On the Frobenius Complexity of Stanley-Reisner Rings

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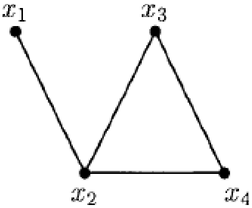
Stanley-Reisner Rings and Simplicial Complexes



0-simplices: x_1, x_2, x_3, x_4
1-simplices: $x_1x_2, x_2x_3, x_3x_4, x_2x_4$
2-simplices: $x_2x_3x_4$

Non-faces : $\{x_1, x_3\}$, $\{x_1, x_4\}$
Facets: $\{x_1, x_2\}$, $\{x_2, x_3, x_4\}$

Figure: Simplicial complex Δ

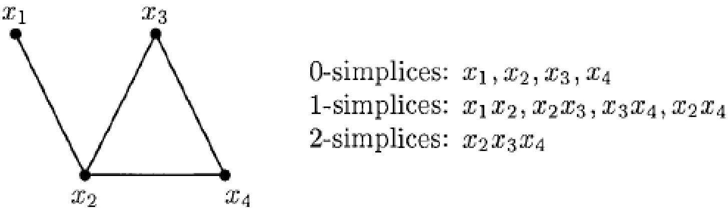


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Figure: Simplicial complex Δ

The Stanley-Reisner ring associated to our simplicial complex Δ is given by

$$k[\Delta] = \frac{k[x_1, x_2, x_3, x_4]}{(x_1x_3, x_1x_4)}.$$

Frobenius Operators

Let (R, m, k) a local ring of characteristic p.
Let $F : R \to R$ be the Frobenius map, that is $F(r) = r^p$. We have that

$$(a+b)^p = a^p + b^p,\ (a \cdot b)^p = a^p \cdot b^p,$$

for all $a, b \in R$. Therefore, the Frobenius map is a ring homomorphism.
Let $F^e : R \to R$ be the e-th iteration of the Frobenius map, that is $F^e(r) = r^q$, where $q = p^e, e \in \mathbb{N}$.

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Let $F^e : R \to R$ be the e-th iteration of the Frobenius map, that is $F^e(r) = r^q$, where $q = p^e, e \in \mathbb{N}$.
For any $e \geq 0$, we let $R^{(e)}$ be the R-algebra defined as follows: as a ring $R^{(e)}$ equals R while the R-algebra structure is defined by $rs = r^q s$, for all $r \in R$, $s \in R^{(e)}$.
In the same way, starting with an R-module M, we can define a new R-module $M^{(e)}$.

Frobenius Operators

Let $E_R := E_R(k)$ denote the injective hull of the residue field k.
An e-**th Frobenius operator(action)** on E_R is an additive map $\phi : E_R \to E_R$ such that $\phi(rz) = r^q \phi(z)$, for all $r \in R$ and $z \in E_R$.
The collection of e-th Frobenius operators(actions) on E_R is an R-module, denoted by $\mathcal{F}^e(E_R)$.

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Definition (The Frobenius Algebra of Operators)

The **algebra of the Frobenius operators** on E_R is defined by

$$\mathcal{F}(E_R) = \bigoplus_{e \geq 0} \mathcal{F}^e(E_R).$$

This is a $\mathbb{N}$-graded noncommutative ring under composition of maps and due to Matlis duality, its zero degree component is R.

The Frobenius Algebra of Operators

- If (R, m, k) is d-dimensional, local and Gorenstein ring, $E_R \cong H^d_m(R)$ and $\mathcal{F}(H^d_m(R))$ is generated by the canonical action F on $H^d_m(R)$. In this case, the Frobenius complexity of the ring R is $-\infty$.

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Question(Lyubeznik, Smith - 1999)

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The Frobenius Algebra of Operators

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Question(Lyubeznik, Smith - 1999)

Is $\mathcal{F}(E_R)$ always finitely generated as a ring over R?

- In 2009, Katzman gave an example of a ring R such that $\mathcal{F}(E_R)$ is not finitely generated as a ring over R. The ring is $R = k[x, y, z]/(xy, xz)$. The Frobenius complexity of the ring R equals 0.

The Frobenius Algebra of Operators

- Katzman raised the finite generation question for the determinantal ring of $2x2$ minors in a $2x3$ matrix. Enescu and Yao showed that the Frobenius complexity of determinantal rings can be positive, irrational and depends upon the characteristic.

The Frobenius Algebra of Operators

- Katzman raised the finite generation question for the determinantal ring of $2x2$ minors in a $2x3$ matrix. Enescu and Yao showed that the Frobenius complexity of determinantal rings can be positive, irrational and depends upon the characteristic.
- In 2012, Àlvarez, Boix and Zarzuela completely described what happens in the case of Stanley-Reisner rings: when finite generation occurs, then $\mathcal{F}(E_R)$ is principally generated.

The Frobenius Algebra of Operators

Problem

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The Frobenius Algebra of Operators

Problem

Find ways to measure the generation of $\mathcal{F}(E_R)$ over R.

- Enescu and Yao were motivated by the finite generation question to introduce a new invariant of a local ring of prime characteristic, called the Frobenius complexity.
- In the case when this Frobenius algebra is infinitely generated over R, they used the complexity sequence $\{c_e\}_{e\geq 0}$ in order to describe how far it is from being finitely generated.

Frobenius Complexity

Let $G_e := G_e(\mathcal{F}(E_R))$ be the subring of $\mathcal{F}(E_R)$ generated by elements of degree less or equal to e.
Note that $G_{e-1} \subseteq G_e$, for all e.
Moreover, $(G_e)_i = \mathcal{F}(E_R)_i$, for all $0 \leq i \leq e$ and $(G_{e-1})_e \subseteq \mathcal{F}(E_R)_e$.
We will denote the minimal number of homogeneous generators of G_e as a subring of $\mathcal{F}(E_R)$ over $\mathcal{F}(E_R)_0 = R$ by k_e.

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Proposition [Enescu, Yao]

The minimal number of generators of the R-module $\dfrac{\mathcal{F}(E_R)_e}{(G_{e-1})_e}$ equals $k_e - k_{e-1}$, for all e.

Frobenius Complexity

Definition [Enescu, Yao]

The sequence $\{k_e\}_e$ is called the **growth sequence** for $\mathcal{F}(E_R)$. The **complexity sequence** is given by $\{c_e = k_e - k_{e-1}\}_e$. The **complexity** of $\mathcal{F}(E_R)$ is

$$cx(\mathcal{F}(E_R)) = inf\{n > 0 : c_e = O(n^e)\}.$$

Frobenius Complexity

Definition [Enescu, Yao]

The **Frobenius complexity** of the ring R is defined by

$$cx_F(R) = log_p(cx(\mathcal{F}(E_R))).$$

- It is easy to note that $\mathcal{F}(E_R)$ is finitely generated as a ring over R if and only if $cx(\mathcal{F}(E_R))) = 0$ if and only if $\{c_e\}_{e \geq 0}$ is eventually zero. In this case, $cx_F(R) = -\infty$.

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- If the sequence $\{c_e\}_{e\geq 0}$ is bounded by above, but not eventually zero, $cx(\mathcal{F}(E_R))) = 1$. Hence, $cx_F(R) = 0$.

Frobenius Complexity

Let k be a field of characteristic p, $S = k[[x_1, \ldots, x_n]]$ and $q = p^e$, for $e \geq 0$. Let $I \leq S$ be an ideal in S and $R = S/I$.
We denote $I^{[q]} = (i^q : i \in I)$.

Proposition(Fedder)

There exists an isomorphism of R-modules:

$$\mathcal{F}^e(E_R) \cong \frac{I^{[q]} :_S I}{I^{[q]}}.$$

Frobenius Complexity

For any $e \geq 0$ denote $K_e := (I^{[p^e]} :_S I)$ and

$$L_e := \sum_{1 \leq \beta_1, \ldots, \beta_s < e, \beta_1 + \ldots + \beta_s = e} K_{\beta_1} K_{\beta_2}^{[p^{\beta_1}]} \cdots K_{\beta_s}^{[p^{\beta_1 + \cdots + \beta_{s-1}}]}.$$

Proposition(Katzman)

For any $e \geq 1$, let $\mathcal{F}_{<e}$ be the R-subalgebra of $\mathcal{F}(E_R)$ generated by $\mathcal{F}^0(E_R), \ldots, \mathcal{F}^{e-1}(E_R)$. Then

$$\mathcal{F}_{<e} \cap \mathcal{F}^e(E_R) = L_e.$$

Therefore, $(G_{e-1})_e(\mathcal{F}(E_R)) \cong \dfrac{L_e + I^{[q]}}{I^{[q]}}$.

Frobenius Complexity

Let k be a field of characteristic p, $S = k[[x_1, \ldots, x_n]]$ and $q = p^e$, for $e \geq 0$. Let $I \leq S$ be an ideal in S and $R = S/I$.

Definition [Enescu, Yao]

The **complexity sequence** $\{c_e\}_e$ of the ring R is given by

$$c_e = \mu_S \left(\frac{I^{[q]} :_S I}{L_e + I^{[q]}} \right).$$

The case of Stanley-Reisner Rings

Let $I \leq S$ be a square-free monomial ideal in S and $R = S/I$ the Stanley-Reisner ring associated to I.

Theorem (Àlvarez Montaner, Boix and Zarzuela)

The Frobenius algebra $\mathcal{F}(E_R)$ associated to a Stanley-Reisner ring R is either principally generated or infinitely generated.

Remark

Therefore, for Stanley-Reisner rings, the Frobenius complexity $cx_F(R)$ is either $-\infty$ or 0.

Main Results

Definition

We define J_q to be the unique minimal monomial ideal satisfying the equality

$$(I^{[q]} : I) = I^{[q]} + J_q + (\mathbf{x}^1)^{q-1},$$

where $(\mathbf{x}^1)^{q-1} = x_1^{q-1} \cdots x_n^{q-1}$.

Main Results

Example

Let $I = (x_1x_5, x_2x_5, x_2x_3, x_2x_4)$. Then

$$(I^{[q]} : I) = (x_1^q x_5^q, x_2^q x_5^q, x_2^q x_3^q, x_2^q x_4^q, x_1^{q-1}x_2^{q-1}x_5^q, x_2^q x_3^{q-1}x_4^{q-1}x_5^{q-1},$$

$$x_1^{q-1}x_2^{q-1}x_4^q x_5^{q-1}, x_1^{q-1}x_2^{q-1}x_3^q x_5^{q-1}, x_1^{q-1}x_2^{q-1}x_3^{q-1}x_4^{q-1}x_5^{q-1})$$

and therefore

$$J_q = (x_1^{q-1}x_2^{q-1}x_5^q, x_2^q x_3^{q-1}x_4^{q-1}x_5^{q-1}, x_1^{q-1}x_2^{q-1}x_4^q x_5^{q-1}, x_1^{q-1}x_2^{q-1}x_3^q x_5^{q-1}).$$

Main Results

Remark

The complexity sequence $\{c_e\}_{e \geq 0}$ is bounded by above since $c_e \leq \mu_S(J_p) + 1$, for any $e \geq 0$.

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Remark

In the case when $\mathcal{F}(E_R)$ is principally generated, it is generated by $(\mathbf{x}^1)^{p-1}$. When $\mathcal{F}(E_R)$ is infinitely generated, $\mathcal{F}^1(E_R)$ has $\mu + 1$ minimal generators, μ of them being the minimal generators of J_p and $(\mathbf{x}^1)^{p-1}$. Each graded piece $\mathcal{F}^e(E_R)$ adds up μ new generators coming from J_q.

Main Results

Main Theorem(-)

Let k be a field of characteristic p, $S = k[[x_1, \ldots, x_n]]$ and $q = p^e$, for $e \geq 0$. Let $I \leq S$ be a square-free monomial ideal in S and $R = S/I$ its Stanley-Reisner ring. Then,

$$\{c_e\}_{e\geq 0} = \{0, \mu + 1, \mu, \mu, \mu, \ldots\},$$

where $\mu := \mu_S(J_p)$.

Our result generalizes the work of Àlvarez Montaner, Boix and Zarzuela and settles an open question mentioned by Àlvarez Montaner in one of his papers.

Àlvarez Montaner defined the generating function of a skew R-algebra using the complexity sequence.

Definition

The generating function of $\mathcal{F}(E_R)$ is defined as

$$\mathcal{G}_{\mathcal{F}(E_R)}(T) = \sum_{e \geq 0} c_e T^e.$$

Main Results

Corollary

Let k be a field of characteristic p, $S = k[[x_1, \ldots, x_n]]$ and $q = p^e$, for $e \geq 0$. Let $I \leq S$ be a square-free monomial ideal in S, $R = S/I$ its Stanley-Reisner ring.

Then the generating function of the Frobenius algebra of operators is

$$\mathcal{G}_{\mathcal{F}(E_R)}(T) = (\mu + 1)T + \sum_{e \geq 2} \mu T^e = \frac{(\mu+1)T - T^2}{1 - T}.$$

Proof

Note that $c_0 = 0$. Using the Main Theorem, we have that $c_1(R) = \mu + 1$ and $c_e = \mu$, for every $e \geq 2$.

Applications of my result

- Our theorem describes the complexity sequence of any Stanley-Reisner ring. Moreover, we show that the complexity sequence is independent on the characteristic of the ring in this case.

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- Our theorem describes the complexity sequence of any Stanley-Reisner ring. Moreover, we show that the complexity sequence is independent on the characteristic of the ring in this case.
- We showed that the Frobenius complexity sequence, which is a positive characteristic invariant of our ring is in fact a combinatorial invariant introduced by Àlvarez Montaner, Boix and Zarzuela, the number of maximal free pairs of the simplicial complex associated to our ring.

References

I. Ilioaea
On the Frobenius Complexity of Stanley-Reisner Rings
Communications in Algebra, 2020.

F. Enescu, Y. Yao
The Frobenius Complexity of a Local Ring of Prime Characteristic
Journal of Algebra, 459 (2003), 133 – 156.

A. Boix, J. Montaner, S. Zarzuela
Frobenius and Cartier Algebras of Stanley-Reisner Rings
J. Algebra, 358, (2012), 162–177.

References

A. Boix, S. Zarzuela
Frobenius and Cartier Algebras of Stanley-Reisner Rings II
Acta Mathematica Vietnamica, 324, (2019), 1–16.

J. Àlvarez Montaner
Generating functions associated to Frobenius algebras
Linear Algebra and its Applications, 528, (2019), 310–326.

Thank you!