serre bundle $\mathcal{O}(1)$ for the ruled surface $U \times \mathbf{P}^1$.

We assume here that $k = \mathbf{C}$. The exponential sequence

$$0 \longrightarrow \underline{\mathbf{Z}} \longrightarrow \mathcal{O}_{X^{\mathrm{an}}} \longrightarrow \mathcal{O}_{X^{\mathrm{an}}}^* \longrightarrow 0$$

gives

$$\begin{array}{cccc} H^1(X, \mathcal{O}_{X^{\mathrm{an}}}) & \longrightarrow \operatorname{Pic}(X^{\mathrm{an}}) & \stackrel{c^1}{\longrightarrow} & H^2(X, \underline{\mathbb{Z}}) & \longrightarrow & H^2(X, \mathcal{O}_{X^{\mathrm{an}}}) \\ & \parallel & \parallel & & \parallel \\ & H^1(X, \mathcal{O}_X) & & \operatorname{Pic}(X) & & H^2(X, \mathcal{O}_X) & \stackrel{\mathrm{Step 1}}{\longrightarrow} & 0 \end{array}$$

where the vertical equalities are by GAGA (Theorem 9.11). Step 1 implies $H^2(X, \mathcal{O}_{X^{an}}) = 0$, and so the first Chern class map

$$c^1 \colon \operatorname{Pic}(X) \longrightarrow H^2(X, \underline{\mathbb{Z}})$$
 (10.5)

is surjective. Therefore, it is enough to find $h \in H^2(X, \mathbb{Z})$ such that $h \cdot b = 1$, where $b = c^1(\mathcal{O}_X(F))$, and

$$h \cdot b = \deg((h \cup b) \cap \eta_X).$$

We also know from Poincaré duality 9.8 that the map

$$\frac{H^{2}(X, \mathbf{Z})}{\operatorname{tors}} \times \frac{H^{2}(X, \mathbf{Z})}{\operatorname{tors}} \longrightarrow \mathbf{Z}$$

$$(\alpha, \beta) \longmapsto \alpha \cdot \beta$$
(10.6)

is a perfect pairing. So we need to find something that pairs to get 1 with b. Let

$$I = \left\{ a \cdot b \in \mathbf{Z} \mid a \in H^2(X, \mathbf{Z}) \right\} \subseteq \mathbf{Z},$$

which is an ideal in **Z**. It cannot be the zero ideal, since b is the class of a fiber, hence $h \cdot b > 0$ if $h = c^1$ (ample line bundle). Then, $I = d\mathbf{Z}$ for some d > 0, since **Z** is a PID. Now consider the map

$$\begin{array}{ccc} H^2(X, \mathbf{Z}) \Big/_{\text{tors}} & \longrightarrow & \mathbf{Z} \\ a & \longmapsto & \frac{1}{d} (a \cdot b) \end{array}$$

Poincaré duality 9.8 implies that in fact, there exists $\beta \in H^2(X, \mathbb{Z})$ such that

$$\frac{1}{d}(a \cdot b) = (a \cdot \beta)$$

for all $a \in H^2(X, \mathbb{Z})$, so $b - d\beta$ is torsion by Poincaré duality 9.8 since the pairing in (10.6) is perfect. Thus,

$$b \cdot \kappa = -2 = d(\kappa \cdot \beta) \qquad (b^2) = 0 = (\beta^2),$$

where we write $\kappa = c^1(\omega_X)$. Moreover, for any $\alpha \in H^2(X, \mathbb{Z})$, the quantity $(\alpha^2) + (\alpha \cdot \kappa)$ is *even*, since • $(\alpha^2) + (\alpha \cdot \kappa)$ is linear modulo 2;

• Surjectivity of c^1 in (10.5) implies it is enough to check the assertion for $\alpha \in \text{im}(c^1) = H^2(X, \mathbb{Z})$;

• By the adjunction formula (10.4), it is okay for $c^1(\mathcal{O}(D))$, where D is an irreducible curve.

Applying this for $\alpha = \beta$, we obtain that $(\kappa \cdot \beta)$ is even, and so d = 1. This implies there exists H such that $(H \cdot F) = 1$.

Step 3. Standard application of semicontinuity and base change theorems

The map $f: X \to C$ is flat by [Har77, Ch. III, Prop. 9.7]. We know f is smooth over x, and so since smoothness is an open property [Har77, Ch. III, Exc. 10.2], there exists $U \ni x$ such that f is smooth over U. Now let $F_y = f^{-1}(y)$ denote the fiber over $y \in U$. Then, by the semicontinuity theorem,

$$h^{0}(F_{y}, \mathcal{O}_{F_{y}}) \leq h^{0}(F_{x}, \mathcal{O}_{F_{x}}) = 1$$
$$h^{1}(F_{y}, \mathcal{O}_{F_{y}}) \leq h^{1}(F_{x}, \mathcal{O}_{F_{x}}) = 0$$

which imply that F_y is connected and that $F_y \simeq \mathbf{P}^1$ for all $y \in U$, respectively.

Now let H be the divisor on X obtained from Step 2, and let $\mathscr{E} = f_*\mathcal{O}_X(H)$. We will apply base change to this sheaf \mathscr{E} . Note for every $y \in U$, $(H \cdot F_y) = (H \cdot F) = 1$, and so $\mathcal{O}_X(F)|_{F_y} \simeq \mathcal{O}_{\mathbf{P}^1}(1)$. In particular, $h^0(\mathcal{O}_X(H)|_{F_y}) = 2$. By Grauert's theorem [Har77, Ch. III, Cor. 12.9], we see that $\mathscr{E}|_U$ is locally free of rank 2, and commutes with base change in the sense that for every $y \in U$, the canonical map

$$\mathscr{E}_y \otimes k(y) \xrightarrow{\sim} H^0(F_y, \mathcal{O}_X(H)|_{F_y})$$

is an isomorphism.

• The canonical map $f^* \mathscr{E} \to \mathcal{O}_X(H)$ is surjective on $f^{-1}(U)$: By Nakayama's lemma (cf. [Har77, Ch. II, Exc. 5.8]), it is enough to check surjectivity after restricting to each F_y for all $y \in U$. Note F_y fits into the cartesian square



and so by flat base change [Har77, Ch. III, Prop. 9.3], we have a natural isomorphism

$$(\mathscr{E}_y \otimes k(y)) \otimes \mathcal{O}_{F_y} \cong H^0(\mathcal{O}_X(H)|_{F_y}) \otimes \mathcal{O}_{F_y}.$$

Thus, the map

is surjective, since $\mathcal{O}_X(H)|_{F_y} \simeq \mathcal{O}_{\mathbf{P}^1}(1)$ is globally generated.

• The surjection $f^*\mathscr{E} \to \mathcal{O}_X(H)$ above induces a morphism

$$g\colon f^{-1}(U)\longrightarrow \mathbf{P}(\mathscr{E}|_U)$$

over U, such that the restriction of g to each fiber F_y is the morphism $F_y \to \mathbf{P}^1$ defined by $\mathcal{O}(H)|_{F_y} \simeq \mathcal{O}_{\mathbf{P}^1}(1)$. But this morphism is an isomorphism for all $y \in U$. Thus, g is an isomorphism. \Box

This proves the Noether–Enriques theorem Theorem 10.5. The key technical ingredient for Castelnuovo's theorem Theorem 10.1 is the following proposition:

Proposition 10.6. If X is a minimal smooth projective surface such that $p_2(X) = 0 = q(X)$, then there exists a curve $C \hookrightarrow X$ such that $C \simeq \mathbf{P}^1$ and $(C^2) \ge 0$.

Note 10.7. This is not obvious even if you knew Castelnuovo's theorem.

Recall 10.8. X is minimal if there is no curve $C \simeq \mathbf{P}^1$ on X such that $(C^2) = -1$. If there is such a C, then there is a morphism $f: X \to Y$, where Y is a smooth projective surface, and f is the blowup of Y at a point p whose exceptional divisor E is C [Har77, Ch. V, Thm. 5.7]. We can then repeat this process until there are no more (-1)-curves; note we only need to repeat finitely many times since

$$\operatorname{rank}(\operatorname{Pic}(X) / \equiv) = \operatorname{rank}(\operatorname{Pic}(Y) / \equiv) - 1,$$

where \equiv here denotes numerical equivalence, and rank(Pic(X)/ \equiv) is finite (this is nontrivial, but not too hard to prove in characteristic zero; cf. [Har77, Ch. V, Excs. 1.7–8]). Thus, after a finite number of steps, you obtain a minimal surface. For Castelnuovo's criterion (Theorem 10.1), you are allowed to replace X with this blown down surface, and so we may assume without loss of generality that X is minimal. Proof of Castelnuovo's Theorem 10.1 assuming Proposition 10.6. One implication is trivial: We know that if X is rational, then $p_m(X) = 0$ for all $m \ge 1$ and $h^i(X, \mathcal{O}_X) = 0$ for all $i \ge 1$ by Proposition 1.10 and Remark 1.11(3).

Conversely, suppose $p_2(X) = 0 = q(X)$. After performing finitely many blow-downs, we may assume that X is minimal, and so we may apply Proposition 10.6 to show that there exists a curve $C \subseteq X$ that is isomorphic to \mathbf{P}^1 such that $(C^2) = d \ge 0$. We then claim that C moves in a linear system without base curves, i.e., $h^0(\mathcal{O}_X(C)) \ge 2$. Consider

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(C) \longrightarrow \mathcal{O}_X(C)|_C \longrightarrow 0$$

$$\downarrow \wr$$

$$\mathcal{O}_{\mathbf{P}^1}(d)$$

Since $H^1(X, \mathcal{O}_X) = 0$, we obtain the short exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_X) \longrightarrow H^0(\mathcal{O}_X(C)) \longrightarrow H^0(\mathcal{O}_{\mathbf{P}^1}(d)) \longrightarrow 0,$$

which implies that $h^0(\mathcal{O}_X(C)) = 1 + h^0(\mathcal{O}_{\mathbf{P}^1}(d)) \ge 2$. Consider $V \subseteq H^0(X, \mathcal{O}_X(C))$ where dim V = 2, such that V contains an equation defining C. Since C is irreducible, the linear system |V| has no base components. Now consider the rational map it defines:

$$\varphi \colon X \dashrightarrow \mathbf{P}^1.$$

If |V| is basepoint-free, then φ is a morphism such that $\varphi^{-1}(x) = C \simeq \mathbf{P}^1$ for some $x \in \mathbf{P}^1$, so Theorem 10.5 implies there exists $U \ni x$ such that $\varphi^{-1}(U) \simeq U \times \mathbf{P}^1$, and so X is rational. Suppose now that |V| has a basepoint at $p \in X$, and let π : Bl_p $X \to X$ be the blowup of X at p, with exceptional divisor p. By assumption, $p \in \mathrm{Bs}|V|$, and so for all $D \in |V|$, $p \in D$ implies $\pi^*D = E$ + other stuff. Now since C is smooth, $p \in C$ has multiplicity 1, and so $\pi^*C = \tilde{C} + E$, where \tilde{C} is the strict transform of C, isomorphic to \mathbf{P}^1 . Hence the rational map $\varphi \circ \pi$ is defined by the base-curve free linear system given by $\pi^*|V| - E$, and one element of this linear system is the curve $\tilde{C} \simeq \mathbf{P}^1$. We can therefore repeat.

It therefore suffices to show that after blowing up at base points a finite number of times, φ becomes a morphism. We know that there is a diagram



where g is birational and Y is smooth, projective such that $\varphi \circ g$ is a morphism (e.g., take a resolution for the graph of φ). Also, by assumption, $p \notin \text{Dom}(\varphi)$, and so $P \notin \text{Dom}(g^{-1})$. There is then a factorization $h: Y \to \text{Bl}_p X$ such that $g = \pi \circ h$ [Har77, Ch. V, Prop. 4.3]. We can then factor g as a composition of r blowups, and we have

$$(K_Y^2) = (K_X^2) - r$$

for some r. Thus, after at most r steps, we can make φ a morphism.

The meat of Castelnuovo's Theorem 10.1 is actually Proposition 10.6; we will spend most of next time proving it. We will then explain the examples of Shioda [Shi74].

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11.1 Rationality and unirationality for surfaces (continued)

Recall we want to prove Castelnuovo's rationality criterion:

Theorem 10.1 (Castelnuovo). Let X be a smooth projective surface. Then, X is rational if and only if $p_2(X) = 0 = q(X)$.

We reduced this to the following:

Proposition 10.6. If X is a smooth minimal projective surface such that $p_2(X) = 0 = q(X)$, then there is a curve $C \subseteq X$ such that $C \simeq \mathbf{P}^1$ and $(C^2) \ge 0$.

The proof will be in three cases depending on the sign of (K_X^2) . We isolate the $(K_X^2) < 0$ case.

Lemma 11.1. If X is a smooth projective minimal surface such that $(K_X^2) < 0$, then for every a > 0, there exists $D \ge 0$ such that $(K_X \cdot D) \le -a$ and $|K_X + D| = \emptyset$.

Proof. The proof proceeds in two steps.

Step 1. Show that it is enough to find $E \ge 0$ such that $(K_X \cdot E) < 0$.

After replacing E by one of its components, we may assume that E is a prime divisor. We first claim that $(E^2) \ge 0$. By the adjunction formula (10.4), we have

$$\underbrace{2p_a(E) - 2}_{\ge 2} = \underbrace{(K_X \cdot E)}_{<0} + (E^2).$$

This implies $(E^2) \ge -1$. Moreover, if $(E^2) = -1$, then $p_a(E) = 0$; this implies $E \simeq \mathbf{P}^1$, which contradicts minimality of X. Hence, $(E^2) \ge 0$.

Now consider the linear system $|aE + nK_X|$. This is nonempty if n = 0, and as $n \to \infty$,

$$(aE + nK_X) \cdot E = a(E^2) + n(K_X + E) \longrightarrow -\infty.$$
(11.1)

Since E is a prime divisor and $(E^2) \ge 0$, we have that $(E \cdot \text{effective}) \ge 0$. Hence (11.1) implies $|aE + nK_X| = \emptyset$ for $n \gg 0$. Therefore, there exists $n \ge 0$ such that $|aE + nK_X| \ne \emptyset$, and $|aE + (n+1)K_X| = \emptyset$.

Finally, any divisor $D \in |aE + nK_X|$ satisfies the conditions in Lemma 11.1, since $|D + K_X| = \emptyset$, and

$$(D \cdot K_X) = \left((aE + nK_X) \cdot K_X \right) = a(E \cdot K_X) + n(K_X^2) \le -a.$$

Step 2. We find E as in Step 1.

Let H be an effective very ample divisor.

- If $(H \cdot K_X) < 0$, take E = H.
- If $(H \cdot K_X) = 0$, take $E \in |K_X + nH|$ for $n \gg 0$. Then, $(K_X \cdot E) = (K_X^2) < 0$.

We are therefore left with the case when $(H \cdot K_X) > 0$.

Let r_0 be such that $((H + r_0 K_X) \cdot K_X) = 0$, so

$$r_0 = \frac{(H \cdot K_X)}{-(K_X^2)} > 0.$$

Then,

$$(H + r_0 K_X)^2 = \left((H + r_0 K_X) \cdot H \right) = (H^2) + \frac{(H \cdot K_X)}{-(K_X^2)} \cdot (H \cdot K_X) > 0,$$

since both terms on the right are positive by amplitude. Let $r > r_0$ close to r_0 with $r \in \mathbf{Q}$. Then,

$$(H + rK_X)^2 > 0,$$
 $(H + rK_X) \cdot K_X < 0,$ $(H + rK_X) \cdot H > 0.$ (11.2)

We now claim some multiple of $H + rK_X$ has sections. Write $r = \frac{p}{q}$, where $p, q \in \mathbb{Z}_{>0}$. Let $D_m = mq(H + rK_X)$. By Riemann-Roch (10.3),

$$\chi(\mathcal{O}_X(D_m)) = \chi(\mathcal{O}_X) + \frac{1}{2} (D_m \cdot (D_m - K_X))$$
$$= \chi(\mathcal{O}_X) + \frac{1}{2} (m^2 q^2 (H + rK_X)^2 - mq(H + rK_X) \cdot K_X) \longrightarrow \infty$$

as $m \to \infty$, since each term in the parentheses is positive by (11.2). Now by Serre duality,

$$h^2(\mathcal{O}_X(D_m)) = h^0(\mathcal{O}_X(K_X - D_m)),$$

which is zero for $m \gg 0$ since as $m \to \infty$,

$$(K_X - D_m) \cdot H = (K_X \cdot H) - mq \underbrace{(H \cdot rK_X) \cdot H}_{>0} \longrightarrow -\infty.$$

Combined with the Riemann–Roch calculation above, this implies $h^0(\mathcal{O}_X(D_m)) > 0$ for $m \gg 0$. Now choose $E \in |D_m|$ for $m \gg 0$. Then, by (11.2) we have

$$(E \cdot K_X) = mq((H + rK_X) \cdot K_X) < 0,$$

and so E satisfies the conditions in Step 1.

Proof of Proposition 10.6. The proof proceeds in two steps, the second of which splits into different cases depending on the sign of (K_X^2) .

Step 1. Show that it is enough to find $D \ge 0$ on X such that $(K_X \cdot D) < 0$ and $|K_X + D| = \emptyset$.

Suppose we have such a D. Then, there is a component C of D that is a prime divisor such that $(K_X \cdot C) < 0$. Since $D - C \ge 0$ and $|K_X + D| = \emptyset$, we also have that $|K_X + C| = \emptyset$. We claim that C satisfies the conditions in Proposition 10.6.

We first show $C \simeq \mathbf{P}^1$. The Euler–Poincaré characteristic of \mathcal{O}_X is

$$\chi(\mathcal{O}_X) = 1 - q(X) + p_1(X) = 1 \tag{11.3}$$

by hypothesis. By Riemann–Roch (10.3) for $\mathcal{O}_X(-C)$, we have

$$\chi(\mathcal{O}_X(-C)) = 1 + \frac{1}{2}(-C) \cdot (-C - K_X) = 1 + \frac{1}{2}(C \cdot (C + K)) = p_a(C) \ge 0$$

by the adjunction formula (10.4). Since C is effective, $|-C| = \emptyset$, and so $h^0(\mathcal{O}_X(-C)) = 0$. By Serre duality,

$$h^2(\mathcal{O}_X(-C)) = h^0(\mathcal{O}_X(K_X+C)) = 0$$

by the fact that $|K_X + C| = \emptyset$. Combined with the Riemann–Roch calculation above, this implies $-h^1(\mathcal{O}_X(-C)) = p_a(C)$. But $p_a(C) \ge 0$, and so we must have $p_a(C) = 0$ hence $C \simeq \mathbf{P}^1$.

Now by the adjunction formula (10.4), we have $-2 = (C^2) + (C \cdot K_X)$, and $(C \cdot K_X) \leq -1$ so that $(C^2) \geq -1$. But we cannot have $(C^2) = -1$ by minimality, and so $(C^2) \geq 0$.

Step 2. We find D as in Step 1.

Case 1.
$$(K_X^2) < 0$$

In this case, we are done by Lemma 11.1.

Case 2. $(K_X^2) = 0.$

We first show $|-K_X| \neq \emptyset$. Applying Riemann–Roch (10.3) for $\mathcal{O}_X(-K_X)$,

$$\chi(\mathcal{O}_X(-K_X)) = \chi(\mathcal{O}_X) + \frac{1}{2}(-K_X)(-K_X - K_X) = 1 + (K_X^2) = 1,$$
(11.4)

using that $\chi(\mathcal{O}_X) = 1$ by (11.3). This implies that $|-K_X| \neq \emptyset$ because

$$h^2(\mathcal{O}_X(-K_X)) = h^0(\mathcal{O}_X(2K_X)) = p_2(X) = 0$$

by assumption.

Now let H be an effective very ample divisor on X, and consider $|H + nK_X|$ for $n \ge 0$.

- $|H + nK_X| \neq \emptyset$ for n = 0 since $H \ge 0$.
- $|H + nK_X| = \emptyset$ for $n \gg 0$: we have that $(H + nK_X) \cdot H \to -\infty$ as $n \to \infty$, since $|-K_X| \neq \emptyset$ implies $(H \cdot K_X) < 0$.

Therefore, there exists $n \ge 0$ such that $D \in |H + nK_X| \ne \emptyset$, and $|H + (n+1)K_X| = \emptyset$. This satisfies the conditions in Step 1, since $|D + K_X| \ne \emptyset$ and $(D \cdot K_X) = (H \cdot K_X) < 0$. Case 3. $(K_X^2) > 0$.

The Riemann-Roch calculation for $\mathcal{O}_X(-K_X)$ in (11.4) implies

$$h^0(\mathcal{O}_X(-K_X)) - h^1(\mathcal{O}_X(-K_X)) = 1 + (K_X^2) \ge 2,$$

so $h^0(\mathcal{O}_X(-K_X)) \geq 2$.

We claim we may assume that every $G \in |-K_X|$ is a prime divisor. Otherwise, for some G, there exists a decomposition G = A + B where A, B > 0, and

$$(G \cdot K_X) = -(K_X^2) < 0.$$

After possibly switching A and B, we have that $(K_X \cdot A) < 0$ and

$$|K_X + A| = |A + (-A - B)| = |-B| = \emptyset.$$

Thus, A satisfies the condition in Step 1.

Hence, we will assume that all elements in $|-K_X|$ are prime divisors. Fix $G \in |-K_X|$. For every H effective on X, we have that

- $|H + nK_X| \neq \emptyset$ for n = 0 since $H \ge 0$.
- $|H + nK_X| = \emptyset$ for $n \gg 0$: we have that

$$(H + nK_X) \cdot (\text{ample}) = H \cdot (\text{ample}) - n(G \cdot (\text{ample})) \longrightarrow -\infty$$

as $n \to \infty$.

Thus, as in the proof of Case 2 there exists $n \ge 0$ such that $|H + nK_X| \ne \emptyset$ and $|H + (n+1)K_X| = \emptyset$. There are two subcases:

Subcase A. For some choice of $H \ge 0$, the number $n \ge 0$ as above is such that $H + nK_X \not \sim 0$.

Choose $E = \sum n_i C_i \in |H + nK_X|$ such that $E \neq 0$. We claim that $(K_X \cdot E) \leq 0$. But

$$(K_X \cdot E) = -(G \cdot E) \le 0$$

since $(G^2) = (K_X^2) > 0$, and the fact that G is prime implies $(G \cdot E) \ge 0$. Thus, there exists a component $C = C_i$ such that $(K_X \cdot C) \le 0$. Moreover, since $E - C \ge 0$ and $|E + K_X| = \emptyset$, we have $|C + K_X| = \emptyset$. Now if $(K_X \cdot C) < 0$, then C satisfies the condition in Step 1.

We want to rule out the possibility that $(K_X \cdot C) = 0$. Suppose this equality holds. Then, by the adjunction formula (10.4), we have

$$2p_a(C) - 2 = (C^2), (11.5)$$

and combined with Riemann-Roch (10.3) for $\mathcal{O}_X(-C)$, we obtain

$$\chi(\mathcal{O}_X(-C)) = 1 + \frac{1}{2}(-C) \cdot (-C - K_X) = p_a(C).$$

On the other hand, $h^0(\mathcal{O}_X(-C)) = 0$ since C is effective, and $h^2(\mathcal{O}_X(-C)) = h^0(\mathcal{O}_X(K_X + C)) = 0$. This implies $-h^1(\mathcal{O}_X(-C)) = p_a(C)$, hence $p_a(C) = 0$ and $C \simeq \mathbf{P}^1$. Then, (11.5) says $(C^2) = -2$. Next, Riemann-Roch (10.3) for $\mathcal{O}_X(-K_X - C)$ gives

$$\chi (\mathcal{O}_X(-K_X - C)) = 1 + \frac{1}{2}(-K_X - C) \cdot (-2K_X - C)$$
$$= 1 + \frac{1}{2} (2(K_X^2) + (C^2))$$
$$= 1 + (K_X^2) + \frac{1}{2} (C^2)$$
$$= (K_X^2) \ge 1.$$

Now $h^2(\mathcal{O}_X(-K_X-C)) = h^0(\mathcal{O}_X(2K_X+C)) \leq h^0(\mathcal{O}_X(K_X+C)) = 0$ where the inequality is by the fact that $|-K_X| \neq \emptyset$. Thus, the Riemann–Roch calculation above implies $h^0(\mathcal{O}_X(-K_X-C)) > 0$. Letting $C' \in |-K_X-C|$, we have that $C' \neq 0$ since otherwise $h^0(\mathcal{O}_X(K_X+C)) \neq 0$. This contradicts the fact that every divisor in $-K_X$ is prime: $C' + C \in |-K_X|$.

Subcase B. For all choices of $H \ge 0$, the number $n \ge 0$ as above is such that $H + nK_X \sim 0$.

The assumption implies that $\operatorname{Pic}(X)$ is generated by $\mathcal{O}(K_X)$. Since $\operatorname{Pic}(X)$ is not a torsion group (by intersection theory, for example), we see that $\operatorname{Pic}(X) \simeq \mathbf{Z} \cdot \mathcal{O}(K_X)$.

We now need to assume $k = \mathbf{C}$. The exponential sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{O}_{X^{\mathrm{an}}} \longrightarrow \mathcal{O}_{X^{\mathrm{an}}}^* \longrightarrow 0.$$

By GAGA (Theorem 9.11), we have that the sequence

$$0 = H^1(X, \mathcal{O}_X) \longrightarrow \operatorname{Pic}(X) \xrightarrow{c^1} H^2(X, \mathbf{Z}) \longrightarrow H^2(X, \mathcal{O}_X) = 0$$

is exact, hence c^1 is an isomorphism. By Poincaré duality 9.8, we have a perfect pairing

$$H^{i}(X, \mathbf{Z}) /_{\text{tors}} \times H^{n-i}(X, \mathbf{Z}) /_{\text{tors}} \longrightarrow \mathbf{Z}.$$

Thus, $(K_X^2) = 1$. Now we apply Noether's formula (10.2):

$$1 = \chi(\mathcal{O}_X) = \frac{1}{12} \left((K_X^2) + \chi_{\text{top}}(X) \right) = \frac{1}{12} \left(1 + b_0 - b_1 + b_2 - b_3 + b_4 \right) = \frac{1}{12} \left(1 + 2 - 4q(X) + 1 \right) = 3,$$

a contradiction.

Next time we will discuss Shioda's examples of unirational but not rational surfaces in positive characteristic [Shi74]. We will then start preparing for the Artin–Mumford example [AM72]. The goal will be to find an invariant that detects unirationality. To do so, we will need to compute cohomology of projective space and of blowups, which will use the Thom isomorphism to replace relative simplicial cohomology groups.

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Exercise 12.1. In the proof of both Castelnuovo's criterion for rationality (Theorem 10.1) and the Noether– Enriques Theorem 10.5, show that both results hold when k is algebraically closed of characteristic zero, but not necessarily the complex numbers. This should be true by the Lefschetz principle, but is a bit subtle.

12.1 Shioda's examples of unirational but not rational surfaces

We conclude our study of the theory of rationality for surfaces with examples due to Shioda [Shi74] of unirational but not rational surfaces in positive characteristic.

Theorem 12.2. Suppose char k = p > 2. Then, the surface in \mathbf{P}^3 defined by

$$x_0^{p+1} - x_1^{p+1} = x_2^{p+1} - x_3^{p+1}$$
(12.1)

is a smooth unirational surface.

Remarks 12.3.

- (1) If k has the appropriate (p+1)th roots of unity, then the surface defined by (12.1) is a Fermat surface.
- (2) The proof of Theorem 12.2 can be modified to show that if $k = \overline{k}$ and p > 2, then the Fermat surface

$$x_0^n + x_1^n + x_2^n + x_3^n = 0$$

is unirational if there exists m such that $p^m \equiv -1 \pmod{n}$. The case of Theorem 12.2 is when m = 1.

- (3) The surfaces defined above are not rational, since a smooth surface of degree $d \ge 4$ in \mathbf{P}^3 is not rational (Example 1.13). Thus, Shioda's examples give counterexamples to the Lüroth problem for smooth surfaces in positive characteristic.
- (4) [Shi74] does not give examples when char(k) = p = 2.

So far, we have shown unirationality by exhibiting dominant morphisms from projective space, i.e., by explicitly parametrizing the varieties in question. The proof of Theorem 12.2 instead boils down to a manipulation of equations.

Proof. We denote Y to be the surface defined by (12.1). First, by the Jacobian criterion, the surface Y is smooth. It remains to show that Y is unirational.

First, we perform the change of coordinates

 $y_1 = x_0 + x_1,$ $y_2 = x_0 - x_1,$ $y_3 = x_2 + x_3,$ $y_4 = x_2 - x_3.$

Then (12.1) becomes

$$y_1 y_2 (y_1^{p-1} + y_2^{p-1}) = y_3 y_4 (y_3^{p-1} + y_4^{p-1}),$$
(12.2)

since the left-hand side is

$$y_1y_2(y_1^{p-1} + y_2^{p-1}) = (x_0 + x_1)(x_0 - x_1)((x_0 + x_1)^{p-1} + (x_0 - x_1^{p-1}))$$

= $(x_0 - x_1)(x_0 + x_1)^p + (x_0 + x_1)(x_0 - x_1)^p$
= $(x_0 - x_1)(x_0^p + x_1^p) + (x_0 + x_1)(x_0^p - x_1^p)$
= $2(x_0^{p+1} - x_1^{p+1}).$

Working in the chart $y_4 \neq 0$, the equation (12.2) becomes

$$y_1 y_2 (y_1^{p-1} + y_2^{p-1}) = y_3 (y_3^{p-1} + 1).$$
(12.3)

Now after a change of coordinates

$$Spec(k[y_1, u, v]) \longrightarrow Spec(k[y_1, y_2, y_3])$$
$$y_1u \longleftrightarrow y_2$$
$$uv \longleftrightarrow y_3$$

which is birational since $u = y_2/y_1$ and $v = y_1y_3/y_2$, the equation (12.3) becomes

$$y_1^2 u(y_1^{p-1} + y_1^{p-1}u^{p-1}) = uv(u^{p-1}v^{p-1} + 1).$$

The strict transform of Y is defined in Spec(k[y, u, v]) by

$$y_1^{p+1}(1+u^{p-1}) = v(u^{p-1}v^{p-1}+1)$$

Note that u, v are algebraically independent over k, since $\operatorname{trdeg}_k k(Y) = 2$ and

$$y_1^{p+1} = \frac{v(u^{p-1}v^{p-1}+1)}{1+u^{p-1}}.$$
(12.4)

We now introduce an inseparable extension, which is what you would expect to cause issues in positive characteristic. Let $t = y_1^{1/p}$. Then, (12.4) implies that $y_1^{p+1} \notin k(u, v)^p$, and so $t \notin k(Y)$. Now let K = k(Y)(t), which is an inseparable extension of k(Y).

Claim 12.4. k(Y)(t) is purely transcendental over k.

Note this implies there is a dominant rational map $\mathbf{P}^2 \dashrightarrow Y$.

In K, (12.4) implies the relation

$$t^{p(p+1)}(1+u^{p-1}) = v(u^{p-1}v^{p-1}+1)$$

which is equivalent to

$$u^{p-1}(t^{p+1}-v)^p = v - t^{p(p+1)}.$$

Let $s = u(t^{p+1} - v)$ so that

$$s^{p} = u(v - t^{p(p+1)}) = \frac{s(v - t^{p(p+1)})}{t^{p+1} - v}.$$

Then,

$$s^{p}t^{p+1} - s^{p}v = sv - st^{p(p+1)}$$

which holds if and only if $v(s+s^p) = s^p t^{p+1} + st^{p(p+1)}$. Now $v \in k(s,t)$ and $u = s/(t^{p+1}-v) \in k(s,t)$ implies K = k(s,t). Since $\operatorname{trdeg}_k(K/k) = 2$, the fact that s, t are algebraically independent gives the claim. \Box

Remark 12.5. The main point of this example is that inseparable extensions are what cause issues for surfaces in characteristic p, in contrast to the characteristic zero case. One says that X is separably unirational if it admits a dominant, separable rational map from a projective space. With this restriction, Lüroth's problem can be answered affirmatively: any separably unirational smooth surface is in fact rational by applying Remark 1.11(2) and using Castelnuovo's criterion (Theorem 10.1).

12.2 More topology

To prepare for the higher-dimensional theory, we need to review more material on the topology of algebraic varieties.

Goal 12.6. We want to understand the examples of Artin–Mumford [AM72]. The key birational invariant used is the torsion subgroup of H^3 , which detects non-stable rationality. To prove it is indeed a birational invariant, we need to study the cohomology of \mathbf{P}^n and how cohomology changes under blowups.

One difficulty is that we want to use relative cohomology. We will avoid relative cohomology by using results form topology: one can deduce relative cohomology from the cohomology of the complement.

The Mayer–Vietoris Sequence 12.7. If X is a topological space, and there exist U, V open such that $U \cup V = X$, then there exists an exact sequence

$$\cdots \longrightarrow H^q(X,A) \xrightarrow{(j_1^*,j_2^*)} H^q(U,A) \oplus H^q(V,A) \xrightarrow{i_1^*-i_2^*} H^q(U \cap V,A) \longrightarrow H^{q+1}(X,A) \longrightarrow \cdots$$

where the maps are



There exists a similar sequence for homology.

Example 12.8. You can compute the cohomology of the sphere S^n by induction using the Mayer–Vietoris sequence: S^n has a cover by two copies of the *n*-dimensional disc, whose intersection is S^{n-1} .

Two results on fibrations We state some theorems about fibrations, i.e., maps of topological spaces which locally look like products. We will restrict ourselves to the case of algebraic varieties to ensure there exists finite open covers satisfying certain properties.

Definition 12.9. A continuous map $f: X \to Y$ is *locally trivial* with fiber F if there exists an open cover $Y = \bigcup_{\alpha} U_{\alpha}$ such that $f^{-1}(U_{\alpha}) \simeq U_{\alpha} \times F$ over U_{α} for all α .

Example 12.10. (Topological) vector bundles are locally trivial.

Leray–Hirsch Theorem 12.11. Suppose $f: X \to Y$ is locally trivial with fiber F. Assume $H^i(F, \mathbb{Z})$ is finitely generated and torsion-free for all i. Suppose $\alpha_1, \ldots, \alpha_r \in H^*(X, \mathbb{Z})$ are elements such that for all $y \in Y$, the elements $i_y^*(\alpha_1), \ldots, i_y^*(\alpha_r)$ give a basis for $H^*(X, \mathbb{Z})$, where $i_y: f^{-1}(y) \hookrightarrow X$ is the inclusion. Then, we have the following isomorphism of abelian groups:

$$\mathbf{Z}^r \otimes_{\mathbf{Z}} H^*(Y, \mathbf{Z}) \longrightarrow H^*(X, \mathbf{Z})$$
$$(u_1, \dots, u_r) \otimes \beta \longmapsto \left(\sum_{i=1}^r u_i \alpha_i\right) \cup f^*(\beta).$$

There are many proofs of the Leray–Hirsch Theorem 12.11; for example, it follows from the Leray spectral sequence in sheaf cohomology. The proof using Mayer–Vietoris, however, is the simplest.

Sketch of proof. If $X = Y \times F$, then it is easy to deduce Theorem 12.11 from the Künneth formula 9.5.

Now suppose there are finitely many open subsets U_1, \ldots, U_s such that $f^{-1}(U_i) \simeq U_i \times F$ over U_i and $Y = \bigcup_i U_i$. Then, you can argue by induction on X using the Mayer–Vietoris sequence 12.7.

To prove Theorem 12.11 in general, you can use a limiting argument. However, we will only need the case where a finite cover as stated exists for applications. The Leray–Hirsch Theorem 12.11 will allow us to understand the cohomology of projective bundles, and particular that of \mathbf{P}^n .

We now turn to a "more interesting" result.

Recall 12.12. If X is a topological space, then a topological (real) vector bundle E is oriented if it is oriented on a trivialization, such that the orientations match on intersections of elements of the open cover giving the trivialization. In this case, denoting $\pi: E \to X$ to be the structure map for the vector bundle, we have that $E_x := \pi^{-1}(x)$ is an oriented r-dimensional **R**-vector space, where $r = \operatorname{rank} E$.

Now consider the zero section $X \hookrightarrow E$, and suppose $r \ge 1$. Then,

$$H^r(\mathbf{R}^r, \mathbf{R}^r \setminus \{0\}, \mathbf{Z}) \simeq \mathbf{Z}$$

and choosing a generator is equivalent to choosing an orientation. The long exact sequence in cohomology implies

$$H^{r}(\mathbf{R}^{r}, \mathbf{R}^{r} \smallsetminus \{0\}, \mathbf{Z}) \simeq \widetilde{H}^{r-1}(\mathbf{R}^{r-1} \smallsetminus \{0\}, \mathbf{Z}) \simeq \widetilde{H}^{r-1}(S^{r-1}, \mathbf{Z}) \simeq \mathbf{Z},$$

since $\mathbf{R}^{r-1} \setminus \{0\}$ is homotopy equivalent to S^{r-1} . Here, \widetilde{H}^* denotes reduced cohomology, which is the cokernel of the canonical map $\mathbf{Z} \hookrightarrow H^*$.

The Thom Isomorphism 12.13. Let E be an oriented, real vector bundle on X of rank $r = \operatorname{rank} E \ge 1$. (1) There exists a Thom class $\eta_E \in H^r(E, E \smallsetminus X, \mathbf{Z})$ such that for all $x \in X$, denoting

$$E_{(x)} \xrightarrow{j_x} E$$

$$\uparrow \qquad \qquad \uparrow$$

$$E_{(x)} \smallsetminus \{0\} \xrightarrow{} E \smallsetminus X$$

we have that

$$j_x^*(\eta_E) \in H^r(E_{(x)}, E_{(x)} \smallsetminus \{0\}, \mathbf{Z})$$

is the cohomology class corresponding to the orientation on $E_{(x)}$.

(2) For every closed subset $Z \hookrightarrow X$, the map

$$H^{i}(X, X \smallsetminus Y, \mathbf{Z}) \longrightarrow H^{i+r}(E, E \smallsetminus Y, \mathbf{Z})$$
$$\alpha \longmapsto \pi^{*}(\alpha) \cup \eta_{E}$$

is an isomorphism for all $i \in \mathbf{Z}$, called the Thom isomorphism. In particular, $H^i(E, E \setminus Y, \mathbf{Z}) = 0$ for all i < r.

We won't prove this, but it is not hard.

Idea of proof. If E is trivial, then one can deduce both assertions from the Künneth formula 9.5.

Otherwise, if X is covered by finitely many open subsets such that E is trivial over each of them, then you can argue by induction on these subsets using the Mayer–Vietoris sequence 12.7. \Box

We will primarily use the Thom isomorphism 12.13 in the following special case. Suppose X is an oriented C^{∞} (real) manifold, and let Y be a closed, oriented submanifold. The normal exact sequence

$$0 \longrightarrow T_Y \longrightarrow T_X|_Y \longrightarrow N \longrightarrow 0$$

where $N = N_{Y/X}$ is the normal bundle for Y in X implies $\det(T_X|_Y) \simeq \det(T_Y) \otimes \det(N)$. Since $T_X|_Y$ and T_Y are both oriented, this gives an orientation of the real vector bundle N.

In this situation, there exists an open neighborhood U of Y which restricts to Y in a trivial sense, and the topology of U is that of $N_{Y/X}$. More precisely, we have the following:

Tubular Neighborhood Theorem 12.14. There exists an open neighborhood U of Y in X which retracts onto Y, and a homeomorphism $U \simeq N$ making the diagram

$$\begin{array}{cccc} Y & \stackrel{i}{\longrightarrow} U & \stackrel{r}{\longrightarrow} Y \\ & & & & & \\ \| & & & & \\ V & \stackrel{j}{\longrightarrow} N & \stackrel{\pi}{\longrightarrow} Y \end{array}$$

commute, where $j: Y \hookrightarrow N$ is the zero section.

Suppose now that $W \hookrightarrow Y$ is a closed subset, and let r be the real codimension of Y in X. Then,

$$H^{i}(Y, Y \smallsetminus W) \simeq H^{i+r}(N, N \smallsetminus W) \qquad \text{(by the Thom isomorphism 12.13)}$$
$$\simeq H^{i+r}(U, U \smallsetminus W) \qquad \text{(by Theorem 12.14)}$$
$$\simeq H^{i+r}(X, X \smallsetminus W) \qquad \text{(by Excision (9.1))}$$

Thus, $H^i(Y, Y \smallsetminus W) \simeq H^{i+r}(X, X \smallsetminus W)$.

In particular, if W = Y, we have isomorphisms $H^i(Y, \mathbb{Z}) \simeq H^{i+r}(X, X \setminus Y, \mathbb{Z})$. Thus, the long exact sequence for $(X, X \setminus Y)$ comes

$$\cdots \longrightarrow H^{i-r}(Y, \mathbf{Z}) \longrightarrow H^{i}(X, \mathbf{Z}) \longrightarrow H^{i}(X \smallsetminus Y, \mathbf{Z}) \longrightarrow H^{i-r+1}(Y, \mathbf{Z}) \longrightarrow \cdots .$$
(12.5)

Terminology 12.15. In this case, if $f: Y \hookrightarrow X$ is the inclusion map, then

$$\begin{array}{ccc} H^{i-r}(Y, \mathbf{Z}) & \stackrel{f_*}{\longrightarrow} & H^i(X, \mathbf{Z}) \\ & & \swarrow \\ & & & & \\ & & & \\ & & & \\ & & & H^i(X, X \smallsetminus Y) \end{array}$$

is called the Gysin map corresponding to f.

Remark 12.16. If $f: Y \to X$ is any C^{∞} map between compact, oriented real manifolds, and $r = \dim X - \dim Y$, we also get a map

$$\begin{array}{ccc} H^{i}(Y, \mathbf{Z}) & & \stackrel{f_{*}}{\longrightarrow} & H^{i+r}(X, \mathbf{Z}) \\ & & & \downarrow & & \downarrow \\ PD & & & \downarrow & PD \\ H_{\dim Y - i}(Y, \mathbf{Z}) & \stackrel{f_{*}}{\longrightarrow} & H_{\dim X - i - r}(X, \mathbf{Z}) \end{array}$$

where we note $\dim X - i - r = \dim Y - i$. If f is an inclusion, this is the Gysin map defined above.

Remark 12.17. Let $f: X \to Y$ be a surjective morphism of complete smooth algebraic varieties over \mathbf{C} , such that dim $X = \dim Y$. Then, the pushforward $f: H_p(X) \to H_p(Y)$ is such that $f_*(\mu_X) = (\deg f)\mu_Y$, where μ_X, μ_Y are the fundamental classes for X, Y, respectively. The projection formula 9.4 and Poincaré duality 9.8 then imply

$$f_*(f^*(\alpha)) = \deg(f) \cdot \alpha$$

for all $\alpha \in H^*(Y, \mathbb{Z})$. Thus, f^* is injective if f is birational or after tensoring with \mathbb{Q} .

Fact 12.18. f^* is injective for non-compact X and Y under the same conditions above. To prove this, one passes through Borel–Moore homology (which is Poincaré dual to cohomology).

Next time, we will investigate the cohomology of \mathbf{P}^n and blowups, as well as for projective bundles (using the Leray–Hirsch Theorem 12.11). This will imply that H^3_{tors} is the same for varieties that are stably birational. We will then be able to study the Artin–Mumford examples [AM72].

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13.1 Some cohomology computations

We work over the complex numbers \mathbf{C} . We start with some computations of cohomology for projective space, projective bundles, and blowups.

Proposition 13.1. For every $n \ge 1$, we have a ring isomorphism

$$\begin{aligned} \mathbf{Z}[x]/(x^{n+1}) & \xrightarrow{\sim} & H^*(\mathbf{P}^n, \mathbf{Z}) \\ \\ \bar{x} & \longmapsto & c^1(\mathcal{O}(1)) \end{aligned}$$

Proof. We induce on *n*. The case n = 0 is trivial, since in this case $\mathbf{P}^0 = \{ pt \}$.

For the induction step, consider a hyperplane $\mathbf{P}^{n-1} \hookrightarrow \mathbf{P}^n$, with complement \mathbf{C}^n . We saw last time that the long exact sequence on cohomology for $(\mathbf{P}^n, \mathbf{P}^n \smallsetminus \mathbf{P}^{n-1})$ becomes (by the Thom isomorphism 12.13)

 $\cdots \longrightarrow H^{i-1}(\mathbf{P}^n \smallsetminus \mathbf{P}^{n-1}) \longrightarrow H^{i-2}(\mathbf{P}^{n-1}) \longrightarrow H^i(\mathbf{P}^n) \longrightarrow H^i(\mathbf{P}^n \smallsetminus \mathbf{P}^{n-1}) \longrightarrow \cdots$

as in (12.5). Now note that $H^i(\mathbf{P}^n \setminus \mathbf{P}^{n-1}) = 0$ for i > 0, and so

$$H^{1}(\mathbf{P}^{n}, \mathbf{Z}) = 0$$

$$H^{i}(\mathbf{P}^{n}, \mathbf{Z}) \simeq H^{i-2}(\mathbf{P}^{n-1}, \mathbf{Z})$$

By the induction hypothesis, we have that $H^i(\mathbf{P}^n, \mathbf{Z}) \simeq \mathbf{Z}$ if $0 \le i \le 2n$ if i is even, and 0 otherwise.

We now need to check that for $0 \leq i \leq n$, $H^{2i}(\mathbf{P}^n, \mathbf{Z})$ is generated by h^i , where $h = c^1(\mathcal{O}(1))$. We know that $h^n = (\mathcal{O}(1)^n) = 1$ by the compatibility of the cup product with intersection product, and so in particular, $h^n \neq 0$. Given $0 \leq i \leq n$, since $h^i \in H^{2i}(\mathbf{P}^n, \mathbf{Z}) \simeq \mathbf{Z}$ is nonzero, it follows that for every $\alpha \in H^{2i}(X, \mathbf{Z})$, there exists $r \in \mathbf{Q}$ such that $\alpha = r \cdot h^i$. On the other hand, since $\alpha \in H^{2i}(X, \mathbf{Z})$, it follows that $\alpha \cdot h^{n-i} \in \mathbf{Z}$, and so

$$\alpha \cdot h^{n-i} = r \cdot h^i h^{n-i} = r \in \mathbf{Z}.$$

Thus, h^i is a generator of $H^{2i}(X, \mathbf{Z})$.

Corollary 13.2. If E is a rank n+1 (algebraic) vector bundle on X, and $\pi: \mathbf{P}(E) \to X$ is the corresponding projective bundle, then there is a ring isomorphism

$$H^*(\mathbf{P}(E), \mathbf{Z}) \longleftarrow H^*(X)[x]/(x^{n+1})$$
$$\sum \pi^*(\alpha_i) c^1 (\mathcal{O}(1))^i \longleftarrow \sum \alpha_i x^i$$

Proof. It is clear that this is a ring homomorphism, and so it is enough to show that it is an isomorphism of abelian groups. For this, we apply the Leray–Hirsch Theorem 12.11 for the classes $c^1(\mathcal{O}(1))^i$, where $0 \le i \le n$. By Proposition 13.1, these satisfy the hypothesis of Leray–Hirsch Theorem 12.11.

We are now ready to discuss the blowup. Let X be a smooth projective variety, and consider a smooth closed subvariety Y of codimension r in X. Consider the blowup along Y:

Proposition 13.3. For every $p \ge 0$, we have an isomorphism

$$H^{p}(\widetilde{X}, \mathbf{Z}) \xleftarrow{\sim} H^{p}(X, \mathbf{Z}) \oplus \bigoplus_{q=1}^{r-1} H^{p-2q}(Y, \mathbf{Z})$$
$$f^{*}(\alpha) + \sum_{q=1}^{r-1} j_{*} \left(c^{1} (\mathcal{O}(1))^{q-1} \cup \varphi^{*}(\beta_{q}) \right) \xleftarrow{\sim} (\alpha, \beta_{1}, \dots, \beta_{r-1})$$

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Proof. Consider the long exact sequence on cohomology:

$$\cdots \longrightarrow H^{p}(\widetilde{X}, \widetilde{X} \smallsetminus E) \longrightarrow H^{p}(\widetilde{X}) \longrightarrow H^{p}(\widetilde{X} \smallsetminus E) \longrightarrow H^{p+1}(\widetilde{X}, \widetilde{X} \smallsetminus E) \longrightarrow \cdots$$

$$\uparrow f^{*} \qquad \uparrow f^{*} \qquad \uparrow f^{*} \qquad \uparrow f^{*} \qquad \uparrow f^{*} \qquad (13.1)$$

$$\cdots \longrightarrow H^{p}(X, X \smallsetminus Y) \longrightarrow H^{p}(X) \longrightarrow H^{p}(X \smallsetminus Y) \longrightarrow H^{p+1}(X, X \smallsetminus Y) \longrightarrow \cdots$$

Since $\varphi \colon E \to Y$ is a projective bundle with fiber \mathbf{P}^{r-1} , Corollary 13.2 implies we have an isomorphism

$$H^{p-2}(E) \xleftarrow{\sim} \bigoplus_{q=1}^{r} H^{p-2q}(Y)$$
$$\sum_{q=1}^{r} c^{1} (\mathcal{O}(1))^{q-1} \cup \varphi^{*}(\beta_{q}) \xleftarrow{\sim} (\beta_{1}, \dots, \beta_{r})$$

Moreover, the Thom isomorphism 12.13 implies we have a diagram

$$\begin{array}{ccc} H^{p}(\widetilde{X},\widetilde{X}\smallsetminus E) & \stackrel{\sim}{\longrightarrow} & H^{p-2}(E) \\ f^{*} & & \uparrow^{c^{1}}(\mathcal{O}(1))^{r-1} \cup \varphi^{*}(-) \\ H^{p}(X,X\smallsetminus Y) & \stackrel{\sim}{\longrightarrow} & H^{p-2r}(Y) \end{array}$$

where you can check the map on the right makes the diagram commute. Since f is a birational morphism between smooth varieties X, \tilde{X} , we see that the Gysin morphism

$$f^* \colon H^p(X) \longrightarrow H^p(\widetilde{X})$$

is injective by Remark 12.17 and Fact 12.18. Proposition 13.3 then follows from a diagram chase using the diagram (13.1).

13.2 An invariant that detects non-stable-rationality

We now introduce an invariant that can detect when a variety is not stably rational.

Definition 13.4. Two varieties X, Y are stably birational if there exist $m, n \ge 0$ such that $X \times \mathbf{P}^m$ and $Y \times \mathbf{P}^n$ are birational.

Example 13.5. X is stably rational if and only if X is stably birational to a point.

Proposition 13.6. If X and Y are smooth, complete complex varieties which are stably birational, then $H^3(X, \mathbf{Z})_{\text{tors}} \simeq H^3(Y, \mathbf{Z})_{\text{tors}}$.

Since a point has no H^3 , Proposition 13.6 immediately implies the following:

Corollary 13.7. If X is stably birational, then $H^3(X, \mathbf{Z})_{\text{tors}} = 0$.

Proof of Proposition 13.6. We need to check the following:

(1) $H^3(X \times \mathbf{P}^n, \mathbf{Z})_{\text{tors}} \simeq H^3(X, \mathbf{Z})_{\text{tors}};$

(2) If X and Y are smooth, complete, birational varieties, then $H^3(X, \mathbb{Z})_{\text{tors}} \simeq H^3(Y, \mathbb{Z})_{\text{tors}}$.

For (1), since $H^i(\mathbf{P}^n, \mathbf{Z})$ is finitely generated free abelian group, the Künneth formula 9.5 implies

$$H^{3}(X \times \mathbf{P}^{n}, \mathbf{Z}) \simeq \bigoplus_{i+j=3} \left(H^{i}(X, \mathbf{Z}) \otimes_{\mathbf{Z}} H^{j}(\mathbf{P}^{n}, \mathbf{Z}) \right) \simeq H^{3}(X, \mathbf{Z}) \oplus H^{1}(X, \mathbf{Z})$$

But $H^1(X, \mathbb{Z})$ has no torsion in general (Consequences 9.3), and so $H^3(X \times \mathbb{P}^n, \mathbb{Z})_{\text{tors}} \simeq H^3(X, \mathbb{Z})_{\text{tors}}$.

For (2), the weak factorization theorem [Abr+02, Thm. 0.1.1] implies X and Y are connected by a chain of blow-ups of smooth complete varieties along smooth subvarieties. Hence, it suffices to show that if Y is the blowup of X along Z, where Z is smooth, then $H^3(Y)_{\text{tors}} \simeq H^3(X)_{\text{tors}}$. Denoting

$$E \xrightarrow{j} Y$$
$$\downarrow \qquad \qquad \downarrow^{f}$$
$$Z \xrightarrow{} X$$

we have, by Proposition 13.3,

$$H^3(Y, \mathbf{Z}) \simeq H^3(X, \mathbf{Z}) \oplus H^1(Z, \mathbf{Z}).$$

Since $H^1(Z, \mathbf{Z})$ has no torsion, we see that $H^3(X, \mathbf{Z})_{\text{tors}} \simeq H^3(Y, \mathbf{Z})_{\text{tors}}$.

One thing to note is that for higher cohomology groups, these arguments would break down since there would be torsion contributions from other cohomology groups.

Remark 13.8. When Artin and Mumford wrote their paper, weak factorization was not yet known. Instead, they use the following strong version of Hironaka's theorem [Hir64]: Let $\varphi \colon X \to Y$ be a rational map, where X is smooth and X, Y are complete. Then, there exists a sequence of smooth blowups



Remark 13.9. The same proof implies that $H^2(X, \mathbb{Z})_{\text{tors}}$ is the same for stably birational varieties, since $H^2(X, \mathbb{Z})_{\text{tors}} = 0$ already for rationally connected varieties.

Recall that a smooth projective variety X is rationally connected if any two general points on X lie in the image of some map $\mathbf{P}^1 \to X$. Unirational varieties are rationally connected, since two general points in the dominating rational variety are connected by a \mathbf{P}^1 . It is also true that rationally connected varieties are simply connected [Cam91, Thm. 3.5]: First, one can show that they have finite fundamental group [Cam91, Thm. 2.2]. Rationally connected varieties also satisfy $h^0((\Omega_X^j)^{\otimes m}) = 0$. Thus, $\chi(\mathcal{O}_X) = 1$. If $\pi_1(X) \neq 1$, then there would exist a rationally connected étale cover of X, but this would contradict that $\chi(\mathcal{O}_X) = 1$. Thus, $H^2(X, \mathbf{Z})_{\text{tors}} = 0$ by the Universal Coefficient Theorem 9.2.

Remark 13.10. If $X \subseteq \mathbf{P}^n$ is a smooth hypersurface, where $n \ge 4$, then $H^3(X, \mathbf{Z})_{\text{tors}} = 0$. This follows from Weak Lefschetz, which says

$$H_i(X, \mathbf{Z}) \xrightarrow{\sim} H_i(\mathbf{P}^n, \mathbf{Z})$$
 for $i < n - 1$.

The right-hand side is torsion-free, so $H_2(X, \mathbf{Z})_{\text{tors}} = 0$ for n > 3. Thus, the Universal Coefficient Theorem 9.2 implies $H^3(X, \mathbf{Z})_{\text{tors}} = 0$.

This shows that $H^3(X, \mathbb{Z})_{\text{tors}}$ cannot be used to prove non-rationality. The method of Voisin is interesting because it passes through *singular* hypersurfaces to compute $H^3(X, \mathbb{Z})_{\text{tors}}$.

13.3 How to produce examples with $H^3(X, \mathbf{Z})_{\text{tors}} \neq 0$

We briefly describe the idea for how to show $H^3(X, \mathbf{Z})_{\text{tors}} \neq 0$.

The idea is that $H^3(X, \mathbb{Z})_{\text{tors}} \neq 0$ if there exists a map $f: Y \to X$ such that for all $x \in X$, $f^{-1}(x) \simeq \mathbb{P}^n$, but such that $Y \not\simeq \mathbb{P}(E)$ over X for any vector bundle E (this holds, for example, if $f: Y \to X$ does not have a rational section). One can show that in this case, f is locally trivial in the analytic or the étale topology.

In fact, such objects are parametrized by $\check{H}^1(X^{\mathrm{an}}, \mathrm{PGL}_{n+1,X})$, which we note still makes sense even though PGL_{n+1} is non-abelian. We then have an exact sequence

This gives a map $\check{H}^1(\operatorname{PGL}_{n+1,X}) \to H^2(X^{\operatorname{an}}, \mathcal{O}^*_{X^{\operatorname{an}}})_{\operatorname{tors}}$, and the exponential sequence gives

 $0 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{O}_{X^{\mathrm{an}}} \longrightarrow \mathcal{O}_{X^{\mathrm{an}}}^* \longrightarrow 0$

which gives a map $H^2(\mathcal{O}_{X^{\mathrm{an}}}^*)_{\mathrm{tors}} \to H^3(X, \mathbb{Z})_{\mathrm{tors}}$. We want our assumption to imply that this is not zero. If this is zero, then our class in $\check{H}^1(X, \mathrm{PGL}_{n+1,X})$ comes from $\check{H}^1(X, \mathrm{GL}_{n+1,X})$, which implies that $Y \simeq \mathbf{P}(E)$ for some holomorphic vector bundle E on X. We get a contradiction if X is complete (by GAGA), since in this case E is algebraic. Note that since we will be working with non-complete varieties, we will need to introduce étale cohomology.

Next time we will talk about the geometric construction of the Artin–Mumford example, then review facts from étale cohomology needed to prove non-stable-rationality.

14 March 7

Today, we will start discussing the geometry of the Artin–Mumford example. We will later need to review some facts about étale cohomology and on Brauer groups.

14.1 The Artin–Mumford example

The Artin–Mumford is constructed from a linear system of quadrics in \mathbf{P}^3 . We start by studying the complete linear system of quadrics.

14.1.1 The complete linear system of quadric hypersurfaces in P^3

We assume $k = \overline{k}$, and char $(k) \neq 2$.

Let $\mathbf{P} = \mathbf{P}(H^0(\mathbf{P}^3, \mathcal{O}(2))^*) \simeq \mathbf{P}^9$ be the projective space parametrizing quadric hypersurfaces in \mathbf{P}^3 , and let G be the Grassmann variety parametrizing lines in \mathbf{P}^3 , which is a four-dimensional, smooth, projective, rational variety. Then, we have an incidence correspondence $I \subset \mathbf{P} \times G$:

$$\begin{array}{ccc} I^{10} & \stackrel{q}{\longrightarrow} & G^4 \\ p \\ \downarrow & & \\ \mathbf{P} \end{array}$$

$$(14.1)$$

Since I is a projective bundle over G of relative dimension 6, we see that I is a smooth, projective of dimension 10. Given a system of coordinates x_0, \ldots, x_3 on \mathbf{P}^3 , we may write any nonzero element of $H^0(\mathbf{P}^3, \mathcal{O}(2))$ uniquely as

$$f = \sum_{i,j=0}^{3} a_{ij} x_i x_j \qquad \text{where } a_{ij} = a_{ji}.$$

Definition 14.1. The rank of f (or of f = 0) is equal to rank (a_{ij}) , the rank of the matrix (a_{ij}) .

If f has rank r, then there exists a system of coordinates such that $f = x_0^2 + \cdots + x_{r-1}^2$, which is a smooth quadric if and only if the rank is equal to 4.

Using this notion of rank, we may stratify \mathbf{P} by closed subsets of the form

$$W_i = \{ \text{quadrics of rank} \le i \},\$$

in which case $W_1 \subseteq W_2 \subseteq W_3 \subseteq W_4 = \mathbf{P}$.

Proposition 14.2.

- (1) $\dim W_1 = 3$, $\dim W_2 = 6$, $\dim W_3 = 8$;
- (2) W_2 is an irreducible subset of **P** of degree 10;
- (3) W_3 is an integral hypersurface of degree 4 in **P** such that all points of $W_3 \setminus W_2$ are smooth points of W_3 ;
- (4) For every $P \in W_2 \setminus W_1$, the tangent cone $C_p(W_3)$ is isomorphic to the hypersurface $x_1^2 + x_2^2 + x_3^2$ in \mathbf{A}^9 .

Proof. We start by showing (1). First,

$$W_1 = \{ \text{double planes} \} \subseteq \mathbf{P}_1$$

and so there is a bijective map

$$\{ \text{planes in } \mathbf{P}^3 \} \longrightarrow W_1$$
$$\stackrel{|\mathcal{C}}{\mathbf{P}^3}$$

Thus, W_1 is irreducible of dimension 3. Next,

 $W_2 = \{ \text{unions of 2-planes} \} \subseteq \mathbf{P},$

and so there is a generically 2 : 1 map

$$\mathbf{P}^3 \times \mathbf{P}^3 \stackrel{\varphi}{\longrightarrow} W_2,$$

and W_2 is irreducible of dimension 6. Finally, W_3 is an integral hypersurface in $\mathbf{P} \simeq \mathbf{P}^9$ defined by an equation of degree 4, and so W_3 has dimension 8.

We now show the assertion that $\deg(W_2) = 10$ in (2). The morphism $\varphi \colon \mathbf{P}^3 \times \mathbf{P}^3 \to W_2$ is of degree 2 (this is clear in characteristic zero, but the characteristic *p* case requires some work, which we leave as an exercise), and so we have

$$\deg(W_2) = \frac{1}{2} (\varphi^* \mathcal{O}(1))^6 = \frac{1}{2} (\mathcal{O}(1,1))^6 = \frac{1}{2} (\mathcal{O}(1,0) + \mathcal{O}(0,1))^6 = \frac{1}{2} \binom{6}{3} = 10.$$

For (3), we have already seen that W_3 is an integral hypersurface of degree 4, and so it suffices to show that all points of $W_3 \setminus W_2$ are smooth points in W_3 . In characteristic zero, this follows by generic smoothness: $W_3 \setminus W_2$ contains a smooth point, and $W_3 \setminus W_2$ has a transitive group action, so every point in $W_3 \setminus W_2$ is smooth. We also give a direct argument, which works in arbitrary characteristic. Let $P \in W_3 \setminus W_2$; after a change of coordinates, we may assume that P corresponds to the quadric $x_0^2 + x_1^2 + x_2^2$. The equation of W_3 at P in the affine chart $a_{00} \neq 0$ is

Thus, the tangent space of W_3 at P is defined by $a_{33} = 0$ as a subspace of the tangent space of **P** at P, and so P is a smooth point of W_3 .

For (4), let $P \in W_2 \setminus W_1$. After change of coordinates, we may assume that P corresponds to the quadric $x_0^2 + x_1^2$, in which case the equation of W_3 at P in the affine chart $a_{00} \neq 0$ is

$$\begin{vmatrix} 1 & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} - 1 & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{vmatrix} \in - \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix} + \mathfrak{m}_P^3$$

The tangent cone is therefore $x_1^2 + x_2^2 + x_3^2 = 0$ in $\mathbf{A}^9 \simeq T_P(W_3)$.

It remains to show the assertion that W_2 is an irreducible subset of **P** in (2). We saw that the singular locus of $W_3 = \{\det(a_{ij}) = 0\}$ is contained in W_2 ; since the singular locus is closed, we see that the singular locus of W_3 is in fact equal to W_2 . The locus W_2 is therefore cut out by the partial derivatives of $\det(a_{ij}) = 0$, which one can check produces an irreducible subset of W_3 .

14.1.2 Geometry of the map p in (14.1)

If $Q \in \mathbf{P} \setminus W_3$, then $p^{-1}(Q)$ is the disjoint union of two \mathbf{P}^1 's, since after a change of coordinates,

$$Q = \operatorname{im}(\mathbf{P}^1 \times \mathbf{P}^1 \stackrel{i}{\hookrightarrow} \mathbf{P}^3)$$

has two families of lines on Q, $i(\{P\} \times \mathbf{P}^1)_{P \in \mathbf{P}^1}$ and $i(\mathbf{P}^1 \times \{P\})_{P \in \mathbf{P}^1}$.

Note 14.3. If L_1 is one family of lines, and L_2 is the other family, then $L_1 \cap L_2 \neq \emptyset$.

Proposition 14.4.

- (1) The morphism $p: I \to \mathbf{P}$ is smooth over $\mathbf{P} \setminus W_3$, hence $p^{-1}(Q) = \mathbf{P}^1 \sqcup \mathbf{P}^1$ for all $Q \in \mathbf{P} \setminus W_3$;
- (2) There exists a prime divisor R on I such that $p^*(W_3) = 2R$.

Proof. For (1), we follow the strategy in Theorem 6.6(*ii*). Let $(Q, L) \in I$; after change of coordinates,

$$L = (x_0 = x_1 = 0),$$
 $Q = (x_0 f_0 + x_1 f_1 = 0).$

If $g_i = f_i|_L$, in order to get that p is étale at (Q, L), it is enough to show

$$\begin{aligned} H^0(L,\mathcal{O}(1)) \oplus H^0(L,\mathcal{O}(1)) & \longrightarrow H^0(L,\mathcal{O}(2)) \\ (u,v) & \longmapsto ug_0 + vg_1 \end{aligned}$$

is surjective. So suppose this were to fail. Then, g_0, g_1 are proportional, hence there exits some point $P \in L$ such that $g_0(P) = 0 = g_1(P)$, and so $\operatorname{mult}_P Q \ge 2$, a contradiction. We have therefore shown (1).

For (2), let $D = p^*(W_3)$, in which case we have the diagram



We want to show that the support of D is irreducible and has multiplicity 2. We first describe

$$D \cap I_L \hookrightarrow I_L \coloneqq q^{-1}(L).$$

As before, choose coordinates such that $L = (x_0 = x_1 = 0)$. Then, I_L is the projective space of symmetric matrices

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & 0 & 0 \\ a_{03} & a_{13} & 0 & 0 \end{pmatrix}$$

in $\mathbf{P} \times \{L\}$. This implies $D \cap I_L$ is defined by

$$\begin{vmatrix} a_{02} & a_{03} \\ a_{12} & a_{13} \end{vmatrix}^2$$

This implies that $D \cap I_L$ is an irreducible set of dimension 5. Thus, $R = D_{red}$ is irreducible, hence R is a prime divisor on I. There are then two possibilities:

• D = 2R;

• D = R and the fibers of $R \to G$ are everywhere nonreduced.

The second case is not possible: this is clear in characteristic zero (by generic smoothness), and in characteristic p, we simply note that our calculation works over the generic point, which would imply that the generic fiber of $R \to G$ is not reduced, which is a contradiction.

Remark 14.5. If $Q \in W_3 \setminus W_2$, then $p^{-1}(Q)$ is not reduced, whereas $p^{-1}(Q)_{\text{red}} \simeq \mathbf{P}^1$. This is because Q is a projective cone over a smooth conic C contained in a plane in \mathbf{P}^3 , and so

$$p^{-1}(Q)_{\mathrm{red}} \longrightarrow C \simeq \mathbf{P}^1$$

 $L \longmapsto L \cap \mathrm{plane}$

is an isomorphism; the semicontinuity theorem implies that we must have $p^{-1}(Q)$ nonreduced, for it is a specialization of the disjoint union of two \mathbf{P}^1 's we had in Proposition 14.4(1).

Exercise 14.6. Consider the diagram



Show that if $Q \in W_3 \setminus W_2$, then the fiber $(p|_R)^{-1}(Q)$ is reduced, hence $\simeq \mathbf{P}^1$.

14.1.3 Introduction to cyclic covers

Recall from §13.3 that our goal is to find a morphism of varieties such that all of the fibers are \mathbf{P}^{1} 's. We are almost in this setting, but Proposition 14.4(1) shows we have two \mathbf{P}^{1} 's in each fiber instead of one. The idea now is to factor this through a ramified double cover of \mathbf{P} , and so we start with some background on cyclic covers of this sort.

Let X be a variety, and let $\mathscr{L} \in \operatorname{Pic} X$ be a line bundle. Let m > 1 be an integer, and assume char $(k) \nmid m$. Let $s \in H^0(X, \mathscr{L}^m)$, which corresponds to a morphism

$$\varphi_s \colon \mathscr{L}^{-m} \longrightarrow \mathcal{O}_X.$$

The *m*-cyclic cover of X corresponding to s is constructed as follows. Let

$$\mathscr{S} = \bigoplus_{i \ge 0} \mathscr{L}^{-i} y^i \Big/ \mathscr{J},$$

where \mathscr{I} is the ideal sheaf locally generated by $uy^m - \varphi_s(u)$ where u is a local section of \mathscr{L}^{-m} . We then take

$$Y = \mathscr{S}pec(\mathscr{S}) \stackrel{\pi}{\longrightarrow} X.$$

Locally on $U \subseteq X$, we have a trivialization $\mathscr{L}|_U \simeq \mathcal{O}_U$ inducing an isomorphism $\mathscr{L}^m|_U \simeq \mathcal{O}_U$, mapping $s|_u \mapsto g \in \mathcal{O}(U)$. Then,

$$\pi^{-1}(U) \simeq \operatorname{Spec} \frac{\mathcal{O}(U)[y]}{(y^m - g)}.$$
(14.3)

We therefore see that π is finite and flat.

Example 14.7. We give the simplest example of a cyclic cover. If $s = t^m$ for $t \in H^0(\mathscr{L})$, then Y is the disjoint union of m copies of X, and π is the identity on each of them.

Proposition 14.8.

- (1) π is étale over $X \setminus D$, where D = Z(s) is the zero locus of s.
- (2) There is an effective Cartier divisor R on Y such that $\pi^*D = mR$, and π induces an isomorphism $R \xrightarrow{\sim} D$.
- (3) If X is smooth, D irreducible, smooth, then Y is smooth.
- (4) If X is smooth, and D is a prime divisor, then Y is irreducible and reduced.

Proof. For (1), the local description in (14.3) and the fact that $\operatorname{char}(k) \nmid m$ implies that π is étale over $X \setminus D$.

For (2), since $\mathcal{O}(\pi^{-1}(U))$ is free over $\mathcal{O}(U)$, we have that g is a nonzerodivisor in $\mathcal{O}(\pi^{-1}(U))$, hence y is a nonzero divisor on $\mathcal{O}(\pi^{-1}(U))$. Now let R be the effective Cartier divisor defined by (y). Then, $\pi^*D = mR$, and $R \simeq D$.

For (3), X is smooth and π is étale over $X \setminus D$, and so $Y \setminus R$ is smooth. Since $R \simeq D$ and D is smooth, we see that R is smooth. This implies that Y is smooth around R (since R is a smooth Cartier divisor in Y), and so Y is smooth.

For (4), suppose Y has two irreducible components Y_1, Y_2 . π is flat and finite, and so we see that both Y_1, Y_2 surject onto X. Since $\pi|_R \colon R \to D$ is a bijection, we see that $R \subseteq Y_1 \cap Y_2$. By (3), $\pi^{-1}(Y \setminus D_{\text{sing}})$ is smooth, and

$$\emptyset \neq R \smallsetminus \pi^{-1}(D_{\text{sing}}),$$

which implies there exists $x \in Y_1 \cap Y_2$ such that Y is smooth at x, which is a contradiction. Thus, Y is irreducible. For reducedness, (14.3) implies Y is Cohen–Macaulay, and generically reduced by (1), hence Y is reduced.

Remark 14.9. The construction of cyclic covers is functorial, in the following sense: if $f: X' \to X$ is a morphism of varieties such that $s' = f^*(s) \in H^0(X', f^*\mathscr{L}^m)$ is nonzero, there exists a Cartesian diagram



where π' is the *m*th cyclic cover corresponding to s'. This is trivial to see: if \mathscr{S}' is the algebra on X' corresponding to s', then $\mathscr{S}' \simeq f^*(\mathscr{S})$.

On a complete variety, we will show that the construction only depends on the divisor Z(s), not s. We will then talk about how the factorization needed for the Artin–Mumford example works.

15 March 9

As a reminder, there are no classes next week.

15.1 The Artin–Mumford example (continued)

We will today finish discussing the geometry of the Artin–Mumford example.

Recall 15.1. For a variety X, and a section $0 \neq s \in H^0(X, \mathscr{L}^m)$, we constructed a *m*-cyclic cover

$$Y \\ \pi \downarrow \\ X$$

where $\pi_*\mathcal{O}_Y = \bigoplus_{i=0}^{m-1} \mathscr{L}^{-i}$. This map is finite, flat; it is also étale away from Z(s).

Remark 15.2. If ζ_m is the group of *m*th roots of 1 in *k*, then ζ_m acts on *Y*, where $\alpha \in \zeta_m$ acts on \mathscr{L}^{-i} by α^i . Then, $X \simeq Y/\zeta_m$. To show this, note that π is finite, hence affine, and so it suffices to check that $\mathcal{O}_X \hookrightarrow \pi_* \mathcal{O}_Y$ embeds \mathcal{O}_X as the subsheaf of ζ_m -invariant sections of $\pi_* \mathcal{O}_Y$.

Remark 15.3. If X is complete, then (up to isomorphism) π only depends on the divisor D = Z(s). For suppose s' is such that Z(s) = Z(s'). Then, $s' = \lambda s$ for some $\lambda \in k^*$. Since k is algebraically closed, there exists $\alpha \in k^*$ such that $\alpha^m = \lambda$. The multiplication on the *i*th component by α^i (or α^{-i}) gives an isomorphism between the two sheaves of \mathcal{O}_X -algebras.

Remark 15.4. Suppose that X is complete, and D, Y are as in Remark 15.3. Let $p: X' \to X$ be a morphism from a complete variety X' such that $p^*(D) = mT$ for some effective divisor T on X'. Then, there exists a commutative diagram



Proof. We have a Cartesian diagram

$$\begin{array}{ccc} Y' & \xrightarrow{q} & Y \\ \pi' & \stackrel{\neg}{} & & \downarrow \pi \\ X' & \longrightarrow & X \end{array}$$

where π' is a cyclic cover corresponding to p^*D . Since $p^*D = mT$, we have that Y' is the disjoint union of m copies of X' (Example 14.7). If $\sigma: X' \to Y'$ is a section of π' , we may take $g = q \circ \sigma$.

We now return to the situation we are interested in. Consider

$$I \xrightarrow{q} G = G(2, 4)$$

$$\downarrow^{p}$$

$$\mathbf{P}^{9} \simeq \mathbf{P} = \text{complete linear system of quadrics in } \mathbf{P}^{3}$$

We saw that $p^*W_3 = 2R$ for some effective Cartier divisor R on I by Proposition 14.4(2). Thus, by the universal property in Remark 15.4, if $Y \xrightarrow{f} \mathbf{P}$ is the cyclic 2-cover corresponding to W_3 , then there exists $g: I \to Y$ such that we have a factorization



We then claim the following:

Claim 15.5. If $y \in Y$ is such that $f(y) \notin W_2$, then $g^{-1}(y) \simeq \mathbf{P}^1$.

- This is clear for $y \notin W_3$.
- If $y \in W_3 \setminus W_2$, then

$$g^{-1}(y) \underbrace{(p|_R)^{-1}(f(y))}_{p^{-1}(f(y))}$$

The two schemes have the same reduced structure, but $(p|_Y)^{-1}(f(y)) \simeq \mathbf{P}^1$, hence $g^{-1}(y) \simeq \mathbf{P}^1$. Now let $\Pi \subset \mathbf{P}$ be a general linear subspace of dimension 3.

Proposition 15.6. For Π general, the following will hold:

- (1) Π is basepoint-free;
- (2) $S := \Pi \cap W_3 \subseteq \Pi$ is an irreducible and reduced quartic surface, $\Gamma := \Pi \cap W_2$ is a reduced set of ten points, and $\Pi \cap W_1 = \emptyset$;
- (3) $S \setminus \Gamma$ is smooth, and for every $P \in \Gamma$, S has a node at Γ (i.e., the projectivized tangent cone of S at P is a smooth conic in \mathbf{P}^2);
- (4) $p^{-1}(\Pi)$ is a smooth, irreducible variety of dimension 4;
- (5) The projection $I \to G$ induces a birational map $p^{-1}(\Pi) \to G$.

Proof. Since four general quadrics in \mathbf{P}^3 do not intersect, we have (1).

For (2), since $\dim(W_1) = 3$, $\dim(W_2) = 6$, $\deg(W_2) = 10$, and $\dim(W_3) = 8$, $\deg(W_3) = 4$, we see that for Π general, we have the required properties, except possibly the irreducibility and reducedness of S.

Since $(W_3)_{\text{sing}} \subseteq W_2$, then Bertini implies that for general Π , the singular locus $S_{\text{sing}} \subseteq \Gamma$. This implies that S is irreducible and reduced. Now recall that for all $Q \in W_3 \setminus W_2$, $C_Q(W_3) \subseteq \mathbf{A}^3$ is defined by $x_1^2 + x_2^2 + x_3^2 = 0$. Thus, if $Q \in W_3 \setminus W_2$ and Π is a general codimension 6 linear subspace with $\Pi \ni Q$, then S has a node at Q.

Exercise 15.7. Use an argument as in the proof of Bertini to show that for all Π general, and for all $Q \in S \cap W_2$, then Q is a node of S. Recall that one shows Bertini by using an incidence correspondence for containing the tangent plane at a point, and that a dimension count shows that a general choice avoids this issue; see [Har77, Ch. II, Thm. 8.18].

This finishes (3).

For (4), the smoothness of $p^{-1}(\Pi)$ follows from Kleiman–Bertini (assuming that char(k) = 0), and connectedness holds for all Π [Laz04, Thm. 3.3.3] (Exercise: check this directly in our situation, i.e., check that given the morphism



that $g^{-1}(\Pi)$ is connected.)

Finally, for (5), we have the diagram



Fix a line $L \in G$, and consider $q^{-1}(L) \hookrightarrow \mathbf{P} = \mathbf{P}^9$. Then, $q^{-1}(L)$ is a linear subspace of dimension 6, and for Π general, we have that $q^{-1}(L) \cap \Pi$ is one reduced point. The fiber of $p^{-1}(\Pi) \to G$ over L is therefore one reduced point. Since G is smooth, we therefore have that the map is birational.

Form now on, we assume that Π satisfies these properties. We have



We know that I' is a smooth, connected variety of dimension 4, and that f' is the 2-cyclic cover of Π ramified along S. Since S is reduced and irreducible, and $S_{\text{sing}} = \Gamma$, then Y' is reduced and irreducible, and $Y'_{\text{sing}} \subseteq (g')^{-1}(\Gamma)$; this is a set of ten points.

Claim 15.8. If $P \in (g')^{-1}(\Gamma)$, then P is a node of Y' (that is, the projectivized tangent cone is a smooth quadric in \mathbf{P}^3).

Proof. Working locally, let U contain the image of P in Π such that

$$\mathcal{O}((f')^{-1}(U)) \simeq \frac{\mathcal{O}(U)[y]}{(y^2 - s)}$$

and (s = 0) = S. We therefore get the claim via the fact that S has a node at the image of P.

Exercise 15.9. Show that $S \subseteq \mathbf{P}^3$ is a quartic symmetroid, i.e., there are linear forms ℓ_{ij} with $1 \leq i, j \leq 4$, with $\ell_{ij} = \ell_{ji}$, such that $S = (\det(\ell_{ij}) = 0)$.

Now let V be the blowup of Y' at the ten nodes, with exceptional divisor E. Then, E is the disjoint union of E_i , $1 \le i \le 10$, which are isomorphic to the projectivized tangent cone at the corresponding point in S. By assumption, this is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. In particular, they are smooth, so E is smooth, hence V is smooth in a neighborhood of E. Thus, V is smooth.

Definition 15.10. V is the Artin–Mumford threefold.

Note 15.11. V is unirational: we had a birational morphism $I' = p^{-1}(G) \to G$, which implies I' is rational, and the fact that $I' \to Y'$ is surjective and Y' birational to V implies V is unirational.

Theorem 15.12 (Artin–Mumford). V is not stably rational.

We will use the following that we will review later.

Recall 15.13. $f: X \to Y$ is a \mathbf{P}^n -fibration if f is flat, and $f^{-1}(y) \simeq \mathbf{P}^n$ for all $y \in Y$ closed. We will show the following: If Y is smooth, then to every such f, we associate an element $\alpha(f) \in Br(Y)$ such that $\alpha(f) = 0$ if and only if X is a projective bundle over Y.

• If $k = \mathbf{C}$, we have a surjective morphism

$$Br(Y) \twoheadrightarrow H^3(Y, \mathbf{Z})_{tors},$$

which is an isomorphism if c^1 : $\operatorname{Pic}(Y) \to H^2(Y, \mathbb{Z})$ is surjective.

Lemma 15.14. Let $d \ge 2$ and |V| is a basepoint-free linear system of hypersurfaces of degree d on \mathbf{P}^n . We then have the following picture:



where \mathscr{H} is the universal hyperplane. Then, p has no rational section.

We will prove the Lemma next time.

Proof of Theorem 15.12. Recall that we have the cartesian square

$$\begin{array}{ccc} I' & \longrightarrow & Y' \\ \cup & & \cup \\ I'_0 & \longrightarrow & Y'_0 = (Y')_{\rm sm} \end{array}$$

All fibers of $I'_0 \to Y'_0$ are isomorphic to \mathbf{P}^1 , and I'_0, Y'_0 smooth imply $I'_0 \to Y'_0$ is a \mathbf{P}^1 -fibration (it is flat since the fibers all have the same dimension).

Claim 15.15. This has no rational section.

If we have such a section, we may assume that the rational section is defined on the locus where $g': Y' \to \Pi$ is étale. Such a section takes a point $y \in Y'$ to a point $(g'(y), L_y)$, where L_y is contained in the quadric corresponding to g'(y). Let $\varphi: Y' \to Y'$ be the isomorphism over Π that interchanges the two fibers. Now consider the map $y \mapsto L_y \cap L_{\varphi(y)}$. These two are lines on the same quadric, but in different families, hence $L_y \cap L_{\varphi(y)}$ is one point on the quadric corresponding to g'(y). Then, y and $\varphi(y)$ map to the same point, and so we get a rational map $\sigma: \Pi \dashrightarrow \mathbf{P}^3$ such that $\sigma(Q)$ is a point on the quadric corresponding to Q. This contradicts the Lemma. Therefore, we have a \mathbf{P}^1 -fibration $I'_0 \to Y'_0$ with no rational sections. In particular, it is not a \mathbf{P}^1 -bundle. This implies we have a nontrivial element of $\operatorname{Br}(Y'_0)$.

It remains to show:

- (1) $\operatorname{Pic}(Y'_0) \to H^2(Y'_0, \mathbb{Z})$ is surjective, which would imply that $H^3(Y'_0, \mathbb{Z})_{\operatorname{tors}} \neq 0$.
- (2) $H^3(V, \mathbf{Z})_{\text{tors}} \simeq H^3(Y'_0, \mathbf{Z})_{\text{tors}}.$
- (3) Lemma 15.14.

It will take a couple of lectures to prove these statements and the material about Brauer groups that we need.

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We might have a few more lectures after the semester ends. We won't be able to cover birational rigidity, but we will try to do examples using Voisin's techniques, such as those of Totaro and Hassett–Tschinkel.

16.1 The Artin–Mumford example (continued)

Lemma 15.14. Let $V \subset H^0(\mathbf{P}^n, \mathcal{O}(d))$, $d \geq 2$ be a basepoint-free linear system. If we have the universal hypersurface



in |V|, then the map f induced by projection has no rational sections.

In the proof, we will argue in terms of cohomology over the complex numbers. But the argument can be adapted in the language of Chow groups, which works over an arbitrary ground field.

Proof over **C**. After replacing |V| by a suitable linear subsystem, we may assume that $\dim |V| = n$. Now giving a rational section s of f is equivalent to giving a closed irreducible subvariety W of H such that $W \to |V|$ is birational (take $W = \overline{\operatorname{im}(s)}$). We will show this cannot happen using the cohomology of \mathscr{H} .

The cohomology of $|V| \times \mathbf{P}^n$ can be determined by the Künneth formula 9.5. Let $\alpha = c^1(\mathrm{pr}_1^* \mathcal{O}(1))$, and $\beta = c^1(\mathrm{pr}_2^* \mathcal{O}(1))$, which both lie in $H^2(|V| \times \mathbf{P}^n, \mathbf{Z})$. The Künneth formula and the description of the cohomology of \mathbf{P}^n (Proposition 13.1) implies $H^*(|V| \times \mathbf{P}^n, \mathbf{Z})$ has a basis $\alpha^i \beta^j$ where $0 \le i, j \le n$. Now \mathscr{H} is an effective Cartier divisor on $|V| \times \mathbf{P}^n$ with $\mathcal{O}(\mathscr{H}) = \mathcal{O}(1, d)$. Since |V| is basepoint-free,



is a projective subbundle of relative dimension n-1. The cohomology of \mathbf{P}^n plus the cohomology of a projective bundle (Corollary 13.2) imply $H^*(\mathcal{H}, \mathbf{Z})$ has a basis given by $\varphi^*(\alpha^i \beta^j)$ for $0 \le j \le n, 0 \le i \le n-1$. Given $\psi \colon W \hookrightarrow \mathcal{H}$ we can write

$$\psi_*(\eta_W) = \sum_{0 \le i \le n-1} c_i \varphi^*(\alpha^i \beta^{n-1-i}) \in H_{2n}(\mathscr{H}).$$

This implies that

$$\varphi_*\psi_*(\eta_W) = \sum_{i=0}^{n-1} c_i \varphi_*(\varphi^*(\alpha^i \beta^{n-1-i})) = \sum_{i=0}^{n-1} c_i \alpha^i \beta^{n-1-i}(\alpha + d\beta).$$

Note that

$$p_*(\alpha^i \beta^j) = \begin{cases} \alpha^i & \text{if } j = n \\ 0 & \text{if } j \neq n \end{cases}$$

and so $p_*\varphi_*\psi_*(\eta_W) = dc_0 \in \mathbb{Z} \simeq H^0(|V|)$. Since $W \to |V|$ is birational, $p_*\varphi_*\psi_*(\eta_W) = \eta_{|V|} = 1 \in H^0(|V|)$. This contradicts the hypothesis that $d \ge 2$.

It would be nice to have a more geometric argument, without using cohomology.

We now return to the setting of the Artin–Mumford example, where Lemma 15.14 found use.

Recall 16.1. We recall the construction of the Artin–Mumford example; see Proposition 15.6 for proofs of the assertions that follow. We considered a general three-dimensional linear system Π in \mathbf{P} , the projective space of quadrics in \mathbf{P}^3 . Using the stratification by rank on $\Pi \subset \mathbf{P}$, we had the following diagram:



where

- $S = \{ \text{singular quadrics in } \Pi \}$ is an irreducible, reduced quartic surface,
- $\Gamma = \{ \text{quadrics in } \Pi \text{ with rank } \leq 2 \}$ is a set of ten points, and
- Y' is the double cover of Π ramified along S.

 $S \smallsetminus \Gamma$ is smooth, and for every $P \in \Gamma$, S has a node at P. Its preimage $Y'_0 = (g')^{-1}(\Pi \smallsetminus \Gamma)$ is also smooth. If $Q \in (g')^{-1}(P)$ for $P \in \Gamma$, then Q is a node of Y', i.e., its tangent cone is the cone over a smooth quadric. Letting

$$E \subset V$$
$$\downarrow \qquad \qquad \downarrow^{\pi}$$
$$\Gamma \subset Y'$$

be the blowup of the points over Γ in Y', we have that E is the union of ten connected components, each of them isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. π gives an isomorphism $V_0 = \pi^{-1}(Y'_0) \xrightarrow{\sim} Y'_0$. V is a smooth projective threefold. Since $I' \to G(2, 4)$ was birational, I' is rational, hence V is unirational.

Goal 16.2. Show that V is not stably rational by showing that $H^3(V, \mathbf{Z})_{\text{tors}} \neq 0$. This implies that V is not stably rational.

We saw that the fibers of $I' \to Y'$ over Y'_0 are isomorphic to \mathbf{P}^1 . Lemma 15.14 implies that this map has no rational section. This implies that $\operatorname{Br}(Y'_0) \neq \{1\}$, and we will show that if $\operatorname{Pic}(Y'_0) \to H^2(Y'_0, \mathbf{Z})$ is surjective, then $\operatorname{Br}(Y'_0) \simeq H^3(Y'_0, \mathbf{Z})_{\operatorname{tors}}$.

Claim 16.3. This map $\operatorname{Pic}(Y'_0) \to H^2(Y'_0, \mathbf{Z})$ is surjective.

Proof. $Y'_0 \simeq V_0 = V \smallsetminus E$, and we have the commutative diagram

$$\begin{array}{ccc} \operatorname{Pic} V & \stackrel{\sim}{\longrightarrow} & H^2(V, \mathbf{Z}) \\ & & & & \downarrow^{\operatorname{res}} \\ \operatorname{Pic} V_0 & \longrightarrow & H^2(V_0, \mathbf{Z}) \end{array}$$

The top horizontal map is an isomorphism, since the exponential sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{O}_V^{\mathrm{an}} \longrightarrow \mathcal{O}_V^{\mathrm{an}*} \longrightarrow 0$$

plus GAGA and the fact that V is unirational, imply that

$$H^1(V, \mathcal{O}_V^{\mathrm{an}}) = 0 = H^2(V, \mathcal{O}_V^{\mathrm{an}}).$$

Hence it is enough to show that res is surjective. We use the long exact sequence on cohomology and the Thom isomorphism as in (12.5) to obtain the sequence

$$H^0(E, \mathbf{Z}) \longrightarrow H^2(V, \mathbf{Z}) \xrightarrow{\text{res}} H^2(V_0, \mathbf{Z}) \longrightarrow H^1(E, \mathbf{Z})$$

Since E is the disjoint union of ten $\mathbf{P}^1 \times \mathbf{P}^1$, by the Künneth formula and the fact that $H^1(\mathbf{P}^1) = 0$, we see that $H^1(E, \mathbf{Z}) = 0$. Hence the restriction map res is surjective.

We therefore conclude that $H^3(Y'_0, \mathbf{Z})_{\text{tors}} = H^3(V_0, \mathbf{Z})_{\text{tors}} \neq 0$. Another piece of the same long exact sequence gives

and we note that $\mathbf{Z}^{\oplus 20}$ has no torsion. Thus, $H^3(V, \mathbf{Z})_{\text{tors}} = H^3(V_0, \mathbf{Z})_{\text{tors}} \neq 0$, hence V is not stably rational.

16.2 Introduction to étale cohomology

Before moving forward, we need to prove some statements about Brauer groups. To do so, we will need some facts about étale cohomology. We will at least state everything that we need.

Since the Zariski topology evaluated on a constant sheaf is always trivial, we want something to give nontrivial cohomology groups for \mathbf{Z} , for example.

Idea 16.4. Define an improved version of the Zariski topology that recovers, for example, the singular cohomology in the case of a complex algebraic variety.

Idea 16.5. Replace open subsets of X by étale maps to X.

Note 16.6. If $f: X \to Y$ is an étale morphism of complex algebraic varieties, then $f^{an}: X^{an} \to Y^{an}$ is a local isomorphism.

Fix X to be a noetherian scheme. We want to look at the category

$$\mathsf{\acute{E}t}(X) = \{ f \colon Y \to X \mid f \text{ \'etale} \}.$$

Example 16.7.

- (1) If U is open in X, then $U \to X$ is $\acute{\mathsf{Et}}(X)$.
- (2) A finite Galois cover $f: Y \to X$ of a finite group G is in $\acute{\mathsf{Et}}(X)$. Recall that a finite Galois cover is a finite étale morphism $f: Y \to X$ such that G acts on Y over X (on the right) such that

$$\begin{split} & \coprod_{g \in G} Y_g \overset{\sim}{\longrightarrow} Y \times_X Y \\ & Y_g \longrightarrow Y \times_X Y \\ & y \longmapsto (y, y \cdot g) \end{split}$$

and each $Y_q = Y$.

Note 16.8. For every finite étale map $f: Y \to X$, there exists Z such that

$$Z \xrightarrow{f} Y \xrightarrow{} X$$
finite Galois cover

Remark 16.9. If



are in $\acute{\mathsf{Et}}(X)$, then every morphism $Y \to Z$ making the diagram commute is étale.

Definition 16.10 (Étale topology on X). We say that a family $(U_i \stackrel{f_i}{\to} U)$ of objects of $\acute{\mathsf{Et}}(U)$ is an étale cover if each $U_i \stackrel{f_i}{\to} U$ is étale, and $U = \bigcup_{i \in I} f_i(U_i)$. This notion of a cover form a Grothendieck topology, since it satisfies the following properties:

1. $(U \xrightarrow{\text{isom}} V)$ is a cover;

- 2. If $(U_i \to U)_i$ and $(V_{ij} \to U_i)_j$ are covers, then $(V_{ij} \to U)_{i,j}$ is a cover;
- 3. If $(U_i \to U)_i$ is a cover and $V \to U$ is in $\acute{\mathsf{Et}}(X)$, then $(U_i \times_U V \to V)_i$ is a cover.

16.2.1 Presheaves and sheaves in the étale topology

This notion of a Grothendieck topology allows one to define the notions of presheaves and sheaves on the étale topology.

Definition 16.11. A presheaf of abelian groups (resp. sets, groups) is a contravariant functor

 $\mathscr{F} \colon \acute{\mathsf{Et}}(X) \longrightarrow$ corresponding category.

This functor \mathscr{F} is a sheaf if for every cover $(U_i \to U)_i$, the diagram

$$\mathscr{F}(U) \longrightarrow \prod_{i} \mathscr{F}(U_{i}) \Longrightarrow \prod_{i,j} \mathscr{F}(U_{i} \times_{U} U_{j})$$

is an equalizer diagram.

Notation 16.12. If \mathscr{F} is a presheaf, $s \in \mathscr{F}(U)$, and $V \to U$ in $\acute{\mathsf{Et}}(X)$, then we write $s|_V$ for the image of s in $\mathscr{F}(V)$.

Examples 16.13.

(1) Let \mathcal{M} be a quasicoherent sheaf on X. Consider the functor

$$W(\mathscr{M}) \colon \acute{\mathsf{E}t}(X) \longrightarrow \mathsf{Ab}$$
$$(Y \xrightarrow{f} X) \longmapsto \Gamma(Y, f^*\mathscr{M})$$

Fact 16.14. By faithfully flat descent, $W(\mathcal{M})$ is a sheaf.

(2) (Constant sheaves) If A is an abelian group, then consider the functor

$$\begin{array}{l} A_X \colon {\rm \acute{E}t}(X) \longrightarrow {\rm Ab} \\ (Y \xrightarrow{f} X) \longmapsto A^{\#{\rm conn \ comp \ of \ }Y} \end{array}$$

where on the right-hand side, the exponent denotes a direct product. This defines a sheaf.

(3) Given any scheme $p: Y \to X$ over X, we get a presheaf

$$\mathbf{Y} \colon \acute{\mathsf{Et}}(X) \longrightarrow \mathsf{Set}$$
$$(U \xrightarrow{f} X) \longmapsto \operatorname{Hom}_X(U, Y)$$

By faithfully flat descent, this is indeed a sheaf. If Y is a group scheme over X, then this defines a sheaf of groups.

Examples 16.15.

(1) Let $\mathbf{G}_m = \operatorname{Spec}(\mathbf{Z}[t, t^{-1}])$, so that $\operatorname{Hom}(R, \mathbf{G}_m) \simeq \mathcal{O}(R)^*$ with a group structure under multiplication. Given X, let

$$\mathbf{G}_{m,X} \coloneqq \mathbf{G}_m \times_{\operatorname{Spec} \mathbf{Z}} X \longrightarrow X.$$

Then, for all $f: Y \to X$ in $\acute{\mathsf{Et}}(X)$, we have that $\mathbf{G}_m(Y) = \mathcal{O}(Y)^*$. (2) Let

$$\mu_n \coloneqq \operatorname{Spec}\left(\frac{\mathbf{Z}[t,t^{-1}]}{(t^n-1)}\right)$$

so that $\operatorname{Hom}(R,\mu_n) = \{f \in \mathcal{O}(R) \mid f^n = 1\}$. Similarly, we let $\mu_{n,X} \coloneqq \mu_n \times_{\operatorname{Spec} \mathbf{Z}} X$.

Note 16.16. Suppose k is a field, char(k) $\nmid n$, and k is separably closed. If A is a k-algebra that is also a domain, then $\{f \in A \mid f^n = 1\} = \{f \in k \mid f^n = 1\}$. This implies that $\mu_{n,X}$ is isomorphic to $(\mathbf{Z}/n\mathbf{Z})_X$, once we choose an isomorphism $\mathbf{Z}/n\mathbf{Z} \simeq \{f \in k \mid f^n = 1\}$.

(3) Let k be a field, and consider $X = \operatorname{Spec} k$. Then, $\operatorname{\acute{Et}}(X)$ consists of disjoint unions of $\operatorname{Spec} K$, where K/k is a finite separable extension. If \mathscr{F} is a sheaf on X, then we may put

$$M_K \coloneqq \mathscr{F}(\operatorname{Spec} K)$$
if K/k is finite separable. Functoriality implies G(K/k) acts on M_K . \mathscr{F} is a sheaf if and only if for every L/K, we have $M_K = (M_L)^{G(L/K)}$. In this case, we define $M := \lim_{k \to \infty} M_K$. This carries a continuous action of the Galois group $G(k^s/k) = G$. The assignment $\mathscr{F} \mapsto (M,$ continuous *G*-action) gives an equivalence of categories between sheaves on $(\operatorname{Spec} k)_{\text{ét}}$ and sets with continuous *G*-action.

Next time we will talk about exact sequences, cohomology, and then we can talk about Brauer groups. Once we finish this discussion about Brauer groups, we will then talk about Chow groups, emphasizing two things: the specialization map, and actions of correspondences.

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17.1 Introduction to étale cohomology (continued)

We continue with some background material on étale cohomology. But first, we will state what we used before from descent theory.

17.1.1 Some statements from descent theory

This follows [Mil80, Ch. 1.2].

Definition 17.1. A morphism $f: X \to Y$ of schemes is faithfully flat if it is flat, and is surjective on points.

Exercise 17.2. If $X = \operatorname{Spec} B \to \operatorname{Spec} A = Y$, then f is faithfully flat if and only if B is a faithfully flat A-module, i.e., every complex

$$M' \longrightarrow M \longrightarrow M'$$

of A-modules is exact if and only if

$$M' \otimes_A B \longrightarrow M \otimes_A B \longrightarrow M'' \otimes_A B$$

is exact.

Proposition 17.3. If $A \to B$ is faithfully flat, and M is a module over A, then

$$0 \longrightarrow M \xrightarrow{i} M \otimes_A B \longrightarrow M \otimes_A (B \otimes_A B)$$
$$m \longmapsto m \otimes 1$$
$$m \otimes b \longmapsto m \otimes (b \otimes 1 - 1 \otimes b)$$

is exact.

Idea. Apply $-\otimes_A B$; this reduces to the same question for $B \xrightarrow{j} B \otimes_A B$ given by $b \mapsto b \otimes 1$, which has a section $p: B \otimes_A B \to B$, i.e., $p \circ j = id$.

This can be used to recover sheaves on Y from sheaves on X, and similarly for morphisms:

Proposition 17.4. If $f: X \to Y$ is faithfully flat of finite type, then f is a strict epimorphism, i.e., for every scheme Z, the sequence

$$\operatorname{Hom}(Y,Z) \longrightarrow \operatorname{Hom}(X,Z) \Longrightarrow \operatorname{Hom}(X \times_Y X,Z)$$
$$\varphi \longmapsto \varphi \circ f$$

is exact.

It is easy to deduce this from Proposition 17.3 if X, Y, Z are affine; otherwise, you have to patch things together. Proposition 17.3 implies that if \mathscr{F} is a quasi-coherent sheaf on X, the assignment sending $U \to X$ étale to $\Gamma(U, f^*\mathscr{F})$ gives an étale sheaf as we stated in Fact 16.14. The other example we had is of representable sheaves: **Proposition 17.5.** If $W \to X$, then the assignment sending $Y \to X$ étale to $\operatorname{Hom}_X(Y, W)$ gives an étale sheaf.

Example 17.6. The scheme $\mathbf{G}_m \to X$ gives an étale sheaf on X.

Proposition 17.3 also implies the following:

Proposition 17.7. If $f: X \to Y$ is quasi-compact and faithfully flat, then to give a sheaf \mathscr{M} on Y is equivalent to giving a quasi-coherent sheaf \mathscr{N} on X, together with an isomorphism $\varphi: p_1^* \mathscr{N} \xrightarrow{\sim} p_2^* \mathscr{N}$ that satisfies a compatibility condition

$$p_{13}^*(\varphi) = p_{23}^*(\varphi) \circ p_{12}^*(\varphi)$$

where notation is as below:

$$X \times_Y X \times_Y X \xrightarrow[p_{13}]{p_{13}} X \times_Y X \xrightarrow[p_{23}]{p_{13}} X \xrightarrow[p_{23}]{p_{13}} Y$$

Note 17.8. Given \mathscr{M} , we have $\mathscr{N} = f^*(\mathscr{M})$, and φ comes from the fact that both $p_1^*\mathscr{N}$ and $p_2^*\mathscr{N}$ are isomorphic to pullback of \mathscr{M} by $f \circ p_1 = f \circ p_2$.

17.1.2 Presheaves and sheaves in the étale cohomology (continued)

Let $\mathsf{Psh}_{\mathrm{\acute{e}t}}$ denote the category of étale presenaves of abelian groups. It is an abelian category, and the inclusion

$$\mathsf{Sh}_{\mathrm{\acute{e}t}} \hookrightarrow \mathsf{Psh}_{\mathrm{\acute{e}t}}$$

of the category of étale sheaves of abelian groups has a left adjoint functor associating an étale sheaf to an étale presheaf. This adjoint functor can be used to show that $Sh_{\acute{e}t}$ has cokernels, and so it is an abelian category.

Note 17.9. A complex $\mathscr{F}' \xrightarrow{\alpha} \mathscr{F} \xrightarrow{\beta} \mathscr{F}''$ of étale sheaves is exact if for all $Y \to X$ in $\acute{\mathsf{Et}}(X)$ and $s \in \mathscr{F}(Y)$ such that $\beta(s) = 0$, we have that for all $y \in Y$, there exists $Z \to Y$ containing y in its image such that $s|_Z \in \operatorname{im}(\mathscr{F}'(Z) \to \mathscr{F}(Z)).$

Example 17.10 (Kummer sequence). Suppose n is invertible on X. We then claim we have an exact sequence

$$0 \longrightarrow \mu_n \longrightarrow \mathbf{G}_m \stackrel{\cdot n}{\longrightarrow} \mathbf{G}_m \longrightarrow 0$$

where $\mu_n(Y) = \{f \in \mathcal{O}(Y) \mid f^n = 1\}$, and $\mathbf{G}_m(Y) = \{f \in \mathcal{O}(Y) \mid f \text{ invertible}\}.$

For surjectivity of $\mathbf{G}_m \xrightarrow{\cdot n} \mathbf{G}_m$, let $U = \operatorname{Spec}(A) \to X$ be étale, and consider $a \in A^* = \mathbf{G}_m(U)$. Then, the ring homomorphism $A \to B = A[x]/(x^n - a)$ is étale and induces a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ which is étale and surjective, and in B, there exists $t \in B^*$ such that $a = t^n$.

We now discuss cohomology, which is possible because of the following facts:

- $\mathsf{Sh}_{\mathrm{\acute{e}t}}(X)$ has enough injectives.
- If $Y \to X$ is in $\acute{\mathsf{Et}}(X)$, then $\{\mathscr{F} \mapsto \mathscr{F}(Y)\}$ is a left exact functor whose right-derived functors are written $H^i_{\acute{et}}(Y,\mathscr{F})$.

Remark 17.11. If \mathscr{F} is an étale sheaf on X, then \mathscr{F} also gives a sheaf on X with respect to the Zariski topology, and we have canonical maps

$$H^i_{\operatorname{Zar}}(X,\mathscr{F}) \longrightarrow H^i_{\operatorname{\acute{e}t}}(Y,\mathscr{F}).$$

Examples 17.12.

- 1. If \mathscr{F} is a quasi-coherent sheaf on X, and $W(\mathscr{F})$ is the induced étale sheaf sending $f: Y \to X$ étale to $\Gamma(Y, f^*\mathscr{F})$, then the canonical map $H^i(X, \mathscr{F}) \to H^i_{\acute{e}t}(X, W(\mathscr{F}))$ is an isomorphism.
- 2. If $\mathscr{F} = \mathbf{G}_m$, then

$$\operatorname{Pic}(X) = H^{1}(X, \mathcal{O}_{X}^{*}) = H^{1}_{\operatorname{Zar}}(X, \mathbf{G}_{m}) \longrightarrow H^{1}_{\operatorname{\acute{e}t}}(X, \mathbf{G}_{m})$$

is an isomorphism (we will show this in a more general setting later, for arbitrary GL_n).

3. If X = Spec(k), then there is a correspondence

$$\left\{\begin{array}{c} \text{étale sheaves} \\ \text{on } X \end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c} k \text{-vector spaces } V \\ \text{with a continuous action} \\ \text{of } G = G(k^{\text{sep}}/k) \end{array}\right\}$$

such that the correspondence $\mathscr{F} \leftrightarrow M$ satisfies $\Gamma(X, \mathscr{F}) = M^G$. Thus, the corresponding cohomology groups are the Galois cohomology groups.

The interesting new invariants come from the cohomology of constant sheaves, which are not quasi-coherent.

Theorem 17.13. If X is a smooth complex algebraic variety, and A is a finite abelian group, then there is an isomorphism

$$H^i_{\text{ét}}(X, A) \simeq H^i(X^{\text{an}}, A),$$

where on the right-hand side, we have singular cohomology.

We will postpone the discussion of non-commutative sheaves.

17.2 Brauer group

Definition 17.14. If X is a noetherian scheme, then the *(cohomological)* Brauer group is

$$\operatorname{Br}(X) \coloneqq H^2_{\operatorname{\acute{e}t}}(X, \mathbf{G}_m)_{\operatorname{tors}}.$$

Fact 17.15. If X is a smooth variety over a field k, then $H^2_{\text{\acute{e}t}}(X, \mathbf{G}_m)$ is torsion (we will not need this). There is a proof in Milne's book.

We now have to do two things: connect Brauer groups with \mathbf{P}^n -fibrations, and then connect it with torsion in singular cohomology.

17.2.1 Connection with singular cohomology

We now work over the complex numbers \mathbf{C} , and let X be a smooth algebraic variety.

Notation 17.16. If A is an abelian group, we denote $A_n = \{a \in A \mid na = 0\}$.

Proposition 17.17. For each n > 0, we have an exact sequence

$$0 \longrightarrow \operatorname{Pic}(X)/n \cdot \operatorname{Pic}(X) \xrightarrow{\alpha} H^2(X, \mathbf{Z}/n\mathbf{Z}) \longrightarrow \operatorname{Br}(X)_n \longrightarrow 0$$

Proof. We use the Kummer exact sequence

$$0 \longrightarrow \mu_n \longrightarrow \mathbf{G}_m \stackrel{\cdot n}{\longrightarrow} \mathbf{G}_m \longrightarrow 0$$

and the fact that $\mu_n \simeq \mathbf{Z}/n\mathbf{Z}$ after choosing a primitive *n*th root of unity. Part of the long exact sequence has

$$\begin{array}{ccc} H^1(X, \mathbf{G}_m) & \stackrel{\cdot n}{\longrightarrow} & H^1(X, \mathbf{G}_m) & \longrightarrow & H^2_{\text{\acute{e}t}}(X, \mathbf{Z}/n\mathbf{Z}) & \longrightarrow & H^2_{\text{\acute{e}t}}(X, \mathbf{G}_m) & \stackrel{\cdot n}{\longrightarrow} & H^2_{\text{\acute{e}t}}(X, \mathbf{G}_m) \\ & & & & & & \\ \gamma & & & & & \\ \text{Pic } X & & & & H^2(X, \mathbf{Z}/n\mathbf{Z}) \end{array}$$

Remark 17.18. The map α is given by taking L to the image of $c^1(L)$ via $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}/n\mathbb{Z})$.

We used the following in the Artin–Mumford example:

Proposition 17.19. We have a surjective morphism

$$\operatorname{Br}(X) \longrightarrow H^3(X, \mathbf{Z})_{\operatorname{tors}}$$

which is an isomorphism if c^1 : Pic $X \to H^2(X, \mathbb{Z})$ is surjective.

Proof. For n > 0, we have the short exact sequence in the top row below:

It is clear that $\operatorname{im}(\alpha) \subseteq \operatorname{im}(\beta)$ and this is an equality if c^1 is surjective. We therefore get a morphism $\operatorname{coker}(\alpha) \to \operatorname{coker}(\beta)$, which is an isomorphism if c^1 is surjective. Now taking the long exact sequence for cohomology for

$$0 \longrightarrow \mathbf{Z} \xrightarrow{\cdot n} \mathbf{Z} \longrightarrow \mathbf{Z}/n\mathbf{Z} \longrightarrow 0$$

we obtain

$$H^{2}(X, \mathbf{Z}) \xrightarrow{\cdot n} H^{2}(X, \mathbf{Z}) \xrightarrow{\beta} H^{2}(X, \mathbf{Z}/n\mathbf{Z}) \longrightarrow H^{3}(X, \mathbf{Z}) \xrightarrow{\cdot n} H^{3}(X, \mathbf{Z})$$

hence $\operatorname{coker}(\beta) \simeq H^3(X, \mathbb{Z})_n$. We therefore get a surjective morphism $\operatorname{Br}(X)_n \twoheadrightarrow H^3(X, \mathbb{Z})_n$, which is an isomorphism if c^1 is surjective. Taking the union on both sides, we obtain a surjective morphism $\operatorname{Br}(X) \twoheadrightarrow H^3(X, \mathbb{Z})_{\operatorname{tors}}$, which is an isomorphism if c^1 is surjective.

17.2.2 Brauer groups and Pⁿ-fibrations

The last ingredient needed for the Artin–Mumford example is the following:

Goal 17.20. Attach to a \mathbf{P}^n -fibration $f: X \to Y$ an element in the Brauer group Br(Y), which is trivial if and only if f is isomorphic to a projective bundle.

The idea is to prove \mathbf{P}^n -fibrations are locally trivial in the étale topology, and use this to associate a certain cohomology class to the \mathbf{P}^n -fibration.

Recall 17.21. Suppose $k = \overline{k}$. Then, $f: X \to Y$ is a \mathbf{P}^n -fibration if f is flat, proper, and $f^{-1}(y) \simeq \mathbf{P}^n$ for all $y \in Y$.

Proposition 17.22. Given a \mathbf{P}^n -fibration $f: X \to Y$ and a point $y \in Y$, there is an étale morphism $\varphi: V \to Y$ such that $y \in im(\varphi)$, and an isomorphism $X \times_Y V \simeq V \times_k \mathbf{P}^n$ over V, i.e., an isomorphism fitting into the commutative diagram



The key point is that \mathbf{P}^n is *rigid*, i.e., it has no nontrivial infinitesimal deformations. By a general result [Ser06, Cor. 1.2.15], this follows from the fact that \mathbf{P}^n is smooth, and $H^1(\mathbf{P}^n, T_{\mathbf{P}^n}) = 0$. We will argue directly.

Step 1. Let (R, \mathfrak{m}, k) be a local ring, let X be proper flat over R, and consider the cartesian diagram



If $X_0 \simeq \mathbf{P}_k^n$, then, $X \simeq \mathbf{P}_R^n$.

Proof. For $s \ge 0$, we let



We want to show by induction on $s \ge 0$ that there exists $\mathscr{L}_s \in \operatorname{Pic}(X_s)$ such that

- (1) $\mathscr{L}_{s}|_{X_{s-1}} \simeq \mathscr{L}_{s-1}$ for $s \ge 1$; (2) $H^{0}(X_{s}, \mathscr{L}_{s}) \to H^{0}(X_{s-1}, \mathscr{L}_{s-1})$ is surjective for $s \ge 1$;
- (3) $(X_s, \mathscr{L}_s) \simeq (\mathbf{P}^n_{R/\mathfrak{m}^{s+1}}, \mathcal{O}(1)).$

We have \mathscr{L}_0 by hypothesis, since we only need (3) to be satisfied.

We therefore want to prove the inductive step; to do so, we will compare the Picard groups on X_{s-1} and X_s to construct \mathscr{L}_s from \mathscr{L}_{s-1} . Using flatness of X over R, we have a short exact sequence

 $0 \longrightarrow \mathfrak{m}^{s}/\mathfrak{m}^{s+1} \otimes_{R} \mathcal{O}_{X_{0}} \longrightarrow \mathcal{O}_{X_{s}} \longrightarrow \mathcal{O}_{X_{s-1}} \longrightarrow 0$

of sheaves on \mathcal{O}_X , where we note $\mathfrak{m}^s/\mathfrak{m}^{s+1}\otimes_R \mathcal{O}_{X_0} = I_{X_{s-1}}/I_{X_s}$. Since $X_0 \simeq \mathbf{P}^n$, we have

$$H^1(X_0, \mathcal{O}_{X_0}) = H^2(X_0, \mathcal{O}_{X_0}) = 0,$$

and so this short exact sequence induces an exact sequence

$$0 \longrightarrow \mathfrak{m}^{s}/\mathfrak{m}^{s+1} \otimes_{R} \mathcal{O}_{X_{0}} \longrightarrow \mathcal{O}_{X_{s}}^{*} \longrightarrow \mathcal{O}_{X_{s-1}}^{*} \longrightarrow 0.$$

The associated long exact sequence implies $\operatorname{Pic}(X_s) \xrightarrow{\sim} \operatorname{Pic}(X_{s-1})$ via restriction, i.e., there exists \mathscr{L}_s such that (1) is satisfied. Tensoring with \mathscr{L}_s yields the exact sequence

$$H^0(X_s, \mathscr{L}_s) \longrightarrow H^0(X_{s-1}, \mathscr{L}_{s-1}) \longrightarrow \mathfrak{m}^s/\mathfrak{m}^{s+1} \otimes H^1(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)) = 0$$

which implies (2) is satisfied. In particular, we get sections

$$s_0,\ldots,s_n\in H^0(X_s,\mathscr{L}_s)$$

lifting a basis of $H^0(X_{s-1}, \mathscr{L}_{s-1})$ over R/\mathfrak{m}^s , and so we have a diagram

The surjectivity of the left and right vertical maps implies s_0, \ldots, s_n generate \mathscr{L}_s , and so we have a morphism

$$\varphi_s \colon X_s \longrightarrow \mathbf{P}^n_{R/\mathfrak{m}^{s+1}}$$

We know that this becomes an isomorphism after restricting to Spec R/\mathfrak{m}^s . Using the fact that X_s is flat over R/\mathfrak{m}^{s+1} , this implies that φ_s is an isomorphism [Stacks, Tag 0CF4].

We will continue next time to show the assertion still holds after going to the completion by using the theorem on formal functions, and then we have to descend to an étale map by using Artin approximation. \Box

18 March 28

Brauer groups (continued) 18.1

We hope to finish the Brauer group stuff today, then move on to Chow groups.

18.1.1Brauer groups and P^n -fibrations (continued)

Recall 18.1. Let $X \to \operatorname{Spec}(R)$ be a flat morphism, where R is complete and local, and suppose we have a commutative diagram



We want to show that any such X is isomorphic to \mathbf{P}_{B}^{n} .

So far, we have shown that for all $n \ge 0$, there exists $\mathscr{L}_r \in \operatorname{Pic}(X_r)$ such that

- $\mathscr{L}_r|_{X_{r-1}} \simeq \mathscr{L}_{r-1},$ $H^0(X_r, \mathscr{L}_r) \twoheadrightarrow H^0(X_{r-1}, \mathscr{L}_{r-1}),$ $s_0^{(r)}, \ldots, s_n^{(r)} \in H^0(X_r, \mathscr{L}_r)$ lift corresponding sections of \mathscr{L}_{r-1} , and define an isomorphism

$$(X_r, \mathscr{L}_r) \simeq (\mathbf{P}^n_{R/\mathfrak{m}^{r+1}}, \mathcal{O}(1))$$

Then, the Grothendieck existence theorem [EGAIII₁, Thm. 5.1.4] implies there exists $\mathscr{L} \in \operatorname{Pic}(X)$ such that $\mathscr{L}|_{X_r} \simeq \mathscr{L}_r$ for all r. Since R is complete, the theorem on formal functions implies

$$H^0(X,\mathscr{L}) \simeq \underline{\lim} H^0(X_r,\mathscr{L}_r),$$

and so in particular, we get sections $s_0, \ldots, s_n \in \Gamma(X, \mathscr{L})$ that lift $s_0^{(r)}, \ldots, s_n^{(r)}$. They must generate \mathscr{L} . We therefore get a morphism $\varphi \colon X \to \mathbf{P}_R^n$. Since the induced morphisms $X_r \to \mathbf{P}_{R/\mathfrak{m}^{r+1}}^n$ are isomorphisms, we see that φ is an isomorphism [EGAIII₁, Thm. 5.4.1].

We want to use this to show that if we have a \mathbf{P}^n -fibration, then it is étale-locally trivial.

Theorem 18.2. If $f: X \to Y$ is a \mathbf{P}^n -fibration, then for all $y \in Y$, there is an étale morphism $V \to Y$ such that y is in the image of V, such that $X \times_Y V \to V$ is isomorphic over V to $\mathbf{P}^n \times V$.

We have everything that we need, modulo the Artin approximation theorem.

Proof. Consider



The previous result implies $X_y \simeq \mathbf{P}^n \times \operatorname{Spec} \widehat{\mathcal{O}}_{Y,y}$. Now consider the henselization $\mathcal{O}_{Y,y}^h$.



where we recall that the henselization is

$$\mathcal{O}_{Y,y}^h = \lim_{(\overrightarrow{B},\mathfrak{q})} B_\mathfrak{q},$$

where the limit is over pairs (B, \mathfrak{q}) such that Spec $B \to Y$ is étale, $\mathfrak{q} \in \text{Spec } B$ maps to y. Artin Approximation [Art69] implies that $X_y^h \simeq \mathbf{P}^n \times \operatorname{Spec} \mathcal{O}_{Y,y}$. This implies that in fact, there exists (B, \mathfrak{q}) as in the direct limit, such that $X \times_Y \operatorname{Spec} B \to \operatorname{Spec} B$ is isomorphic to $\mathbf{P}^n \times \operatorname{Spec} B$ over $\operatorname{Spec} B$.

The goal now is to show that given a \mathbf{P}^n -fibration, we get a class in the Brauer group.

Introduction to first cohomology of non-abelian sheaves 18.2

We want to define

$$\dot{H}^1_{\text{\acute{e}t}}(X,G).$$

As in the usual case of coherent cohomology, we can define a version of Čech cohomology for the étale topology. But we want this to work for a sheaf G of not necessarily abelian groups.

Definition 18.3. Let $\mathcal{U} = (U_i \to X)_i$ be an étale cover of X. Let $U_{ij} = U_i \times_X U_j$, and $U_{ijk} = U_i \times_X U_j \times_X U_k$. Then, a 1-cocycle of G with respect to \mathcal{U} is a collection of elements $u_{ij} \in G(U_{ij})_{i,j}$ such that $u_{ij}|_{U_{ijk}} = u_{ik}|_{U_{ijk}} \cdot u_{kj}|_{U_{ijk}}$. Two cocycles $(u_{ij}), (v_{ij})$ are cohomologous if there exists a family $(p_i \in G(U_i))_i$ such that $u_{ij} = p_i|_{U_{ij}}v_{ij}p_j^{-1}|_{U_{ij}}$ for all i, j. The set of 1-cocycles up to this equivalence relation is $\check{H}^1_{\acute{e}t}(\mathcal{U}/X, G)$. This is a pointed set, with distinguished element represented by $(1_{G(U_{ij})})_{i,j}$. We then define

$$\check{H}^{1}_{\mathrm{\acute{e}t}}(X,G) \coloneqq \varinjlim_{\mathcal{U}} \check{H}^{1}_{\mathrm{\acute{e}t}}(\mathcal{U}/X,G).$$

If G is a sheaf of abelian groups, this is an abelian group, and there is a canonical isomorphism

$$\check{H}^1_{\mathrm{\acute{e}t}}(X,G) \simeq H^1_{\mathrm{\acute{e}t}}(X,G).$$

Note such an isomorphism holds for arbitrary topological spaces, but just for H^1 . Given suitable assumptions, such an isomorphism holds for higher cohomology as well.

- We list some properties of $\check{H}^1_{\mathrm{\acute{e}t}}(X, -)$:
- This is functorial: If $G \to G'$ is a morphism of sheaves of groups, then there is an induced morphism $\check{H}^1_{\acute{e}t}(X,G) \to \check{H}^1_{\acute{e}t}(X,G')$.
- Suppose $1 \to G' \to G \to G'' \to 1$ is an exact sequence, i.e., for every $U \to X$ étale, we have $G'(U) = \ker(G(U) \to G''(U))$, and for every $s \in G''(U)$ and every $x \in U$, there exists an étale neighborhood $V \to U$ of x such that $s|_V \in \operatorname{im}(G(V) \to G''(V))$. Then, there is an exact sequence

$$0 \longrightarrow H^0_{\text{\acute{e}t}}(X,G') \longrightarrow H^0_{\text{\acute{e}t}}(X,G) \longrightarrow H^0_{\text{\acute{e}t}}(X,G'')$$

$$\longrightarrow \check{H}^1_{\text{\acute{e}t}}(X,G') \longrightarrow \check{H}^1_{\text{\acute{e}t}}(X,G) \longrightarrow \check{H}^1_{\text{\acute{e}t}}(X,G'')$$

of pointed sets, i.e., the inverse image of the distinguished element is the image of the previous map.

• If $G'(U) \subseteq Z(G(U))$ for all U, then there is another connecting map $\check{H}^1_{\acute{e}t}(X, G'') \to H^2_{\acute{e}t}(X, G')$ such that the sequence stays exact.

Remark 18.4. For every G, we have a canonical map

$$\check{H}^1_{\operatorname{Zar}}(X,G) \longrightarrow \check{H}^1_{\operatorname{\acute{e}t}}(X,G)$$

Proposition 18.5. If $G = \mathbf{GL}_n$, then the map

$$\check{H}^1_{\operatorname{Zar}}(X,G) \longrightarrow \check{H}^1_{\operatorname{\acute{e}t}}(X,G)$$

is a bijection of pointed sets.

Note that the left-hand side is the pointed set of isomorphism classes of rank n vector bundles. Last time, we stated the case n = 1, in which case we had an isomorphim of groups, and the left-hand side was Pic(X).

This is a bit of an involved statement. This is very special for \mathbf{GL}_n , and fails for PGL, for example.

Sketch of Proof. There is a Leray spectral sequence connecting the two sites. Modulo that, the key ingredient is the following: If $X = \operatorname{Spec} A$, where A is a local Noetherian ring, then $\check{H}^1_{\operatorname{\acute{e}t}}(X, \operatorname{GL}_n) = \{*\}$. So it is enough just to show that if $f: \operatorname{Spec} B \to \operatorname{Spec} A$ is an étale cover, and $g \in \operatorname{GL}_n(\operatorname{Spec}(B \otimes_A B))$ is a 1-cocycle, then g is cohomologous to (1).

$$\operatorname{Spec}(B \otimes_A B \otimes_A B) \xrightarrow[p_{23}]{p_{12}} \operatorname{Spec}(B \otimes_A B) \xrightarrow[p_2]{p_1} \operatorname{Spec} B \xrightarrow{f} \operatorname{Spec} A$$

We can interpret g as an isomorphism $p_1^* \mathscr{N} \xrightarrow{\varphi} p_2^* \mathscr{N}$, where $\mathscr{N} = \mathcal{O}_{\operatorname{Spec} B}^{\oplus n}$. The condition for being a cocycle is a compatibility condition between the pullbacks of φ via p_{12}, p_{13}, p_{23} . Faithfully flat descent implies that there is a quasi-coherent sheaf \mathscr{M} on Spec A such that $f^* \mathscr{M} \simeq \mathcal{O}_{\operatorname{Spec} B}^{\oplus n}$ and φ is the canonical isomorphism. Now f is faithfully flat, so there exists \mathscr{M} locally free on Spec A, where A is local, such that $\mathscr{M} \simeq \mathcal{O}_{\operatorname{Spec} A}^{\oplus n}$. It is then easy to deduce that g is cohomologous to (1). The moral of the story is that Zariski-triviality is the same as étale-triviality for GL_n .

Proposition 18.6. We have a natural bijection

$$\begin{cases} Isomorphism \ classes \\ of \ \mathbf{P}^n \text{-} fibrations \end{cases} \simeq \check{H}^1_{\acute{e}t}(X, \mathrm{PGL}_{n+1})$$

where the distinguished element on the left-hand side is $X \times \mathbf{P}^n$.

Proof. Let $\mathcal{U} = (U_i \to X)_i$ be an étale cover, and let $g = (g_{ij})$ be a 1-cocycle corresponding to this cover. Let $X_i = U_i \times \mathbf{P}^n$. For all i, j, we have

$$X_i \times_{U_i} U_{ij} = U_{ij} \times \mathbf{P}^n \xrightarrow{\sim} U_{ij} \times \mathbf{P}^n = X_i \times_{U_j} U_{ij}$$
$$(x,t) \longmapsto (x,g_{ij}(x)t)$$

By faithfully flat descent, this data gives a unique morphism $f: Y \to X$ such that $Y \times_X U_i \simeq X_i$. It is then clear that this is a \mathbf{P}^n -fibration. The fact that we have a bijection between the two sets follows from the theorem saying that every \mathbf{P}^n -fibration is étale locally trivial and the fact that $\operatorname{Isom}_U(U \times \mathbf{P}^n, U \times \mathbf{P}^n) \simeq \operatorname{Hom}(U, \operatorname{PGL}_{n+1}(k))$.

You can do the same thing with vector bundles. However, there are more automorphisms of \mathbf{A}^n , so you need to restrict the class of objects on the left-hand side.

Remark 18.7. Similarly, if we consider "étale vector bundles," i.e., morphisms $f: Y \to X$ such that there exists an étale cover $(U_i \to X)_i$ such that $Y \times_X U_i \to U_i$ is isomorphic to $U_i \times \mathbf{A}^n$ over \mathbf{A}^n , such that the glueing isomorphisms $U_{ij} \times \mathbf{A}^n \to U_{ij} \times \mathbf{A}^n$ are given by morphisms $U_{ij} \to \mathrm{GL}_n$. Then,

$$\begin{cases} \text{Isomorphism classes of rank } n \\ \text{étale vector bundles} \end{cases} \xrightarrow{\sim} \check{H}^{1}_{\text{ét}}(X, \text{GL}_{n+1}) \\ & \swarrow \\ & & & \swarrow \\ \\ & & & \downarrow \\ \end{cases} \\ \begin{cases} \text{Isomorphism classes of rank } n \\ \text{vector bundles} \end{cases} \xrightarrow{\sim} \check{H}^{1}_{\text{Zar}}(X, \text{GL}_{n+1}) \end{cases}$$

Remark 18.8. An example of \mathbf{P}^n -fibrations are projective bundles $\mathbf{P}(\mathscr{E}) \to X$.

Theorem 18.9. If $char(k) \nmid n + 1$, then we have a map

$$\begin{cases} Isomorphism \ classes \\ of \ \mathbf{P}^n \text{-} fibrations \end{cases} \xrightarrow{\alpha} Br(X)$$

such that $\alpha^{-1}(0) = \{ projective \ bundles \} / isom.$

Remark 18.10. $\operatorname{PGL}_{n+1}(k)$ is the distinguished affine open subset of $\mathbf{P}(M_{n+1}(k)^*)$ defined by the det $(a_{ij}) \neq 0$. In particular, we have an exact sequence of sheaves

$$0 \longrightarrow \mathbf{G}_m \longrightarrow \mathrm{GL}_{n+1} \longrightarrow \mathrm{PGL}_{n+1} \longrightarrow 0,$$

which is exact even in the Zariski topology, since the map

$$\operatorname{GL}_{n+1}(k) \longrightarrow \operatorname{PGL}_{n+1}(k)$$

is locally trivial in the Zariski topology, with fiber k^* .

Proof of Theorem 18.9. Since $\mathbf{G}_m(U) \subseteq Z(\mathrm{GL}_n(U))$ for all U, there is an exact sequence of pointed sets

$$\check{H}^{1}_{\mathrm{\acute{e}t}}(X, \mathrm{GL}_{n+1}) \longrightarrow \check{H}^{1}_{\mathrm{\acute{e}t}}(X, \mathrm{PGL}_{n+1}) \xrightarrow{\alpha} H^{2}_{\mathrm{\acute{e}t}}(X, \mathbf{G}_{m}).$$

Claim 18.11. $\operatorname{im}(\alpha) \subseteq H^2_{\operatorname{\acute{e}t}}(X, \mathbf{G}_m)_{\operatorname{tors}} = \operatorname{Br}(X).$

We have a commutative diagram:

Note that the top sequence is exact in the étale topology, since $SL_{n+1}(k) \to PGL_{n+1}(k)$ is étale and surjective (this is where the characteristic assumption is used). This implies that $im(\alpha) \subseteq im(H^2_{\acute{e}t}(X, \mu_{n+1}) \to H^2_{\acute{e}t}(X, \mathbf{G}_m)) \subseteq Br(X)$.

Finally, given $f: Y \to X$ a \mathbf{P}^n -fibration, we have $\alpha(f) = 0$ if and only if f lies in the image of $\check{H}^1_{\text{\acute{e}t}}(X, \operatorname{GL}_{n+1}) \simeq \{\operatorname{rank}(n+1)\text{-vector bundles on } X\}/\text{isom.}$ But this is equivalent to saying that f is isomorphic to some $\mathbf{P}(\mathscr{E})$.

This completes the story for the Artin–Mumford example.

We are now moving to the last part of the course. We will review Chow groups; we will recall the definitions and results that we need. It is not hard to go through all the details on your own. We will also talk about the specialization map (to go from the generic to special fiber), and also about the actions of correspondences, for example to prove that two birational varieties have isomorphic Chow groups of 0-cycles.

19 March 30

19.1 Introduction to Chow groups

We work over a field k, and all schemes will be separated and of finite type over k.

Definition 19.1. If X/k, then the group of *p*-cycles is

 $Z_p(X) \coloneqq$ free abelian group on p-dimensional irreducible closed subsets of X.

If $V \subset X$ is such a closed subset, then we denote its class in $Z_p(X)$ as [V]. We also define

$$Z_*(X) \coloneqq \bigoplus_{p \ge 0} Z_p(X).$$

Example 19.2. Let $Y \subset X$ have pure dimension, and let $\dim(Y) = p$. Then,

$$[Y] = \sum_{\substack{V \subset Y \\ \text{irred. comp.}}} \ell(\mathcal{O}_{Y,V}) \cdot [V] \in Z_p(X).$$

Example 19.3. Let $W \subset X$ be an integral subscheme, where dim X = p + 1. Let $\varphi \in k(W) \setminus \{0\}$. Then, we define

$$\operatorname{div}(\varphi) = \sum_{\substack{V \subset W \\ \operatorname{codim } 1}} \operatorname{ord}_V(\varphi)[V] \in Z_p(X),$$

where denoting $R = \mathcal{O}_{W,Y}$, which is a local domain of dimension 1, and writing $\varphi = \frac{a}{b}$ for $a, b \in R$, we define

$$\operatorname{ord}_V(\varphi) = \ell(R/(a)) - \ell(R/(b)).$$

If R is a DVR, this matches the usual definition. One can show the following:

- This definition of ord_V is independent of a, b;
- There are only finitely many V such that $\operatorname{ord}_V(\varphi) \neq 0$;
- $\operatorname{ord}_V(\varphi\psi) = \operatorname{ord}_V(\varphi) + \operatorname{ord}_V(\psi).$

Definition 19.4. We say that $\alpha, \beta \in Z_p(X)$ are rationally equivalent $(\alpha \sim_{\text{rat}} \beta)$ if

$$\alpha - \beta = \sum_{i=1}^{n} n_i \operatorname{div}(\varphi_i)$$

where $n_i \in \mathbb{Z}$, and $0 \neq \varphi_i$ is a rational function on some (p+1)-dimensional integral subscheme. Then, the *pth Chow group of X* is

$$CH_p(X) \coloneqq Z_p(X) / \sim_{\mathrm{rat}} .$$

We also denote

$$CH_*(X) \coloneqq \bigoplus_{p \ge 0} CH_p(X).$$

Example 19.5.

- 1. If dim X = n, then $CH_n(X) = Z_n(X)$ is free with basis given by the irreducible components of X of dimension n.
- 2. If X is normal of dimension n, then $Z_{n-1}(X)$ is the group of Weil divisors on X, and $CH_{n-1}(X) = Cl(X)$.

One should think about Chow groups as similar to homology, not cohomology. There is a cycle class map from $CH_* \to H_*$, but note that Chow groups are actually very subtle (including CH_0).

19.1.1 Operations on Chow groups

Proper push forward Let $f: X \to Y$. We then define a map

$$\begin{split} f_* \colon Z_p(X) &\longrightarrow Z_p(Y) \\ [V] &\longmapsto \begin{cases} \deg(V/f(V)) \cdot [f(V)] & \text{if } \dim f(V) = \dim V \\ 0 & \text{otherwise} \end{cases} \end{split}$$

where V is an irreducible closed set in X. We then extend by linearity. We also have a map

$$f_*: Z_*(X) \longrightarrow Z_*(Y).$$

We want to descend this definition to Chow groups. One can show that if $W \subset X$ has dimension p + 1, and $\varphi \in k(W) \setminus \{0\}$, then we have

$$f_*(\operatorname{div}(\varphi)) = \begin{cases} \operatorname{div}(\operatorname{Norm}(\varphi)) & \text{if } \dim f(W) = \dim W \\ 0 & \text{otherwise} \end{cases}$$

Here, the norm is taken with respect to the finite field extension $k(f(W)) \hookrightarrow k(W)$. We therefore get homomorphisms

$$f_* \colon CH_p(X) \longrightarrow CH_p(Y),$$

$$f_* \colon CH_*(X) \longrightarrow CH_*(Y).$$

Example 19.6. If X is a complete variety, then there is a proper map $X \to \operatorname{Spec} k$, and so there is a degree morphism

$$\deg \colon CH_0(X) \longrightarrow CH_0(\operatorname{Spec} k) \simeq \mathbf{Z}$$
$$\sum_{i=1}^r n_i[p_i] \longmapsto \sum_{i=1}^r n_i \deg(k(p_i)/k)$$

for points $p_i \in X$. If $k = \overline{k}$, then the right-hand side is just the number of points (counted with multiplicity) in X.

Remark 19.7. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ for f, g proper, then $(g \circ f)_* = g_* \circ f_*$ by definition (use that degrees of field extensions are multiplicative).

Flat pullback Suppose $f: X \to Y$ is flat of relative dimension d. This means all fibers have a fixed pure dimension d. For example, if Y is irreducible and X has pure dimension, then a flat morphism f has relative dimension dim X - dim Y. By definition, if $Z \subseteq Y$ has pure dimension p, then $f^{-1}(Z)$ has pure dimension p + d. We can then define:

$$f^* \colon Z_p(Y) \longrightarrow Z_{p+d}(X)$$
$$[V] \longmapsto [f^{-1}(V)]$$

where $f^{-1}(V)$ is the scheme-theoretic inverse image; we extend by linearity. One can show that if Z is any scheme of pure dimension p, then

$$f^*([Z]) = [f^{-1}(Z)].$$
(19.1)

Also, one can show that $f^*(\alpha) \sim_{\text{rat}} 0$ if $\alpha \sim_{\text{rat}} 0$, so we get $f^* \colon CH_p(Y) \to CH_{p+d}(X)$. Note 19.8. If $g \colon Y \to W$ is flat of relative dimension e, then

$$f^* \circ g^* = (g \circ f)^* \colon CH_p(W) \longrightarrow CH_{p+d+e}(X).$$

There is a more complicated operation given by Gysin maps, but we will do this next time.

We give some examples. The easiest proper map is a closed immersion, in which case you just view the subvariety as living in the larger space, and the easiest flat morphism is an open immersion, in which case pullback is just restriction.

Example 19.9. Suppose Y is a closed subscheme of X, and let $U = X \setminus Y$. Let $i: Y \hookrightarrow X$, $j: U \hookrightarrow X$ be the inclusion maps. Then, we have an exact sequence

$$Z_p(Y) \xrightarrow{i_*} Z_p(X) \xrightarrow{j^*} Z_p(U) \longrightarrow 0,$$

which you can check using the definitions. This immediately implies that on Chow groups, we have an exact sequence

$$CH_p(Y) \xrightarrow{i_*} CH_p(X) \xrightarrow{j^*} CH_p(U) \longrightarrow 0.$$

The only thing you need to check is exactness at $CH_p(X)$ (Exercise). This is an analogue of the long exact sequence on cohomology. However, it is only a piece: ideally, you would want to continue it to the left.

Remark 19.10. Proper push-forward and flat pull-back are compatible in the following sense: Suppose have a diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g}_{\text{flat}} & Y \\ \text{rel dim n} & Y \end{array}$$

then $g^* f_* = f'_* g'^*$.

Example 19.11. For every $n \ge 1$ and every X, the projection $\pi: X \times \mathbf{A}^n \to X$ (which is flat of relative dimension n) induces a map $\pi^*: CH_{p-n}(X) \to CH_p(X \times \mathbf{A}^n)$ that is surjective. In particular, if $X = \{\text{pt}\}$, then $CH_i(\mathbf{A}^n) = 0$ for all $i \ne n$, and $CH_n(\mathbf{A}^n) \cong \mathbf{Z}$.

Proof. By induction on n, it suffices to consider the case n = 1. The exact sequence in Example 19.9 implies that if $Y \hookrightarrow X$ is a closed subscheme, and $U = X \setminus Y$, then the assertion for Y and U imply the assertion for X. By induction on dimension, we may assume that X is affine.

Let $V \hookrightarrow X \times \mathbf{A}^1$ be irreducible and reduced of dimension p. Let



Then, $V \hookrightarrow W \times \mathbf{A}^1$. If dim(W) = p - 1, then $V = W \times \mathbf{A}^1$, so $[V] = f^*([W])$. Otherwise, if dim(W) = p, let $R = \mathcal{O}(W)$. Then, V is defined by an ideal $\mathfrak{q} \subseteq R[t]$. The map $V \to W$ in the diagram



is dominant, so $\mathfrak{q} \cap R = (0)$, and for some $f \in R[t]$, we have $\mathfrak{q} \cdot K[t] = f \cdot K[t]$, which is a nonzero prime ideal. Then,

$$\operatorname{div}(f) = [V] + \sum_{i=1}^{r} n_i \cdot [W_i \times \mathbf{A}^1]$$

for some irreducible closed $W_i \subseteq W$.

Note the proof is exactly what you do for class groups. We now calculate the Chow groups of \mathbf{P}^n .

Example 19.12 (Chow groups of \mathbf{P}^n). We have a closed immersion $\mathbf{P}^{n-1} \hookrightarrow \mathbf{P}^n$ with complement \mathbf{A}^n . Then, Examples 19.9 and 19.11 imply that $CH_i(\mathbf{P}^{n-1}) \twoheadrightarrow CH_i(\mathbf{P}^n)$ for $i \neq n$, and so by induction on n, if L_i is an *i*-dimensional linear subspace of \mathbf{P}^n , $0 \leq i \leq n$, then $CH_i(\mathbf{P}^n) = \mathbf{Z} \cdot [L_i]$.

Claim 19.13. $[L_i]$ is a free generator of $CH_i(\mathbf{P}^n)$.

If i = n, this is trivial, and the case when i = n - 1 is well-known. If $i \le n - 2$, then suppose

$$d[L_i] = \sum_{j=1}^r n_j \cdot \operatorname{div}(\varphi_j),$$

where φ_j are rational functions on some Y_j of dimension i + 1. Let $Y = \bigcup_{j=1}^r Y_j$, and let $W \subseteq \mathbf{P}^n$ be a linear subspace dimension n - i - 2 such that $W \cap Y = \emptyset$. The projection with center W then induces a morphism $f: Y \to \mathbf{P}^{i+1}$, which is proper since Y is complete. Then,

$$f_*(d[L_i]) = d \cdot [\text{hyperplane}] \sim_{\text{rat}} 0,$$

which implies d = 0 by using the i = n - 1 case.

Other constructions We also discuss other constructions, which are more elementary.

Remark 19.14. If K/k is a field extension, then there is a morphism

$$Z_p(X) \longrightarrow Z_p(X_K)$$
$$[V] \longmapsto [V_K]$$

where $X_K := X \times_{\text{Spec } k}$ Spec K, which induces a morphism $CH_p(X) \to CH_p(X_K)$. This is very related to flat pullback, but is not exactly the same, since K/k can be an infinite extension. This morphism is not necessarily an isomorphism: for example, even if $CH_0(X) = \mathbb{Z}$, we could have a much larger $CH_0(X_K)$ after taking an infinite extension K/k.

Remark 19.15. If X, Y are schemes over k, then given $V^p \subset X$ and $W^q \subset Y$ irreducible, one can take $[V \times W] \in \mathbb{Z}_{p+q}(X \times Y)$. This gives a morphism of cycle groups

$$Z_p(X) \otimes Z_q(Y) \longrightarrow Z_{p+1}(X \times Y),$$

which descends to a map of Chow groups

$$CH_p(X) \otimes CH_q(Y) \longrightarrow CH_{p+1}(X \times Y).$$

This is compatible with proper pushforward and flat pullback.

19.1.2 Chern classes

We will leave Gysin maps for next time, and now talk about Chern classes. We want a way to act on Chow groups using some analogue of the cap product.

If \mathscr{E} is a rank *n* locally-free sheaf on *X*, then the Chern classes $c_i(\mathscr{E})$ are defined as maps

$$c_i(\mathscr{E}) \cap -: CH_p(X) \longrightarrow CH_{p-1}(X).$$

Case of line bundles The Chern class $c_1(L)$ for a line bundle L is given by a homomorphism

$$CH_p(X) \longrightarrow CH_{p-1}(X)$$

uniquely characterized by:

(1) If $L = \mathcal{O}(D)$ for an effective Cartier divisor D on X, and $V \subset X$ is irreducible, reduced, and of dimension p such that $V \not\subseteq \text{Supp}(D)$, then

$$c_1(L) \cap [V] = [D \cap V]$$

- (2) $c_1(L \otimes L') = c_1(L) + c_1(L').$
- (3) (Projection formula) If $f: Y \to X$ is proper, then

$$f_*(c_1(f^*L) \cap \alpha) = c_1(L) \cap f_*(\alpha)$$

for every $\alpha \in CH_*(Y)$.

(4) If $f: Y \to X$ is flat of relative dimension d, then

$$c_1(f^*L) \cap f^*(\alpha) = f^*(c_1(L) \cap \alpha)$$

for every $\alpha \in CH_*(X)$.

(5) If $L, L' \in \operatorname{Pic} X$, then

$$c_1(L) \cap \left(c_1(L') \cap \alpha\right) = c_1(L') \cap \left(c_1(L) \cap \alpha\right)$$

for all $\alpha \in CH_*(X)$.

Using first Chern class actions of line bundles, one can compute the Chow groups of a projective bundle.

Proposition 19.16. If \mathscr{E} is a rank r locally free sheaf on X, and $\pi \colon \mathbf{P}(\mathscr{E}) \to X$ is the associated projective bundle, then

$$\bigoplus_{i=0}^{r-1} CH_{p-r}(X) \longrightarrow CH_p(\mathbf{P}(\mathscr{E}))$$

$$(\alpha_0, \dots, \alpha_{r-1}) \longmapsto \sum_{i=0}^{r-1} c_1(\mathcal{O}(1))^{r-1-i} \cap \pi^* \alpha$$

is an isomorphism.

This is similar to what we had for singular cohomology. You can prove something similar for blowups.

Chern classes of arbitrary vector bundles We start with the following basic fact:

Fact 19.17. The description of $CH(\mathbf{P}(\mathscr{E}))$ implies that there exist unique linear maps $c_i(\mathscr{E}): CH_p(X) \to CH_{p-i}(X)$ such that

$$\sum_{i=0}^{r} (-1)^{i} c_1 \left(\mathcal{O}(1) \right)^{r-i} \cap \left(c_i(\mathscr{E}) \cap \alpha \right) = 0$$

for all $\alpha \in CH_*(X)$, where $c_0(\mathscr{E}) = \mathrm{id}$ and $c_i(\mathscr{E}) = 0$ for all i > r.

The basic property of Chern classes is the following multiplicative formula:

The Whitney formula 19.18. Given a short exact sequence

$$0 \longrightarrow \mathscr{E}' \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}'' \longrightarrow 0$$

of locally free sheaves, we have

$$c_p(\mathscr{E}) = \sum_{i+j=p} c_i(\mathscr{E}')c_j(\mathscr{E}'')$$

where the product on the right-hand side is given by composition of maps on Chow groups. In particular, if \mathscr{E} has a filtration with successive quotients $L_1, \ldots, L_r \in \operatorname{Pic} X$, then $c_p(\mathscr{E})$ is the pth symmetric function of $c_1(L_1), \ldots, c_1(L_r)$.

Remark 19.19. By taking iterates of projective bundles, there exists a smooth morphism $f: Y \to X$ of some relative dimension, such that $f^* \mathscr{E}$ has a filtration with rank 1 quotients, and $f^*: CH_*(X) \to CH_*(Y)$ is injective. This is what you use to prove properties of Chern classes by reducing to the case of when all given vector bundles have such a filtration.

Next time, we will discuss the most interesting operation defining intersection products. We will then discuss correspondences.

20 April 4

20.1 Introduction to Chow groups (continued)

Last time we discussed the definition of Chow groups, proper pushforwards, and flat pullbacks.

Exercise 20.1. If $f: X \to Y$ is finite and flat of degree d (so that $\operatorname{rk}_{\mathcal{O}_Y} f_*\mathcal{O}_X = d$), then

$$f_*(f^*(\alpha)) = d\alpha$$

for all $\alpha \in CH_*(Y)$.

Exercise 20.2. Let $f: X \to Y$ be a morphism, and suppose Y is an integral scheme. Consider

$$\begin{array}{ccc} X_{\eta} & & & X \\ & & \downarrow & & \downarrow^{f} \\ \operatorname{Spec} k(\eta) & & & Y \ni \eta \text{ gen pt} \end{array}$$

Then,

$$A_p(X_\eta) \simeq \varinjlim_U A_{p+\dim(Y)} (f^{-1}(U)),$$

where the direct limit runs over all open sets $U \subset Y$ with transition maps

$$A_{p+\dim(Y)}(f^{-1}(V)) \longrightarrow A_{p+\dim(Y)}(f^{-1}(U))$$

for all open inclusions $U \subseteq V$ given by restriction. We will use this pretty often.

The interesting operation on Chow groups is the intersection product.

20.1.1 Refined Gysin maps

Goal 20.3. Define "intersection operations."

Suppose $i: X \hookrightarrow Y$ is a regular embedding of codimension d, which is a closed immersion that is locally defined by a regular sequence of length d. Given any cartesian diagram

$$\begin{array}{ccc} X' & \longleftrightarrow & Y' \\ \downarrow & \neg & \downarrow \\ X & \stackrel{i}{\longleftrightarrow} & Y \end{array}$$

we will define a Gysin map

$$i^{!} \colon CH_{*}(Y') \longrightarrow CH_{*}(X')$$

 $CH_{p}(Y') \longrightarrow CH_{p-d}(X')$

We will not define this since it is a bit involved, but will mention a special case. Note that you cannot define this on the level of cycles: you need to work up to rational equivalence.

Example 20.4 (Special Case). If $V \subseteq Y'$ is irreducible and reduced such that $V \times_Y X \to V$ is also a regular embedding of codimension d, then $i!([V]) = [V \times_Y X]$.

Properties 20.5.

(1) Compatibility with proper pushforwards and flat pullbacks. Consider the commutative diagram with cartesian squares



where i is regular of codimension d. If f is proper, then

$$i^! f_* = g_* i^! \colon A_p(Y'') \longrightarrow A_{p-d}(X').$$

If f is flat of relative dimension m, then

$$i^! f^* = g^* i^! \colon A_p(Y') \longrightarrow A_{p-d+m}(X').$$

If j is another regular embedding of codimension d, then

$$i^! = j^! \colon A_p(Y'') \longrightarrow A_p(X'').$$

(2) Functoriality. If

is a commutative diagram with cartesian squares, such that i is a regular embedding of codimension d, and j is a regular embedding of codimension e, so that $j \circ i$ is a regular embedding of codimension d + e, then

$$(j \circ i)^! = i^! \circ j^! \colon A_*(Z') \longrightarrow A_*(X')$$

(3) Commutation of Gysin maps. Consider the following diagram with cartesian squares:

If i is a regular embedding of codimension d, and j is a regular embedding of codimensine, then

$$i^! \circ j^! = j^! \circ i^! \colon A_*(V) \longrightarrow A_*(U').$$

(4) Self-intersection formula. Suppose $i: X \hookrightarrow Y$ is a regular embedding of codimension d. Let $\alpha \in A_p(X)$.

$$i^{!}i_{*}(\alpha) = c_{d}(N_{X/Y}) \cap \alpha \in A_{p-d}(X),$$

where $N_{X/Y}$ is the normal bundle of X in Y is of rank d. More generally, given a diagram

$$\begin{array}{ccc} X' & \stackrel{i'}{\longleftarrow} & Y' \\ f & & \downarrow \\ X & \stackrel{i}{\longleftarrow} & Y \end{array}$$

we have

$$i'i'_*(\alpha) = c_d(f^*N_{X/Y}) \cap \alpha.$$

Exercise 20.6. Use this to show the following: Let \mathscr{E} be a locally free, rank d sheaf on Y, and let $s \in \Gamma(Y, \mathscr{E})$ such that $X = Z(s) \stackrel{i}{\hookrightarrow} Y$ is a regular embedding of codimension d (i.e., s is locally defined by a regular sequence). If $\alpha \in A_p(Y)$, then

$$i_*i^!(\alpha) = c_d(\mathscr{E}) \cap \alpha.$$

We can use this to define a ring structure on Chow groups. Suppose that X is a smooth variety over a field k of dimension n. Since X is smooth, we see that $\Delta_X \colon X \hookrightarrow X \times X$ is a regular embedding of codimension n (it is a closed embedding of smooth varieties). It is also a commutative algebra result that given a closed embedding, checking it is a regular embedding amounts to checking the conormal sheaf is locally free.

We now define an operation on $CH_*(X)$:

$$\alpha \in CH_p(X), \ \beta \in CH_q(X) \rightsquigarrow \alpha \cdot \beta = \Delta^!(\alpha \times \beta) \in CH_{p+q-n}(X).$$

It is usually more convenient to use indexing by codimension, that is, $CH^p(X) \coloneqq CH_{n-p}(X)$. Even though this looks like Poincaré duality, there is not really an interpretation as duals: it really is just a definition. The intersection product respects the grading on $CH^*(X)$, and properties of Gysin maps imply $CH^*(X)$ is a commutative graded ring with identity element $[X] \in CH^0(X) = CH_n(X)$.

Remark 20.7. This intersection product is easy to compute when intersecting two cycles that intersect properly: given

$$\alpha = \sum_i m_i [V_i], \ \beta = \sum n_j [W_j]$$

such that $\dim(V_i \cap W_j) = \dim V_i + \dim W_j - n \ (\geq \text{always holds})$, then

$$\alpha \cdot \beta = \sum m_i n_j [V_i \cap W_j].$$

Remark 20.8. This is enough to compute intersection products on quasi-projective varieties, by using the following

Lemma 20.9 (Moving lemma). Given cycles u, b, we can find $v' \sim_{\rm rat} v$ such that u and v' intersect properly.

This is not obviously well-defined. The point of Fulton's book [Ful98] is that using deformation to the normal cone, you *can* prove the intersection product is well-defined.

20.1.2 Pullback by lci morphisms

Suppose we deal only with varieties that can be embedded in smooth varieties, e.g. quasi-projective varieties and smooth varieties. A morphism $f: X \to Y$ is *lci of relative dimension d* if it factors as

$$X \stackrel{i}{\hookrightarrow} W \stackrel{p}{\longrightarrow} Y$$

with p smooth of relative dimension n, and i is a regular embedding of codimension n - d. One can extend the definition of Gysin maps to this setting: denoting



then

$$f^! = i^! \circ (p')^* \colon CH_p(Y') \longrightarrow CH_{p+d}(X').$$

Fact 20.10.

- This is independent of factorization.
- If f is both lci of relative dimension d and flat of relative dimension d, then $f! = (f')^*$.

We will mostly be interested in morphisms $f: X^n \to Y^m$ between smooth varieties. Then, f is lci of relative dimension n - m: Consider

$$\begin{array}{ccc} X & \stackrel{i}{\longleftrightarrow} X \times Y \xrightarrow{p_2} Y \\ x & \longmapsto (x, f(x)) \end{array}$$

Then, p_2 is smooth since X is smooth, and $X, X \times Y$ smooth (hence regular) implies *i* is a regular embedding. In this case, we will write f^* for $f^!$.

$$f^* \colon A_p(Y) \longrightarrow A_{p+n-m}(X)$$
$$\parallel \qquad \qquad \parallel$$
$$A^{n-p}(Y) \longrightarrow A^{m-p}(X)$$

Fact 20.11. $f^*: A^*(Y) \to A^*(X)$ is a homomorphism of graded rings. Moreover,

- If f is flat of relative dimension q, then f^* coincides with the previous definition of pullback.
- Projection formula: If f is proper, then

$$f_*(f^*(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta).$$

Remark 20.12. If X is a smooth variety, and \mathscr{E} is a rank r vector bundle on X, then

$$c_i(\mathscr{E}) \cap \alpha = \left(c_i(\mathscr{E}) \cap [X]\right) \cdot \alpha$$

Thus, one usually identifies this map $c_i(\mathscr{E}): CH_*(X) \to CH_*(X)$ with $c_i(\mathscr{E}) \cap [X] \in CH^i(X)$.

20.1.3 Connection with singular cohomology

We now describe the connection with singular cohomology. Suppose X is a complete variety over \mathbf{C} . Then, there is a cycle map

$$Z_p(X) \longrightarrow H_{2p}(X, \mathbf{Z})$$
$$\sum n_i[V_i] \longmapsto \sum n_i \eta_{V_i}$$

where η_{V_i} is the class of V_i in $H_{2p}(X, \mathbf{Z})$. This induces a group homomorphism

$$CH_p(X) \xrightarrow{\mathrm{cl}} H_{2p}(X, \mathbf{Z}).$$

- (1) This is compatible with pushforward f_* .
- (2) This is compatible with
 - pullback (and Gysin maps)
 - multiplication
 - for X smooth.

There is always a map to Borel–Moore homology when X is not complete.

This is as much as we wanted to say about general intersection theory.

20.2Correspondences

We will be a bit more detailed about what we will use: correspondences. We assume all varieties are smooth, and most of the time complete.

Definition 20.13. A correspondence from X to Y, written $\alpha: X \vdash Y$, is an element in $CH_*(X \times Y)$.

Using

$$\begin{array}{c} X \times Y \xrightarrow{\sim} Y \times X \\ (x, y) \longmapsto (y, x) \end{array}$$

a correspondence $\alpha \colon X \vdash Y$ yields a correspondence $\alpha' \colon Y \vdash X$. It is clear that $(\alpha')' = \alpha$.

Composition of correspondences Given $\alpha: X \vdash Y$ and $\beta: Y \vdash Z$ such that Y is complete, there is a correspondence $\beta \circ \alpha \colon X \vdash Z$, where denoting



we have

$$\beta \circ \alpha \coloneqq (p_{13})_* \big(p_{12}^*(\alpha) \cdot p_{23}^*(\beta) \big).$$

Fact 20.14. Composition of correspondences is associative, that is, given $\gamma: Z \vdash W$, where Y, Z are complete,

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha)$$

Example 20.15. Consider $f: X \to Y$. Let $\alpha = [\Gamma_f]$, where $\Gamma_f \subseteq X \times Y$ is the graph of f. (1) If $\beta: Y \vdash Z$, and Y is complete, then

$$\beta \circ \alpha = (f \times 1_Z)^*(\beta).$$

(2) If $\gamma: W \vdash X$, and X is complete, then

$$\alpha \circ \gamma = (1_W \times f)_*(\gamma).$$

Proof of (2). Consider the diagram

$$W \times X \times Y$$

$$\downarrow^{p_{12}} \qquad \qquad \downarrow^{p_{23}}$$

$$W \times X \qquad \qquad \downarrow^{p_{13}} \qquad \qquad X \times Y$$

$$W \times Y$$

We have the diagram

$$\begin{array}{cccc} W \times X & \stackrel{j}{\longrightarrow} W \times X \times Y \\ q & & \downarrow \\ X & \stackrel{i}{\longleftarrow} X \times Y \end{array}$$

Then, we have

$$p_{12}^{*}(\gamma) \cdot p_{23}^{*}([\Gamma_{f}]) = p_{12}^{*}(\gamma) \cdot \underbrace{p_{23}^{*}(i_{*}([X]))}_{=j_{*}([W \times X])} \stackrel{\text{proj. formula}}{=} j_{*}(j^{*}p_{12}^{*}(\gamma)) = j_{*}(\gamma),$$

since $p_{12} \circ j = \text{id}$. We then have

$$\alpha \circ \gamma = (p_{13})_* (j_*(\gamma)) = (1_W \times f)_* (\gamma).$$

Next time, we will discuss how correspondences act on Chow groups and cohomology groups. We will then show that birational varieties have the same CH_0 . We will then discuss the specialization map.

21 April 6

We are considering having extra classes on Tuesday, April 25 from 10–12, and Friday, April 28 from 10–12.

Today we would like to finish the material on correspondences. Our goal will eventually be to study quartic threefolds, and Kollár's method of finding interesting examples via reduction modulo p.

21.1 Correspondences (continued)

In what follows, every variety is a complete smooth variety over k. Then, a correspondence $\alpha: X \vdash Y$ is an element $\alpha \in CH^*(X \times Y)$. One can define compositions of correspondences, and this composition is associative.

Example 21.1. $CH^*(X \times X)$ has a new ring structure where addition is the usual one, and multiplication is given by composition of correspondences. The identity element is $[\Delta_X]$.

21.1.1 Actions of correspondences on Chow groups

Let $\alpha \in CH^*(X \times Y)$. Then, we have

$$\alpha_* \colon CH^*(X) \longrightarrow CH^*(Y),$$

where denoting



we have $\alpha_*(u) = p_{2*}(p_1^*(u) \cdot \alpha)$ and $\alpha^* = (\alpha')_* \colon CH^*(Y) \to CH^*(X)$.

Note 21.2. This is a special case of composition of correspondences: Given α as above, and $u \in CH^*(X)$, consider u as the correspondence Spec $k \vdash X$. Then, $\alpha_*(u) = \alpha \circ u$, since



Similarly, if $u \in CH^*(Y)$ is considered as a correspondence Spec $k \vdash Y$, then $\alpha^*(u) = u \circ \alpha$.

A consequence of associativity of composition of correspondences is that if $\alpha \colon X \vdash Y$ and $\beta \colon Y \vdash Z$, then $\beta_* \circ \alpha_* = (\beta \circ \alpha)_*$, and $\alpha^* \circ \beta^* = (\beta \circ \alpha)^*$. Thus, we can think of α as a "generalized map" between X and Y. Remark 21.3. Pushforward and pullbacks by morphisms are a special case: Let $f \colon X \to Y$, and let $\alpha = [\Gamma_f]$ be the class associated to the graph of f. We claim that $f_* = \alpha_*$. Let $u \in CH^*(X)$, so that

$$\alpha_*(u) = \alpha \circ u,$$

where u is considered as a correspondence Spec $k \vdash X$. Last time, we showed that

$$\alpha \circ u = (1_{\operatorname{Spec} k} \times f)_*(u) = f_*(u).$$

Similarly, $\alpha^* = f^*$.

21.1.2 CH_0 is a birational invariant

Theorem 21.4. If X and Y are stably birational smooth, complete varieties over k, then $CH_0(X) \simeq CH_0(Y)$.

If there is a birational *morphism* connecting the two varieties, then the morphism can be used to pushforward and pullback. This was the method employed in the first proof of Theorem 21.4 by Colliot-Thélène, which works in the presence of resolution of singularities. The present proof is from [Ful98, Ex. 16.1.11].

Lemma 21.5. Suppose X is a variety over k, and let $z = \sum n_i[p_i]$ be a zero-cycle, where all the p_i are regular points on X. Then, given any nonempty open subset $U \subseteq X$, there is a zero cycle z' supported on U such that $z \sim_{\text{rat}} z'$.

Proof. We can assume that z is one regular point [p]. If $n = \dim X = 0$, then there is nothing to show. If $n \ge 1$, then there exists a dimension 1 subvariety $C \subseteq X$ such that $p \in C$ is a regular point, and such that $C \cap U \ne \emptyset$. After replacing (X, U) by $(C, C \cap U)$, we may assume X is of dimension 1. We may also assume that X is affine; otherwise, X is complete, and it is enough to show the assertion for $(X \setminus \{q\}, U \setminus \{q\})$ for some $q \in U$. Let $X = \operatorname{Spec} R$, and let \mathfrak{m} be the maximal ideal corresponding to p. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be the ideals corresponding to the points in $X \setminus U$. Then, $R_{\mathfrak{m}}$ is a DVR. We may find an element

$$\varphi \in \mathfrak{m} \smallsetminus \left((\mathfrak{m}^2 R_{\mathfrak{m}} \cap R) \cup \mathfrak{m}_1 \cup \cdots \cup \mathfrak{m}_r \right)$$

by prime avoidance. Then, $\operatorname{div}(\varphi) = [p] + \sum m_i[q_i]$, where the latter term is supported on U.

Note that by replacing U with $U \cap X_{reg}$, we can ensure that z' is a sum of regular points.

Proof of Theorem 21.4. Denote $\pi: X \times \mathbf{P}^n \to X$. Step 1. We have an isomorphism

$$CH_0(X) \xrightarrow{\sim} CH_0(X \times \mathbf{P}^n)$$
$$u \longmapsto c_1(\mathcal{O}(1))^n \cap \pi^*(u)$$
$$\pi_*(w) \longleftrightarrow w$$

Step 2. Suppose X and Y are birational, and choose open subsets $U \subseteq X$ and $V \subseteq Y$ such that there is an isomorphism $\varphi \colon U \xrightarrow{\sim} V$. Let $W = \overline{\Gamma}_{\varphi} \subseteq X \times Y$ be the closure of the graph of φ in $X \times Y$, and let $\alpha = [W]$. Then, denoting $n = \dim X = \dim Y$, we have maps

$$\alpha_* \colon CH_0(X) \longrightarrow CH_0(Y)$$

$$\alpha'_* \colon CH_0(Y) \longrightarrow CH_0(X)$$

We claim these are mutually inverse.

We have $\alpha'_* \circ \alpha_* = (\alpha' \circ \alpha)_*$. Restricting to $U \times X$, we have

$$\alpha' \circ \alpha|_{U \times X} = \alpha' \circ [\Gamma_{U \to Y}] = [\Gamma_{U \hookrightarrow X}].$$

Thus, $(\alpha' \circ \alpha - [\Delta_X])|_{U \times X} = 0$. We have an inclusion

$$U \times X \stackrel{j}{\hookrightarrow} X \times X \stackrel{i}{\longleftrightarrow} (X \smallsetminus U) \times X$$

and so

$$\alpha' \circ \alpha = [\Delta_X] + i_*(\beta)$$

for some $\beta \in CH_n((X \setminus U) \times X)$. This implies $\alpha'_* \circ \alpha_* = \operatorname{id} + (i_*(\beta))_*$. On the other hand, we claim $(i_*(\beta))_* = 0$ on $CH_0(X)$. Let $z \in CH_0(X)$, and write $z \sim_{\operatorname{rat}} z'$ for a zero-cycle z' which is supported on U. Then,

$$(i_*(\beta))_*(z) = (i_*(\beta))_*(z') = p_{2*}(p_1^*(z') \cdot i_*(\beta)).$$

But $p_1^*(z') \cdot i_*(\beta) = 0$ since $p_1^*(z')$ and $i_*(\beta)$ are supported on disjoint closed sets in $X \times X$. This implies $\alpha'_* \circ \alpha_* = \text{id}$. \Box

Remark 21.6. The isomorphism of the theorem commutes with the degree maps, which we recall are defined for complete X as the map

deg:
$$CH_0(X) \longrightarrow \mathbf{Z}$$

 $\sum n_i[p_i] \longmapsto \sum n_i \deg(k(p_i))$

In Step 1, we have a commutative diagram

$$CH_0(X \times \mathbf{P}^n) \xrightarrow{(\operatorname{pr})_*} CH_0(X)$$

$$\downarrow$$

$$CH_0(\operatorname{Spec} k)$$

In Step 2, if X and Y are birational, we used the isomorphism $[W]_*$ where $W \subseteq X \times Y$ is the closure of the graph of a birational map. But denoting



we have

$$\deg_{Y} \circ [W]_{*} = g_{*} \circ [W]_{*} = [\Gamma_{g}]_{*} \circ [W]_{*} = [\Gamma_{g} \circ W]_{*} = ((\mathrm{pr}_{1})_{*}(W))_{*} = [X]_{*} = \deg_{X},$$

where we denote



One can also use the moving lemma to move zero-cycles to the locus where X and Y are isomorphic.

21.1.3 Actions of correspondences on singular cohomology

We now mention how correspondences act on singular cohomology, or really any Weil cohomology theory. Let $k = \mathbb{C}$. Then, $\alpha: X \vdash Y$ gives a class $cl(\alpha) \in H^*(X \times Y, \mathbb{Z})$, and we get a map

$$\alpha_* \colon H^*(X, \mathbf{Z}) \longrightarrow H^*(Y, \mathbf{Z})$$
$$u \longmapsto p_{2*} \left(p_1^*(u) \cdot \operatorname{cl}(\alpha) \right)$$

such that $\alpha^* = (\alpha')_*$. We also have a commutative diagram

since cl commutes with pushforward and pullback. Many of the properties we had before for the action of correspondences on Chow groups extend to this setting, that is, e.g.,

(1) $(\alpha \circ \beta)_* = \alpha_* \circ \beta_*$ as maps on cohomology.

(2) If $f: X \to Y$ is a morphism, then $[\Gamma_f]_* = f_*$ as maps on cohomology.

This can be used to show that stably birational varieties have the same torsion in H_3 (although you have to be a bit careful).

21.1.4 The specialization map

One can develop the theory of Chow groups in a more general setting, e.g., where one works over a regular scheme S (i.e., a noetherian scheme such that the local rings are all regular), where the absolute notion of dimension is replaced by the relative one: if V is an irreducible and reduced scheme of finite type over S, then letting $W = \overline{\text{image of } V \text{ in } S}$,

$$\dim_S(V) \coloneqq \operatorname{trdeg}(k(V)/k(W)) - \operatorname{codim}_S(W).$$

Using the fact that regular local rings are universally caternary, we have that

• If $V_1 \subset V_2$ is an inclusion of irreducible, reduced schemes that are finite type over S, then

$$\dim_S(V_2) = \dim_S(V_1) + \operatorname{codim}_{V_2}(V_1).$$

• If $f: V \to W$ is a dominant morphism of integral schemes of finite type over S, then

$$\dim_S(V) = \dim_S(W) + \operatorname{trdeg}(k(V)/k(W)).$$

Thus, letting X be a finite type, separated scheme over S, we have

 $Z_p(X/S) :=$ free abelian group on irreducible closed subsets of X with $\dim_S(\cdot) = p$.

All results extend to this setting. For us, the main interest in this is to compare Chow groups over the special and generic fibers.

Suppose we have regular schemes S, Z such that $Z \hookrightarrow S$ is a closed embedding (hence a regular closed embedding) of codimension r. Let $S^0 = S \setminus Z$. If X is a scheme over S, then we may consider the commutative diagram of cartesian squares

$$\begin{array}{cccc} X_Z & \stackrel{i_X}{\longleftrightarrow} & X \stackrel{j_X}{\supset} & X^0 \\ \downarrow & & f \downarrow & & \downarrow \\ Z & \stackrel{i}{\longleftarrow} & S & \stackrel{j}{\supset} & S^0 \end{array}$$

As before, we have an exact sequence

Suppose $N_{Z/S} \simeq \mathcal{O}_Z^{\oplus r}$, so that $c_r(N_{Z/S}) = 0$, and so $i! \circ (i_X)_* = 0$. We therefore get a unique induced map $\sigma: CH_p(X^0/S^0) \to CH_p(X_Z/Z)$, such that $\sigma([V \cap X^0]) = i!([V])$.

We will consider the (very) special case where



where (R, \mathfrak{m}, k) is a DVR with fraction field $K = \operatorname{Frac}(R)$. The normal bundle $N_{Z/S} = \mathcal{O}_Z$ is trivial. We get a specialization map

$$\sigma \colon CH_p(X_K) \longrightarrow CH_p(X_k)$$
$$[V] \longmapsto [\overline{V}_k]$$

where \overline{V} is the closure of V in X, by using the description of the Gysin map for divisors (Example 20.4).

Remark 21.7. These specialization maps commute with proper pushforwards, flat pullbacks, and pullbacks by complete intersection maps.

We will use this to deduce properties of the generic fiber from the special fiber. We will then discuss decomposition of the diagonal and show that such a decomposition is equivalent to the triviality of CH_0 . This is due to Bloch and Srinivas. Voisin showed that this can be used to deduce rationality results after degenerating varieties to special ones. Finally, we will discuss quartic threefolds.

22 April 11

We will have two extra classes, both in East Hall 3096:

- Tuesday, April 25, 10–12; and
- Friday, April 28, 10–12.

22.1 Decomposition of the diagonal

This is the last technique in our study of rationality.

Let k be a field. All schemes X today will be of finite type and separated over k. If X is complete, recall that we had the degree morphism

deg:
$$CH_0(X) \longrightarrow \mathbf{Z}$$

 $\sum n_i[P_i] \longmapsto \sum n_i \deg(k(P_i)/k)$

We start with some important definitions:

Definition 22.1. A complete scheme X over k is CH_0 -trivial if deg: $CH_0(X) \to \mathbb{Z}$ is an isomorphism. This holds if and only if the following two properties hold:

- There exists a 0-cycle α on X of degree 1 (this is trivially satisfied if $X(k) \neq \emptyset$); and
- $CH_0(X)_0 \coloneqq \ker(\deg \colon CH_0(X) \to \mathbf{Z})$ is trivial.

Definition 22.2. A complete scheme X over k is universally CH_0 -trivial if for every field extension F/k, the base change X_F is CH_0 -trivial over F.

Example 22.3. \mathbf{P}^n is universally CH_0 -trivial, since if F/k is a field extension, $CH_0(\mathbf{P}_F^n)$ is freely generated by [P] for any $P \in \mathbf{P}^n(F)$. Thus, the degree map is an isomorphism.

Note that universal CH_0 -triviality is not the same as requiring CH_0 -triviality for the base change to the algebraic closure of k.

Remark 22.4. If X is CH_0 -trivial, then X is connected, since if P, Q belonged in different connected components of X, the cycle

$$\deg(Q)P - \deg(P)Q \in CH_0(X)_0$$

is nonzero. Thus, if a scheme is universally CH_0 -trivial, then X is geometrically connected. We will often make the assumption that X is geometrically connected in results below.

Definition 22.5. We say that a complete variety X over k has a decomposition of the diagonal if

$$[\Delta_X] = \alpha \times [X] + Z \in CH_n(X \times X) \tag{22.1}$$

for some $\alpha \in CH_0(X)$ such that $\deg(\alpha) = 1$, and some Z that is the class of a cycle supported on $X \times V$ for a proper closed subset $V \subsetneq X$.

Also, we say that X has a rational decomposition of the diagonal is there is some $N \in \mathbb{Z}_{>0}$ and a decomposition

$$N \cdot [\Delta_X] = N(\alpha \times [X]) + Z \in CH_n(X \times X)$$

with α, Z as above.

Example 22.6. The projective space \mathbf{P}^n has a decomposition of the diagonal. We understand $CH_n(\mathbf{P}^n \times \mathbf{P}^n)$ pretty well, since it is a trivial projective bundle over projective space: it is freely generated by $[L_i] \times [L_{n-i}]$, where $L_i \subset \mathbf{P}^n$ denotes a linear subspace of dimension *i*. We can therefore write

$$[\Delta_X] = \sum_{i=0}^n a_i [L_i] \times [L_{n-i}]$$

for some $a_i \in \mathbf{Z}$. Projecting onto the second component $\operatorname{pr}_2 \colon \mathbf{P}^n \times \mathbf{P}^n \to \mathbf{P}^n$ gives the equation

$$[\mathbf{P}^n] = (\mathrm{pr}_2)_* \bigl([\Delta_X] \bigr) = a_0 [\mathbf{P}^n],$$

and so $a_0 = 1$. Thus,

$$[\Delta_X] = [\{\mathrm{pt}\} \times \mathbf{P}^n] + Z,$$

where Z is a class supported on $\mathbf{P}^n \times$ hyperplane.

Note that just as in this example, if a decomposition of the diagonal as in (22.1) exists, then we must have $deg(\alpha) = 1$.

22.1.1 The Bloch–Srinivas theorem and stable rationality

Theorem 22.7 (Bloch–Srinivas). Let X be a smooth, geometrically connected, complete variety over k. Then, the following are equivalent:

- (1) X is universally CH_0 -trivial;
- (2) X has a zero-cycle of degree 1, and $CH_0(X_K)_0 = 0$, where K = k(X) is the function field;
- (3) X has a decomposition of the diagonal.

Proof. $(1) \Rightarrow (2)$ is trivial. For $(2) \Rightarrow (3)$, consider the cartesian square

$$\begin{array}{ccc} X_K & \xrightarrow{j} & X \times X \\ \downarrow & & \downarrow^{\operatorname{pr}_2} \\ \operatorname{Spec} K & \longrightarrow & X \end{array}$$

Denoting $n = \dim X$, we can define a flat pullback map

$$j^* \colon Z_n(X \times X) \longrightarrow Z_0(X_K)$$
$$[V] \longmapsto [j^{-1}(V)]$$

We have two cycles $j^*([\Delta_X]), j^*(\alpha \times [X]) \in Z_0(X_K)$, which are both of degree 1. The hypothesis says that

$$j^*([\Delta_X]) \sim_{\operatorname{rat}} j^*(\alpha \times [X]).$$

Since the rational equivalence is defined over finitely many elements in k(X), we therefore see that there exists an open subset $U \subseteq X$ such that if we denote $j_U \colon X \times U \hookrightarrow X \times X$, then

$$j_U^*([\Delta_X] - \alpha \times [X]) \sim_{\mathrm{rat}} 0.$$

Thus, $[\Delta_X] - \alpha \times [X] \sim_{\text{rat}} Z$, where Z is supported on $X \times (X \setminus U)$. (3) \Rightarrow (1). We start with the decomposition

$$[\Delta_X] = \alpha \times [X] + Z \in CH_n(X \times X).$$

We will consider each term as correspondences and let them act on $CH_0(X)$. First, recall that if $\Gamma_f \subseteq X \times Y$ is the graph of $f: X \to Y$, then $[\Gamma_f]_* = f_*$ and $[\Gamma_f]^* = f^*$. We therefore have an equality of maps

$$[\Delta_X]^* = (\alpha \times [X])^* + Z^* \colon CH_0(X) \longrightarrow CH_0(X).$$

Since Δ_X is the graph of the identity, we see that $[\Delta_X]^* = 1_{CH_0(X)}$.

For the other terms, we first note that for all $\beta \in CH_0(X)$, we have that

$$(\alpha \times X)^*(\beta) = p_{1*}(p_2^*(\beta) \cdot \alpha \times X) = p_{1*}(\alpha \times \beta) = \deg(\beta)\alpha.$$

For Z, we first note that if W is supported on a closed subset $V \times X$ with $V \subsetneq X$, then $W_* = 0$ on CH_0 (by using the Moving Lemma 21.5). Taking W = Z, we see that $[Z]^* = 0$. We conclude that for every $\beta \in CH_0(X)$, we have

$$\beta = \deg(\beta)\alpha,$$

and so $CH_0(X)_0 = 0$. Since α is a cycle of degree 1, this equality also implies that deg: $CH_0(X) \to \mathbb{Z}$ is an isomorphism. Finally, if F/k is any field extension, a decomposition of the diagonal on X induces a decomposition of the diagonal on X_F . We therefore conclude that X is universally CH_0 -trivial. \Box

We now make some comments about some variants of Theorem 22.7. The same proof shows that if X is as in Theorem 22.7, then the following are equivalent:

(1) There is a zero-cycle of degree 1, and N > 0, such that for all field exentensions F/k, we have

$$N \cdot CH_0(X_F)_0 = 0;$$

(2) There is a zero-cycle of degree 1 and N > 0 such that

$$N \cdot CH_0(X_{k(X)})_0 = 0;$$

(3) There exists a rational decomposition of the diagonal (where N can be taken as in (2)).

Corollary 22.8. Let X, Y be stably birational smooth, complete varieties over k. If Y is geometrically connected, and has a decomposition of the diagonal, then the same holds for X. In particular, if X is a stably rational smooth, complete variety, then X has a decomposition of the diagonal.

Proof. By assumption, there exist m, n and open subsets $U \subseteq X \times \mathbf{P}^m$ and $V \subseteq X \times \mathbf{P}^n$ such that $U \simeq V$. Now given any field F/k, the base extensions are isomorphic: $U_F \simeq V_F$. Now V_F is connected, and U_F is dense in $X_F \times \mathbf{P}_F^n$, and so X_F is connected. We saw that we have an isomorphism $CH_0(X_F) \simeq CH_0(Y_F)$ commuting with the degree maps. Then, Theorem 22.7 implies deg: $CH_0(Y_F) \xrightarrow{\sim} \mathbf{Z}$ and so deg: $CH_0(X_F) \xrightarrow{\sim} \mathbf{Z}$. Thus, X has a decomposition of the diagonal.

The last assertion follows from the first one, since \mathbf{P}^n has a decomposition of the diagonal.

Decomposition of the diagonal is therefore a stable birational invariant. It is more useful than CH_0 since it works better in families, e.g. when a family of smooth varieties degenerates to a singular one.

Remark 22.9. Suppose k is perfect, and suppose we are in a setting where we have resolution of singularities (e.g., dim ≤ 3 or char(k) = 0). If there exists a generically finite, dominant rational map $f: \mathbf{P}^n \dashrightarrow X$ of degree N, then X has a rational decomposition of the diagonal, using this N.

Proof. By resolving the singularities of f, we get a morphism $g: Y \to X$ that is generically finite and surjective, where Y is smooth, complete, and rational. Now given any F/k, consider $g_F: Y_F \to X_F$. Note that since X is unirational, $X(k) \neq \emptyset$. We then have maps

$$CH_0(X) \xrightarrow{g_F^*} CH_0(Y) \xrightarrow{(g_F)_*} CH_0(X)$$

and the projection formula says that $(g_F)_*(g_F^*(\beta)) = N\beta$ for all β . This implies that for all $\beta \in CH_0(X_F)$ such that $\deg(\beta) = 0$, we have $\deg(g_F^*(\beta)) = 0$. But Y is rational, and so $g_F^*(\beta) = 0$, and pushing forward, we see that $N\beta = 0$.

This says that a rational decomposition of the diagonal is no big deal; for example, all Fano varieties have this property. On the other hand, if there exist two generically finite dominant rationals maps

$$\mathbf{P}^n \xrightarrow{f_1} X \xleftarrow{f_2} \mathbf{P}^n$$

such that $\deg(f_1), \deg(f_2)$ are relatively prime, then there exists a(n integral) decomposition of the diagonal.

Proposition 22.10. Let X be a smooth, complete, geometrically connected variety over k. If there exists a zero-cycle of degree 1 on X and $CH_0(X_{\overline{k(X)}})_0 = 0$, then X has a rational decomposition of the diagonal.

Remark 22.11. Suppose $k = \overline{k}$, char(k) = 0, and k is uncountable. Recall that a smooth, connected, projective variety X is rationally connected if one of the following equivalent conditions holds:

- For every two general points $x_1, x_2 \in X$, there exists $f: \mathbf{P}^1 \to X$ such that $f(\mathbf{P}^1) \ni x_1, x_2$;
- For every two points $x_1, x_2 \in X$, there is a chain of rational curves joining x_1 and x_2 ;
- There exists $C \simeq \mathbf{P}^1 \hookrightarrow X$ such that $N_{C/X}$ is ample.

The second condition implies that for all $x_1, x_2 \in X$, we have $[x_1] \sim_{\text{rat}} [x_2]$, and so $CH_0(X)_0 = 0$. By using the third condition, we see that if X is rationally connected, then X_F is rationally connected for every algebraically closed extension F/k. In particular, if X is rationally connected over k (e.g., it is Fano), then $CH_0(X_{\overline{k(X)}})_0 = 0$. Thus, X has a rational decomposition of the diagonal.

The last thing we want to say about general decomposition of the diagonal is a criterion for when something does *not* have a decomposition of the diagonal, which involves $H^0(X, \Omega_X^p)$ and $H^3(X, \mathbb{Z})_{\text{tors}}$. The proof will use the action of correspondences on singular cohomology, and a little bit of Hodge theory. We will then discuss behavior in families.

23 April 13

23.1 Decomposition of the diagonal (continued)

Last time, we showed (Theorem 22.7) that if X is a smooth, geometrically connected, complete scheme over a field k, then

- (1) X is universally CH_0 -trivial
- (2) X has a zero-cycle of degree 1 and $CH_0(X_{k(X)})_0 = 0$
- (3) X has a decomposition of the diagonal.

We had left to explain that if α is a zero-cycle on X, then denoting



for all $u \in CH_0(X)$,

$$(\alpha \times X)^*(u) = (p_1)_* (p_1^*(\alpha) \cdot p_2^*(u)) = \alpha \cdot (p_1)_* (p_2^*(u)) = \deg(u) \cdot \alpha$$

where the second equality is by the projection formula, and the third is by compatibility of proper pushforward and flat pullback in cartesian squares (Remark 19.10).

We now give the proof of the following result we stated last time:

Proposition 22.10. Let X be a smooth, geometrically connected, complete scheme over k. Suppose that

• X has a zero-cycle of degree 1, and

•
$$CH_0(X_{\overline{k(X)}})_0 = 0.$$

Then, X has a rational decomposition of the diagonal.

Note that $CH_0(X_{\overline{k(X)}})_0 = 0$ holds for all rationally connected varieties over an algebraically closed field of characteristic zero (Remark 22.11), so the condition is rather easy to satisfy.

Parenthesis 23.1. Let W be a scheme over k. Then, there is a map

$$CH_p(W) \longrightarrow CH_p(W_{\overline{k}})$$
$$[V] \longmapsto [V_{\overline{k}}]$$

and it is easy to see that

$$CH_p(W_{\overline{k}}) = \varinjlim_{\substack{k'/k\\\text{finite}\\08}} CH_p(W_{k'}).$$

Note that if k'/k is finite of degree d, the map $\varphi \colon W_{k'} \to W$ is finite flat of degree d, and so the composition

$$CH_p(W) \xrightarrow{\varphi^*} CH_p(W_{k'}) \xrightarrow{\varphi_*} CH_p(W)$$

is equal to $d \cdot id$.

Proof of Proposition 22.10. Consider the cartesian diagram

$$\begin{array}{ccc} X_{k(X)} & \xrightarrow{g} & X \times X \\ & \downarrow & & \downarrow^{p_1} \\ \operatorname{Spec} k(X) & \longrightarrow & X \end{array}$$

Then, letting α be a zero-cycle of degree 1 on X, we have

$$g^*(\Delta_X) - g^*(\alpha \times [X]) \in CH_0(X_{k(X)})_0.$$

By hypothesis, the pullback of the above to $\overline{k(X)}$ is zero. Thus, there exists a finite extension k'/k(X) of degree d such that the pullback to k' is zero. By Parenthesis 23.1, pushing forward to k(X) implies that

$$d \cdot \left(g^*(\Delta_X) - g^*(\alpha \times [X])\right) = 0.$$

Now arguing as in the proof of Theorem 22.7, there exists some closed proper subset $V \subsetneq X$ such that

$$d([\Delta_X] - (\alpha \times [X])) \sim_{\text{rat}} \text{cycle supported on } X \times V.$$

We now want a criterion to show a scheme does *not* have a decomposition of the diagonal.

Proposition 23.2. Let X be a smooth, complete, complex algebraic variety such that X has a decomposition of the diagonal. Then, the following two conditions hold:

(a) $H^0(X, \Omega^p_X) = 0$ for all $p \ge 1$;

(b) $H^3(X, \mathbf{Z})_{\text{tors}} = 0.$

Note the first condition is satisfied for all rationally connected varieties, in particular for all Fano varieties. In fact, rationally connected varieties satisfy $h^0(X, (\Omega_X^p)^{\otimes m}) = 0$ for all $p \ge 1$ and m > 0, as we stated in Remark 13.8. Thus, the criterion we will use often is (b).

There is another criterion based on unramified cohomology, but in practice, unramified cohomology is hard to compute.

Remark 23.3. Note that (a) implies $H^2(X, \mathcal{O}_X) = 0$ by Hodge symmetry (1.3), which in turn implies that Pic $X \to H^2(X, \mathbb{Z})$ is surjective by using the exponential sequence. Thus, $Br(X) \simeq H^3(X, \mathbb{Z})_{tors}$ by Proposition 17.19, hence Br(X) is trivial by (b).

The idea of the proof of Proposition 23.2 is very simple, and again the idea is to use the action of correspondences on singular cohomology. The only thing that makes it a bit messy is that the subvariety V that shows up in the decomposition of the diagonal may not be smooth, and so we must resolve the singularities of V.

Proof of Proposition 23.2. By assumption, we have a decomposition of the diagonal

$$[\Delta_X] = \{p\} \times X + Z,$$

where $Z \in Z_n(X \times X)$ is supported on $X \times V$ for some proper closed subset $V \subsetneq X$. We first reduce to the case where V is a smooth divisor. Let $f: \tilde{X} \to X$ be an embedded resolution of singularities, i.e., f is a projective and birational morphism such that \tilde{X} is smooth, and such that $E = f^{-1}(V)$ is a divisor with simple normal crossings (we only need that the irreducible components E_1, \ldots, E_r are smooth divisors). Note that we may assume f is an isomorphism over $X \smallsetminus V$. Now rearranging our decomposition, we first have

$$\left(\left[\Delta_X\right] - \left\{p\right\} \times \left[X\right]\right)\Big|_{X \times (X \setminus V)} = 0.$$
(23.1)

We want to translate this into an equation on $X \times \tilde{X}$. Consider the graph $\Gamma_f \subseteq \tilde{X} \times X$ of f, and consider $\Gamma'_f \subseteq X \times \tilde{X}$, the image of the graph under the isomorphism interchanging the two factors. Then, (23.1) implies

$$\left(\left[\Gamma_{f}'\right]-\left\{p\right\}\times\left[\widetilde{X}\right]\right)\Big|_{X\times\left(\widetilde{X}\smallsetminus E\right)}=0.$$

We therefore can write

$$[\Gamma'_f] = \{p\} \times [\widetilde{X}] + \sum_{k=1}^r (1_X \times i_k)_*(z_k),$$

where $i_k \colon E_k \hookrightarrow \widetilde{X}$ is the inclusion, and $z_k \in CH_n(X \times E_k)$. Now consider the action

$$[\Gamma'_f]_* = [\Gamma_f]^* = f^* \colon H^m(X, \mathbf{Z}) \longrightarrow H^m(\widetilde{X}, \mathbf{Z})$$

on cohomology. To compute $[\Gamma'_f]$, we consider each term separately in the decomposition above. First,

$$\left(\{p\}\times [\widetilde{X}]\right)_* = \left([\widetilde{X}]\times \{p\}\right)^*$$

and $[\widetilde{X}] \times \{p\} \subseteq \widetilde{X} \times X$ is the graph of the constant map $\widetilde{X} \to \{p\} \hookrightarrow X$. Since $H^m(\text{pt}, \mathbf{Z}) = 0$ for m > 0, we see that $(\{p\} \times [\widetilde{X}])_* = 0$. Next, for any $u \in H^m(X, \mathbf{Z})$, denoting



we have

$$((1_X \times i_k)_*(z_k))_*(u) = (p_2)_* (p_1^*(u) \cdot \operatorname{cl}((1_X \times i_k)_*(z_k))) = (p_2)_* (1_X \times i_k)_* ((1_X \times i_k)^* p_1^*(u) \cdot \operatorname{cl}(z_k)) = (i_k)_* (\beta_k)$$

where the second equality is by the projection formula, and where the third equality holds by letting

$$\beta_k = (\varphi_k)_* \left((1_X \times i_k)^* p_1^*(u) \cdot \operatorname{cl}(z_k) \right)$$
(23.2)

and observing that the diagram

$$\begin{array}{c} X \times E_k \stackrel{{\scriptstyle (1_X \times i_k)}}{\longrightarrow} X \times \widetilde{X} \\ \varphi_k \downarrow & \downarrow^{p_2} \\ E_k \stackrel{i_k}{\longleftarrow} \widetilde{X} \end{array}$$

commutes. We therefore conclude that for all $u \in H^m(X, \mathbb{Z}), m > 0$, we can write

$$f^*(u) = \sum_{k=1}^r (i_k)_*(\beta_k)$$
(23.3)

with β_k as above. If we tensor this with **C**, we also get the same result for cohomology with **C** coefficients. We then use the following fact about how pullback on cohomology preserves the Hodge decomposition: Note 23.4. If $f: X \to Y$ is a morphism of smooth projective varieties, then

$$f^* \colon H^m(Y, \mathbf{C}) \longrightarrow H^m(X, \mathbf{C})$$

preserves the Hodge decomposition, i.e., f^* restricts to a map

$$H^q(Y, \Omega^p_Y) \longrightarrow H^q(X, \Omega^p_X).$$

This map can be identified with that induced by $f^*\Omega^p_Y \to \Omega^p_X$. Note that this fact is automatic if you interpret the Hodge decomposition in terms of harmonic forms.

Using this, we can describe (via Poincaré Duality 9.8) the behavior of f_* with respect to the Hodge decomposition. In our case, for the map $i_k : E_k \hookrightarrow \widetilde{X}$, we have

$$(i_k)_* \left(H^{p,q}(E_k) \right) \subseteq H^{p+1,q+1}(\widetilde{X}).$$

This implies

$$H^{m,0}(\widetilde{X}, \mathbf{C}) \cap \operatorname{im}\left(\bigoplus_{k=1}^{r} H^{m}(E_{k}, \mathbf{C}) \to H^{m}(\widetilde{X}, \mathbf{C})\right) = 0.$$

Thus, $f^*(H^{m,0}(X)) = 0$ by using the decomposition in (23.3). Since $f^*: H^0(X, \Omega_X^m) \to H^0(X, \Omega_{\widetilde{X}}^m)$ is injective (see the proof of Proposition 1.10), we therefore see that $H^0(X, \Omega_X^m) = 0$. This shows (a).

For (b), we note that if dim X = n, then by Poincaré duality, it is enough to show $H_{2n-3}(X, \mathbb{Z})_{\text{tors}} = 0$. If $u \in H_{2n-3}(X, \mathbb{Z})$ is torsion, then in the decomposition (23.3), the formula (23.2) for β_k implies each β_k is torsion. Since

$$\beta_k \in H_{2n-3}(E_k, \mathbf{Z}) \simeq H^1(E_k, \mathbf{Z})$$

and $H^1(E_k, \mathbb{Z})$ has no torsion (by Consequences 9.3 of the Universal Coefficient Theorem 9.2), we see that $\beta_k = 0$ for all k. Thus, $f^*(u) = 0$ and $f_*(f^*(u)) = u = 0$ since f was birational.

23.1.1 CH_0 -triviality for morphisms

The main advantage to working with decompositions of the diagonal is that it has interesting behavior under families in which smooth varieties degenerate to singular ones. This is useful since these singular degenerations often do not have decompositions because its resolution has torsion in H^3 . Such behavior does not occur for smooth families, since all fibers would be diffeomorphic by Ehresmann's theorem.

The main definition is just a relative version of CH_0 -triviality. We work over a fixed field k.

Definition 23.5. A proper morphism $f: X \to Y$ is CH_0 -trivial if the induced map

$$f_* \colon CH_0(X) \longrightarrow CH_0(Y)$$

is an isomorphism. We say f is universally CH_0 -trivial if under every extension K/k of the ground field, the morphism $f_K \colon X_K \to Y_K$ is CH_0 -trivial.

If $Y = \operatorname{Spec} k$, then this recovers the previous notions of CH_0 -triviality.

We want a criterion for a morphism to be CH_0 -trivial. We will get a criterion for universal CH_0 -triviality as a Corollary.

Proposition 23.6. Let $f: X \to Y$ be proper. Then, f is CH_0 -trivial if the following two conditions hold:

(1) For every $y \in Y$ closed, the fiber X_y is CH_0 -trivial, where we think of X_y as a scheme over $\operatorname{Spec} k(y)$;

(2) For every $\eta \in Y$ with dim $\overline{\{\eta\}} = 1$, the degree map deg: $CH_0(X_\eta) \to \mathbb{Z}$ is surjective.

Proof. Surjectivity of $f_*: CH_0(X) \to CH_0(Y)$: For every $y \in Y$, we know by (1) that there exists a cycle supported on X_y of degree 1 over k(y), whose pushforward to Y is [y]. Since $CH_0(Y)$ is generated by the classes [y] for closed points $y \in Y$, we therefore have surjectivity.

Injectivity of $f_*: CH_0(X) \to CH_0(Y)$: Suppose $\alpha \in Z_0(X)$ is such that $f_*(\alpha) \sim_{\mathrm{rat}} 0$. This implies that we can write

$$f_*(\alpha) = \sum_{i=1}^r n_i \operatorname{div}_{Y_i}(\varphi_i),$$

where $Y_i \subset X$ is irreducible and reduced of dimension 1, and $\varphi_i \in k(Y_i) \setminus \{0\}$. Now by (2), if we denote η_i to be the generic point of Y_i , then there exists a zero-cycle of degree 1 on X_{η_i} . We can write this zero-cycle of degree 1 as

$$\sum_{j} a_{i,j} Y_{i,j} \in CH_0(X_{\eta_i}),$$

where each $Y_{i,j}$ is a one-dimensional irreducible subset that dominates Y_i . After taking degrees, we have

$$\sum_{j} a_{i,j} \deg(Y_{i,j}/Y_i) = 1.$$

Let $\psi_{i,j}$ be the pullback of φ_i to $Y_{i,j}$. Then,

$$f_*\left(\operatorname{div}_{Y_{i,j}}(\psi_{i,j})\right) = \operatorname{div}_{Y_i}\left(\operatorname{Norm}(\psi_{i,j})\right) = \operatorname{div}_{Y_i}\left(\varphi_i^{\operatorname{deg}(Y_{i,j}/Y_i)}\right) = \operatorname{deg}(Y_{i,j}/Y_i) \cdot \operatorname{div}_{Y_i}(\varphi_i).$$

Now let

$$\beta = \alpha - \sum_{i,j} n_i a_{i,j} \operatorname{div}_{Y_{i,j}}(\psi_{i,j}).$$

Then, pushing forward, we obtain

$$f_*(\beta) = f_*(\alpha) - \sum_{i,j} n_i a_{i,j} \operatorname{deg}(Y_{i,j}/Y_i) \cdot \operatorname{div}_{Y_i}(\varphi_i) = 0.$$

Thus, we can write $\beta = \sum_{j=1}^{s} \beta_j$ such that $\operatorname{Supp} \beta_j = \{y_j\}$ for a closed point $y_j \in Y$, and such that $f_*(\beta_j) = 0$. Now (1) implies that $\beta_j \sim_{\operatorname{rat}} 0$ in X_{y_j} for all j, hence $\beta_j \sim_{\operatorname{rat}} 0$ in X. Thus, $\beta \sim_{\operatorname{rat}} 0$, and so $\alpha \sim_{\operatorname{rat}} 0$. \Box

Next time, we will study the way decomposition of the diagonal behaves in families, especially when all fibers in the family are universally CH_0 -trivial. Later, we will use these methods to study Colliot-Thélène and Pirutka's result that very general quartic threefolds are not stably rational [CTP16], and Totaro's generalization of their result [Tot16], which also generalizes some non-rationality results by Kollár [Kol95; Kol00].

24 April 18

The last lecture will be Tuesday, April 25 10–12 in this room.

24.1 Decomposition of the diagonal (continued)

Here is an open question:

Question 24.1. Suppose X/k where $k = \overline{k}$ is a smooth, complete scheme, and K/k is a field extension such that $K = \overline{K}$. Suppose X_K is universally CH_0 -trivial. Is X universally CH_0 -trivial?

It is not clear that this holds. We will get around it since rational connectedness *does* descend from an algebraically closed field extension.

24.1.1 A criterion for universal CH_0 -triviality for morphisms

Last time, we discussed the notion of universally CH_0 -trivial morphisms. We showed (Proposition 23.6) that if $f: X \to Y$ is a proper morphism such that

• For every closed point $y \in Y$, the fiber X_y is CH_0 -trivial; and

• For every $\eta \in Y$ with dim $\overline{\{\eta\}} = 1$, the degree map deg: $CH_0(X_\eta) \to \mathbb{Z}$ is surjective,

then f is CH_0 -trivial, i.e., $f_*: CH_0(X) \xrightarrow{\sim} CH_0(Y)$ is an isomorphism.

Corollary 24.2. If $f: X \to Y$ is proper such that for all $y \in Y$ (including non-closed points), the fiber X_y is universally CH_0 -trivial, then f is universally CH_0 -trivial.

Proof. Suppose K/k is a field extension, and consider the base change $f_K \colon X_K \to Y_K$. We claim this satisfies our previous hypotheses. Given $y' \in Y_K$ mapping to $y \in Y$, we have the following picture:



and $(X_K)_{y'} = (X_y)_{k(y')}$ is CH_0 -trivial since X_y is universally CH_0 -trivial. Then apply Proposition 23.6.

Note that we need to put the condition on all points $y \in Y$, since we cannot control the dimension of the image of $y' \in Y_K$.

Behavior of decomposition of the diagonal in families 24.1.2

We will now state our main theorem for families over a DVR.

Main Theorem 24.3 [Voi15; CTP16]. Consider a commutative diagram with cartesian squares



where (A, \mathfrak{m}, k) is a DVR with fraction field K, and f is a proper, flat morphism such that

- (1) X_{η} is geometrically connected;
- (2) X_0 is geometrically integral and we have a resolution of singularities $g: \widetilde{X}_0 \to X_0$ with \widetilde{X}_0 smooth;
- (3) g is universally CH_0 -trivial;
- (4) There exists a zero-cycle of degree 1 on \widetilde{X}_0 ;

(5) \widetilde{X}_0 is not universally CH_0 -trivial. Then, X_η is not universally CH_0 -trivial. In particular, X_η is not stably rational.

The point is that we will degenerate a variety we are interested in to a variety with mild singularities with a resolution of the form above.

The following is the example that appeared in [Voi15]:

Example 24.4. Let Y be a variety over $k = \overline{k}$ with char $(k) \neq 2$ with only isolated singularities that are all nodes, i.e., the tangent cone at each of these points is an affine cone over a smooth quadric. In this case, the blowup $q: \tilde{Y} \to Y$ at its singular points is a resolution of singularities.

- If $y \in Y$ is a node, the fiber $g^{-1}(Y)$ is a smooth quadric (since the fiber is isomorphic to the projectivized tangent cone at the point), hence rational (since $k = \overline{k}$) and universally CH_0 -trivial.
- For all other points $z \in Y$, the fiber $g^{-1}(z)$ over z is Spec(k(z)).
- By Corollary 24.2, we see that g is universally CH_0 -trivial.

We will see next time an example where there is a whole line of singularities. Voisin's examples were cyclic covers of \mathbf{P}^3 along quadrics with six to ten nodes.

The proof of Main Theorem 24.3 is fairly straightforward. Voisin's proof was fairly involved, but was simplified and generalized by [CTP16].

We will first say in words what the proof is. Suppose that \widetilde{X}_0 is such that the degree map is not injective. Then we have a nontrivial zero-cycle on X_0 . We want something in X_η that maps to this zero-cycle under the specialization map. In order to do that, the trick is to restrict to the open subset where g is an isomorphism, and where f is smooth.

Recall 24.5. In the setting of Main Theorem 24.3, we have a specialization map

$$\sigma \colon CH_p(X_\eta) \longrightarrow CH_p(X_0)$$
$$[V] \longmapsto [\overline{V} \cap X_0]$$

Note that the specialization map σ is defined on the level of cycles, even though the Gysin map $f^{!}$ is not necessarily, and that σ commutes with the degree maps.

We first prove the following technical lemma:

Lemma 24.6. Suppose we have a commutative diagram with cartesian squares



where (A, \mathfrak{m}, k) is a complete DVR with fraction field K, and f is smooth. Given any closed point $x_0 \in W_0$, there exists a closed point $x \in W_{\eta}$ such that $\sigma([x]) = [x_0]$ in $Z_0(W_0)$.

Remark 24.7. Lemma 24.6 still holds if A is only assumed to be a henselian DVR instead of a complete DVR.

This is quite easy if we assume that k is algebraically closed, but we will not assume this.

Proof of Lemma 24.6. Let $k' = k(x_0)$, in which case k'/k is finite. Then, there is an injective local homomorphism $A \hookrightarrow A'$, where A' is another complete DVR with maximal ideal $\mathfrak{m}A'$, inducing $k \hookrightarrow k'$ at the level of residue fields [Mat89, Thm. 29.1]. For example, if A contains a field so that $A \simeq k[t]$, then you can take $A' = k' \llbracket t \rrbracket$. Since A is complete and $A' / \mathfrak{m} A'$ is finitely generated over A/\mathfrak{m} , it follows that A' is finite over A (exercise; the idea is to approximate modulo \mathfrak{m}^n , and use completeness). We then make the following: C

Claim 24.8. There is a morphism
$$\tilde{x}$$
: Spec $A' \to W$ over Spec A such that the diagram

$$\begin{array}{ccc} \operatorname{Spec} A' & \xrightarrow{x} & W \\ & \uparrow & & \uparrow \\ & & & \uparrow \\ \operatorname{Spec} k' & \xrightarrow{x_0} & W_0 \end{array}$$

commutes.

To prove the claim, we use completeness. We may assume that $W = \operatorname{Spec} R$ is affine. We will use smoothness and completeness of A. The diagram on rings is



Since $A \to R$ is smooth, hence formally smooth, and since A' is complete (lift morphisms and use completeness), there is a morphism $R \to A'$ making the two triangles commutative. This proves Claim 24.8.

Now if K' is the fraction field of A', then K'/K is finite, and \tilde{x} induces a morphism Spec $K' \to W_{\eta}$, which determines the closed point $x \in X_{\eta}$. It is easy to see that $[\overline{\{x\}} \cap W_0] = [x_0]$.

In Main Theorem 24.3, we will want to assume A is complete, and do a field extension to use the assumption (5).

Proof of Main Theorem 24.3. We first claim that we may assume that A is complete, and that

$$\deg\colon CH_0(X_0) \longrightarrow \mathbf{Z} \tag{24.1}$$

is not injective. By assumption, there exists a field extension k'/k such that the map deg: $CH_0((\tilde{X}_0)_{k'}) \to \mathbb{Z}$ is not injective. As before, there exists an injective local homomorphism $A \hookrightarrow A'$, where A' is a complete DVR with maximal ideal $\mathfrak{m}A'$, and such that there is a residue field extension $k \hookrightarrow k'$. We now claim that we can replace f by

$$f' \colon X \times_{\operatorname{Spec} A'} \operatorname{Spec} A \longrightarrow \operatorname{Spec} A'.$$

First note that (1) and (4) are trivially preserved under the base change above. For (2), note that $g_{k'}: (\widetilde{X}_0)_{k'} \to (X_0)_{k'}$ is still a resolution of singularities. The generic fiber is $(X_\eta)_{K'}$ for $K' = \operatorname{Frac}(A')$, and is geometrically connected, so that (1) is preserved. Now for (5), if $(X_\eta)_{K'}$ is not universally CH_0 -trivial, then X_η is not universally CH_0 -trivial by definition. Thus, we may replace f by f' to assume that A is complete.

We now want to use the two homomorphisms in the middle row of the following commutative diagram:

$$Z_{0}(\widetilde{X}) \xrightarrow{g_{*}} Z_{0}(X_{0}) \xleftarrow{\sigma} Z_{0}(X_{\eta})$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$CH_{0}(\widetilde{X}_{0}) \xrightarrow{g_{*}} CH_{0}(X_{0}) \xleftarrow{\sigma} CH_{0}(X_{\eta})$$

$$\overbrace{\operatorname{deg}}^{\operatorname{deg}} \operatorname{deg}$$

As indicated, both g_* and σ are defined on the level of cycles, and commute with the degree maps. Since (24.1) is not injective, there exists a cycle class $\alpha \in CH_0(\widetilde{X}_0)$ such that $\alpha \neq 0$ but deg $(\alpha) = 0$. Now choose an open subset $V \subseteq X_0$ such that $g^{-1}(V) \to V$ is an isomorphism, so that in particular, V is smooth over k. Let $U \subseteq X$ be an open subset such that $V = U \cap X_0$. This implies that U is smooth over A in a neighborhood of X_0 , hence after shrinking U, we may assume that $U \to \operatorname{Spec} A$ is smooth. By the Moving Lemma 21.5, we may represent α as the class of a cycle β supported on $g^{-1}(V)$. This implies $g_*(\beta)$ is supported on V_1 Now applying Lemma 24.6 to $U \to \operatorname{Spec} A$, we have that there exists $\gamma \in Z_0(X_\eta)$ supported on U_η such that $\sigma(\gamma) = \beta$. Since g_* is an isomorphism, we see that $g_*(\beta)$ is not rationally equivalent to zero and has degree zero, hence γ is not rationally equivalent to zero. But γ has degree zero, hence X_η is not CH_0 -trivial. \Box

One disadvantage of this formulation is that even though it proves that the generic fiber is not stably rational, the generic fiber is not defined over an algebraically closed field, and in particular may not have rational points, hence not stably rational for trivial reasons. However, there is a short argument that says that if k is algebraically closed, then you can show that the generic fiber is not universally CH_0 -trivial, even after passing to the algebraic closure.

Note that we did not need that X_{η} is smooth. However, this is a useful assumption when you want to deduce things about decomposition of the diagonal.

Corollary 24.9. In the situation of Main Theorem 24.3, assume furthermore that

- k is algebraically closed,
- X_n is smooth, and
- A is excellent (or just Japanese).

Then, $(X_{\eta})_{\overline{K}}$ is not universally CH_0 -trivial.

Proof. Since X_{η} is smooth and complete, universal CH_0 -triviality is equivalent to the existence of a decomposition of the diagonal (Theorem 22.7). We will use the following general fact:

Fact 24.10. If Y is geometrically integral over k is such that $Y_{\overline{k}}$ has a decomposition of the diagonal, then there is some finite extension k'/k such that $Y_{k'}$ has a decomposition of the diagonal.

This follows from

$$CH_n(Y_{\overline{k}} \times Y_{\overline{k}}) = \varinjlim_{\overline{k'/k}} CH_n(Y_{k'} \times Y_{k'})$$

plus the definition of decomposition of the diagonal.

Now if $(X_{\eta})_{\overline{K}}$ is universally CH_0 -trivial, then there exists some K'/K finite such that $(X_{\eta})_{K'}$ is universally CH_0 -trivial. Let A' be the localization at a maximal ideal of the integral closure of A in K'. This implies A' is

a DVR, and the residue field of A' is finite over k so it is equal to k since $k = \overline{k}$. Applying Main Theorem 24.3 for $X \times_{\text{Spec }A} \text{Spec }A' \to \text{Spec }A'$ yields that the generic fiber $(X_{\eta})_{K'}$ is not universally CH_0 -trivial, a contradiction.

On Tuesday, we want to construct an interesting quartic threefolds with a resolution, to show that the set of non-stably rational quartic threefolds is dense.

25 April 25

25.1 Non-stable-rationality of quartic threefolds

The plan today is to explain how decomposition of the diagonal is used in examples to prove non-stablerationality results. The goal is the following:

Theorem 25.1 [CTP16, Thm. 1.17]. If k is an algebraically closed field that strictly contains $\overline{\mathbf{Q}}$, then the set

 $\{H \subseteq \mathbf{P}^4(k) \mid H \text{ smooth, degree 4 hypersurface, not stably rational}\}$

is dense in the set of smooth degree 4 hypersurfaces over k.

The main ingredient will be:

Theorem 25.2 [Huh14; CTP16, App. A]. There is a hypersurface $Z \subseteq \mathbf{P}^4(\overline{\mathbf{Q}})$ of degree 4, which has a resolution of singularities $g: \widetilde{Z} \to Z$ such that

- g_i is universally CH_0 -trivial;
- \widetilde{Z} is not universally CH_0 -trivial, and in fact, $H^3(\widetilde{Z}_{\mathbf{C}}, \mathbf{Z})_{\text{tors}} \neq 0$.

We will first explain why what we did so far and Theorem 25.2 imply Theorem 25.1, and then go back to proving Theorem 25.2.

25.1.1 Rationality and base change

We have not yet discussed how rationality does not depend on the algebraically closed field you work over. Recall that if X is a rational variety over an algebraically closed field k, then X_K is rational for any algebraically closed field extension $k \subseteq K$. The following says that the reverse implication also holds.

Lemma 25.3. Let X, Y be geometrically integral varieties over k and K/k a field extension.

- (1) If X_K, Y_K are birational, then there exists a finite extension $k \subseteq k'$ such that $X_{k'}, Y_{k'}$ are birational.
- (2) If k is algebraically closed, and X_K is (stably) rational, then X is (stably) rational.

This can be proved using a general Hilbert scheme argument, but we will prove this more elementarily.

Proof. For (1), we may assume that K is finitely generated as a field over k, since the birational maps between X and Y involve finitely many elements in K. Thus, we may assume K = k(W) for some variety W/k, and we have open subsets $U \subseteq X \times W$ and $V \subseteq Y \times W$ that are isomorphic as schemes over W, i.e., they fitt into the diagram below:



Now choose a closed point $w \in W$ lying in the image of U under the projection $X \times W \to W$; since U and V are isomorphic over W, the point w lies in the image of V under the projection $Y \times W \to W$ as well. Then, k(w) is a finite extension of k, and by base change along Spec $k(w) \to W$, we get an induced isomorphism

$$U \times_W \operatorname{Spec} k(w) \simeq V \times_W \operatorname{Spec} k(w),$$

which are nonempty varieties over k(w) by choice of w. Thus, $X_{k(w)}$ and $Y_{k(w)}$ are birational.

(2) follows by applying the argument in (1) for $Y = \mathbf{P}^n$ in the rationality statement, and $Y = X \times \mathbf{P}^n$ in the stably rationality statement.

25.1.2 Proof of Theorem 25.1

We now prove Theorem 25.2 implies Theorem 25.1. Let **P** be the parameter space of all hypersurfaces over \overline{Q} , that is,

$$\mathbf{P} = \mathbf{P} \left(H^0 \left(\mathbf{P}_{\overline{\mathbf{O}}}^4, \mathcal{O}(4) \right)^* \right),$$

and let $U \subseteq \mathbf{P}$ be the subset parametrizing smooth hypersurfaces. Consider the universal family



If $t \in \mathbf{P}(k)$, then we denote $H_t \subseteq \mathbf{P}^4(k)$ to be the corresponding hypersurface over k. Note that if $t \in \mathbf{P}(\overline{\mathbf{Q}})$, then $H_t = (H'_t)_k$, where H'_t is the corresponding hypersurface over $\overline{\mathbf{Q}}$.

Step 1. We may assume that $k \subseteq \mathbf{C}$.

Proof. Suppose $V \subsetneq U$ is a closed subset such that for all $t \in U(k) \smallsetminus V(k)$, the corresponding hypersurface H_t is stably rational. The set V is defined by finitely many equations in k. Let $k_0 \subseteq k$ be the algebraic closure of the subfield of k generated by $\overline{\mathbf{Q}}$ and the coefficients of those equations. Then, V is defined over k_0 , and trdeg $\overline{\mathbf{Q}}(k_0) < \infty$, hence k_0 embeds into \mathbf{C} as a subfield. By Lemma 25.3, we have that for all $t \in U(k_0) \smallsetminus V(k_0)$, the corresponding hypersurface H_t is stably rational.

Step 2. Let $s \in \mathbf{P}(\mathbf{Q})$ be the point corresponding to the hypersurface in the statement of Theorem 25.2, and let $L \subseteq \mathbf{P}$ be a smooth curve defined over $\overline{\mathbf{Q}}$ containing s such that $L \cap U \neq \emptyset$. Then, for all $y \in L(k) \setminus L(\overline{\mathbf{Q}})$, the hypersurface H_y is not stably rational.

Proof. Theorem 25.2 says that H'_s has a resolution of singularities $H'_s \to H'_s$ such that

- The resolution $\widetilde{H}'_s \to H'_s$ is universally CH_0 -trivial; and
- $H^3((\widetilde{H}'_s)_{\mathbf{C}}, \mathbf{Z})_{\text{tors}} \neq 0.$

After base changing to k, we have a resolution of singularities $\widetilde{H}_s \to H_s$ such that \widetilde{H}_s is not universally CH_0 -trivial by Proposition 23.2 and Theorem 22.7.

Now suppose that L is a curve as in the statement of Step 2, e.g., a line in **P** joining s to a point in $\mathbf{P}(\overline{\mathbf{Q}})$. Consider the cartesian diagram

$$\begin{array}{c} X & \longrightarrow & \mathcal{H} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathcal{O}_{L,s} & \longrightarrow & L & \longrightarrow & \mathbf{P} \end{array}$$

Since $L \cap U \neq \emptyset$, we know that the generic fiber X of the universal family $\mathscr{H}|_L$ is smooth, and the resolution of the special fiber H_s described above has the properties needed in Main Theorem 24.3. Since $\mathcal{O}_{L,s}$ is a DVR, Main Theorem 24.3 implies that $(X_\eta)_{\overline{k(L)}}$ is not universally CH_0 -trivial, hence not stably rational. Now if $y \in L(k) \smallsetminus L(\overline{\mathbf{Q}})$, then the morphism $\operatorname{Spec} k(y) \to L$ factors through $\operatorname{Spec} \overline{k(L)}$. Thus, H_y is a base extension of $(X_\eta)_{\overline{k(L)}}$. By Lemma 25.3, this implies that H_y is not stably rational.

Step 3. The set of non-stably rational smooth hypersurfaces is dense in \mathbf{P} .

Proof. Suppose there is V as in Step 1. Given any $y \in U(\overline{\mathbf{Q}})$, let L be a line joining y and s. Step 2 implies that $L(k) \smallsetminus L(\overline{\mathbf{Q}})$ is contained in V(k). Since $L(k) \smallsetminus L(\overline{\mathbf{Q}})$ is dense in L(k), we see that $L(k) \cap U(k) \subseteq V(k)$. This implies that $y \in V(k)$. Since $U(\overline{\mathbf{Q}})$ is dense in U(k) (exercise), we have that $\overline{U} \subseteq V$, which contradicts that U is an open dense subset of **P**.

Remarks 25.4.

(1) [CTP16, Thm. 1.20] provides an example of a non-stably-rational smooth quartic threefold over $\overline{\mathbf{Q}}$, but the argument of Theorem 25.1 does not quite apply.

(2) A general result says that given a projective family $\mathscr{X} \to T$ over $k = \overline{k}$ with integral fibers such that T is integral, there exists a countable set $(T_i)_{i\geq 1}$ of closed subsets of T such that

$$\{t \in T(k) \mid \mathscr{X}_t \text{ has a decomposition of the diagonal}\} = \bigcup_{i \ge 1} T_i(k).$$

This was a folklore result that was proved in [CTP16, App. B]. The rough idea is that Chow varieties can parametrize cycles, and since cycles depend on countably many parameters, the set of T_i is countable. This implies that if $k = \overline{k}$ and k is uncountable, then a very general quartic hypersurface in \mathbf{P}^4 does not have a decomposition of the diagonal, since we constructed one such smooth hypersurface. In particular, a very general quartic hypersurface in \mathbf{P}^4 is not stably rational.

This was the story for very general quartic threefolds.

25.1.3 The geometry of quartic symmetroids

It does not make too much sense to do all the computations needed for Theorem 25.2, and so instead we will describe the geometry behind the construction, which is very classical in flavor.

We first remind ourselves of some aspects from the construction of the Artin–Mumford example. We considered

$$W_2 \subset W_3 \subset \mathbf{P}^9 = \text{space of quadrics in } \mathbf{P}^3,$$

where W_i was the set of quadrics of rank $\leq i$ (§14.1.1). We then considered a general three-dimensional linear subspace $\Pi \subset \mathbf{P}^9$, and letting $S = \Pi \cap W_3 \subset \Pi \simeq \mathbf{P}^3$, we saw that S was a quartic hypersurface with ten nodes at $\Pi \cap W_2$ (Proposition 15.6). The Artin–Mumford example then took a double cover of \mathbf{P}^3 along S, and then resolved the singularities (§15.1).

Artin and Mumford [AM72] used explicit things about the equation of S in their argument. We will need this to construct the example we need for Theorem 25.2. The surface S is an example of a quartic symmetroid, which is defined by the vanishing of a 4×4 symmetric matrix of linear forms. The construction goes back to Cayley in 1916.

Choose coordinates on Π such that one of the nodes is P = (0, 0, 0, 1). We want to write down the equation $g(z_0, \ldots, z_3)$ of S. Since $P \in S$ is a node, we have

$$g(z_0, \dots, z_3) = \alpha(z_0, z_1, z_2) z_3^2 + \beta(z_0, z_1, z_2) z_3 + \gamma(z_0, z_1, z_2),$$
(25.1)

where deg $\alpha = 2$, deg $\beta = 3$, deg $\gamma = 4$. Moreover, $(\alpha = 0) \subset \mathbf{P}^2$ is the projectivized tangent cone of S at P, hence is a smooth conic since P is a node.

Claim 25.5 (Cayley). For general II, the discriminant $\beta^2 - 4\alpha\gamma$ factors as $\varepsilon_1 \cdot \varepsilon_2$, where $\varepsilon_1, \varepsilon_2$ are two smooth cubics in \mathbf{P}^2 meeting at nine distinct points. Moreover, ($\varepsilon_i = 0$) is tangent to ($\alpha = 0$) at three points, and the fifteen points involved are distinct.

To show Claim 25.5, the idea is to project away from P to \mathbf{P}^2 . For a general choice of Π , this induces a 2 : 1 map from the blowup of Π at P, which is ramified exactly along the vanishing locus of the discriminant $\beta^2 - 4\alpha\gamma$.

First, we will describe the resolution of S more geometrically. Of course, it can also be defined as the blowup at each of the ten nodes on S, but a more geometric description will be useful.

Notation 25.6. For $t \in \mathbf{P}^9$, we denote by $H_t \subseteq \mathbf{P}^3$ the corresponding quadric. We consider the incidence correspondence

$$R = \left\{ (Q, t) \in \mathbf{P}^3 \times \mathbf{P}^9 \mid Q \in (H_t)_{\text{sing}} \right\} \subseteq \mathbf{P}^3 \times \mathbf{P}^9.$$

This set R is the intersection of four divisors of type (1, 1), since it is defined by the vanishing of partial derivatives. There are projection maps



which are both surjective, where
- $\varphi: R \to \mathbf{P}^3$ is a projective subbundle of codimension four in $\mathbf{P}^3 \times \mathbf{P}^9$, hence R is smooth of dimension 3+9-4=8;
- $\psi: R \to W_3$ is injective over $W_3 \smallsetminus W_2$, hence birational.

Letting $\Pi \subseteq \mathbf{P}^9$ be a general three-dimensional linear subspace, we also define

$$S = \Pi \cap W_3 \subseteq W_3, \qquad \widetilde{S} = \psi^{-1}(S), \qquad S' = \varphi(\widetilde{S}),$$

so that we have the following diagram:



Now we note the following:

- (1) Since Π is general, \widetilde{S} is a smooth subvariety of $\mathbf{P}^3 \times \Pi$, which is cut out by four divisors of type (1, 1). Thus, by the adjunction formula, \widetilde{S} is a smooth K3 surface, and the induced birational map $\widetilde{S} \to S$ is a minimal resolution of singularities that blows up the ten nodes on S.
- (2) Letting q_0, \ldots, q_3 span the linear space corresponding to Π , a point $Q \in \mathbf{P}^3$ lies in S' if and only if there exist $\lambda_0, \ldots, \lambda_3$, not all zero, such that

$$\sum_{i=0}^{3} \lambda_i \frac{\partial q_i}{\partial x_j}(Q) = 0$$

for all j. This is equivalent to saying that

$$\det\left(\frac{\partial q_i}{\partial x_j}\right) = 0,\tag{25.2}$$

where the matrix is a 4×4 matrix of linear forms. Moreover, we have the following:

Fact 25.7. For general II, the hypersurface given by (25.2) is a smooth quartic in \mathbf{P}^3 .

(3) The morphism $\tilde{\varphi} \colon \widetilde{S} \to S'$ is birational, hence an isomorphism since both \widetilde{S} and S' have trivial canonical bundles. We prove that $\deg(\varphi) = 1$ via intersection theory, by computing $(\mathcal{O}_{\widetilde{S}}(1,0)^2)$ in two different ways. First,

$$\left(\mathcal{O}_{\widetilde{S}}(1,0)^2\right) = \left(\varphi^*\mathcal{O}_{S'}(1)^2\right) = 4 \cdot \deg(\varphi)$$

by compatibility of intersection numbers with pullbacks, and since Fact 25.7 implies $4 = (\mathcal{O}_{S'}(1)^2)$. We also have

$$\left(\mathcal{O}_{\widetilde{S}}(1,0)^{2}\right) = \mathcal{O}_{\mathbf{P}^{3} \times \Pi}(1,0)^{2} \cdot \left(\mathcal{O}_{\mathbf{P}^{3} \times \Pi}(1,0) + \mathcal{O}_{\mathbf{P}^{3} \times \Pi}(0,1)\right)^{4} = 4\left(\mathcal{O}(1,0)^{3} \cdot \mathcal{O}(0,1)^{3}\right) = 4$$

by using the description of \widetilde{S} in (1). Combining the two equations gives $\deg(\varphi) = 1$.

We now return to the proof of Claim 25.5. We will need the following:

Fact 25.8. If $L \subset S$ is a line, then there are three nodes on L, which give singular points on S'. Thus, if Π is general, then S' is smooth, and so there cannot be a line on S.

We will use Fact 25.8 when L parametrizes singular quadrics.

Now consider the projection $S \setminus \{P\} \to \mathbf{P}^2$ from the point P. Since P is a node, this induces a morphism $\pi: \operatorname{Bl}_P S \to \mathbf{P}^2$. Since S contains no lines, we see that π is a finite map, and is of degree two. In this case, one can prove the following:

Exercise 25.9. If $\pi: X \to Y$ is a degree two finite surjective morphism, with X normal and Y smooth, then π is a cyclic cover of order two.

Thus, π : Bl_P $S \to \mathbf{P}^2$ is a cyclic cover of order two. The ramification locus, i.e., the locus where π is not étale, is a curve in \mathbf{P}^2 . By looking at the equation of S, you can see that this locus is exactly the vanishing locus of the discriminat $\beta^2 - 4\alpha\gamma$.

We will try to understand the inverse image of this ramification locus in $S \setminus \{P\}$. This is the locus

$$\{t \in S \mid \text{the line } \overline{tP} \text{ is tangent to } S \text{ at } t\} = \{t \in S \mid \text{the line } \overline{tP} \text{ is contained in } T_tS\}$$
 (25.3)

Since Π is general, the intersection $W_3 \cap \Pi = S$ is as transversal as possible. Thus, if t is not a node, then $T_t S = T_t W_3 \cap \Pi$. We did an explicit computation of $T_t W_3$ in coordinates (cf. (14.2)): W_3 is given by

$$\begin{vmatrix} a_{11} + 1 & & \\ & a_{22} + 1 & \\ & & a_{33} + 1 \\ & & & a_{44} \end{vmatrix} = 0$$

and the tangent space is given by $a_{44} = 0$. The quadric $x_0^2 + x_1^2 + x_2^2$ is singular exactly at (0, 0, 0, 1), hence

$$T_t W_3 = \left\{ Q \in \mathbf{P}^9 \mid (H_t)_{\text{sing}} \in H_Q \right\}.$$

This implies that the locus (25.3) when intersected with $S \setminus \{\text{nodes}\}$ is

$$\{t \in S \smallsetminus \{\text{nodes}\} \mid P \in T_t S\},\$$

and by the above, this condition is equivalent to $(H_t)_{sing} \in H_P$, which holds if and only if there exists $(Q, t) \in R$ such that $Q \in H_P$. Thus, the locus in (25.3) is $\widetilde{\psi}(\widetilde{\varphi}^{-1}(S' \cap H_P))$, where we recall the notation



We then need to understand $S' \cap H_P$. P is a node corresponding to a quadric of rank 2, hence H_P is a union of two planes $H_P = L_1 + L_2$, where $L_1, L_2 \subseteq \mathbf{P}^3$ are planes meeting along a line ℓ . It is then easy to see that $\ell \subseteq S'$, and therefore,

$$S' \cap L_i = \ell + \underbrace{\text{residual cubic curve in } \mathbf{P}^2}_{C_i}.$$

These are the two cubic curves we were looking for. Thus, the discriminant curve $\beta^2 - 4\alpha\gamma$ is the union of the images in \mathbf{P}^2 of these two cubic curves. Now since $\mathrm{Bl}_P S$ has precisely nine nodes, it follows that these two cubic curves in \mathbf{P}^3 are smooth, and intersect at nine distinct points.

This concludes the description of the Artin–Mumford example, which is what they start with in [AM72]. This is quite important for the computations in [Huh14; CTP16].

25.1.4 The construction for Theorem 25.2

Let g as in (25.1) define S. [Huh14] takes $Z \subseteq \mathbf{P}^4$ defined by

$$g(x_0, x_1, x_2, x_3) - x_4^2 \delta(x_0, \dots, x_3)$$

with δ is a general form of degree two. [CTP16] take

$$g(x_0, x_1, x_2, x_3) - x_4^2 x_0^2,$$

where x_0 can be replaced by $x_0 + ux_1 + vx_2$ for $u, v \in k$ general. The advantage of the second Z is that it is birational to the Artin–Mumford example, since in the chart $x_0 \neq 0$,

$$Z \cap \{x_0 \neq 0\} = \{g(1, x_1, x_2, x_3) - x_4^2 = 0\} \subseteq \mathbf{A}^4$$

is an affine piece of the double cover of \mathbf{P}^3 ramified along S. Hence, for every resolution $\widetilde{Z} \to Z$, we have that $H^3(\widetilde{Z}, \mathbf{Z})_{\text{tors}} \neq \emptyset$. We then need to show that we can find some resolution $\widetilde{Z} \to Z$ that is universally CH_0 -trivial.

Description of the resolution We have a line

$$L = (x_0 = x_1 = x_2 = 0) \subset Z_{\text{sing}},$$

and $(Z \setminus L)_{\text{sing}}$ consists of exactly nine nodes.

Step 1. Let $g': Z' = \operatorname{Bl}_L Z \to Z$, with exceptional divisor $E' \to L$. One can check in charts that

$$Z'_{\text{sing}} \cap (g')^{-1}(L) = (g')^{-1}(P)_{2}$$

and that this set is a line, which we denote by L'. Also, E' is birational to a smooth quadric, hence is rational. Moreover, $E' \to L$ is a conic bundle, which has a rational section.

Step 2. Let $g'': Z'' = \operatorname{Bl}_{L'} Z' \to Z'$, with exceptional divisor $E'' \to L'$. Then, E'' is smooth, hence Z'' is smooth over L, and moreover, $E'' \to L'$ is a conic bundle over \mathbf{P}^1 , hence E'' is rational by the Noether-Enriques Theorem 10.5.

Step 3. Consider the composition



where $g''': \widetilde{Z} \to Z''$ is the blowup at the remaining nine nodes. The fibers of g are:

- Points (over the open set where g is an isomorphism);
- Smooth quadrics, which are rational (over the nine nodes);
- A smooth quadric with a rational point (over the generic point $\operatorname{Spec} k(L)$ of L);
- Smooth conics (over closed points in $L \setminus \{P\}$);
- The ruled surface E'' (over P).

All of these are rational, hence g is universally CH_0 -trivial by Corollary 24.2. This completes the sketch of the proof of Theorem 25.2.

The general strategy for producing interesting examples is to find explicit descriptions of resolutions like what we did for Theorem 25.2, and use degeneration arguments. This is the strategy employed by Hassett, Pirutka, and Tschinkel in [HPT16a; HPT16b].

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