A CHARACTERISTIC ZERO PROOF OF THE LIFTING THEOREM VIA SESHADRI CONSTANTS

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We give a characteristic zero proof of the following generalization of [Mur19, Thm. C].

Theorem 1. *Let (X,Δ) be an effective log pair such that X is a complex projective normal variety, and such that $K_X+\Delta$ is $\mathbf{Q}$-Cartier. Consider a closed point $x \in X$ such that (X,Δ) is log canonical at x, and suppose that D is a Cartier divisor on X such that $H = D-(K_X+\Delta)$ satisfies*

$$\varepsilon\bigl(\lVert H\rVert;x\bigr) > \operatorname{lct}_x\bigl((X,\Delta);\mathfrak{m}_x\bigr).$$

Then, $\mathcal{O}_X(D)$ has a global section not vanishing at x.

Proof. We proceed in a sequence of steps.

Step 1. *Replacing the moving Seshadri constant with the usual Seshadri constant.*

By definition of $\varepsilon(\lVert H\rVert;x)$ [Mur19, Def. 7.1.1], there exists a birational morphism $f\colon X_1 \to X$ where X_1 is a normal projective variety, such that f is an isomorphism around x, and such that $f^*H \equiv_{\mathbf{R}} A+F$ for $\mathbf{Q}$-Cartier divisors A and F for which

$$\varepsilon(A;x) > \operatorname{lct}_x\bigl((X,\Delta);\mathfrak{m}_x\bigr) =: c \in \mathbf{Q}. \tag{1}$$

Step 2. *Replacing Δ with a $\mathbf{Q}$-divisor with simple normal crossing support.*

We consider a log resolution $g\colon X_2 \to X_1$ of

$$(X_1, f^{-1}_*\Delta + \operatorname{Exc}(f) + F, f^{-1}\mathfrak{m}_x\cdot\mathcal{O}_{X_1}). \tag{2}$$

We denote

$$\begin{array}{ccc} X_2 & \xrightarrow{\ g\ } & X_1 \\ & \searrow^{\mu} & \big\downarrow f \\ & & X \end{array}$$

and let E be the effective divisor defined by $\mu^{-1}\mathfrak{m}_x\cdot\mathcal{O}_{X_2}$. By the long exact sequence on cohomology, it suffices to show that the top horizontal arrow in the commutative diagram

$$\begin{array}{ccc} H^1\bigl(X,\mathfrak{m}_x\cdot\mathcal{O}_X(D)\bigr) & \longrightarrow & H^1\bigl(X,\mathcal{O}_X(D)\bigr) \\ \big\downarrow & & \big\downarrow \wr \\ H^1\bigl(X_2,\mathcal{O}_{X_2}(\mu^*D-E)\bigr) & \longrightarrow & H^1\bigl(X_2,\mathcal{O}_{X_2}(\mu^*D)\bigr) \\ \big\downarrow & & \big\downarrow \wr \\ H^1\bigl(X_2,\mathcal{O}_{X_2}(\mu^*D+G-E)\bigr) & \longrightarrow & H^1\bigl(X_2,\mathcal{O}_{X_2}(\mu^*D+G)\bigr) \end{array} \tag{3}$$

is injective, where G is an arbitrary effective μ-exceptional divisor. Since the right vertical arrows are isomorphisms by the projection formula and the fact that $\mathcal{O}_X \simeq \mu_*\mathcal{O}_{X_2}(G)$ for every effective μ-exceptional divisor G, the commutativity of the diagram implies that it suffices to show that the bottom horizontal arrow is injective for some effective μ-exceptional divisor G.

Date: May 11, 2019.

The goal of the rest of the proof is to maneuver to a setting where Fujino's Kollár-type injectivity theorem [Fuj17, Thm. 5.4.1] applies.

Step 3. *Appropriately choosing G and a divisor $E_x \preceq E$ so that $\mu^*D + G - E_x$ is $\mathbf{Q}$-linearly equivalent to the sum of a log canonical divisor and a big and nef divisor.*

Write

$$K_{X_2} - \mu^*(K_X + \Delta) - cE = P - N$$

where P and N are effective $\mathbf{Q}$-divisors without common components, such that $P - N$ has simple normal crossings support and P is μ-exceptional. We moreover write $N = N_x + N_0$, where $\mu(\operatorname{Supp} N_x) = \{x\}$. Setting $G = \lceil P \rceil$ in (3) and looking at the long exact sequence on cohomology, it suffices to show that the top horizontal arrow in the commutative diagram

$$\begin{array}{ccc}
H^0\big(X_2, \mathcal{O}_{X_2}(\mu^*D + \lceil P \rceil)\big) & \longrightarrow & H^0\big(E, \mathcal{O}_E\big((\mu^*D + \lceil P \rceil)|_E\big)\big) \\
\big\uparrow & & \big\uparrow \wr \\
H^0\big(X_2, \mathcal{O}_{X_2}(\mu^*D + \lceil P \rceil - \lfloor N_0 + g^*F \rfloor)\big) & \longrightarrow & H^0\big(E, \mathcal{O}_E\big((\mu^*D + \lceil P \rceil - \lfloor N_0 + g^*F \rfloor)|_E\big)\big) \\
\big\| & & \big\downarrow \wr \\
H^0\big(X_2, \mathcal{O}_{X_2}(\mu^*D + \lceil P \rceil - \lfloor N_0 + g^*F \rfloor)\big) & \longrightarrow & H^0\big(E_x, \mathcal{O}_{E_x}\big((\mu^*D + \lceil P \rceil - \lfloor N_0 + g^*F \rfloor)|_{E_x}\big)\big)
\end{array}$$

is surjective, where E_x is the sum of components in N_x with coefficient 1. Since the top right vertical arrow is an isomorphism by the fact that $E \cap \operatorname{Supp}(N_0 + g^*F) = \emptyset$, and the bottom right vertical arrow is an isomorphism by the fact that both groups are isomorphic to $\mathbf{C}$ by the Shokurov–Kollár connectedness theorem for log triples [dFEM, Thm. 3.2.3], we see that it suffices to show that the bottom horizontal arrow is surjective. We now set

$$\Delta_2 \coloneqq N_x + N_0 + g^*F - \lfloor N_0 + g^*F \rfloor + \lceil P \rceil - P,$$

which is an effective $\mathbf{Q}$-divisor with simple normal crossing support and coefficients in $(0, 1]$. By the long exact sequence on cohomology, it suffices to show that

$$H^1\big(X_2, \mathcal{O}_{X_2}(\mu^*D + \lceil P \rceil - \lfloor N_0 + F \rfloor - E_x)\big) \longrightarrow H^1\big(X_2, \mathcal{O}_{X_2}(\mu^*D + \lceil P \rceil - \lfloor N_0 + F \rfloor)\big) \tag{4}$$

is injective. We rewrite the divisor on the left-hand side of (4) as

$$\begin{aligned}
\mu^*D + \lceil P \rceil - \lfloor N_0 + g^*F \rfloor - E_x &\sim_{\mathbf{Q}} \mu^*H + \mu^*(K_X + \Delta) + \lceil P \rceil - \lfloor N_0 + g^*F \rfloor - E_x \\
&\sim_{\mathbf{Q}} \mu^*H - g^*F - cE + K_{X_2} + \Delta_2 - E_x \\
&\sim_{\mathbf{Q}} \mu^*H - g^*F - (c + \delta)E + K_{X_2} + \Delta_2 - E_x + \delta E.
\end{aligned} \tag{5}$$

Note that $\Delta_2 - E_x$ has simple normal crossing support and coefficients in $(0, 1]$ and that the coefficients of components appearing in E have coefficients in $(0, 1)$, hence $\Delta_2 - E_x + \delta E$ has simple normal crossings support and coefficients in $(0, 1]$ for all $0 < \delta \ll 1$.

We note that [Fuj17, Thm. 5.4.1] does not yet apply since Fujino's result does not incorporate a big and nef divisor in the statement. We therefore perform the following:

Step 4. *Replacing the big and nef divisor $\mu^*H - g^*F - (c+\delta)E$ in (5) with an appropriate $\mathbf{Q}$-divisor.*

By (1), we first fix a δ as above such that $c + \delta < \varepsilon(A; x)$, in which case we also have

$$\mathbf{B}\big(\mu^*H - g^*F - (c + \delta)E\big) \cap E = \emptyset$$

by [DM19, Lem. 3.3]. By [Laz04a, Prop. 2.1.21], we can choose an integer m sufficiently large and divisible such that

$$\mathbf{B}\big(\mu^*H - g^*F - (c + \delta)E\big) = \operatorname{Bs}\big|m\big(\mu^*H - g^*F - (c + \delta)E\big)\big|_{\mathrm{red}} \eqqcolon S.$$

By Bertini's theorem [Har77, Cor. III.10.9 and Rem. III.10.9.3], we can choose

$$B \in \bigl|m\bigl(\mu^*H - g^*F - (c+\delta)E\bigr)\bigr|$$

such that B intersects E and each component of Δ_2 transversely away from S. Now let $h\colon X_3 \to X_2$ be a log resolution for B and (X_2, Δ_2) that is an isomorphism away from S, which exists by [Kol13, Thm. 10.45.2]. We write

$$h^*B = B' + C \qquad \text{and} \qquad h^*\Delta_2 = h_*^{-1}\Delta_2 + C_1,$$

where B' is a smooth divisor intersecting E transversely, and C, C_1 are supported on $h^{-1}(S)$. Define

$$C' \coloneqq \left\lfloor \frac{1}{m}C + C_1 \right\rfloor \qquad \text{and} \qquad \Delta_3 \coloneqq h^*(\Delta_2 - E_x + \delta E) + \frac{1}{m}h^*B - C'.$$

Note that Δ_3 has simple normal crossing support containing h^*E, and has coefficients in $(0,1]$ by the assumption on the log resolution and by definition of C'. Pulling back (5) to X_3, we then have

$$\begin{aligned} h^*\bigl(\mu^*D + \lceil P\rceil - \lfloor N_0 + g^*F\rfloor - E_x\bigr) &\sim_{\mathbf{Q}} \frac{1}{m}h^*B + h^*(K_{X_2} + \Delta_2 - E_x + \delta E) \\ &\sim_{\mathbf{Q}} h^*K_{X_2} + \Delta_3 + C' \end{aligned}$$

hence

$$h^*\bigl(\mu^*D + \lceil P\rceil - \lfloor N_0 + g^*F\rfloor - E_x\bigr) + K_{X_3/X_2} - C' \sim_{\mathbf{Q}} K_{X_3} + \Delta_3. \tag{6}$$

It now remains to apply [Fuj17, Thm. 5.4.1].

Step 5. *Applying Fujino's Kollár-type injectivity theorem* [Fuj17, Thm. 5.4.1].

We recall that it suffices to show that the top horizontal arrow in the commutative diagram

$$\begin{array}{ccc} H^1\bigl(X_2, \mathcal{O}_{X_2}(\mu^*D + \lceil P\rceil - \lfloor N_0 + F\rfloor - E_x)\bigr) & \longrightarrow & H^1\bigl(X_2, \mathcal{O}_{X_2}(\mu^*D + \lceil P\rceil - \lfloor N_0 + F\rfloor)\bigr) \\ \wr\downarrow & & \wr\downarrow \\ H^1\bigl(X_3, \mathcal{O}_{X_3}\bigl(h^*(\mu^*D + \lceil P\rceil - \lfloor N_0 + F\rfloor - E_x) + K_{X_3/X_2}\bigr)\bigr) & \longrightarrow & H^1\bigl(X_3, \mathcal{O}_{X_3}\bigl(h^*(\mu^*D + \lceil P\rceil - \lfloor N_0 + F\rfloor) + K_{X_3/X_2}\bigr)\bigr) \end{array}$$

is injective. Note that the vertical arrows are isomorphisms by the fact that K_{X_3/X_2} is h-exceptional and by the projection formula, hence it suffices to show the bottom horizontal arrow is injective. By the long exact sequence on cohomology, it suffices to show that the top horizontal arrow in the commutative diagram

$$\begin{array}{ccc} H^0\bigl(X_3, \mathcal{O}_{X_3}\bigl(h^*(\mu^*D + \lceil P\rceil - \lfloor N_0 + F\rfloor) + K_{X_3/X_2}\bigr)\bigr) & \longrightarrow & H^0\bigl(h^*E_x, \mathcal{O}_{h^*E_x}\bigl((\mu^*D + \lceil P\rceil - \lfloor N_0 + g^*F\rfloor + K_{X_3/X_2})|_{h^*E_x}\bigr)\bigr) \\ \uparrow & & \wr\uparrow \\ H^0\bigl(X_3, \mathcal{O}_{X_3}\bigl(h^*(\mu^*D + \lceil P\rceil - \lfloor N_0 + F\rfloor) + K_{X_3/X_2} - C'\bigr)\bigr) & \longrightarrow & H^0\bigl(h^*E_x, \mathcal{O}_{h^*E_x}\bigl((\mu^*D + \lceil P\rceil - \lfloor N_0 + g^*F\rfloor + K_{X_3/X_2} - C')|_{h^*E_x}\bigr)\bigr) \end{array}$$

is surjective. Since the right vertical arrow is an isomorphism by the fact that C' is supported away from $h^*E_x \preceq h^*E$, we see that it suffices to show that the bottom horizontal arrow is surjective. This follows from the $\mathbf{Q}$-linear equivalence (6) and from [Fuj17, Thm. 5.4.1], since Δ_3 has simple normal crossing support and coefficients in $(0,1]$ and h^*E_x is contained in the support of Δ_3. □

References

[dFEM] T. de Fernex, L. Ein, and M. Mustaţă. *Vanishing theorems and singularities in birational geometry.* Dec. 8, 2014. URL: http://homepages.math.uic.edu/~ein/DFEM.pdf. 2

[DM19] Y. Dutta and T. Murayama. "Effective generation and twisted weak positivity of direct images." *Algebra Number Theory* 13.2 (2019), pp. 425–454. DOI: 10.2140/ant.2019.13.425. MR: 3927051. 2

[Fuj17] O. Fujino. *Foundations of the minimal model program.* MSJ Memoirs, Vol. 35. Tokyo: Math. Soc. Japan, 2017. DOI: 10.2969/msjmemoirs/035010000. MR: 3643725. 2, 3

[Har77] R. Hartshorne. *Algebraic geometry.* Grad. Texts in Math., Vol. 52. New York-Heidelberg: Springer-Verlag, 1977. DOI: 10.1007/978-1-4757-3849-0. MR: 463157. 3

[Kol13] J. Kollár. *Singularities of the minimal model program.* With the collaboration of S. J. Kovács. Cambridge Tracts in Math., Vol. 200. Cambridge: Cambridge Univ. Press, 2013. DOI: 10.1017/CBO9781139547895. MR: 3057950. 3

[Laz04a] R. Lazarsfeld. *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series.* Ergeb. Math. Grenzgeb. (3), Vol. 48. Berlin: Springer-Verlag, 2004. DOI: 10.1007/978-3-642-18808-4. MR: 2095471. 2

[Mur19] T. Murayama. *Seshadri constants and Fujita's conjecture via positive characteristic methods.* Ph.D. thesis. University of Michigan, 2019, 206 pp. 1

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