When does a p^{-1}-linear map exist?

Land acknowledgment: I work on the traditional territoria of The Three Fires People:

The Ojibwe (keepers of faith)
The Odawa (keepers of trade)
The Bodéwadmi (keepers of the fire)

I am giving this talk on the land of the Kiikaapoi.

Based on joint work with
Takumi Murayama '20,
Karen Smith '18,
& Takumi Murayama + Karen Smith
in preparation.

Throughout R is a noetherian domain of prime char. $p > 0$ and

$$K := \mathrm{Frac}(R).$$

Frobenius : $\quad F^e : R \longrightarrow F_*^e R \qquad$ [Detects singularities of R]
$(e > 0)$ $\qquad\qquad r \mapsto r^{p^e}$

Here $F_*^e R$ is R as a ring but with R-mod structure given by

$$r \in R,\ x \in F_*^e R \Rightarrow r \cdot x = r^{p^e} x \quad \text{(restriction of scalars)}$$

A p^{-e}-linear map is an R-linear map

$$\varphi : F_*^e R \longrightarrow R.$$

Example : $\quad R = \mathbb{F}_p[x, y]$, then $F_* R$ is a free R-mod with basis

$$x^i y^j, \quad 0 \le i, j \le p-1.$$

$$\varphi : F_* R \longrightarrow R$$

on the basis given by $\varphi(x^i y^j) = \begin{cases} 1 & \text{if } i = 0 = j \\ 0 & \text{otherwise.} \end{cases}$

This is a Frobenius splitting

Why do we care about existence of nonzero p^{-e}-linear maps?

- global variants of such maps on a variety X, especially splittings, imply X satisfies Kodaira vanishing [Mehta - Ramanathan]

  Kodaira vanishing fails in general in char. p (Raynaud)

- used extensively in the theory of test ideals, a prime char. analogue of multiplier ideals

  [Hochster, Huneke, Smith, Hara, Yoshida, Takagi, Watanabe, Lyubeznik, Aberbach, Enescu, Schwede, Blickle, Tucker, Sharp among others]

- used in the study of F-signature, and more recently, its non-local variant.

  [Smith, Van den Bergh, Huneke, Leuschke, Tucker, Aberbach, Enescu, Yao, Singh, De Stefani, Polstra among others]

- existence of "sufficiently many" such maps implies R is Cohen-Macaulay [Hochster - Huneke]

  Strongly F-regular rings.

- If $K = \mathrm{Frac}(R)$ satisfies $[K : K^p] < \infty$, then existence of a nonzero p^{-1}-linear map implies R is excellent [Smith - D]

▶ Large class of rings that behave well under integral closures, completions, openness of regular and other loci.

▶ Deep thms such as Resolution of Singularities conjectured to hold for this class.

Question: When does R have nonzero p^{-e}-linear maps?

Example / Exercise: If $F : R \to F_* R$ is finite, then nonzero p^{-e}-linear maps exist!

If $[K : K^p] < \infty$, then existence of a nonzero p^{-e}-linear maps $\Rightarrow$ Frobenius is finite. [Smith - D]

Above example and its converse give many examples of non-excellent rings with NO nonzero p^{-e}-linear maps.

Folklore: If R is "nice", for example, if R is excellent, then does R admit nonzero p^{-e}-linear maps?

Theorem A [Murayama–D]: For each integer $n \geq 0$, $\exists$

- excellent
- Henselian
- regular local

ring R of Krull dim n that does not admit ANY nonzero p^{-e}-linear map.

Thus, $\exists$ excellent F-pure rings that are NOT F-split.

Answers a long-standing question of Hochster, also raised by others like Smith, Zhang, Schwede, Blickle etc.

Folklore question has positive answer for large class of excellent, but non-F-finite rings.

Theorem B [Murayama–D]: If R is essentially of finite type over a complete local ring, then R has nonzero p^{-1}-linear maps.

Furthermore, for such R, F-pure $\Rightarrow$ F-split.

Open (??) Question: Are there non-excellent local R that admit non-trivial p^{-e}-linear maps?

If we drop local hypothesis then can construct such examples (forthcoming work Murayama - D)

Thm A proof sketch Krull dim 1: We use a construction from rigid analytic geometry.

A NA field $(k, |\,|)$ is a field equipped with

$$|\,| : k \longrightarrow \mathbb{R}_{\geq 0}$$

satisfying

① $|x| = 0 \iff x = 0$
② $|xy| = |x|\,|y|$
③ $|x+y| \leq \max\{|x|, |y|\}$ (ultrametric Δ - inequality)

$(k, |\,|)$ becomes a metric space via $|x-y|$ and we assume k is complete wrt this metric.

For such k have the Tate algebra

$$T_1(k) := \left\{ \sum_{i=0}^{\infty} a_i x^i \in k[[x]] : |a_i| \to 0 \text{ as } i \to \infty \right\}.$$

$\downarrow$

Rigid analytic analogue of $k[x]$.

$T_1(k)$ is
- regular (not local)
- excellent (Kiehl)
- Euclidean domain.

Murayama - D : For $(k, |\ |)$ of char $p > 0$, $T_1(k)$ has a nonzero p^{-e}-linear map $\Longleftrightarrow$ k has a nonzero continuous p^{-e}-linear map.

↓

Gabber / Blaszczok (now Rzepka) - Kuhlmann : $\exists$ NA fields k that DONT admit continuous p^{-1}-linear maps.

This uses non-Archimedean functional analysis.

To get local, Henselian counterexample you localize $T_1(k)$ at the max ideal (X) and then Henselize, for a NA field k given by Gabber / Rzepka - Kuhlmann.

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