A SHORT PROOF OF THE SUBADDITIVITY THEOREM FOR TEST IDEALS

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The subadditivity theorem for test ideals was first proved by Hara and Yoshida using tight closure [HY03]. We give a short proof of the subadditivity theorem for test ideals using Schwede's characterization of the test ideal in terms of F-compatibility [Sch10].

We first review the definition of test ideals for F-finite rings of characteristic $p > 0$. See [ST12] and [TW18, §5] for overviews of the theory. We set the following notation.

Definition 1 (cf. [Sch10, Def. 2.3])**.** A *pair* $(R, \mathfrak{a}_\bullet)$ consists of

(*i*) an F-finite reduced noetherian ring R of characteristic $p > 0$, and
(*ii*) a graded family $\mathfrak{a}_\bullet$ of ideals in R such that $\mathfrak{a}_m \cap R^\circ \neq \emptyset$ for all $m > 0$.

Remark 2. By setting $\mathfrak{a}_m = \mathfrak{a}^{\lceil tm \rceil}$ for a fixed ideal $\mathfrak{a}$ and a real numbers $t > 0$, we recover the more common notion of a pair $(R, \mathfrak{a}^t)$; see [Sch10, §2.1].

We can now define test ideals.

Definition 3 [Sch10, Def. 3.1 and Thm. 6.3]**.** Let $(R, \mathfrak{a}_\bullet)$ be a pair. An ideal $J \subseteq R$ is *uniformly* $(\mathfrak{a}_\bullet, F)$*-compatible* if for every integer $e > 0$ and every $\varphi \in \operatorname{Hom}_R(F^e_* R, R)$, we have

$$\varphi\bigl(F^e_*(J \cdot \mathfrak{a}_{p^e-1})\bigr) \subseteq J.$$

The *test ideal* $\tau(R, \mathfrak{a}_\bullet)$ is the smallest ideal in R that is uniformly $(\mathfrak{a}_\bullet, F)$-compatible and intersects R°. We often drop R from our notation if it is clear from context.

The test ideal exists by [Sch11, Thm. 3.18]. To prove the subadditivity theorem, we will need the following consequence of a lemma of Fedder [Fed83, Lem. 1.6].

Proposition 4 [Sch10, Prop. 3.11]**.** *Let $(R, \mathfrak{a}_\bullet)$ be a pair such that R is regular. Then, an ideal $J \subseteq R$ is uniformly $(\mathfrak{a}_\bullet, F)$-compatible if and only if for all $e \geq 0$, we have $\mathfrak{a}_{p^e-1} \subseteq (J^{[p^e]} : J)$.*

We can now state and prove the subadditivity theorem for test ideals.

Theorem 5 (Subadditivity, cf. [HY03, Thm. 6.10(2)])**.** *Let R be a regular F-finite ring of characteristic $p > 0$. If $(R, \mathfrak{a}_\bullet)$ and $(R, \mathfrak{b}_\bullet)$ are two pairs, then*

$$\tau(\mathfrak{a}_\bullet \cdot \mathfrak{b}_\bullet) \subseteq \tau(\mathfrak{a}_\bullet) \cdot \tau(\mathfrak{b}_\bullet).$$

In particular, if $\mathfrak{a}, \mathfrak{b}$ are two ideals in R intersecting R°, then for all positive real numbers t, s,

$$\tau(\mathfrak{a}^t \cdot \mathfrak{b}^s) \subseteq \tau(\mathfrak{a}^t) \cdot \tau(\mathfrak{b}^s).$$

Proof. The second statement follows from the first by setting $\mathfrak{a}_m = \mathfrak{a}^{\lceil tm \rceil}$ and $\mathfrak{b}_m = \mathfrak{b}^{\lceil sm \rceil}$ as in Remark 2. For the first statement, it suffices to show the chain of inclusions

$$\begin{aligned} \mathfrak{a}_{p^e-1} \cdot \mathfrak{b}_{p^e-1} &\subseteq \bigl(\tau(\mathfrak{a}_\bullet)^{[p^e]} : \tau(\mathfrak{a}_\bullet)\bigr) \cdot \bigl(\tau(\mathfrak{b}_\bullet)^{[p^e]} : \tau(\mathfrak{b}_\bullet)\bigr) \\ &\subseteq \Bigl(\bigl(\tau(\mathfrak{a}_\bullet) \cdot \tau(\mathfrak{b}_\bullet)\bigr)^{[p^e]} : \bigl(\tau(\mathfrak{a}_\bullet)^{[p^e]} \cdot \tau(\mathfrak{b}_\bullet)^{[p^e]}\bigr)\Bigr) \end{aligned}$$

by Proposition 4, since $\tau(\mathfrak{a}_\bullet \cdot \mathfrak{b}_\bullet)$ is the smallest $(\mathfrak{a}_\bullet \cdot \mathfrak{b}_\bullet, F)$-compatible ideal intersecting R° by definition. The first inclusion follows since $\tau(\mathfrak{a}_\bullet)$ and $\tau(\mathfrak{b}_\bullet)$ are uniformly $(\mathfrak{a}_\bullet, F)$- and $(\mathfrak{b}_\bullet, F)$-compatible. The second inclusion holds since, in general, $(I_1 : J_1) \cdot (I_2 : J_2) \subseteq (I_1 I_2 : J_1 J_2)$. □

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