TEST IDEALS FOR PAIRS VIA GENERALIZED TIGHT CLOSURE

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The goal of this talk is to develop a theory of test ideals for pairs $(R, \mathfrak{a}^t)$, and to give some applications that do not mention test ideals. See [ST12] and [TW18, §5] for overviews of the theory. The theory is originally due to Hara–Yoshida [HY03] and Hara–Takagi [HT04], and generalizes the tight closure theory developed by Hochster–Huneke [HH90].

We start by describing our applications. First, recall that the *integral closure* $\overline{\mathfrak{a}}$ of an ideal $\mathfrak{a} \subseteq R$ is the set of all elements $r \in R$ such that r satisfies a polynomial $f(x) = \sum_{i=0}^{d} c_{d-i}x^i \in R[x]$ where $c_0 = 1$ and $c_i \in \mathfrak{a}^i$. A version of the Briançon–Skoda theorem says that if R is regular, then

$$\overline{\mathfrak{a}^{n+\ell-1}} \subseteq \mathfrak{a}^n,$$

where ℓ is the number of generators of $\mathfrak{a}$. This inclusion was shown by Skoda–Briançon for smooth $\mathbf{C}$-algebras [SB74, Thm. 3], by Lipman–Sathaye for all regular rings [LS81, Thm. 1″], and by Hochster–Huneke for weakly F-regular rings [HH90, Thm. 5.4]. We will prove a version of this result in §2. This result has some interesting corollaries, which we will not be able to prove:

(1) If $f \in \mathbf{C}\{z_1, z_2, \ldots, z_n\}$ is a convergent power series in n variables that defines a hypersurface with an isolated singularity at the origin, then $f^n \in (\partial f/\partial z_1, \partial f/\partial z_2, \ldots, \partial f/\partial z_n)$ [SB74, Cor.]. This answers a question of Mather.

(2) If $f_1, f_2, \ldots, f_{n+1} \in R$, where R is regular of dimension n, then

$$f_1^n f_2^n \cdots f_n^n \in (f_1^{n+1}, f_2^{n+1}, \ldots, f_{n+1}^{n+1}).$$

See [Hoc14, Thm. on p. 29].

Second, recall that the *nth symbolic power* of an ideal $\mathfrak{a} \subseteq R$ is

$$\mathfrak{a}^{(n)} \coloneqq R \cap \bigcap_{\mathfrak{p} \in \operatorname{Ass}(\mathfrak{a})} \mathfrak{a}^n R_{\mathfrak{p}}.$$

We will prove the following uniform comparison theorem for symbolic powers on regular rings:

$$\mathfrak{a}^{(hn)} \subseteq \mathfrak{a}^n$$

where h is the maximal height of the associated primes of $\mathfrak{a}$. This uniform comparison theorem is originally due to Ein–Lazarsfeld–Smith for smooth $\mathbf{C}$-algebras [ELS01, Thm. A], to Hochster–Huneke for regular rings containing a field [HH02, Thm. 1.1(a)], and to Ma–Schwede for mixed characteristic regular rings [MS18, Thm. 7.4]. We will prove a version of this result in §3.

Notation. All rings will be commutative with identity, noetherian, and of characteristic $p > 0$. If R is a ring, then R° denotes the complement of the union of the minimal primes of R. We denote the Frobenius morphism by $F\colon R \to F_*R$. For every integer $e \geq 0$, the eth iterate of the Frobenius morphism is denoted by $F^e\colon R \to F^e_*R$.

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1. Definition and preliminaries

We start by defining with the analogue of tight closure and test ideals for pairs $(R, \mathfrak{a}^t)$, following Hara–Yoshida [HY03] and Hara–Takagi [HT04].

Definition 1.1 [HY03, Def. 6.1; HT04, Def. 1.4]**.** Let R be a ring, and let $\mathfrak{a} \subseteq R$ be an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$. Let $\iota\colon N \hookrightarrow M$ be an inclusion of R-modules. For every $t \in \mathbf{R}_{\geq 0}$, the $\mathfrak{a}^t$-*tight closure* is

$$N_M^{*\mathfrak{a}^t} \coloneqq \left\{ x \in M \;\middle|\; \begin{array}{c} \text{there exists } c \in R^\circ \text{ such that for all } e \gg 0, \\ c\mathfrak{a}^{\lceil p^e t\rceil} \otimes x \subseteq \operatorname{im}\bigl(\mathrm{id} \otimes \iota\colon F_*^e R \otimes_R N \to F_*^e R \otimes_R M\bigr) \end{array} \right\}.$$

Now let $E \coloneqq \bigoplus_{\mathfrak{m}} E_R(R/\mathfrak{m})$ be the direct sum of the injective hulls of the residue fields $R/\mathfrak{m}$ for every maximal ideal $\mathfrak{m} \subseteq R$. The (*non-finitistic* or *big*) *test ideal* is

$$\tau(\mathfrak{a}^t) \coloneqq \operatorname{Ann}_R\bigl(0_E^{*\mathfrak{a}^t}\bigr).$$

Now let $\mathfrak{b} \subseteq R$ be another ideal such that $\mathfrak{b} \cap R^\circ \neq \emptyset$. For every $s \in \mathbf{R}_{\geq 0}$, we similarly define

$$N_M^{*\mathfrak{a}^t\mathfrak{b}^s} \coloneqq \left\{ x \in M \;\middle|\; \begin{array}{c} \text{there exists } c \in R^\circ \text{ such that for all } e \gg 0, \\ c\mathfrak{a}^{\lceil p^e t\rceil}\mathfrak{b}^{\lceil p^e s\rceil} \otimes x \subseteq \operatorname{im}\bigl(\mathrm{id} \otimes \iota\colon F_*^e R \otimes_R N \to F_*^e R \otimes_R M\bigr) \end{array} \right\}$$

in which case the test ideal is

$$\tau(\mathfrak{a}^t\mathfrak{b}^s) \coloneqq \operatorname{Ann}_R\bigl(0_E^{*\mathfrak{a}^t\mathfrak{b}^s}\bigr).$$

Setting $\mathfrak{a} = R$ and $t = 1$, one obtains the usual notion of tight closure for modules due to Hochster–Huneke [HH90, Def. 8.2], and the (non-finitistic or big) test ideal $\tau(R)$ defined by Lyubeznik–Smith [LS01, §7].

Remark 1.2. The definition in [HY03] is the analogue of the (finitistic) test ideal defined in [HH90, Def. 8.22] for pairs $(R, \mathfrak{a}^t)$. There is also a version of tight closure for pairs (R, Δ), where Δ is an effective $\mathbf{R}$-Weil divisor [Tak04, Def. 2.1], or triples $(R, \Delta, \mathfrak{a}^t)$ [Sch10, Def. 2.14]. The test ideal $\tau(\mathfrak{a}^t)$ can also be described in terms of an appropriate version of test elements for $\mathfrak{a}^t$-tight closure if R is F-finite [Sch10, Thm. 2.22].

We now state some basic properties of test ideals that we will use often.

Proposition 1.3 (cf. [HY03, Props. 1.3 and 1.11; LS01, Prop. 2.9])**.** *Let R be a ring, and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in R intersecting R°.*

(*i*) $\tau(\mathfrak{a}^t \cdot \mathfrak{b}^s) \cdot \mathfrak{b} \subseteq \tau(\mathfrak{a}^t \cdot \mathfrak{b}^{s+1})$.
(*ii*) *If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\tau(\mathfrak{a}^t) \subseteq \tau(\mathfrak{b}^t)$. Equality holds if $\mathfrak{b} \subseteq \overline{\mathfrak{a}}$.*
(*iii*) *If $s < t$, then $\tau(\mathfrak{a}^s) \supseteq \tau(\mathfrak{a}^t)$.*
(*iv*) *For every $m \in \mathbf{N}$, we have $\tau((\mathfrak{a}^n)^t) = \tau(\mathfrak{a}^{nt})$.*
(*v*) *R is strongly F-regular if and only if $\tau(R) = R$.*

Proof. It suffices to show the corresponding dual statements for $N_M^{*\mathfrak{a}^t}$ by setting $N = 0$ and $M = E$ and taking annihilators, where for (*i*), we use the fact that $\operatorname{Ann}_R(N :_M \mathfrak{a}) \supseteq \operatorname{Ann}_R(N) \cdot \mathfrak{a}$.

For (*i*), we have

$$N_M^{*\mathfrak{a}^t\mathfrak{b}^{s+1}} \subseteq \bigl(N_M^{*\mathfrak{a}^t\mathfrak{b}^s} :_M \mathfrak{b}\bigr) \tag{1}$$

since if $x \in N_M^{*\mathfrak{a}^t\mathfrak{b}^{s+1}}$ with multiplier $c \in R^\circ$, then

$$\begin{aligned} c\mathfrak{a}^{\lceil p^e t\rceil}\mathfrak{b}^{\lceil p^e s\rceil} \otimes \mathfrak{b} \cdot x &= c\mathfrak{a}^{\lceil p^e t\rceil}\mathfrak{b}^{\lceil p^e s\rceil}\mathfrak{b}^{[p^e]} \otimes x \\ &\subseteq c\mathfrak{a}^{\lceil p^e t\rceil}\mathfrak{b}^{\lceil p^e s\rceil + p^e} \otimes x \\ &= c\mathfrak{a}^{\lceil p^e t\rceil}\mathfrak{b}^{\lceil p^e (s+1)\rceil} \otimes x \\ &\subseteq \operatorname{im}\bigl(\mathrm{id} \otimes \iota\colon F_*^e R \otimes_R N \to F_*^e R \otimes_R M\bigr) \end{aligned}$$

for all $e \gg 0$.

The first part of (*ii*) and (*iii*) follow from the fact that $N_M^{*\mathfrak{a}^t} \supseteq N_M^{*\mathfrak{b}^t}$ and $N_M^{*\mathfrak{a}^s} \subseteq N_M^{*\mathfrak{a}^t}$. For the second part of (*ii*), it suffices to show that $N_M^{*\mathfrak{a}^t} \subseteq N_M^{*\mathfrak{b}^t}$. Recall from [HS06, Cor. 1.2.5] that $\mathfrak{b} \subseteq \overline{\mathfrak{a}}$ if and only if there exists an integer $r > 0$ such that

$$\mathfrak{b}^{r+1} = \mathfrak{a}\mathfrak{b}^r,$$

i.e., if and only if $\mathfrak{a}$ is a *reduction* of $\mathfrak{b}$. By [HS06, Rem. 1.2.3], this implies $\mathfrak{b}^{r+s} \subseteq \mathfrak{a}^s$ for every integer $s > 0$. Now consider $x \in N_M^{*\mathfrak{a}^t}$. Setting $s = \lceil p^e t \rceil$, we see that for any choice of $d \in \mathfrak{b}^r \cap R^\circ$ and all $x \in M$, we have

$$\begin{aligned} cd\mathfrak{b}^{\lceil p^e t\rceil} \otimes x &\subseteq c\mathfrak{b}^{\lceil p^e t\rceil + r} \otimes x \\ &\subseteq c\mathfrak{a}^{\lceil p^e t\rceil} \otimes x \\ &\subseteq \operatorname{im}\bigl(\mathrm{id} \otimes \iota\colon F_*^e R \otimes_R N \to F_*^e R \otimes_R M\bigr) \end{aligned}$$

hence $x \in N_M^{*\mathfrak{a}^t}$ implies $x \in N_M^{*\mathfrak{b}^t}$.

For (*iv*), we note that $\lceil p^e t\rceil - 1 \le p^e t \le \lceil p^e t \rceil$, hence multiplying by n throughout and applying ceilings again, we have

$$n\lceil p^e t\rceil - n \le \lceil p^e nt \rceil \le n\lceil p^e t\rceil.$$

We therefore have the inclusions

$$\mathfrak{a}^{n\lceil p^e t\rceil - n} \supseteq \mathfrak{a}^{\lceil p^e nt\rceil} \supseteq \mathfrak{a}^{n\lceil p^e t\rceil}.$$

The right inclusion already implies $N_M^{*\mathfrak{a}^{nt}} \subseteq N_M^{*(\mathfrak{a}^n)^t}$. On the other hand, if $x \in N_M^{*(\mathfrak{a}^n)^t}$ with multiplier $c \in R^\circ$, then for any choice of $d \in \mathfrak{a}^n \cap R^\circ$, we have

$$\begin{aligned} cd\mathfrak{a}^{\lceil p^e nt\rceil} \otimes x &\subseteq cd\mathfrak{a}^{n\lceil p^e t\rceil - n} \otimes x \\ &\subseteq c\mathfrak{a}^{n\lceil p^e t\rceil} \otimes x \\ &\subseteq \operatorname{im}\bigl(\mathrm{id} \otimes \iota\colon F_*^e R \otimes_R N \to F_*^e R \otimes_R M\bigr) \end{aligned}$$

for all $e \gg 0$, hence $N_M^{*(\mathfrak{a}^n)^t} \subseteq N_M^{*\mathfrak{a}^{nt}}$.

Finally, (*v*) follows since R is strongly F-regular if and only if $0_E^* = 0$ [LS01, Prop. 2.9]. ☐

Remark 1.4. Motivated by Proposition 1.3(*v*), we say that $(R, \mathfrak{a}^t)$ is *strongly F-regular* if $\tau(\mathfrak{a}^t) = R$.

The following results are more subtle. The corresponding results for $\tau(R)$ are due to Lyubeznik–Smith [LS01, Thms. 7.1(2) and 7.1(3)], and do not require the full strength of F-finiteness.

Proposition 1.5 [HT04, Props. 3.1 and 3.2]**.** *Let R be a reduced F-*finite *ring, and let $\mathfrak{a} \subseteq R$ be an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$.*

(*i*) *For every multiplicative subset $W \subseteq R$, we have $\tau(\mathfrak{a}^t)W^{-1}R = \tau((\mathfrak{a}W^{-1}R)^t)$.*
(*ii*) *If $(R, \mathfrak{m})$ is a local ring, then $\tau(\mathfrak{a}^t)\widehat{R} = \tau((\mathfrak{a}\widehat{R})^t)$.*

Proof Sketch. Both of these properties follow from the fact (see [HT04, Lem. 2.1]) that

$$\tau(\mathfrak{a}^t) = \sum_{e \ge 0} \sum_{\phi^{(e)}} \phi^{(e)}\bigl(F_*^e(c\mathfrak{a}^{\lceil p^e t\rceil})\bigr),$$

where $\phi^{(e)}$ ranges over all elements of $\operatorname{Hom}_R(F_*^e R, R)$, and $c \in R^\circ$ is an appropriate version of a completely stable big test element for $\mathfrak{a}^t$-tight closure. ☐

Before we can move on to applications, we need the following version of Matlis duality:

Proposition 1.6 (Matlis duality, see [Hoc07, Prop. on p. 242])**.** *Let $(R, \mathfrak{m})$ be a local ring, and let $E = E_R(R/\mathfrak{m})$ be the injective hull of the residue field. If $\mathfrak{a} \subseteq R$ is an ideal, then $\operatorname{Ann}_R(0 :_E \mathfrak{a}) = \mathfrak{a}$. If R is complete and $N \subseteq E$ is a submodule, then $(0 :_E \operatorname{Ann}_R(N)) = N$.*

Proposition 1.6 has the following consequence:

Lemma 1.7. *Let R be a complete local ring, and let $\mathfrak{a} \subseteq R$ be an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$. For every ideal $\mathfrak{b} \subseteq R$ and every $t \in \mathbf{R}_{\geq 0}$, we have*

$$\bigl(0_E^{*\mathfrak{a}^t} :_E \mathfrak{b}\bigr) \supseteq \bigl(0 :_E \tau(\mathfrak{a}^t) \cdot \mathfrak{b}\bigr).$$

Proof. If $x \in (0 :_E \tau(\mathfrak{a}^t) \cdot \mathfrak{b})$, then $\mathfrak{b} \cdot x \subseteq (0 :_E \tau(\mathfrak{a}^t)) = 0_E^{*\mathfrak{a}^t}$ by Matlis duality (Proposition 1.6), hence $x \in (0_E^{*\mathfrak{a}^t} :_E \mathfrak{b})$. □

2. Briançon–Skoda

Our next goal is to prove an application of the machinery developed so far. We first prove a version of the Briançon–Skoda theorem involving test ideals.

Theorem 2.1 (Skoda's theorem, cf. [HT04, Thms. 4.1 and 4.2])**.** *Let R be a reduced F-finite ring or a complete local ring, and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in R intersecting R°. Let ℓ be the number of generators of an ideal $\mathfrak{c}$ such that $\overline{\mathfrak{a}} = \overline{\mathfrak{c}}$. Then, for every $t \in \mathbf{R}_{\geq 0}$, we have*

$$\tau(\mathfrak{a}^\ell \mathfrak{b}^t) = \tau(\mathfrak{a}^{\ell-1}\mathfrak{b}^t) \cdot \mathfrak{a}.$$

Proof. The inclusion $\supseteq$ follows from Proposition 1.3(*i*), hence it suffices to show the reverse inclusion $\subseteq$. If R is not complete local, we can reduce to the complete local case using Proposition 1.5.

By Matlis duality (Proposition 1.6), it suffices to show the chain of inclusions

$$0_E^{*\mathfrak{a}^\ell\mathfrak{b}^t} \supseteq \bigl(0_E^{*\mathfrak{a}^{\ell-1}\mathfrak{b}^t} :_E \mathfrak{a}\bigr) \supseteq \bigl(0 :_E \tau(\mathfrak{a}^{\ell-1}\mathfrak{b}^t) \cdot \mathfrak{a}\bigr).$$

The right inclusion holds by Lemma 1.7, hence it suffices to show the left inclusion.

By (a slight generalization of) the equality statement in Proposition 1.3(*ii*), we may assume that $\mathfrak{a}$ is generated by ℓ elements. Let $x \in (0_E^{*\mathfrak{a}^{\ell-1}\mathfrak{b}^t} :_E \mathfrak{a})$, in which case there exists $c \in R^\circ$ such that

$$c\mathfrak{a}^{p^e(\ell-1)}\mathfrak{b}^{\lceil p^e t\rceil} \otimes \mathfrak{a}x = c\mathfrak{a}^{p^e(\ell-1)}\mathfrak{a}^{[p^e]}\mathfrak{b}^{\lceil p^e t\rceil} \otimes x = 0$$

in $F^e_* R \otimes_R E$ for all $e \gg 0$. Since $\mathfrak{a}$ is generated by ℓ elements, the pigeon-hole principle implies $\mathfrak{a}^{p^e\ell} = \mathfrak{a}^{p^e(\ell-1)}\mathfrak{a}^{[p^e]}$. Thus, we have $c\mathfrak{a}^{p^e\ell}\mathfrak{b}^{\lceil p^e t\rceil} \otimes x = 0$, and therefore $x \in 0_E^{*\mathfrak{a}^\ell\mathfrak{b}^t}$. □

We can now prove a version of the promised application.

Corollary 2.2 (Modified Briançon–Skoda, cf. [HY03, Thm. 2.1 and Rem. 2.2])**.** *Let R, $\mathfrak{a}$, and ℓ be as in Theorem 2.1. Then, we have*

$$\tau(\mathfrak{a}^{n+\ell-1}) \subseteq \mathfrak{a}^n \tag{2}$$

for all $n \geq 0$.

In particular, with R, $\mathfrak{a}$, and ℓ as above, if R is strongly F-regular, then

$$\overline{\mathfrak{a}^{n+\ell-1}} \subseteq \mathfrak{a}^n. \tag{3}$$

Proof. For the first inclusion, we have

$$\tau(\mathfrak{a}^{n+\ell-1}) = \tau(\mathfrak{a}^{n+\ell-2}) \cdot \mathfrak{a} = \cdots = \tau(\mathfrak{a}^{n+\ell-(n+1)}) \cdot \mathfrak{a}^n \subseteq \mathfrak{a}^n$$

by applying Theorem 2.1 n-times. The second inclusion follows from Propositions 1.3(*v*), 1.3(*i*), and 1.3(*ii*) and (2), since

$$\overline{\mathfrak{a}^{n+\ell-1}} = \tau(R) \cdot \overline{\mathfrak{a}^{n+\ell-1}} \subseteq \tau\bigl(\overline{\mathfrak{a}^{n+\ell-1}}\bigr) = \tau(\mathfrak{a}^{n+\ell-1}) \subseteq \mathfrak{a}^n.$$ □

Remark 2.3. The last inclusion (3) for smooth $\mathbf{C}$-algebras was first proved by Skoda and Briançon using analytic methods [SB74, Thm. 3], and was shown for all regular rings by Lipman and Sathaye [LS81, Thm. 1″]. The last inclusion (3) can also be obtained using the usual notion of tight closure, and in fact holds for *weakly* F-regular rings; see [HH90, Thm. 5.4].

3. Subadditivity and Symbolic Powers

For our second application, we first need some more technical results about test ideals.

Proposition 3.1 (cf. [Tak06, Props. 2.1 and 2.2]). *Let R be a ring, and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in R intersecting R°. Then, for every $t, s \in \mathbf{R}_{\geq 0}$, we have*

$$\big(0_E^{*\mathfrak{a}^t\mathfrak{b}^s} :_E \tau(\mathfrak{a}^t)^{*\mathfrak{a}^t}\big) \supseteq \big(0_E^{*\mathfrak{b}^s} :_E \tau(\mathfrak{a}^t)\big).$$

Proof. Let $x \in (0_E^{*\mathfrak{b}^s} :_E \tau(\mathfrak{a}^t))$, in which case there exists $c \in R^\circ$ such that

$$c\mathfrak{b}^{\lceil p^e s\rceil} \otimes \tau(\mathfrak{a}^t)x = c\mathfrak{b}^{\lceil p^e s\rceil}\tau(\mathfrak{a}^t)^{[p^e]} \otimes x = 0$$

in $F^e_* R \otimes_R E$ for all $e \gg 0$. To show that $x \in (0_E^{*\mathfrak{a}^t\mathfrak{b}^s} :_E \tau(\mathfrak{a}^t)^{*\mathfrak{a}^t})$, we want to show that for every element $r \in \tau(\mathfrak{a}^t)^{*\mathfrak{a}^t}$, we have $rx \in 0_E^{*\mathfrak{a}^t\mathfrak{b}^s}$. By definition of $\tau(\mathfrak{a}^t)^{*\mathfrak{a}^t}$, there exists $d \in R^\circ$ such that $d\mathfrak{a}^{\lceil p^e t\rceil}r^{p^e} \subseteq \tau(\mathfrak{a}^t)^{[p^e]}$ for all $e \gg 0$. Thus, we have

$$cd\mathfrak{a}^{\lceil p^e t\rceil}\mathfrak{b}^{\lceil p^e s\rceil} \otimes rx = cd\mathfrak{a}^{\lceil p^e t\rceil}\mathfrak{b}^{\lceil p^e s\rceil}r^{p^e} \otimes x \subseteq c\mathfrak{b}^{\lceil p^e s\rceil}\tau(\mathfrak{a}^t)^{[p^e]} \otimes x = 0,$$

hence $x \in (0_E^{*\mathfrak{a}^t\mathfrak{b}^s} :_E \tau(\mathfrak{a}^t)^{*\mathfrak{a}^t})$. ☐

Theorem 3.2 (Subadditivity, cf. [HY03, Thm. 6.10(2); Tak06, Thm. 2.4]). *Let R be a regular ring that is F-finite or complete local, and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in R intersecting R°. Then, for every $t, s \in \mathbf{R}_{\geq 0}$, we have*

$$\tau(\mathfrak{a}^t\mathfrak{b}^s) \subseteq \tau(\mathfrak{a}^t) \cdot \tau(\mathfrak{b}^s).$$

Proof. We note that if R is not complete local, then we can reduce to the complete local case using Proposition 1.5.

We first claim that it suffices to show that

$$\tau(\mathfrak{a}^t)^{*\mathfrak{a}^t} = R. \tag{4}$$

By Proposition 3.1 and Lemma 1.7, we have

$$0_E^{*\mathfrak{a}^t\mathfrak{b}^s} = \big(0_E^{*\mathfrak{a}^t\mathfrak{b}^s} :_E R\big) = \big(0_E^{*\mathfrak{a}^t\mathfrak{b}^s} :_E \tau(\mathfrak{a}^t)^{*\mathfrak{a}^t}\big) \supseteq \big(0_E^{*\mathfrak{b}^s} :_E \tau(\mathfrak{a}^t)\big) \supseteq \big(0 :_E \tau(\mathfrak{a}^t) \cdot \tau(\mathfrak{b}^s)\big).$$

Taking annihilators, Matlis duality (Proposition 1.6) implies $\tau(\mathfrak{a}^t\mathfrak{b}^s) \subseteq \tau(\mathfrak{a}^t) \cdot \tau(\mathfrak{b}^s)$.

To show (4), it suffices to show that

$$\mathfrak{a}^{\lceil p^e t\rceil} \subseteq \tau(\mathfrak{a}^t)^{[p^e]}$$

for $e \gg 0$ by the definition of $\mathfrak{a}^t$-tight closure. By [HY03, Thms. 1.7(2) and 6.4], since R is regular, 1 is a test element for $\mathfrak{a}^t$-tight closure. Thus, we have

$$0 = \mathfrak{a}^{\lceil p^e t\rceil} \otimes 0_E^{*\mathfrak{a}^t}$$

in $F^e_* R \otimes_R E$. By Matlis duality (Proposition 1.6), we then have

$$0 = \mathfrak{a}^{\lceil p^e t\rceil} \otimes \big(0 :_E \tau(\mathfrak{a}^t)\big) = \mathfrak{a}^{\lceil p^e t\rceil} \cdot \big(0 :_E \tau(\mathfrak{a}^t)^{[p^e]}\big). \tag{5}$$

Here, the second equality follows from the isomorphism

$$F^e_* R \otimes_R E \simeq E$$

coming from the fact that $R \to F^e_* R$ is flat [Kun69, Thm. 2.1] and the description of E as $H^d_{\mathfrak{m}}(R)$. Under this identification, we have $F^e_* R \otimes (0 :_E \tau(\mathfrak{a}^t)) \simeq (0 :_E \tau(\mathfrak{a}^t)^{[p^e]})$. Finally, (5) implies

$$\mathfrak{a}^{\lceil p^e t\rceil} \subseteq \operatorname{Ann}_R\big(0 :_E \tau(\mathfrak{a}^t)^{[p^e]}\big) = \tau(\mathfrak{a}^t)^{[p^e]}$$

by Matlis duality (Proposition 1.6). ☐

We now give the second application of the theory.

Theorem 3.3 (cf. [HH02, Thm. 1.1(a)])**.** *Let R be a regular ring with reduced formal fibers, let $\mathfrak{a} \subseteq R$ be an ideal intersecting R°, and let h be the maximal height of the associated primes of $\mathfrak{a}$. Then, $\mathfrak{a}^{(hn+kn)} \subseteq (\mathfrak{a}^{(k+1)})^n$ for all integers $n \geq 1$ and $k \geq 0$. In particular, $\mathfrak{a}^{(hn)} \subseteq \mathfrak{a}^n$.*

Proof. By replacing R with $R[X]$ for an indeterminate X, we may assume that the residue fields of R are infinite [HH02, Discussion 2.3]. The proof below is written for F-finite regular rings R, although if R is not F-finite, then one may reduce to the case where R is complete local using the assumption on formal fibers and the strategy in [MS18, Thm. 7.4].

Since R is regular, it is also strongly F-regular, hence Proposition 1.3(v) and subadditivity (Theorem 3.2) imply

$$\mathfrak{a}^{(hn+kn)} \subseteq \tau(\mathfrak{a}^{(hn+kn)}) \subseteq \Bigl(\tau\bigl((\mathfrak{a}^{(hn+kn)})^{1/n}\bigr)\Bigr)^n$$

for all $n \geq 0$. It therefore suffices to show $\tau((\mathfrak{a}^{(hn+kn)}R_\mathfrak{p})^{1/n}) \subseteq \mathfrak{a}^{k+1}R_\mathfrak{p}$. We have

$$\tau((\mathfrak{a}^{(hn+kn)}R_\mathfrak{p})^{1/n}) = \tau\bigl((\mathfrak{a}^{hn+kn}R_\mathfrak{p})^{1/n}\bigr) = \tau(\mathfrak{a}^{h+k}R_\mathfrak{p}) \subseteq \mathfrak{a}^{k+1}R_\mathfrak{p}$$

by the definition of symbolic powers, Proposition 1.3(iv), and (2) in the modified Briançon–Skoda theorem (Corollary 2.2), where we use the infinite residue field to use [HS06, Prop. 8.3.7]. ☐

Remark 3.4. The proof here follows [Har05, Thm. 2.21]. This uniform containment theorem was originally proved by Ein–Lazarsfeld–Smith for smooth **C**-algebras using multiplier ideals [ELS01, Thm. A]. Hochster–Huneke generalized their result to regular rings containing a field using tight closure and reduction modulo p [HH02, Thm. 1.1(a)]. The mixed characteristic case (with the addititional assumption on formal fibers) was not proved until recently by Ma–Schwede using perfectoid techniques [MS18, Thm. 7.4].

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