Specialization of Integral Closure of Ideals by General Elements

Based on joint work with Rachel Lynn

Lindsey Hill

Purdue University

June 2020

Basic Definitions

Definition

Let I be an ideal of a ring R. An element $x \in R$ is integral over I if it satisfies an equation of integral dependence of the form

$$x^n + a_1 x^{n-1} + \ldots + a_n = 0$$

with $a_i \in I^i$. The collection of all elements integral over I is the integral closure of I, denoted $\overline{I}$.

Example of Integral Closure

Example

Let $R = k[x, y]$ and $I = (x^3, x^2y, y^3)$. Then $\bar{I} = (x^3, x^2y, xy^2, y^3)$.

- Fact: The integral closure of a monomial ideal is a monomial ideal.
- Notice that xy^2 satisfies $z^2 - (x^2y)(y^3) = 0$, so $(x, y)^3 \subset \bar{I}$.
- Any monomial integral over I has degree at least 3, hence $\bar{I} \subset (x, y)^3$.

Question

Given an integrally closed ideal, can we reduce the height and maintain integral closedness?

An example

Let $R = k[x, y]$ and let $m = (x, y)$.
Notice $m^2 = (x^2, xy, y^2)$ is integrally closed ideal of height two.
Is $\frac{m^2}{(x^2)}$ an integrally closed ideal of $\frac{R}{(x^2)}$?
The answer: No. Notice that x satisfies an equation of integral dependence $z^2 = 0$ in $R/(x^2)$ and therefore, $x \in \overline{m^2/(x^2)} \setminus m^2/(x^2)$.

The generic element approach

Let R be a Noetherian (local) ring and $I = (a_1, \ldots, a_n)$ an R-ideal. Let $T_1, \ldots, T_n$ be variables over R. Recall that $R[T_1, \ldots, T_n]$ and $R(T) = R[T_1, \ldots, T_n]_{m_R R[T]}$ are faithfully flat extensions of R. Then

- $\operatorname{ht} I = \operatorname{ht} IR[T] = \operatorname{ht} IR(T)$
- $\bar{I}R[T] = \overline{IR[T]}$
- $\bar{I}R(T) = \overline{IR(T)}$

and $\alpha = a_1T_1 + a_2T_2 + \ldots + a_nT_n$ is a generic element of $IR[T]$ or $IR(T)$.

A theorem of Itoh (1989)

Let (R, m) be an analytically unramified, Cohen-Macaulay local ring of dimension $d \geq 2$. Let I be a parameter ideal for R. Assume that R/m is infinite. Then there exists a system of generators $x_1, \ldots, x_d$ for I such that if we put $x = \sum_i x_i T_i$ and $I' = IR(T)$, where $R(T) = R[T]_{m[T]}$ with $T = (T_1, \ldots, T_d)$ d indeterminates, then

$$\overline{I'/(x)} = \overline{I'}/(x).$$

A generalization by Hong-Ulrich (2014)

Let R be a Noetherian, locally equidimensional, universally catenary ring such R_{red} is locally analytically unramified. Let $I = (a_1, \ldots, a_n)$ be an R-ideal of height at least 2. Let $R' = R[T_1, \ldots, T_n]$ be a polynomial ring in the variables $T_1, \ldots, T_n$, $I' = IR'$, and $x = \sum_{i=1}^{n} T_i a_i$. Then

$$\overline{I'/(x)} = \overline{I'}/(x).$$

Applications of Hong-Ulrich

1. Enables proofs by induction on the height of an integrally closed ideal.
2. Gives a quick proof of a result proved independently by Huneke and Itoh: Let R be a Noetherian, locally equidimensional, universally catenary ring such that R_{red} is locally analytically unramified. Let I be a complete intersection R-ideal. Then $\overline{I^{n+1}} \cap I^n = \overline{I} I^n$ for all $n \geq 0$.

Specialization by general elements (–, Lynn)

Let (R, m) be a local equidimensional excellent k-algebra, where k is a field of characteristic 0. Let I be an R-ideal of height at least 2 and let x be a general element of I. Then $\overline{I}/(x) = \overline{I/(x)}$.

Main Ingredients of the Proof

1. (Extended) Rees Algebras and Their Integral Closures
2. General Elements and Bertini's Theorems

Rees Algebras

Let R be a ring, I an ideal of R and t a variable over R. The Rees algebra of I is a subring of $R[t]$ defined by

$$R[It] = \oplus_{n \geq 0} I^n t^n.$$

The extended Rees algebra of I is the subring of $R[t, t^{-1}]$ defined as

$$R[It, t^{-1}] = \oplus_{n \in \mathbb{Z}} I^n t^n$$

with $I^n = R$ for $n \leq 0$.

Connections between the Integral Closure of Ideal and the Rees Algebra

Let R be a ring, t a variable over R and I an ideal of R. Then

$$\overline{R[It]}^{R[t]} = R \oplus \overline{I}t \oplus \overline{I^2}t^2 \oplus \overline{I^3}t^3 \oplus \dots$$

and

$$\overline{R[It, t^{-1}]}^{R[t,t^{-1}]} = \dots \oplus Rt^{-2} \oplus Rt^{-1} \oplus R \oplus \overline{I}t \oplus \overline{I^2}t^2 \oplus \dots$$

Bertini's Theorems

Let $I = (x_1, \ldots, x_n)$. Then a general element x_α of I is $x_\alpha = \sum_{i=1}^{n} \alpha_i x_i$ where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is in a Zariski open subset of k^n.

A theorem of Bertini

Let A be a local excellent k-algebra over the field k of characteristic 0 and let $x_1, \ldots, x_n \in m_A$. Let $U \subseteq D(x_1, \ldots, x_n)$ be open, so that for $p \in U$ the ring A_p satisfies Serre's Conditions (S_r) or (R_s) respectively. For general $\alpha \in k^n$ and $p \in U \cap V(x_\alpha)$ the ring $(A/x_\alpha A)_p$ also satisfies the conditions (S_r) or (R_s).

Sketch of the proof

1. Reduce to the case where R is a local normal domain.
2. Define:

$$\mathcal{A} = R[It, t^{-1}]$$

$$\mathcal{B} = \frac{R}{(x)}\left[\frac{I}{(x)}t, t^{-1}\right]$$

$$\overline{\mathcal{A}} = \overline{R[It, t^{-1}]}^{R[t,t^{-1}]}$$

$$\overline{\mathcal{B}} = \overline{\frac{R}{(x)}\left[\frac{I}{(x)}t, t^{-1}\right]}^{\frac{R}{(x)}[t,t^{-1}]}$$

Sketch of the proof

3. Consider the natural map

$$\varphi : \frac{\overline{\mathcal{A}}}{xt\overline{\mathcal{A}}} \to \overline{\mathcal{B}}.$$

   Notice that $\left[\frac{\overline{\mathcal{A}}}{xt\overline{\mathcal{A}}}\right]_1 = \overline{I}/(x)$ and $\left[\overline{\mathcal{B}}\right]_1 = \overline{I/(x)}$. For this reason, it suffices to show that the $C = \text{coker}(\varphi)$ vanishes in degree 1.

4. Define $J = (It, t^{-1})\mathcal{A}$. Show that for $p \in \text{Spec}(\overline{\mathcal{A}}) \setminus V(J\overline{\mathcal{A}})$, φ_p is an isomorphism. In the case where $It \not\subseteq p$, we apply Bertini's Theorem to $\overline{\mathcal{A}}$ to say $\left(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}\right)_p$ is normal, and since the extension $\left(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}\right)_p \to \overline{\mathcal{B}}_p$ is integral, φ_p is an isomorphism.

5. Step 4 implies that $C = H^0_J(C)$. From this, we have an embedding $[C]_n \hookrightarrow [H^2_J(\overline{\mathcal{A}})]_{n-1}$. We use a local cohomology vanishing theorem proved by Hong and Ulrich to say $[C]_1 = 0$.

References