• Computing Syzygies — Juliette Bruce

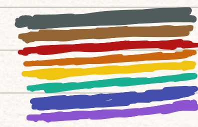
of algebraic varieties.

Where I work and live occupies the ancestral lands of the Ho-Chunk people, and where I am giving this talk from is less than 2 blocks away from where Tony Robinson was murdered by the police.

Black and Indigenous Lives Matter.

Tonight/tomorrow marks the 51st anniversary of the start of the Stonewall Riot/Revolution, a pivotal moment in the fight for LGBTQ+ equality lead by Black, Brown, Indigenous trans women of color. Happy Pride.

Black and Indigenous LGBTQ+ Lives Matter.



$$X \xhookrightarrow{|L|} \mathbb{P}^r$$

sm. proj. variety

- $S = \mathbb{C}[x_0, x_1, \dots x_n]$
- I_X = homg. defining ideal $X \subseteq \mathbb{P}^r$
- $S_X = S/I_X$

$$0 \leftarrow S_X \leftarrow \underbrace{F_0 \leftarrow F_1 \leftarrow \cdots \quad \leftarrow F_p \leftarrow \cdots \quad \leftarrow F_r}_{\text{min. graded free res.}} \leftarrow 0$$

$$\beta_{p,q}(X \subseteq \mathbb{P}^r) = \#\left\{ \begin{matrix} \text{min. gens of } F_p \\ \text{of deg. } q \end{matrix} \right\} = \#\left\{ \begin{matrix} p\text{-syzygies} \\ \text{of deg. } q \end{matrix} \right\}$$

graded Betti #'s / Betti #'s / syzygies

* The syzygies/Betti #'s are extrinsic (e.g. depend on the embedding $X \subseteq \mathbb{P}^r$) *

• **Thm:** (Ein, Green, Lazarsfeld): Let $X \subseteq \mathbb{P}^r$ be a smooth curve, if $\deg(X) \gg 0$ then:

$$\beta_{p,p+1}(X \subseteq \mathbb{P}^r) \neq 0 \iff 0 \leq p \leq r - \mathrm{gon}(X)$$

gonality of X !!

* The Betti #'s capture some of the intrinsic geometry of smooth curves *

• **Central Question:** How is the intrinsic geometry of higher dim. varieties (e.g. dim X ≥ 2) captured by their syzygies?

p-very ample
p-set ample

• **Simpler Question:** What are the syzygies/Betti #'s of $\mathbb{P}^2$?

e.g. next case with simplest geometry.

$$\mathbb{P}^2 \xhookrightarrow{|d|} \mathbb{P}^r$$
$$[x:y:z] \longmapsto [x^d : x^{d-1}y : \cdots \cdots z^d]$$

d'uple Veronese map
X = Image = Veronese surf.

$$\beta_{p,q}(\mathbb{P}^2, d) = \beta_{p,q}(\mathbb{P}^2 \xhookrightarrow{|d|} \mathbb{P}^r)$$

= Betti #'s of d'uple Veronese surface

= Betti #'s of $R^{(d)}$ ← Veronese subring
$R = \mathbb{C}[x,y,z]$

$$\beta_{p,q}(\mathbb{P}^2, d) = \dim_{\mathbb{C}} \mathrm{Tor}_p^S(S_X, \mathbb{C})_q$$

resolve S_X symbolically via g.b. methods.

M2:
d = 1,2 easy
d = 3 few seconds
d = 4 ~5 minutes
d ≥ 5 :(

why? # variables ≈ d^2 !!

① Resolve $\mathbb{C}$ (as an S-module) via Koszul complex.

② Tensor by $S_X \cong R^{(d)}$

avoids having to compute presentation of S_X!

③ Compute dim. of cohomology of complex

e.g.

Compute the ranks of matrices !!

sparse

• But the matrices we are dealing w/ are HUGE!! ← e.g. $10^7 \times 10^7$

• Results: (Bruce, Erman, Goldstein, Yang): Compute the Betti #'s of $\mathbb{P}^2$ for $1 \le d \le 6$ $(0 \le b < d)$ as well as the corresponding:

- multigraded Betti #'s
- Schur Betti #'s
- Boij-Söderberg Decompositions

* Similar work by Greco, Martino and Castryck, Cools, Demeyer, Lemmens *

• How: ① The complex respects the $\mathbb{Z}^3$-multigrading on R
$\Rightarrow$ differentials are block diagonal.

② Sparse numerical linear algebra
← e.g. LU/QR decompositions.

③ high performance, high throughput computing

"super computers"

hundreds/thousands of computers working at once.

e.g. LHC/Human Genome

See: https://syzygydata.com

Schur Veronese for M2

• Conjecture: (Ein, Erman, Lazarsfeld): There exists an explicit set of monomial syzygies $E_{p,q}(\mathbb{P}^2, d)$ such that

$$\beta_{p,q}(\mathbb{P}^2, d) \neq 0 \iff E_{p,q}(\mathbb{P}^2, d) \neq \emptyset$$

* A small subset of very special syzygies is controlling non-vanishing.

• Conjecture: (Bruce, Erman, Goldstein, Yang): With $E_{p,q}(\mathbb{P}^2, d)$ as above:

$$\mathrm{dw}\left\{\begin{matrix}p\text{-syzygies of}\\ \text{deg. } q\end{matrix}\right\} = \mathrm{dw}\, E_{p,q}(\mathbb{P}^2, d).$$

dominant schur weights.