The Fedder Action and a Simplicial Complex of Local Cohomologies

Monica Lewis

joint with Eric Canton

University of Michigan

June 28, 2020

Throughout this talk...

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- All rings have prime characteristic $p > 0$

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- All but one will be commuative and Noetherian

Part 1

What is a Frobenius action?

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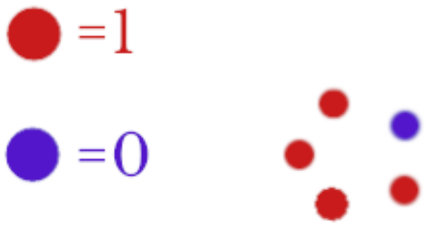

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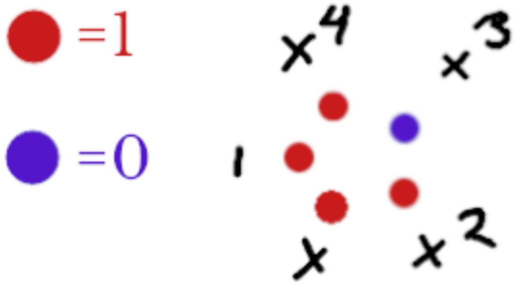

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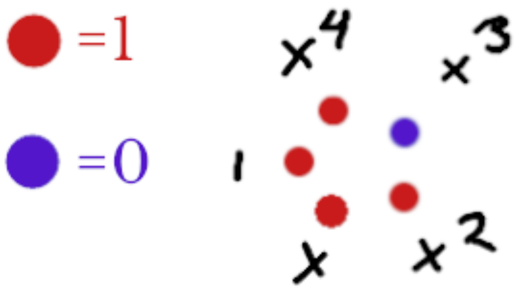

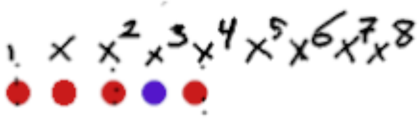

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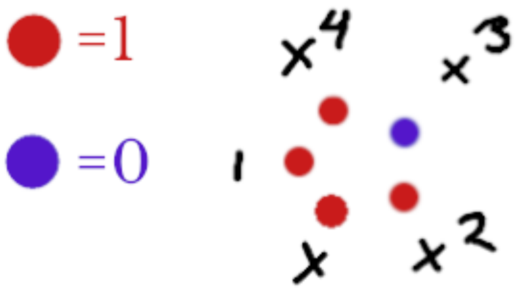

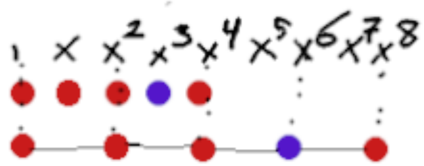

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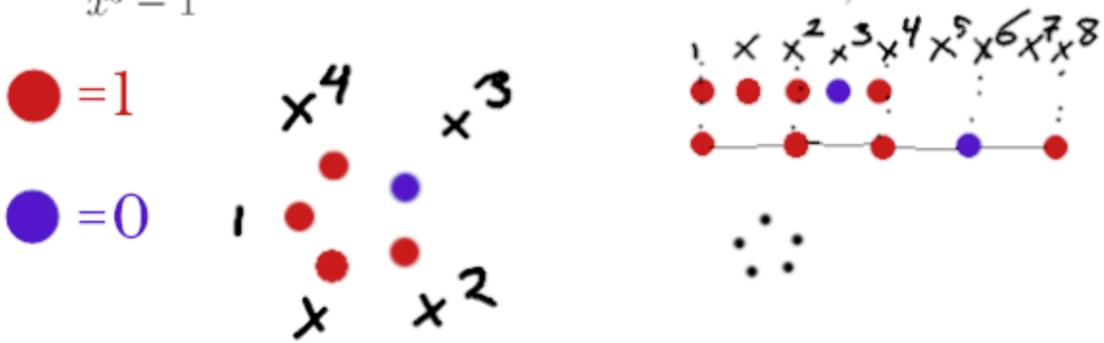

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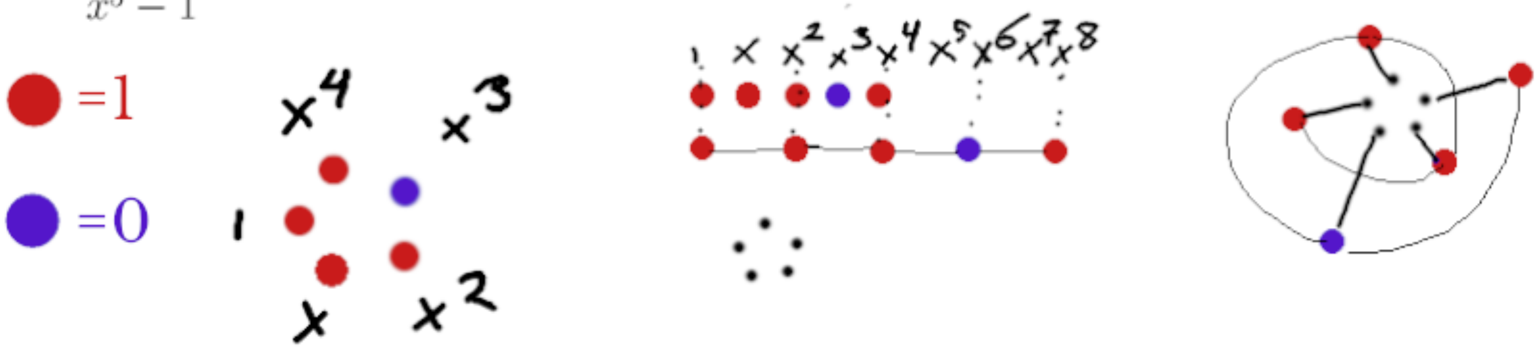

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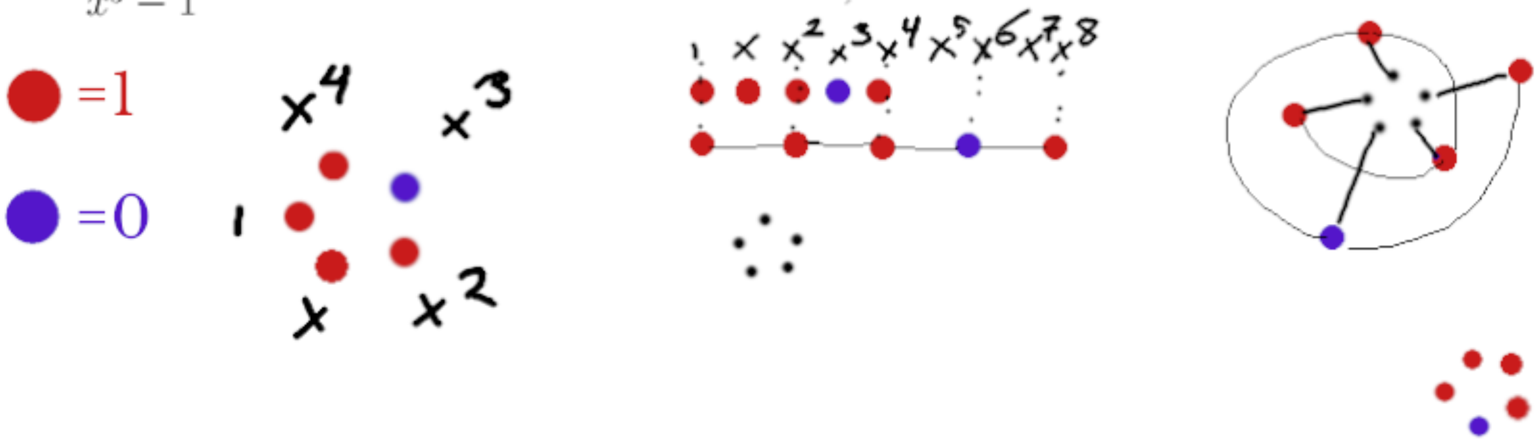

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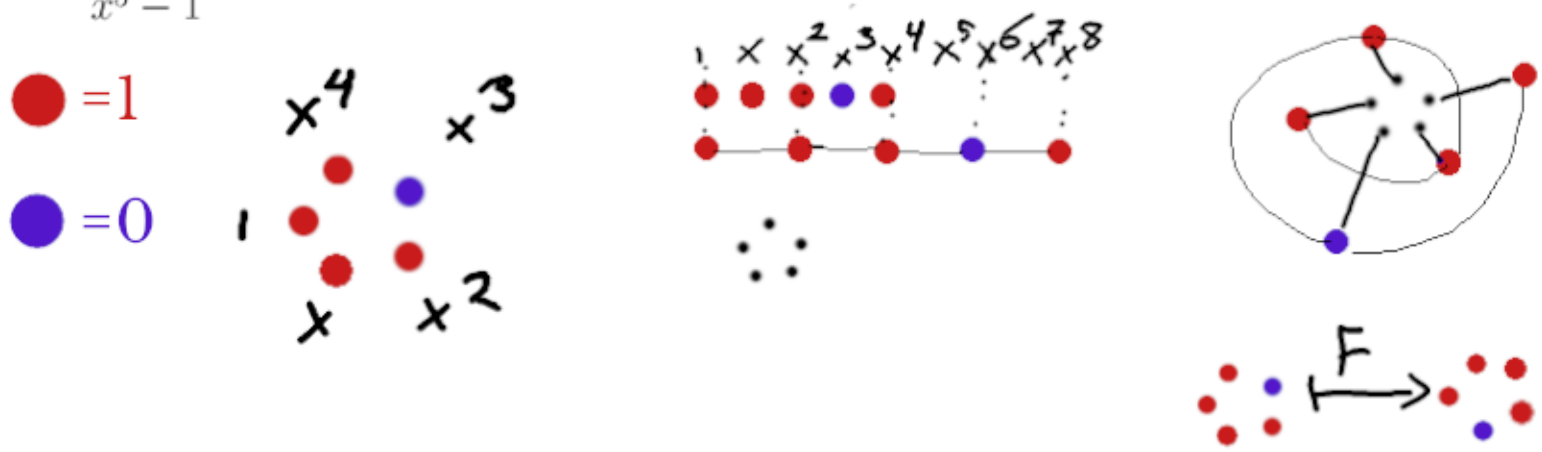

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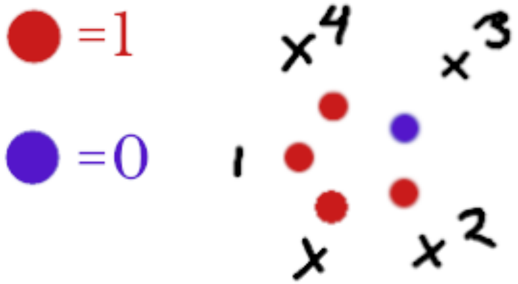

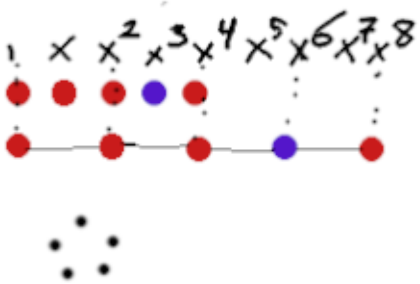

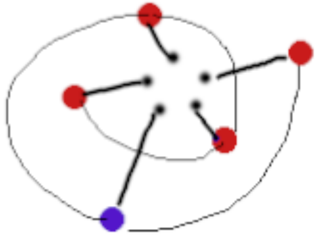

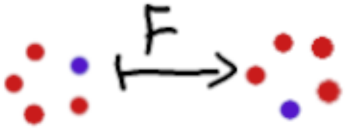

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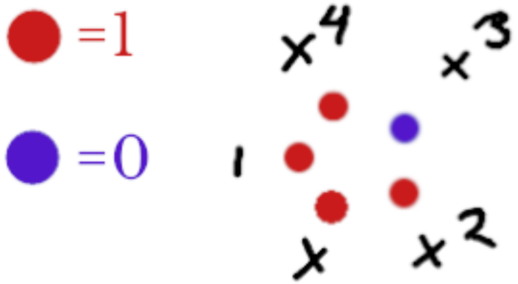

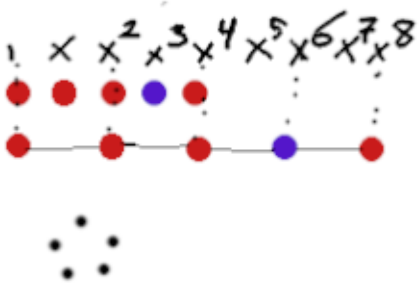

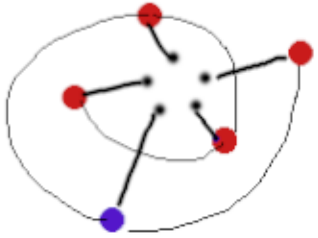

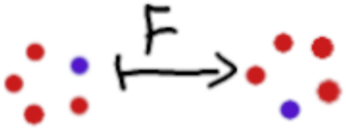

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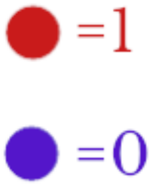

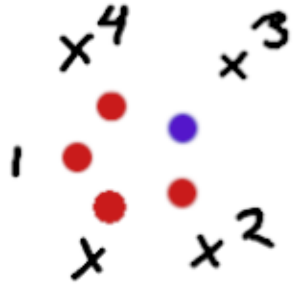

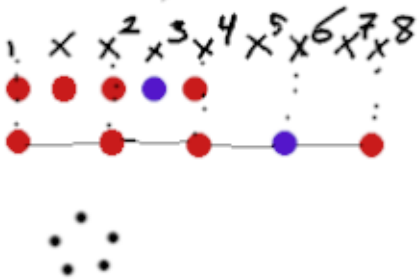

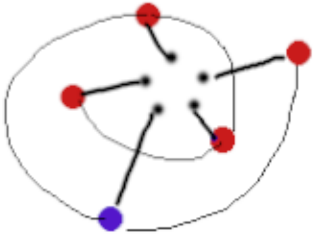

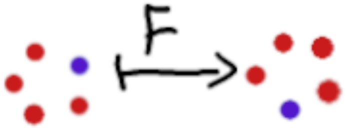

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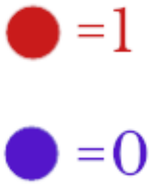

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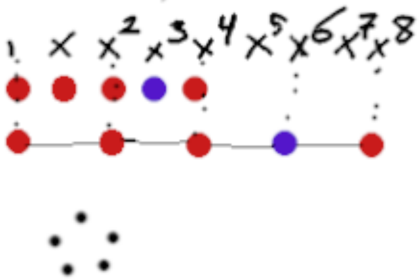
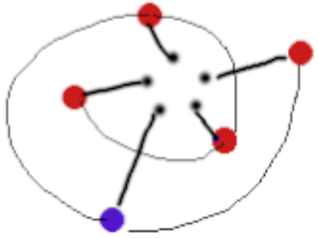
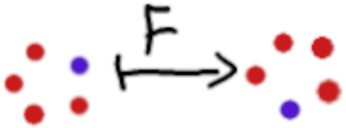
Gives a copy of $\mathbb{F}_{16}$







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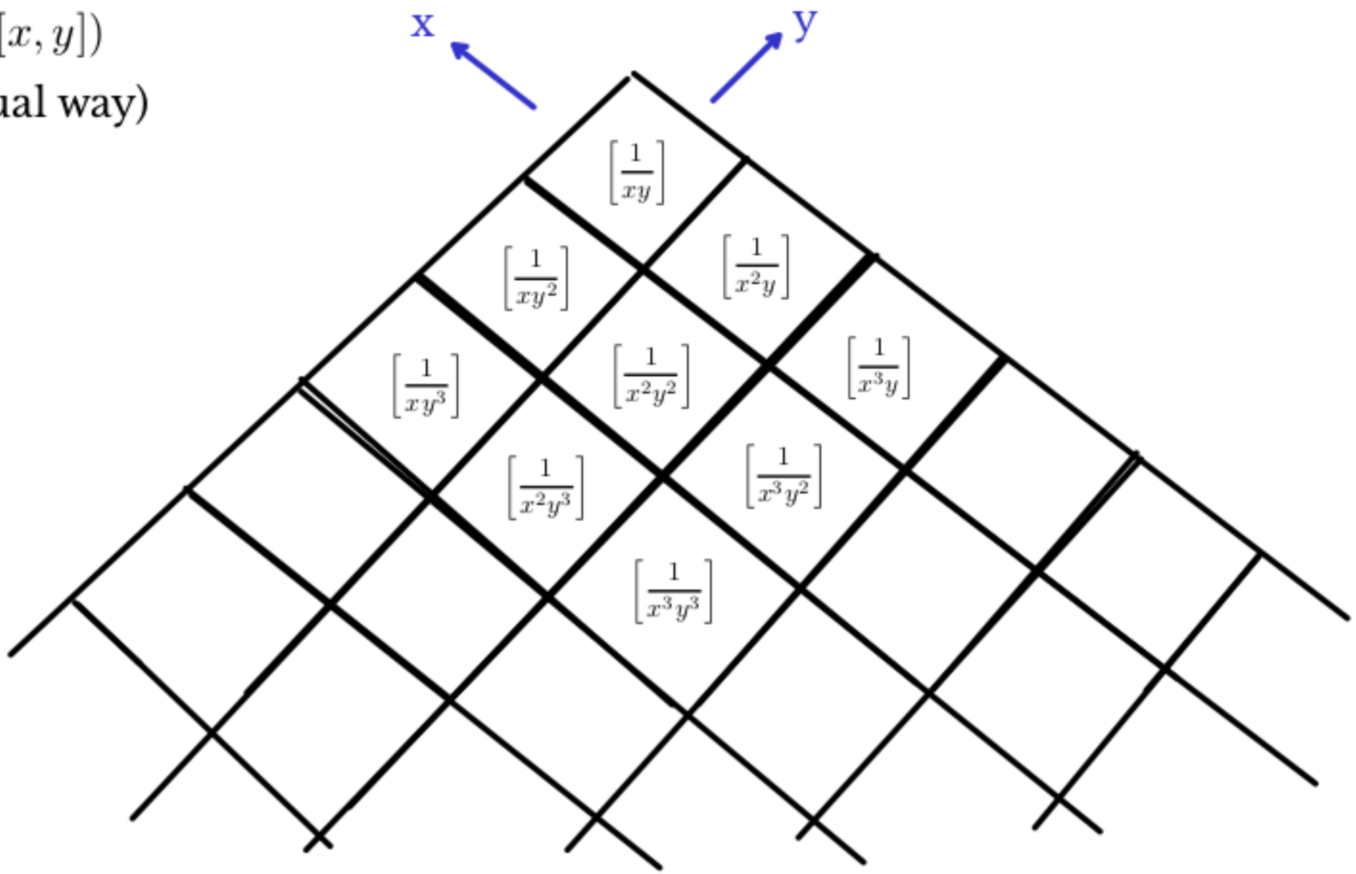
If N is finitely generated over R, and generates M over R<F>, then Supp(M) = Supp(N) is closed

Example: Two different actions on the same R-module...

$M = H^2_{(x,y)}(\mathbb{F}_2[x,y])$

(labeled in the usual way)



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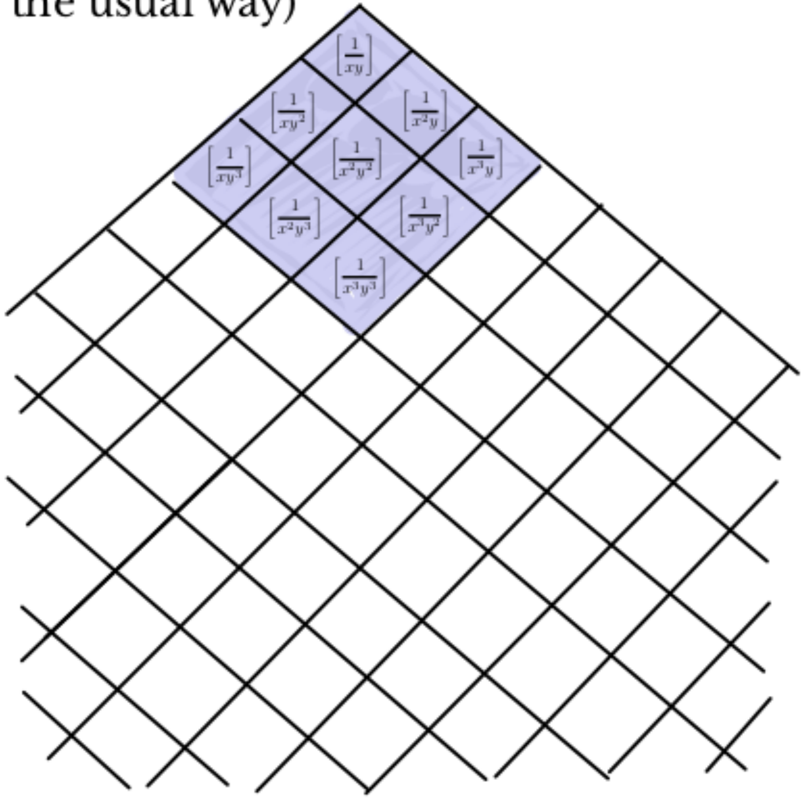

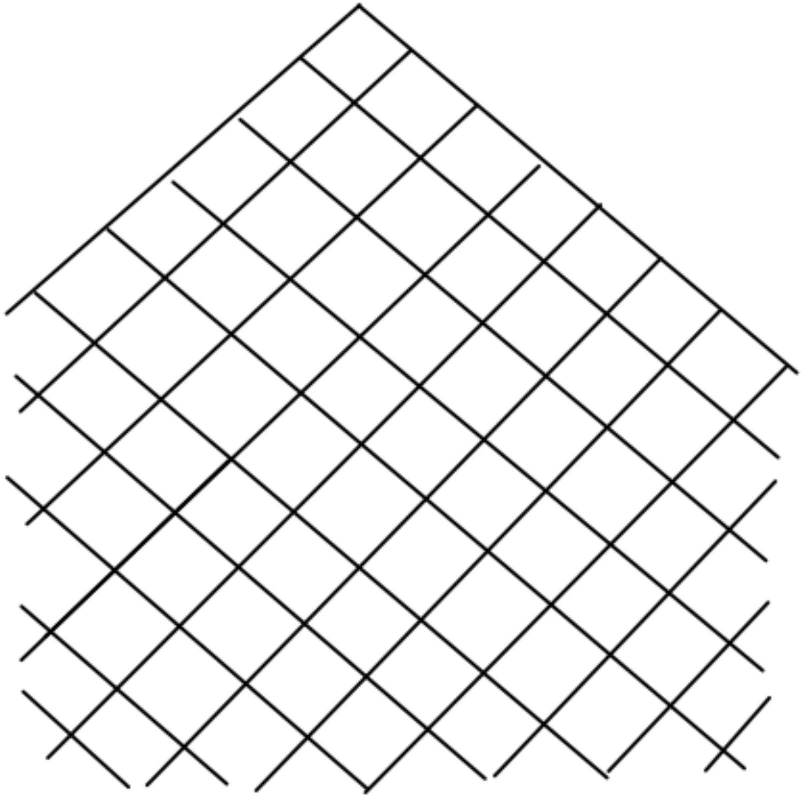

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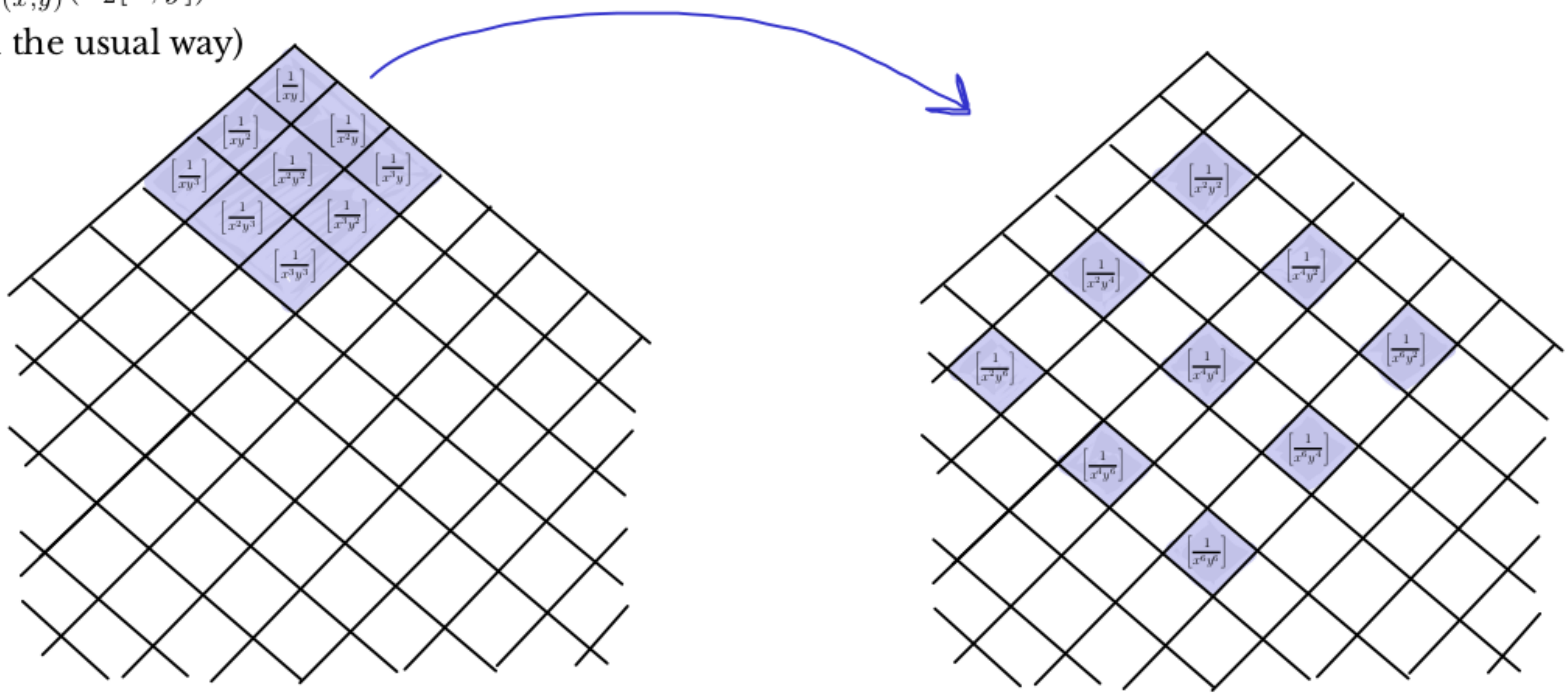

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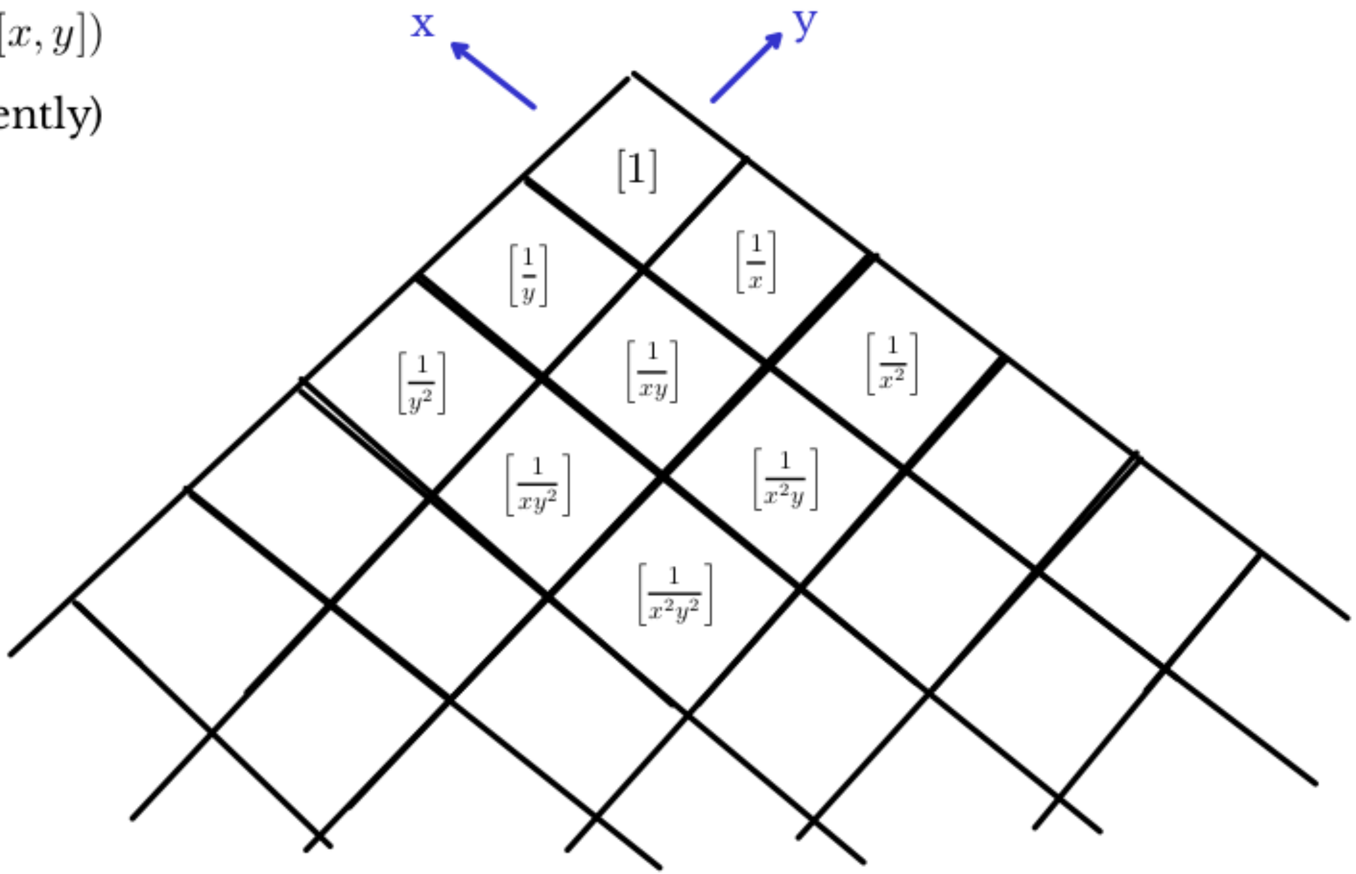

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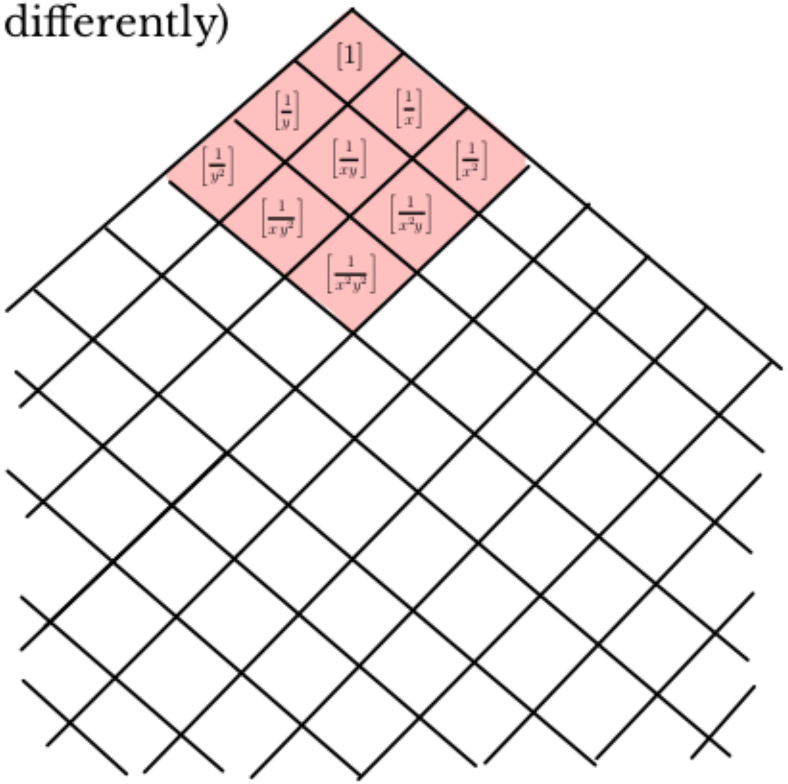

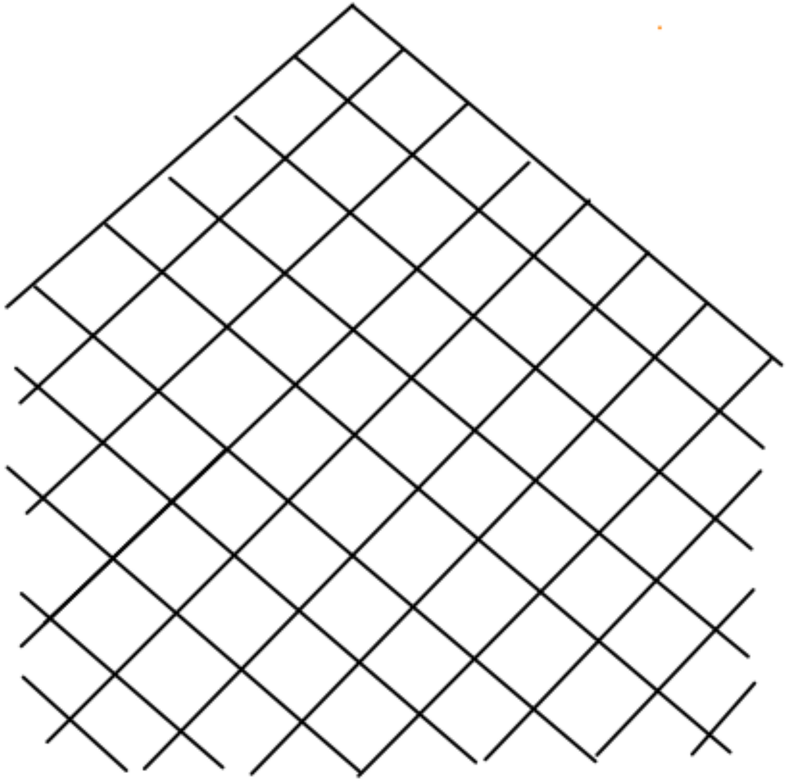

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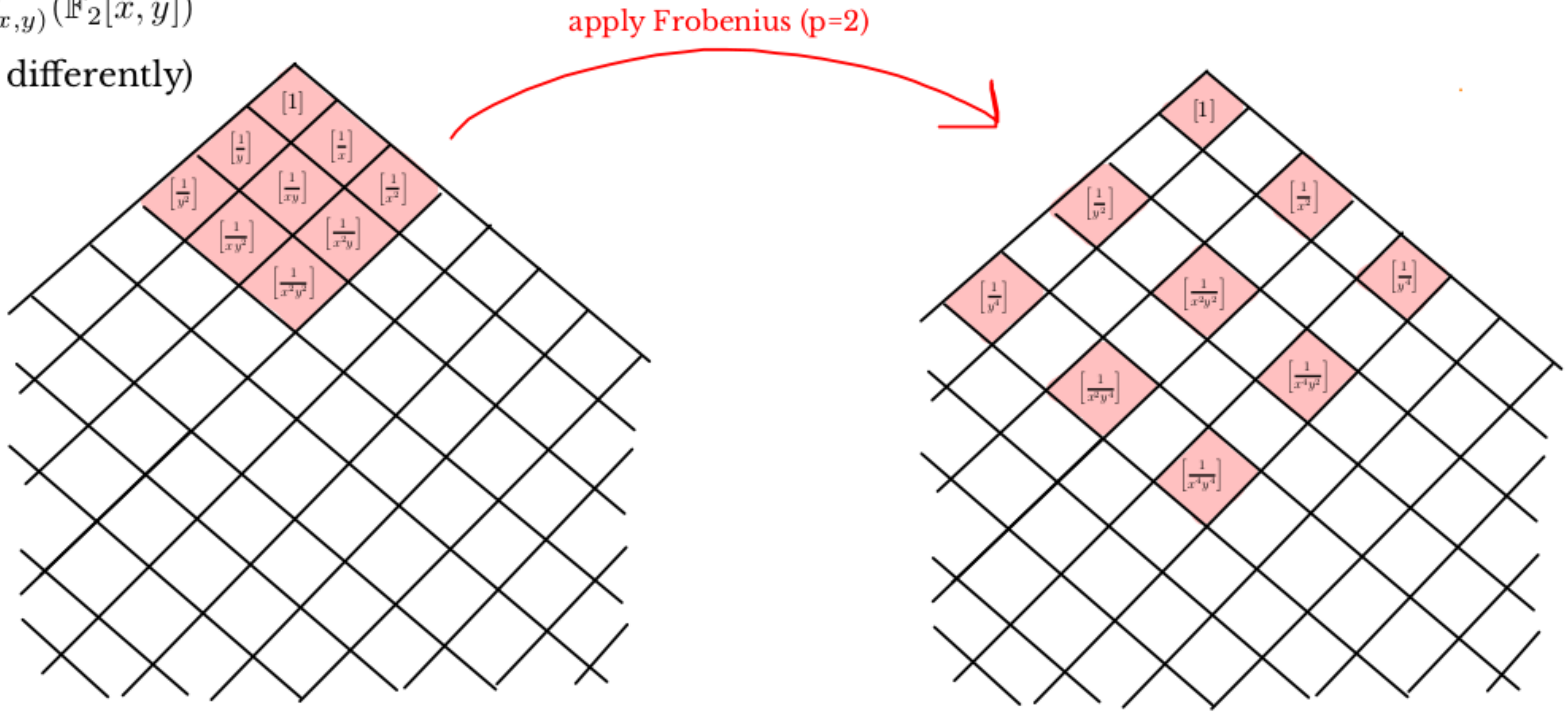

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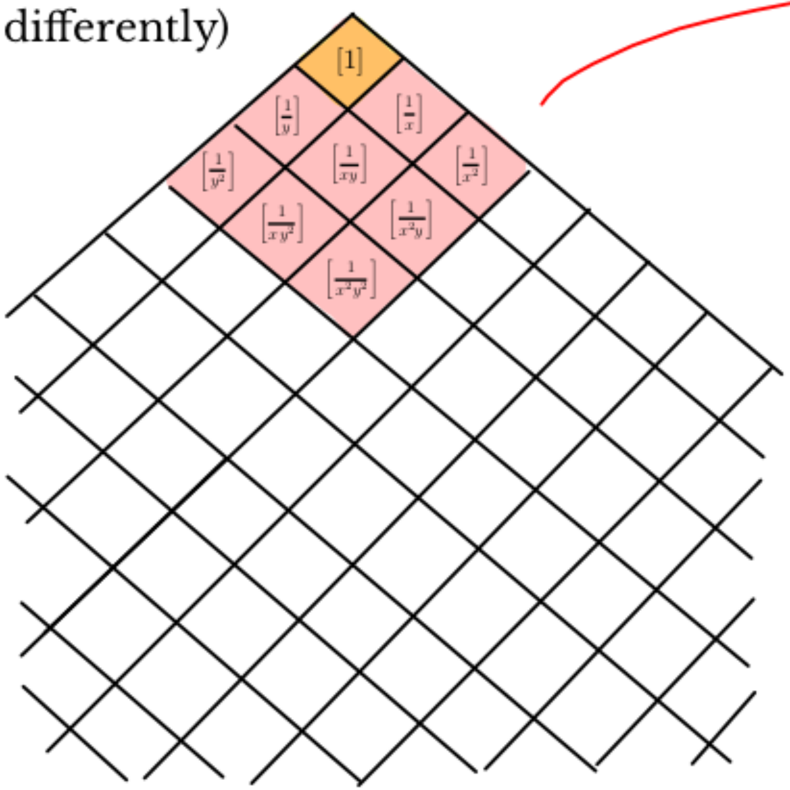

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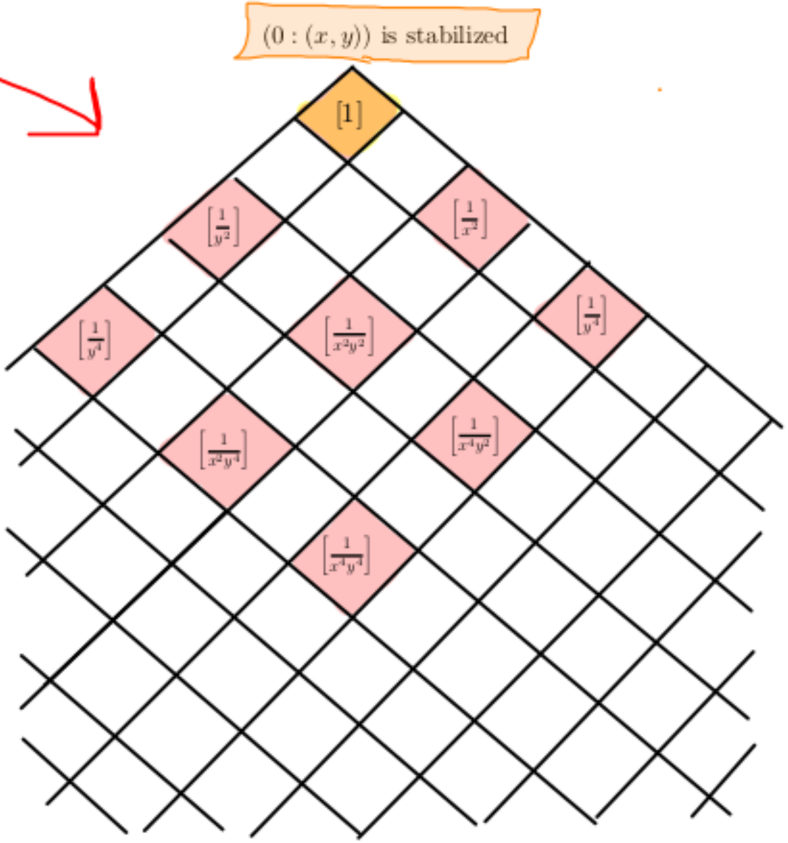

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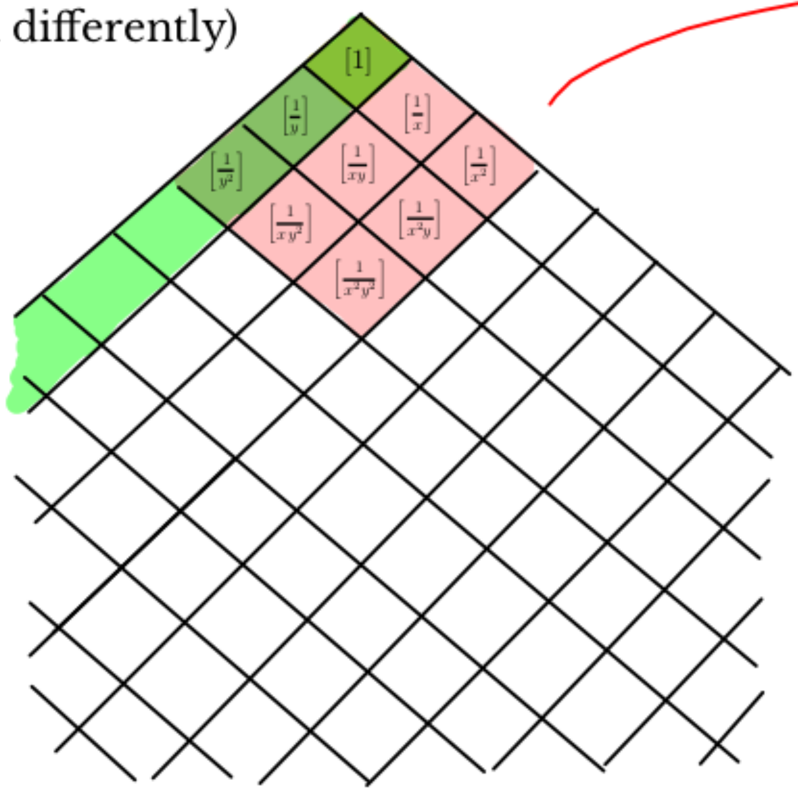

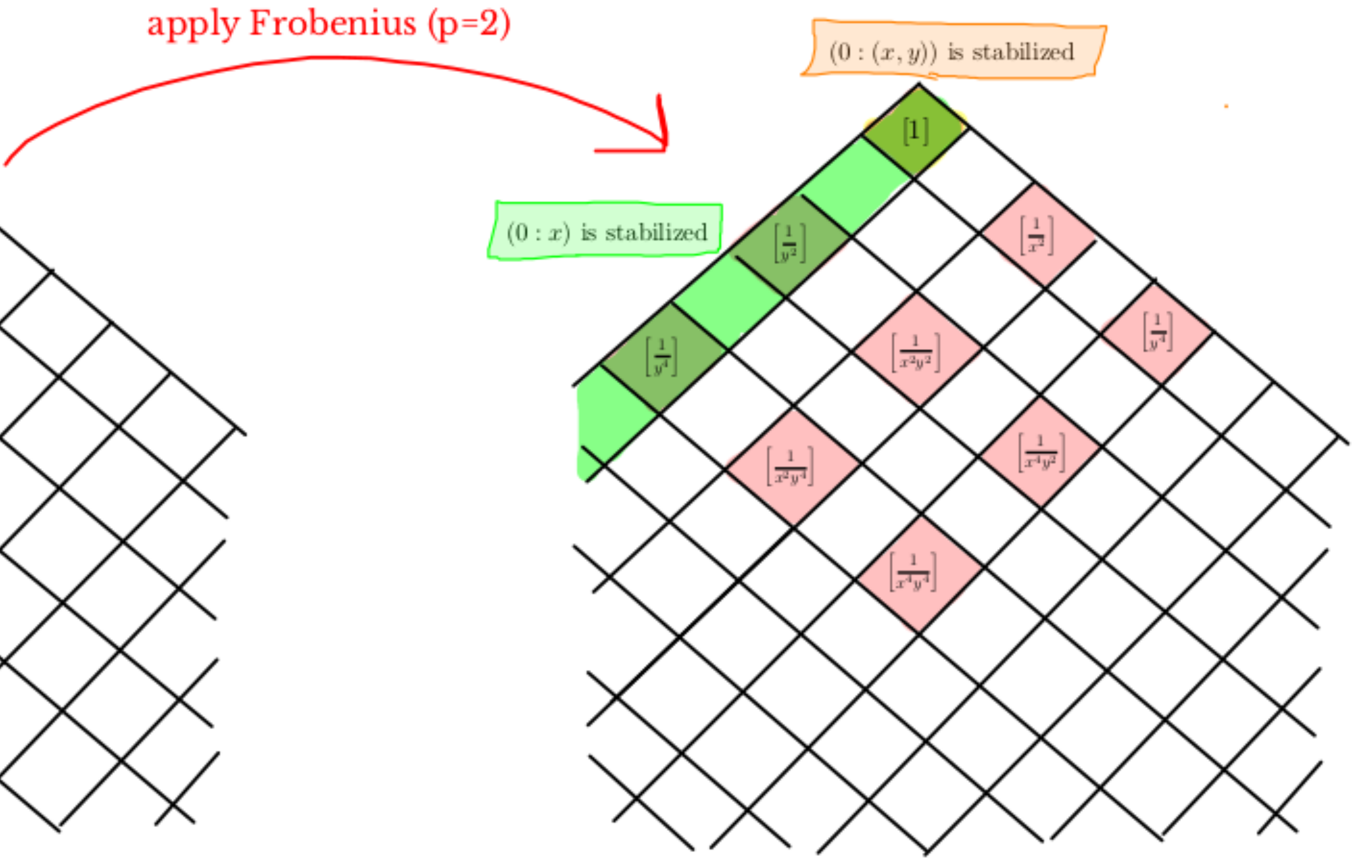

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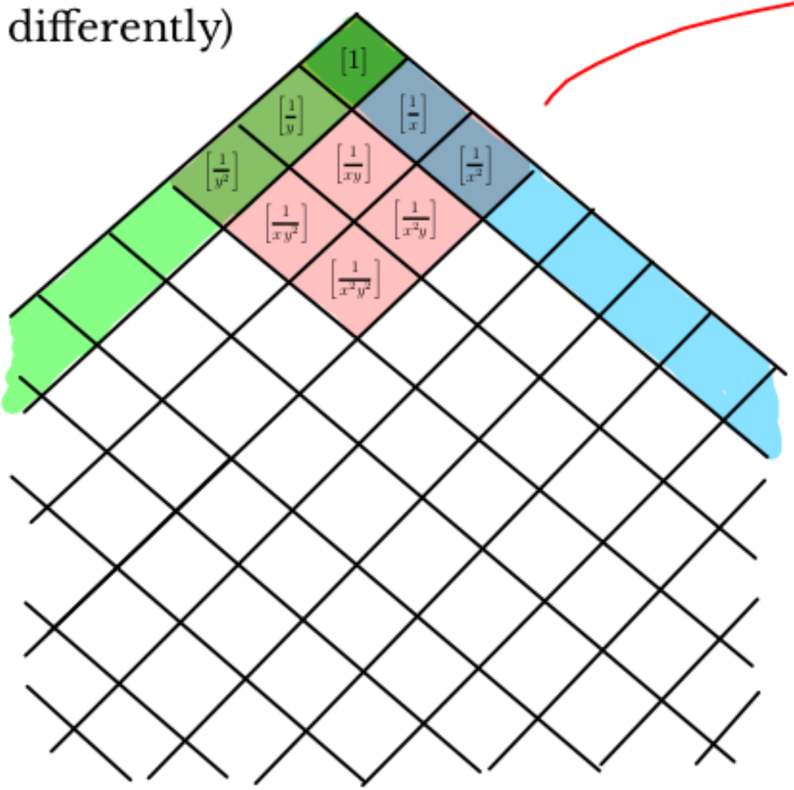

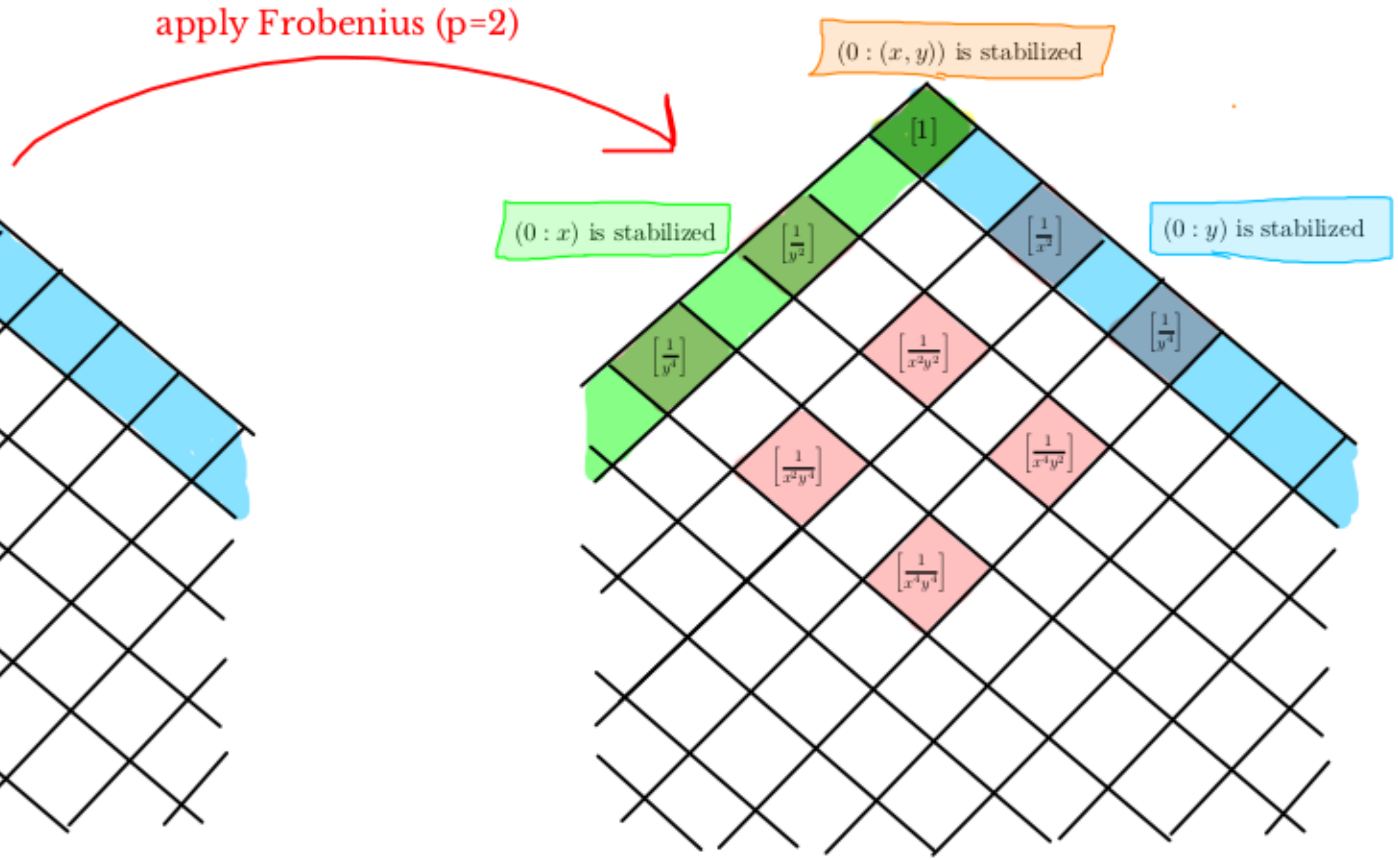

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as an R-module, for $r \in R$, $r \cdot m^{1/q} = (r^q m)^{1/q}$

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as an R-module, for $r \in R$, $r \cdot m^{1/q} = (r^q m)^{1/q}$

(can similarly make sense of the ring R^q and the R^q module M^q)

Some notation:

R is a ring of prime characteristic $p > 0$, M is an R-module

$q = p^e$ for some $e \geq 0$

$$R^{1/q} = \left\{\text{formal symbols } r^{1/q} \text{ for } r \in R\right\} \quad (\simeq R \text{ as a ring})$$

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base change : $\mathrm{Mod}_R \to \mathrm{Mod}_{R^{1/q}}$ via $M \mapsto R^{1/q} \otimes_R M$

Peskine-Szpiro functor $\mathcal{F}_R^e : \mathrm{Mod}_R \to \mathrm{Mod}_R$ via $M \mapsto (R^{1/q} \otimes_R M)^q$

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A morphism from (M, Θ) to (N, Ψ) is an R-linear map $h : M \to N$ such that

$$\begin{array}{ccc} R^{1/p} \otimes_R M & \xrightarrow{1 \otimes h} & R^{1/p} \otimes_R N \\ \Theta \downarrow & & \Psi \downarrow \\ M^{1/p} & \xrightarrow{h^{1/p}} & N^{1/p} \end{array} \quad \text{commutes}$$

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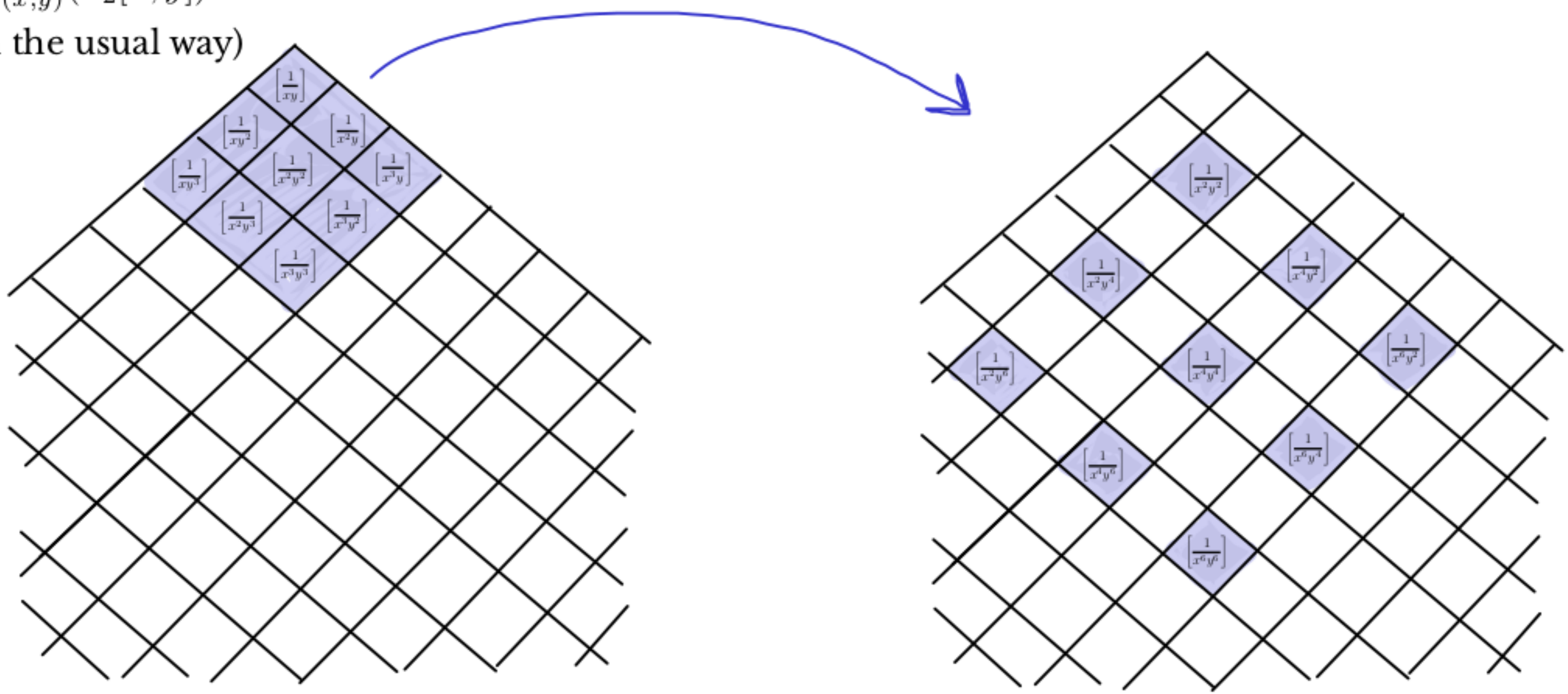
alternatively, $$\begin{array}{ccc} \mathcal{F}_R(M) & \xrightarrow{\mathcal{F}_R(h)} & \mathcal{F}_R(N) \\ {\scriptstyle \Theta}\downarrow & & \downarrow{\scriptstyle \Psi} \\ M & \xrightarrow{h} & N \end{array}$$ commutes

Example: Two different actions on the same R-module...

$M = H^2_{(x,y)}(\mathbb{F}_2[x,y])$

(labeled in the usual way)

apply Frobenius (p=2)



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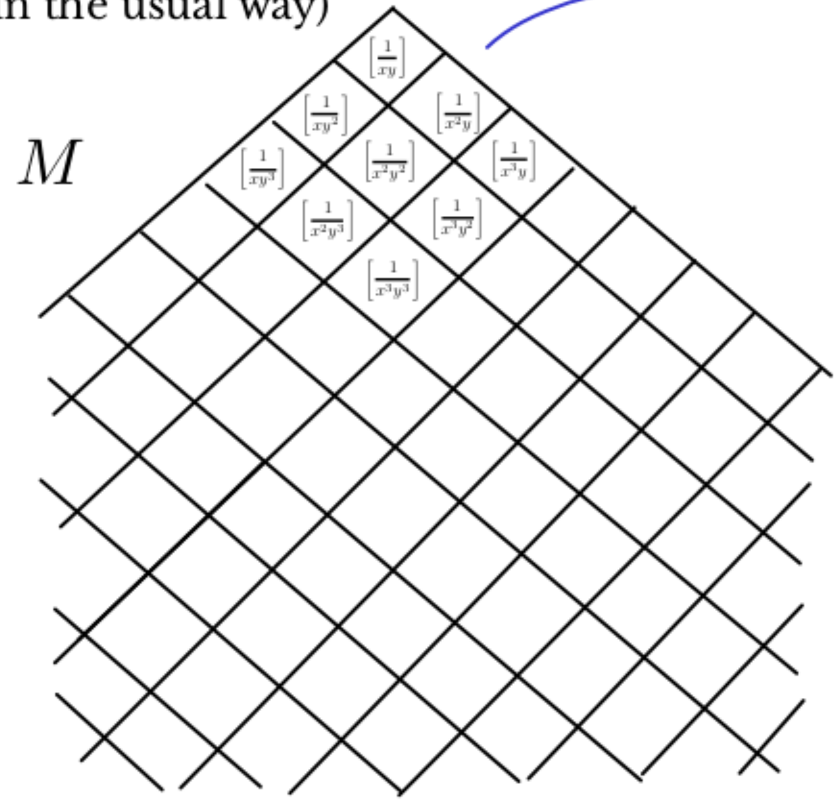
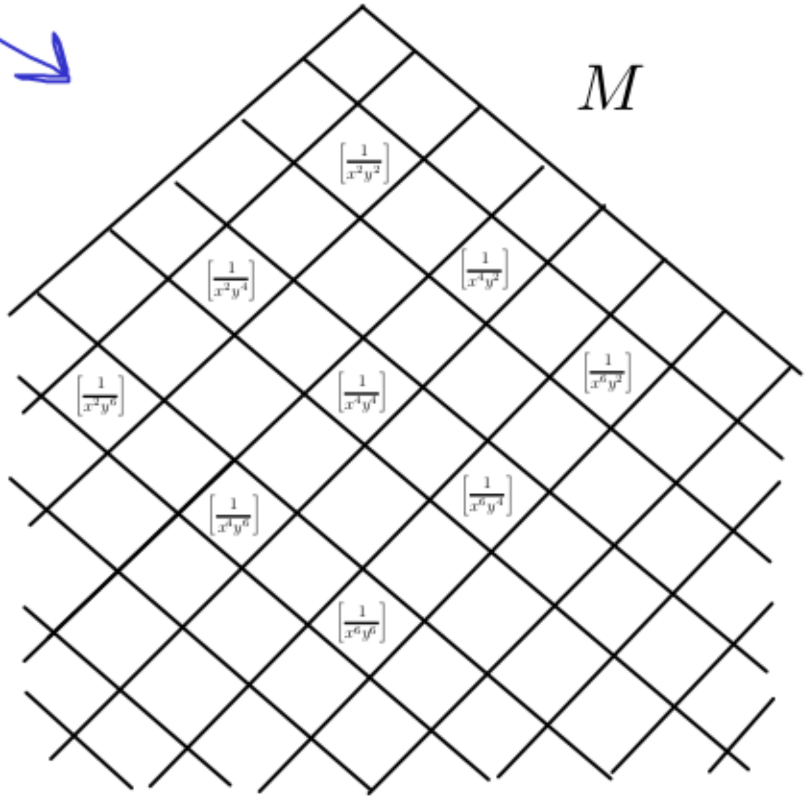
what is the structure map?





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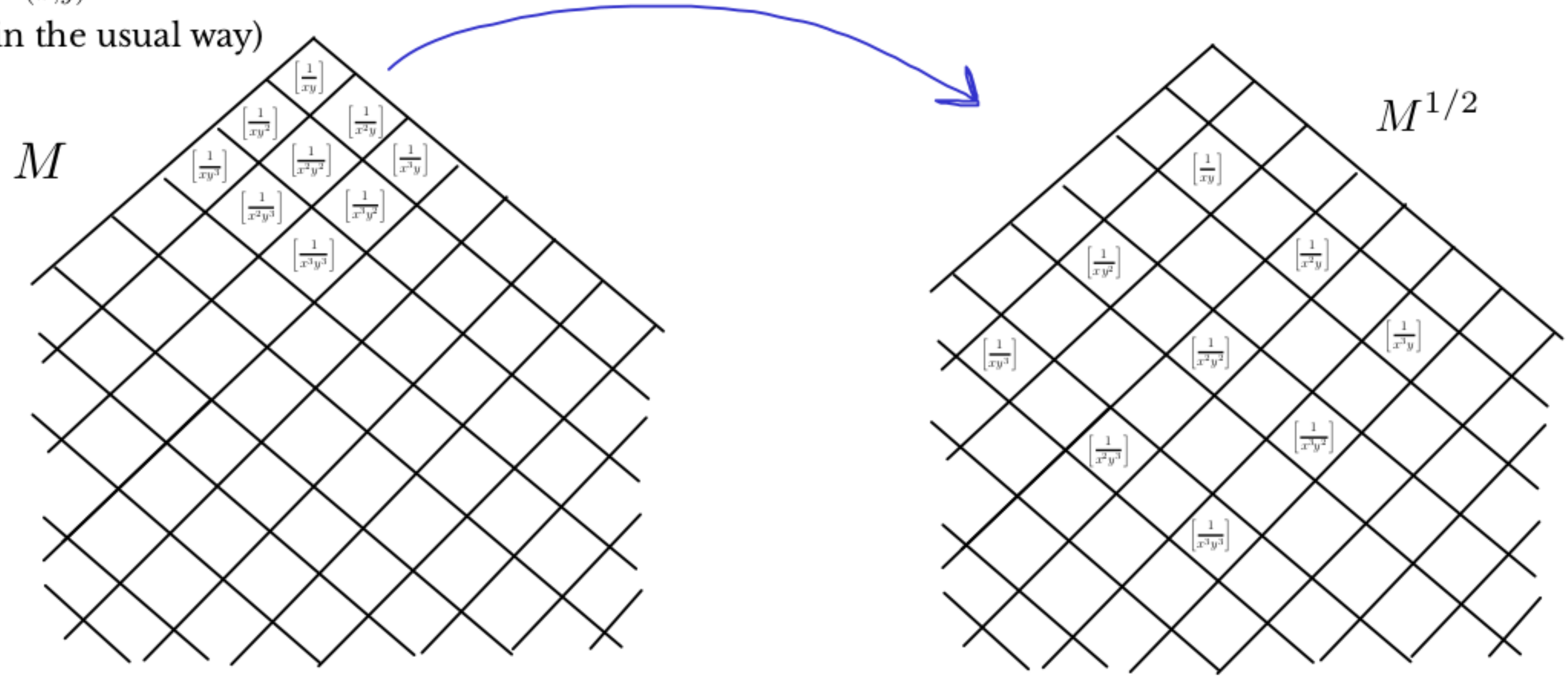

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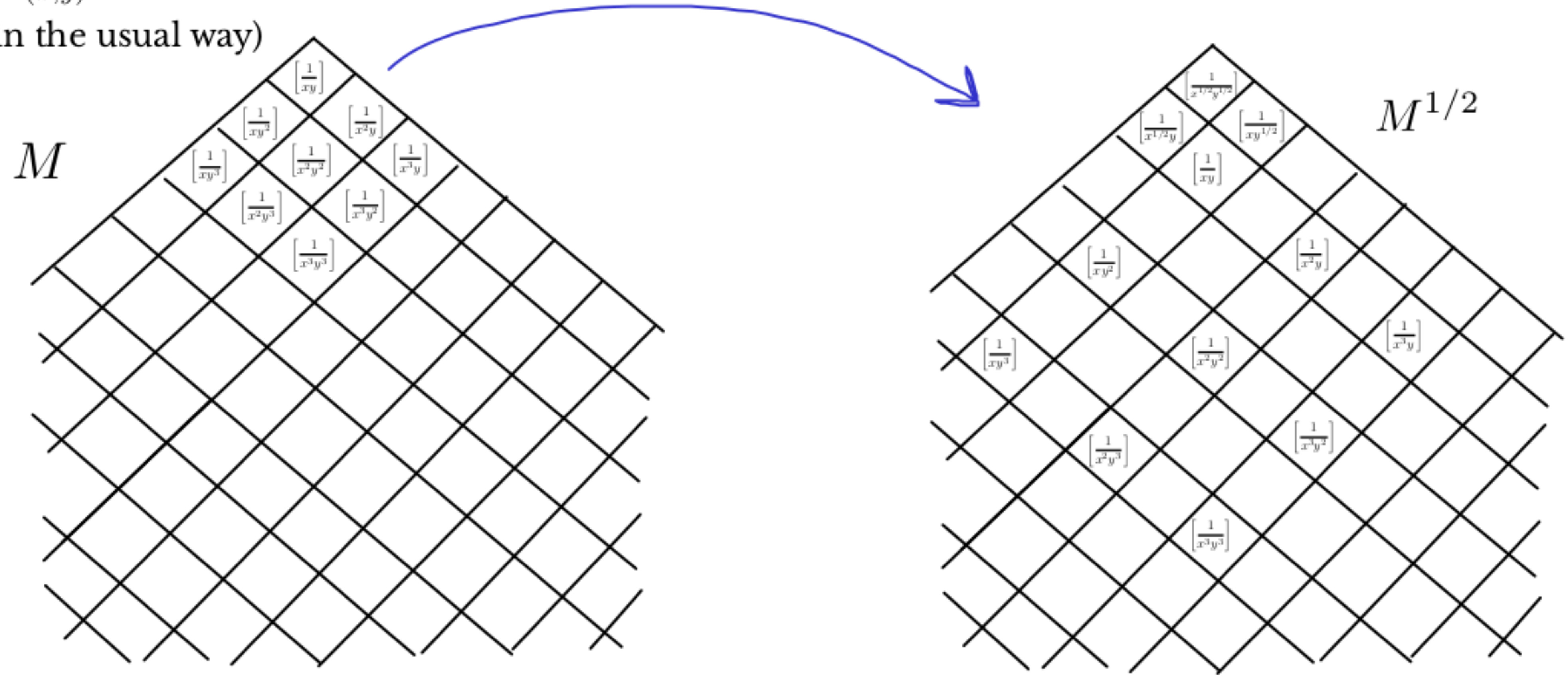

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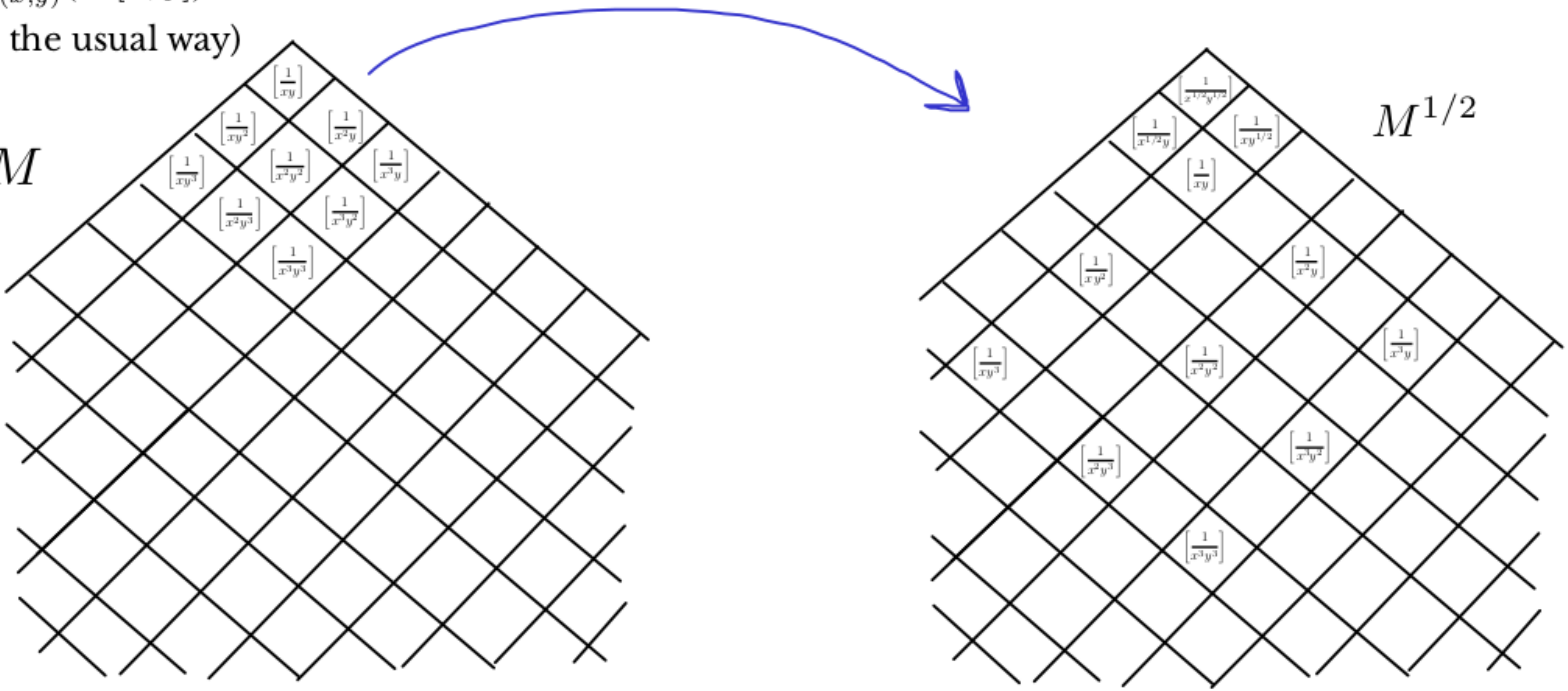
an R-linear map to an $R^{1/2}$-module

M



$M^{1/2}$

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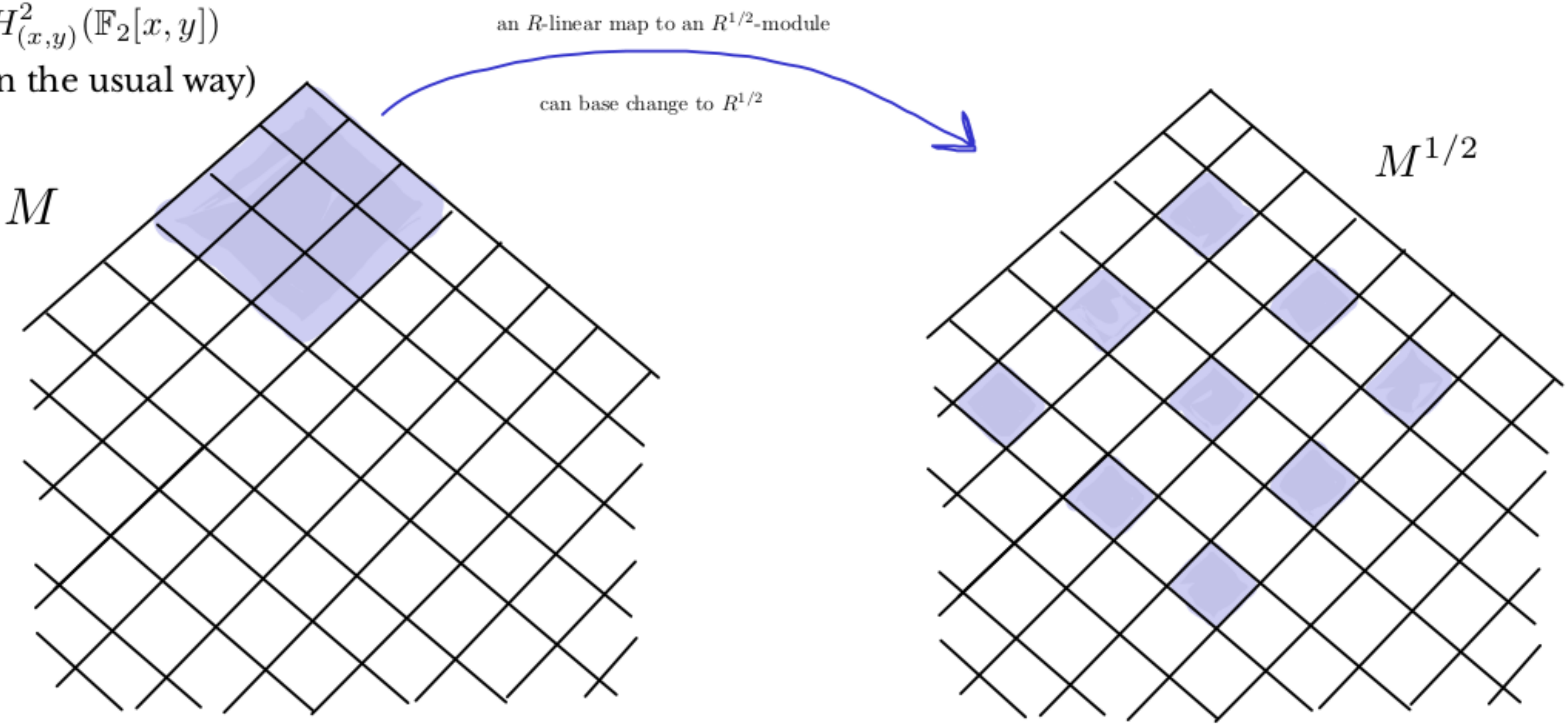

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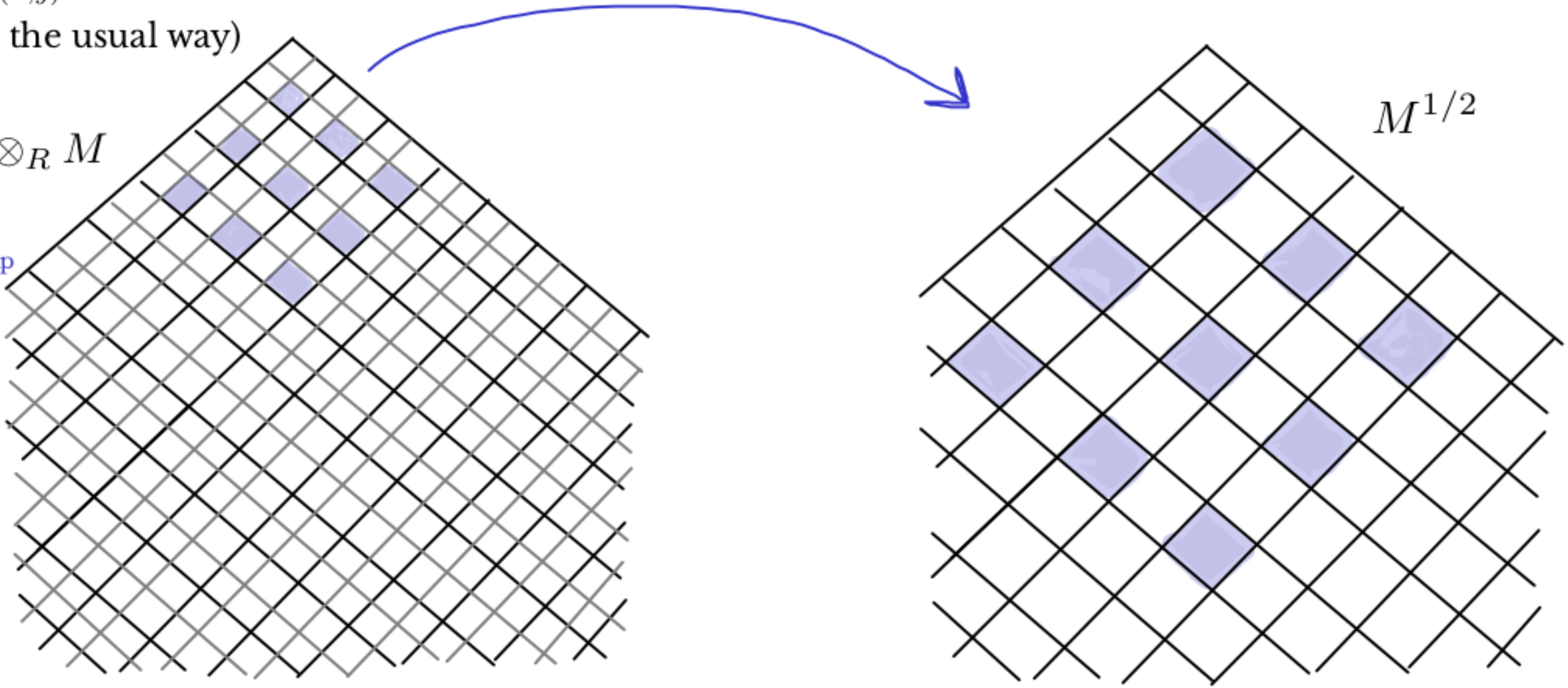

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$M = H^2_{(x,y)}(\mathbb{F}_2[x,y])$

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the $R^{1/2}$-linear structure map $\Theta_1 : R^{1/2} \otimes_R M \to M^{1/2}$

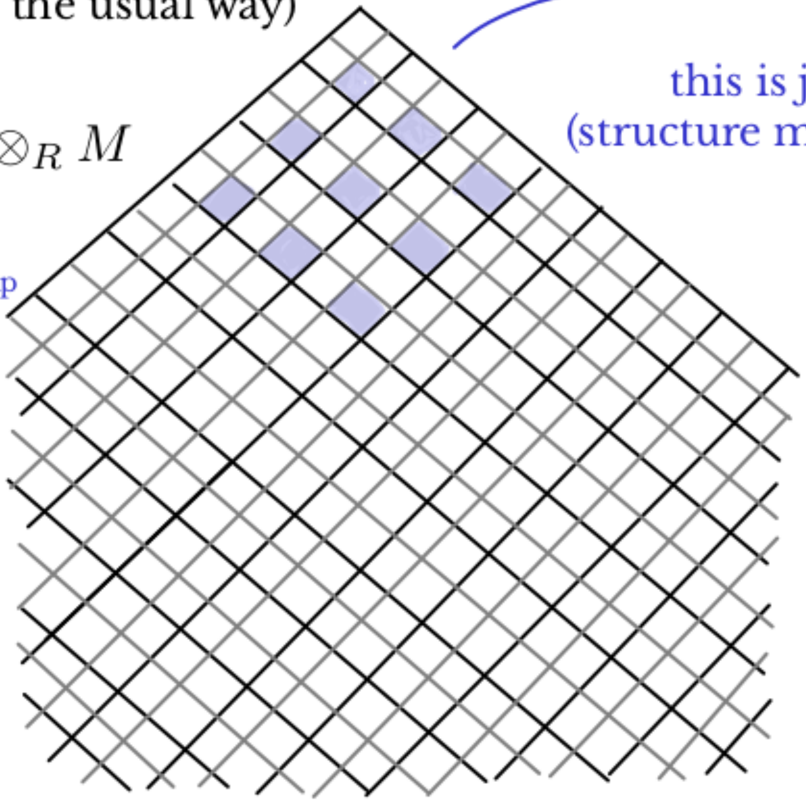
this is just the identity!

(structure map is an isomorphism)

$R^{1/2} \otimes_R M$

=image of M under the map

$M \to R^{1/2} \otimes_R M$



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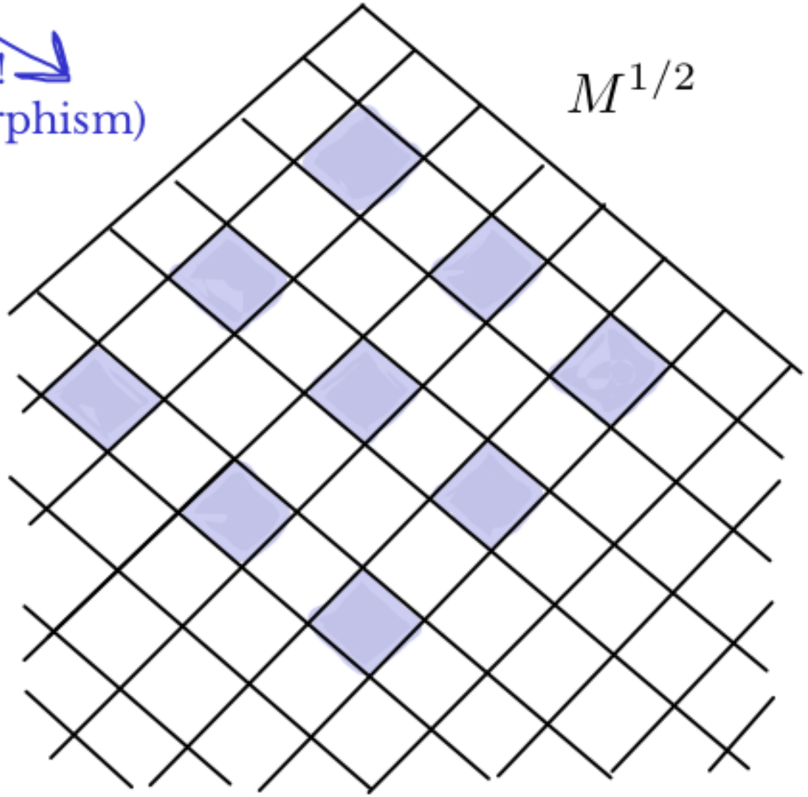

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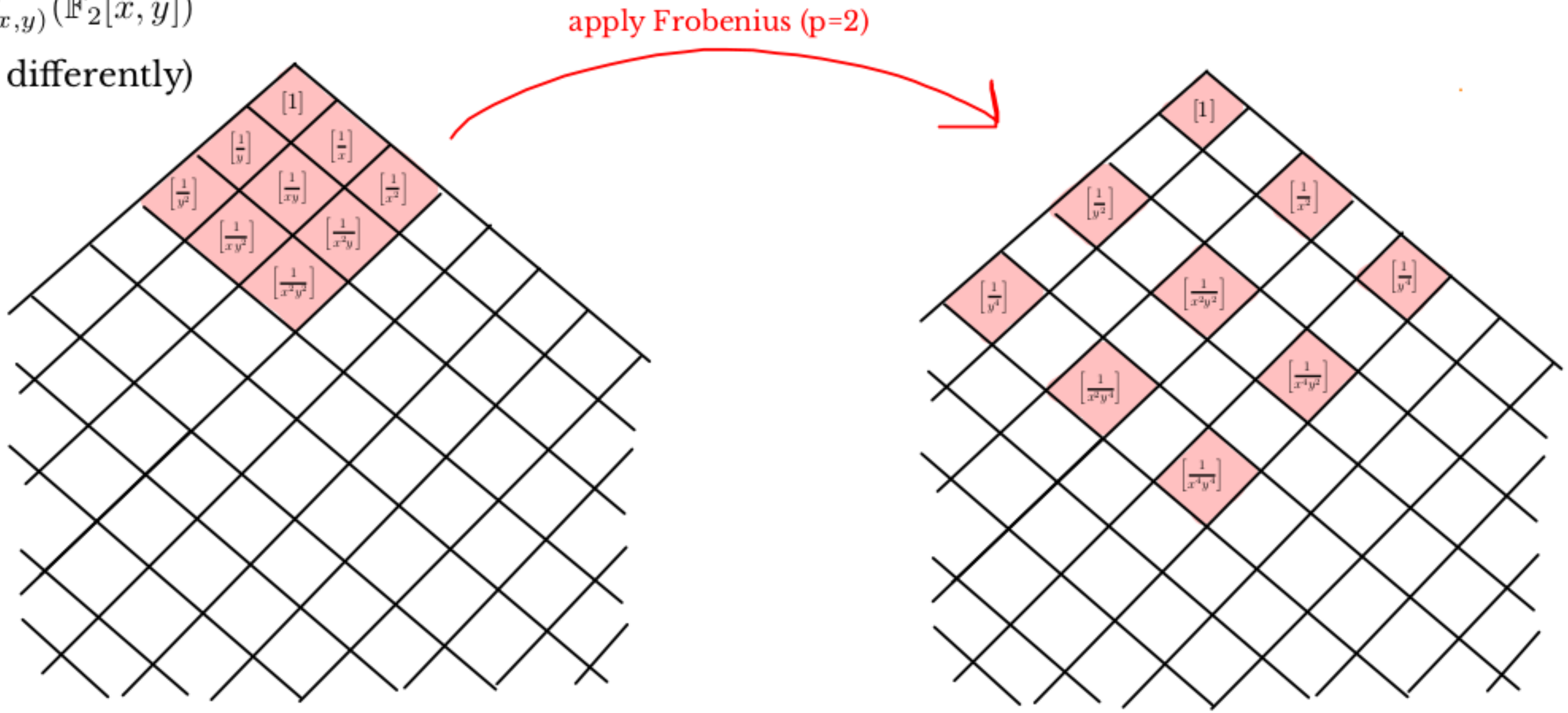

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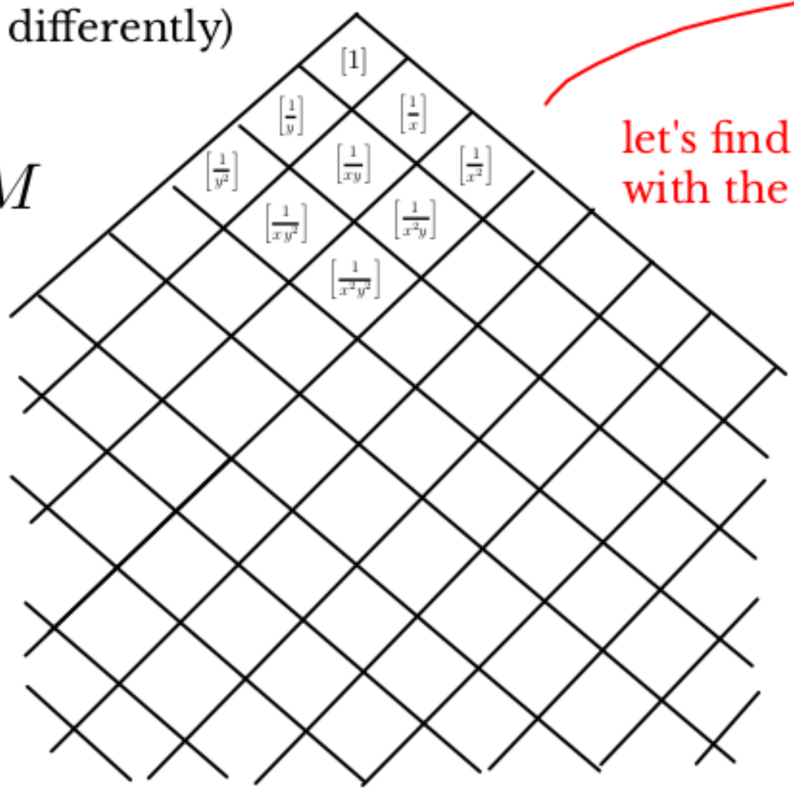
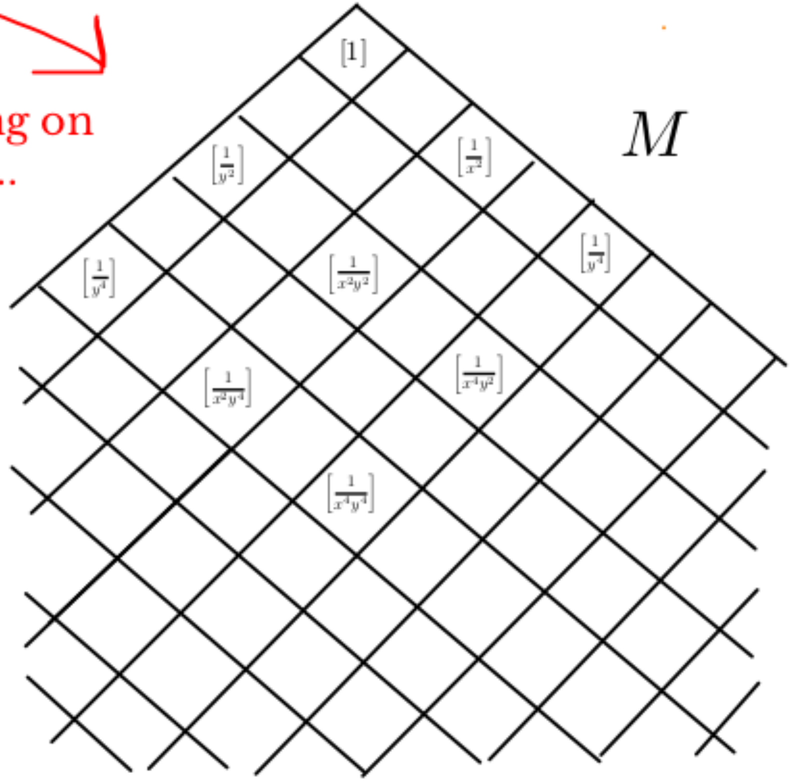
M



let's find out what's going on with the structure map...

M



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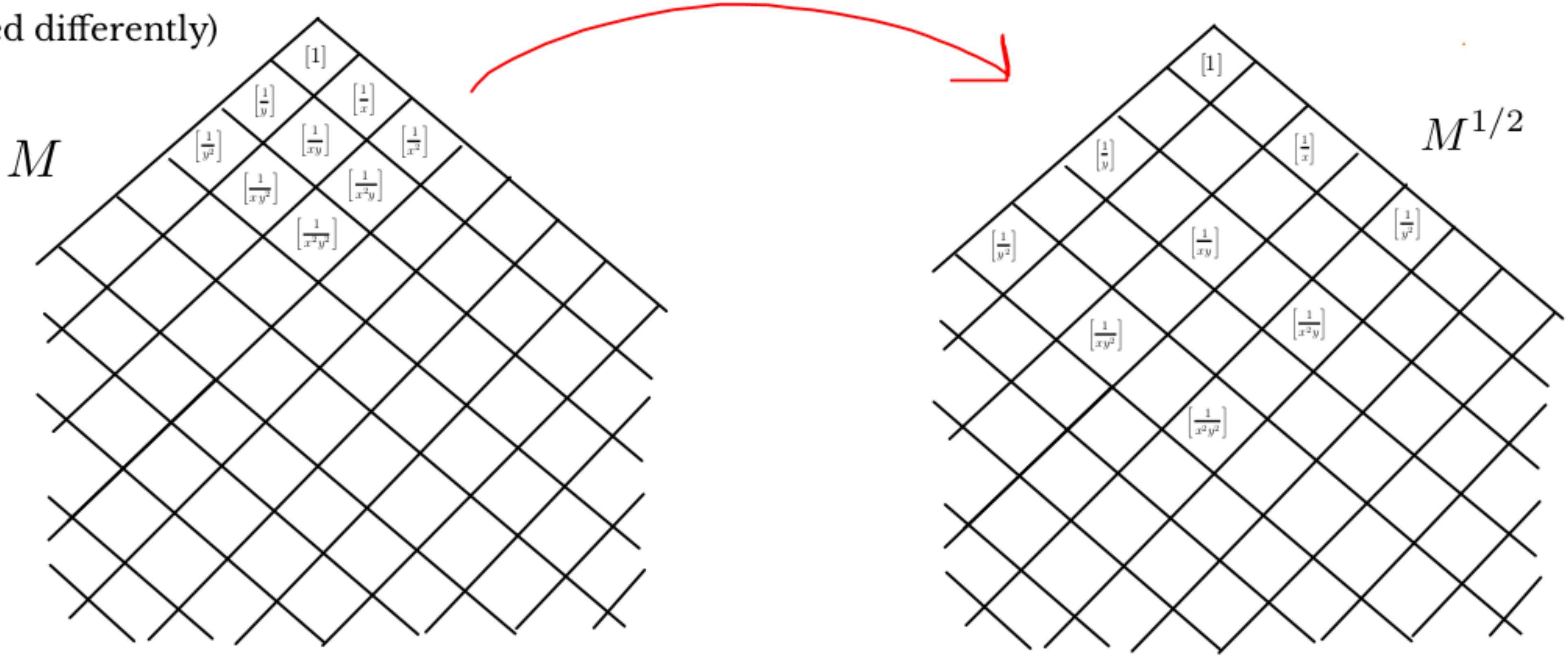

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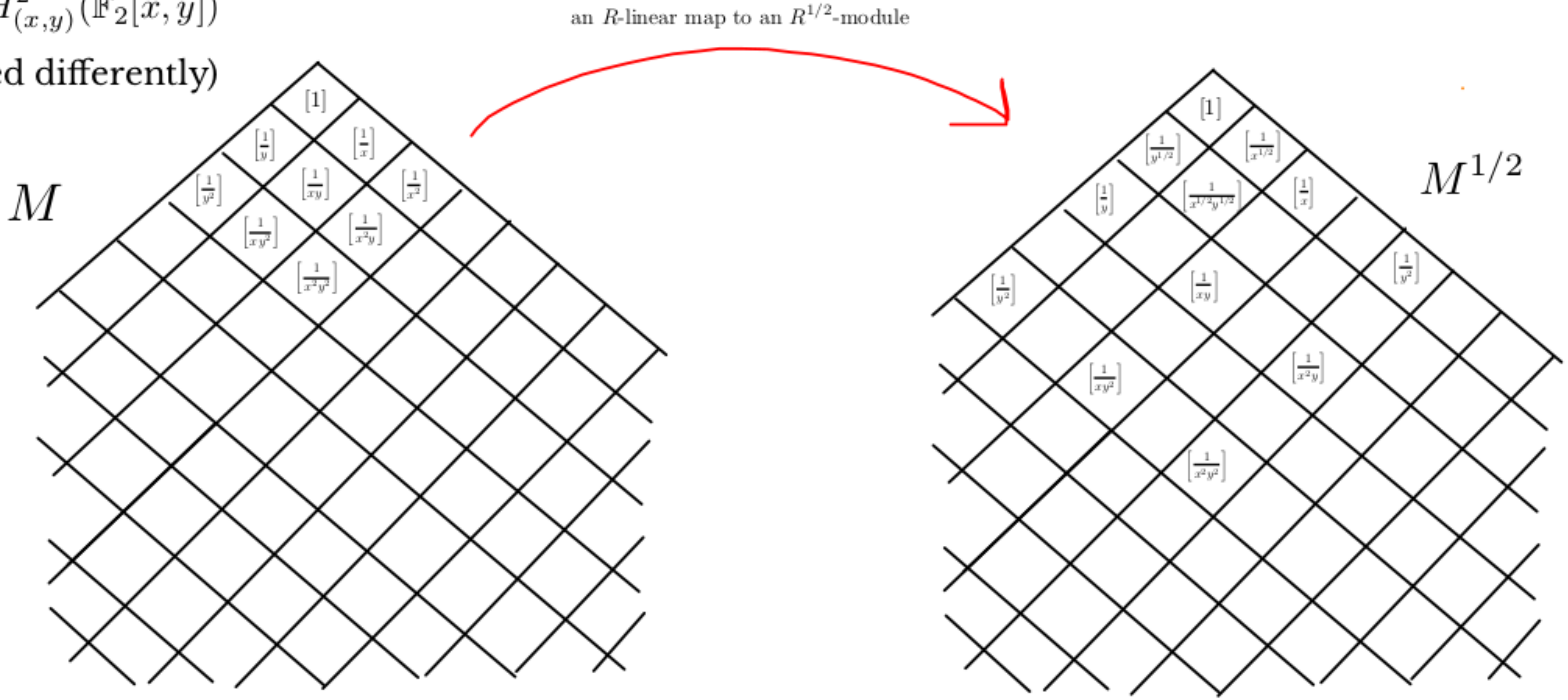

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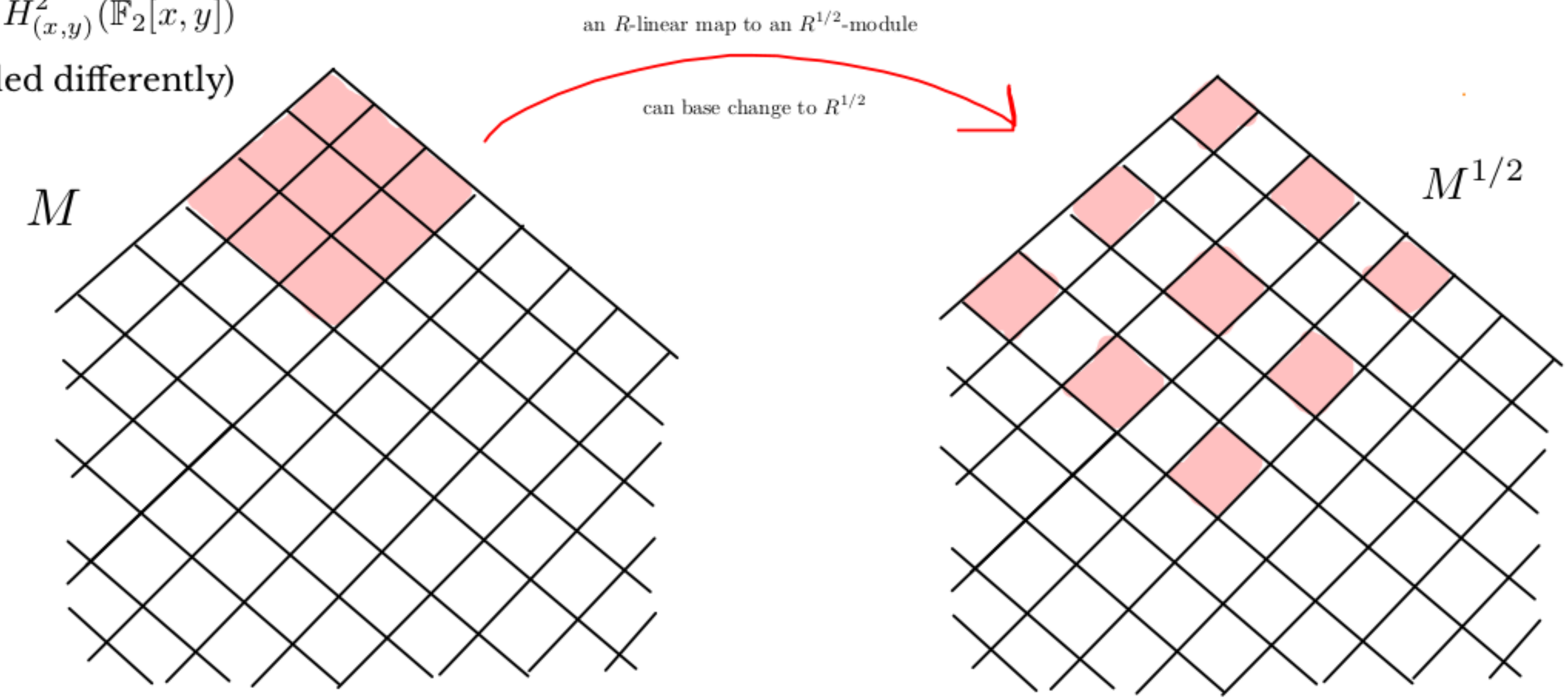

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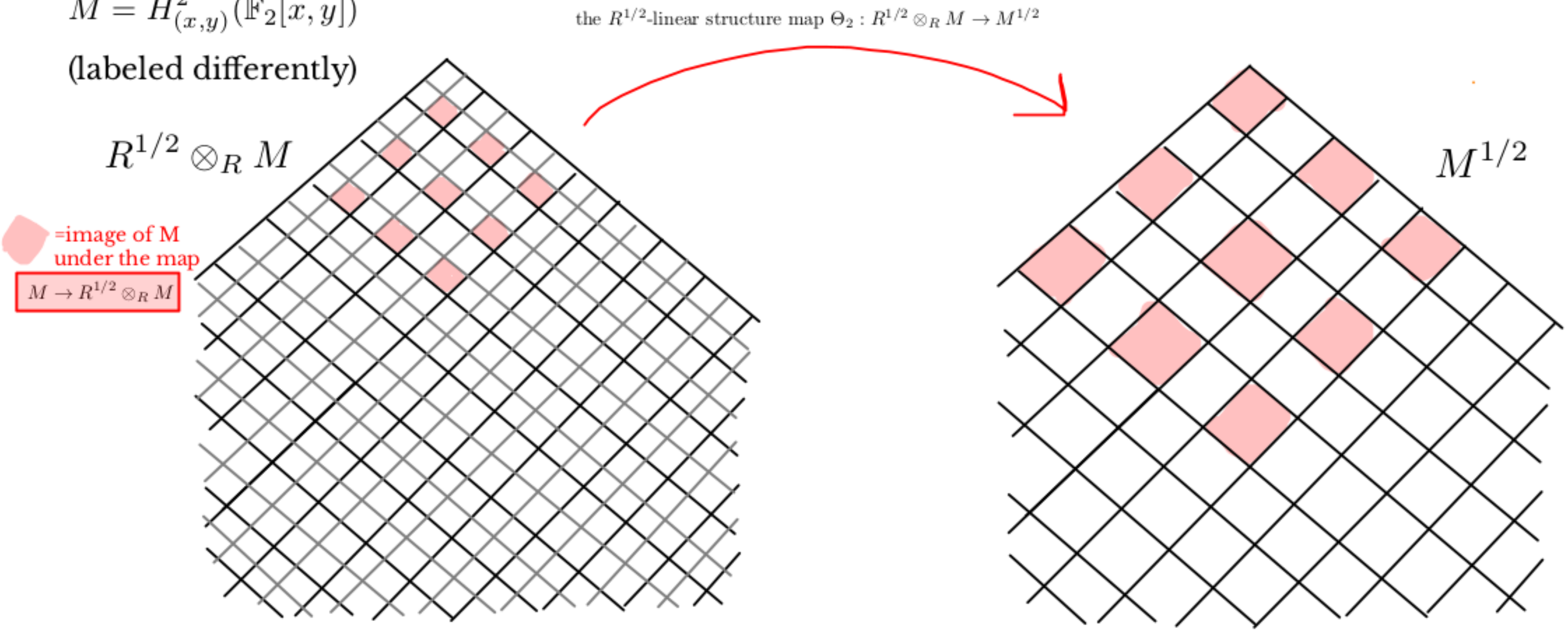

Example: Two different actions on the same R-module...

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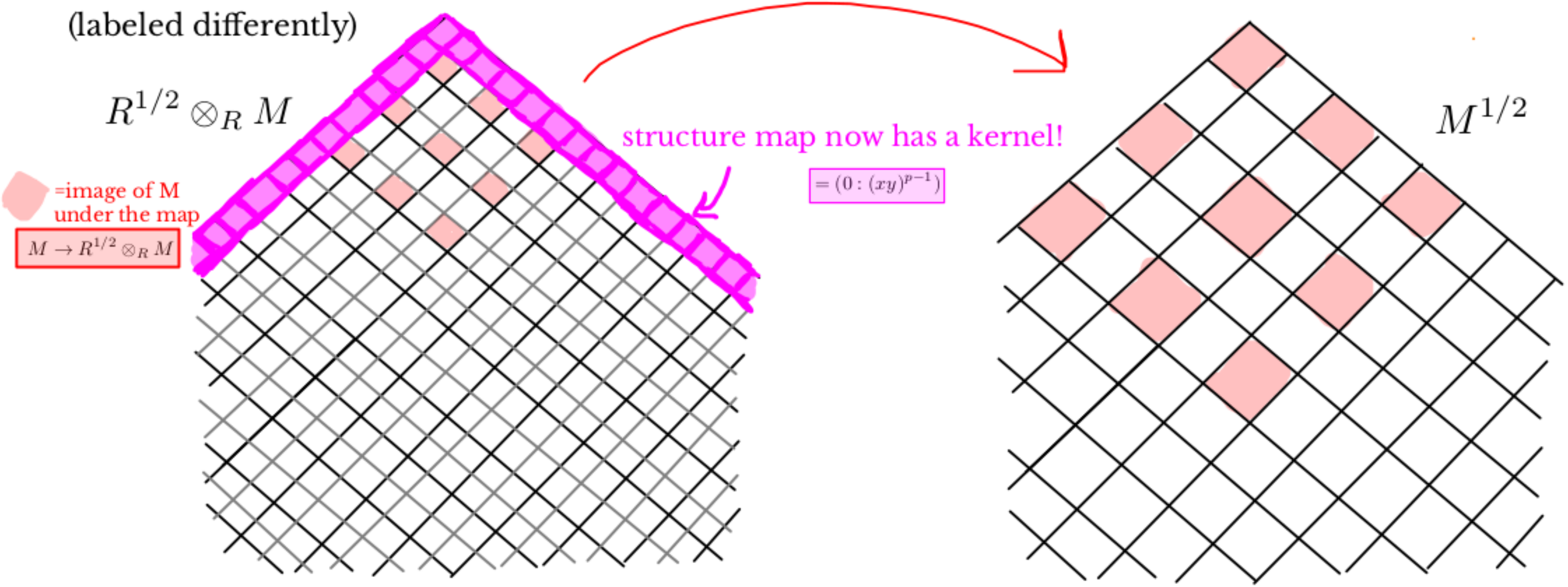
the $R^{1/2}$-linear structure map $\Theta_2 : R^{1/2} \otimes_R M \to M^{1/2}$



Part 2

Using the Frobenius structure

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Definition: An $R\langle F\rangle$-module M is called a *F-module* (in the sense of Lyubeznik) if its structure map is an isomorphism.

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- The category of F-finite F-modules is abelian.
- If N is an $R\langle F\rangle$-submodule of M, and both N and M are Lyubeznik F-modules,

$$M \text{ is finitely generated} \implies N \text{ is finitely generated}$$

- If M is Lyubeznik, then the induced action on $H^i_I(M)$ is Lyubeznik.
- If, additionally, M is F-finite, then $H^i_I(M)$ is also F-finite.

This is not true of finite generation!

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Lyubeznik F-modules

R is a regular ring of prime characteristic $p > 0$, M is an R-module

Definition: An $R\langle F\rangle$-module M is called a *F-module* (in the sense of Lyubeznik) if its structure map is an isomorphism.

$$(\text{that is, } \Theta : \mathcal{F}_R(M) \xrightarrow{\sim} M)$$

(Note: the $R\langle F\rangle$ structure map is the *inverse* of the usual Lyubeznik structure map)

An $R\langle F\rangle$-module that is both (i) finitely generated over $R\langle F\rangle$, and (ii) is a Lyubeznik F-module is called *F-finite*.

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- F-finiteness implies finite length in the category of F-modules.

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Some control is definitely lost: e.g., the set of associated primes can be infinite even if S is a hypersurface.

$$H^2_{(x,y)}\left(\frac{K[u,v,w,x,y,z]}{wu^2x^2 - (w+z)uxvy + zv^2y^2}\right) \text{ [Katzman; 2002]}$$

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but the Frobenius structure can still be quite powerful for studying vanishing questions.

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Lyubeznik F-finite

in particular, finitely generated over R<F>

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(f.g. over R<F>, hence closed support)

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each carries a Frobenius action...

functorial long exact sequence from $\Gamma_I(-)$

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diagram commutes, so maps are compatible with R<F> structures

Lyubeznik F-finite

in particular, finitely generated over R<F>

(f.g. over R<F>, hence closed support)

finite set of associated primes

$$0 \to [\text{image of } H^{i-1}_I(R)] \to H^{i-1}_I(R/J) \to [\text{some submodule of } H^i_I(J)] \to 0$$

Theorem: [Hochster, Núñez-Betancourt; 2017] Let R be a regular ring of characteristic $p > 0$. If Ass $H^i_I(J)$ is finite, then Supp$H^{i-1}_I(R/J)$ is closed.

Ex: Following Hochster and Núñez-Betancourt...

R is regular

$S = R/J$, and $I \subseteq R$ is an ideal

Assume: $H^i_I(J)$ has a finite set of associated primes (e.g. J is a principal ideal).

$$0 \to \mathcal{F}_R(J) \to \mathcal{F}_R(R) \to \mathcal{F}_R(R/J) \to 0$$

$$\| \qquad \| \qquad \|$$

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In other words:

There is evidence to suggest that if $H^i_I(R/J)$ has infinitely many associated primes (the only case where we're asking about closed support), then under certain circumstances, $H^i_I(J)$ necessarily also must have an infinite set of associated primes.

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Should not generally expect to use the latter to control the former.

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Will need to take a brief detour to see where we might find some promising options...

Part 3

The Fedder Action

Recall from Fedder's criterion...

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For simplicity,
$(R, \mathfrak{m}, K)$ is a regular local such that $R \to R^{1/p}$ is a finite map
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$$\left(\frac{(J^{[p]} : J)}{J^{[p]}}\right)^{1/p} \simeq \mathrm{Hom}_{R/J}((R/J)^{1/p}, R/J) \text{ as } (R/J)^{1/p}\text{-modules}$$

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$(R, \mathfrak{m}, K)$ is a regular local ring, $R \to R^{1/p}$ is finite.

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$(R, \mathfrak{m}, K)$ is a regular local ring, $R \to R^{1/p}$ is finite.

$S = R/J$ is a Gorenstein ring, and $g \in R$ gives $(J^{[p]} : J) = g + J^{[p]}$

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F

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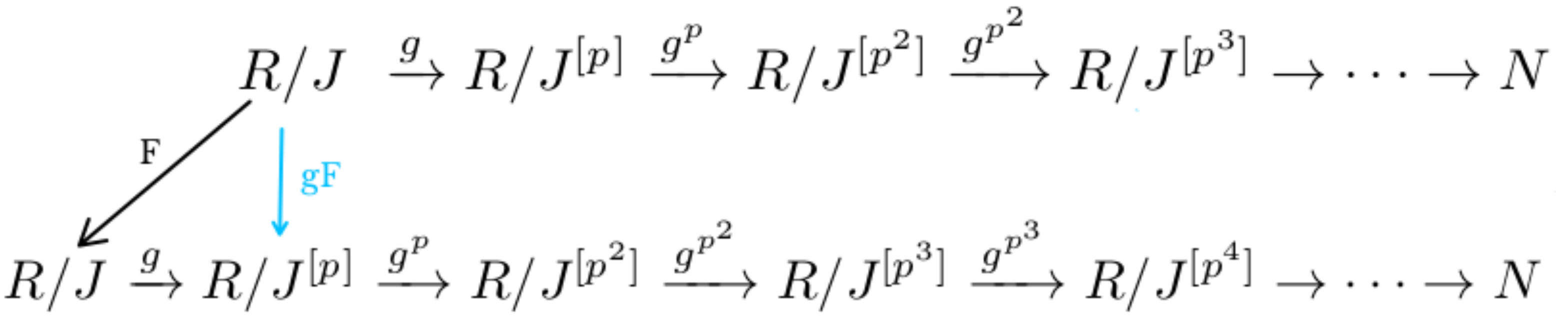

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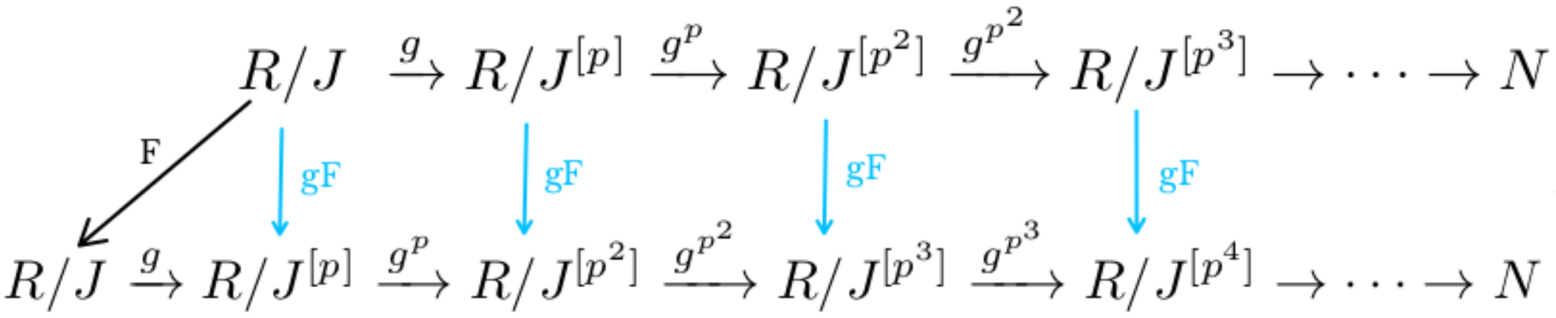

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Call the resulting Frobenius action on the direct limit N sending $n \mapsto gF(n)$ the *Fedder action*.

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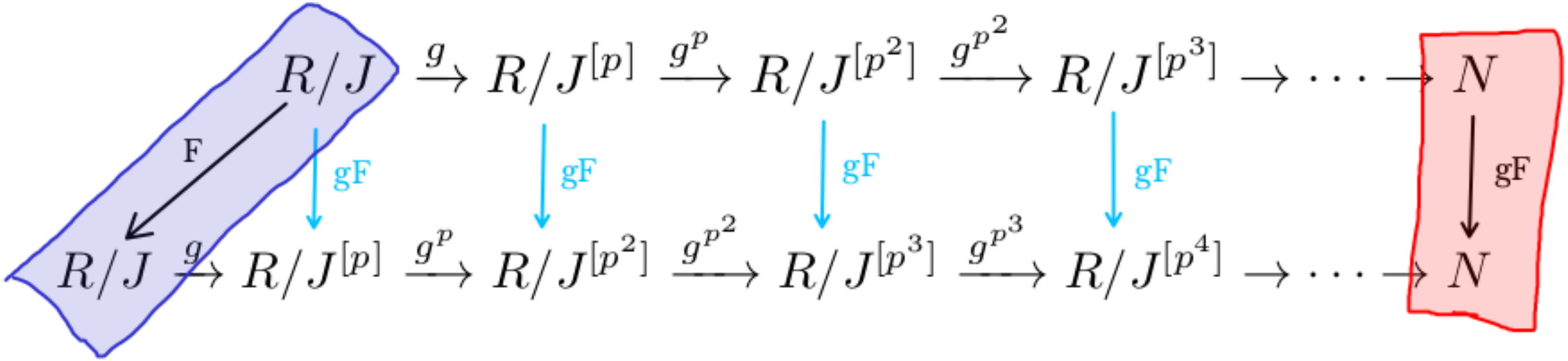

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Let $J = (f_1, \cdots, f_c)$, then $g = f^{p-1}$ where $f = f_1 \cdots f_c$

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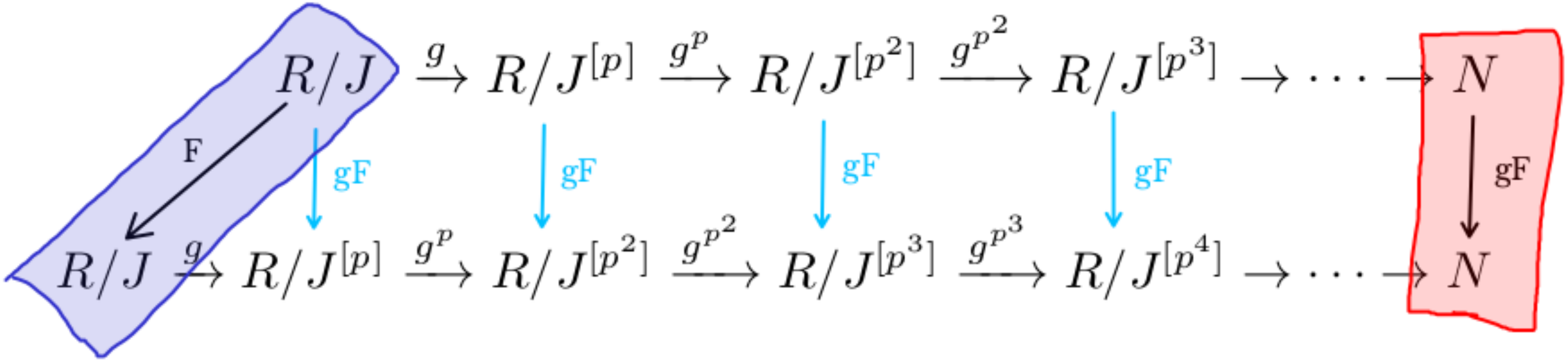

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\swarrow^{F} & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow{\scriptstyle f^{p-1}F} \\
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Call the resulting Frobenius action on the local cohomology $H^c_J(R)$ sending $n \mapsto f^{p-1}F(n)$ the *Fedder action.*

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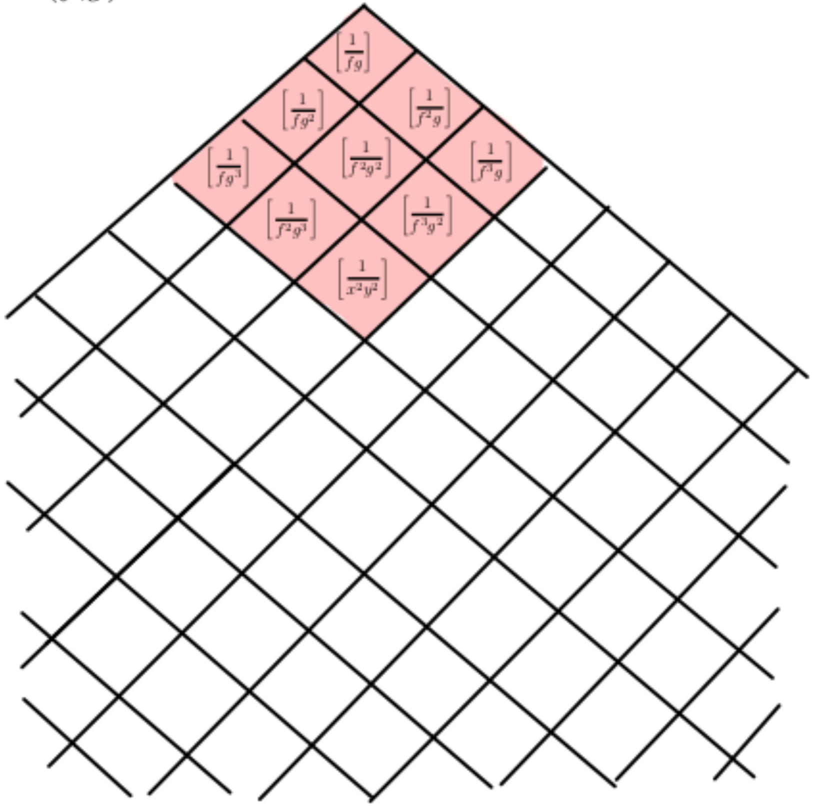
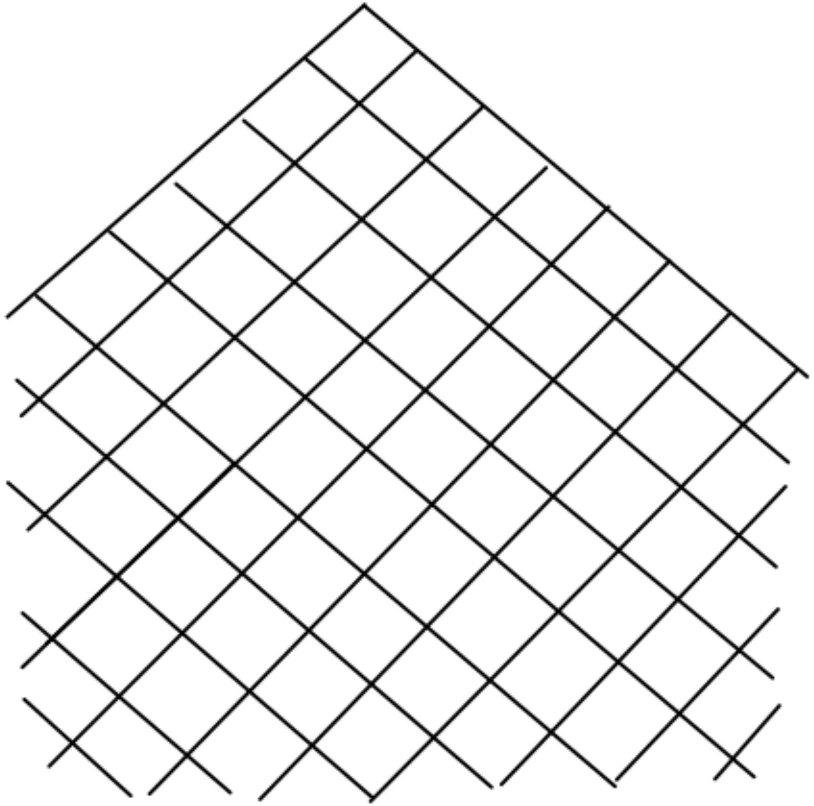
Example in codimension 2

$M = H^2_{(f,g)}(R)_{\text{fed}}$





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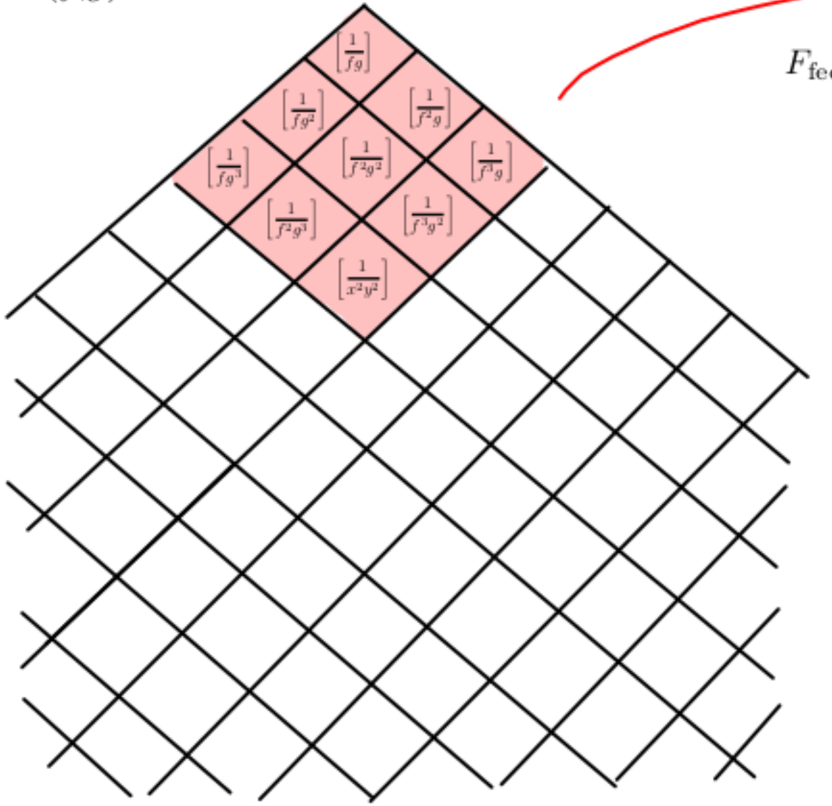
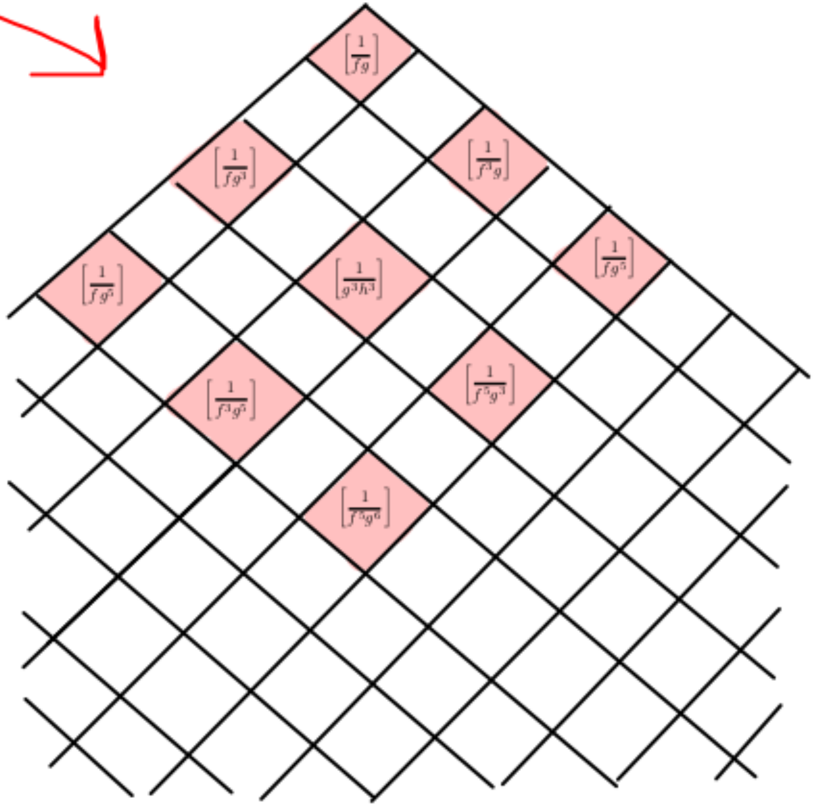


apply Fedder action (p=2)

$F_{\text{fed}} = (fg)^{p-1}F$



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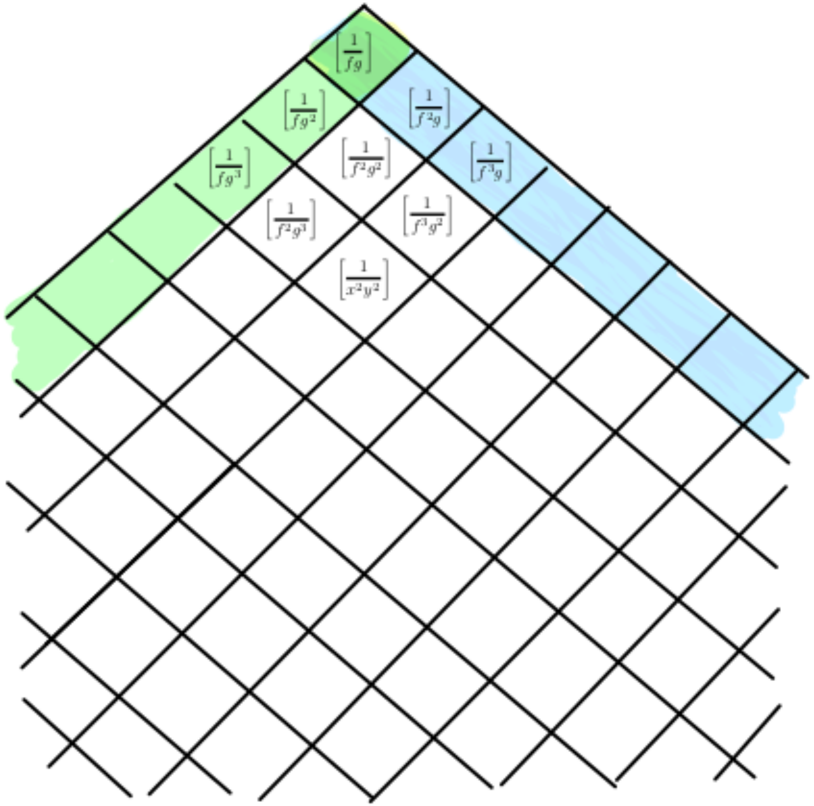

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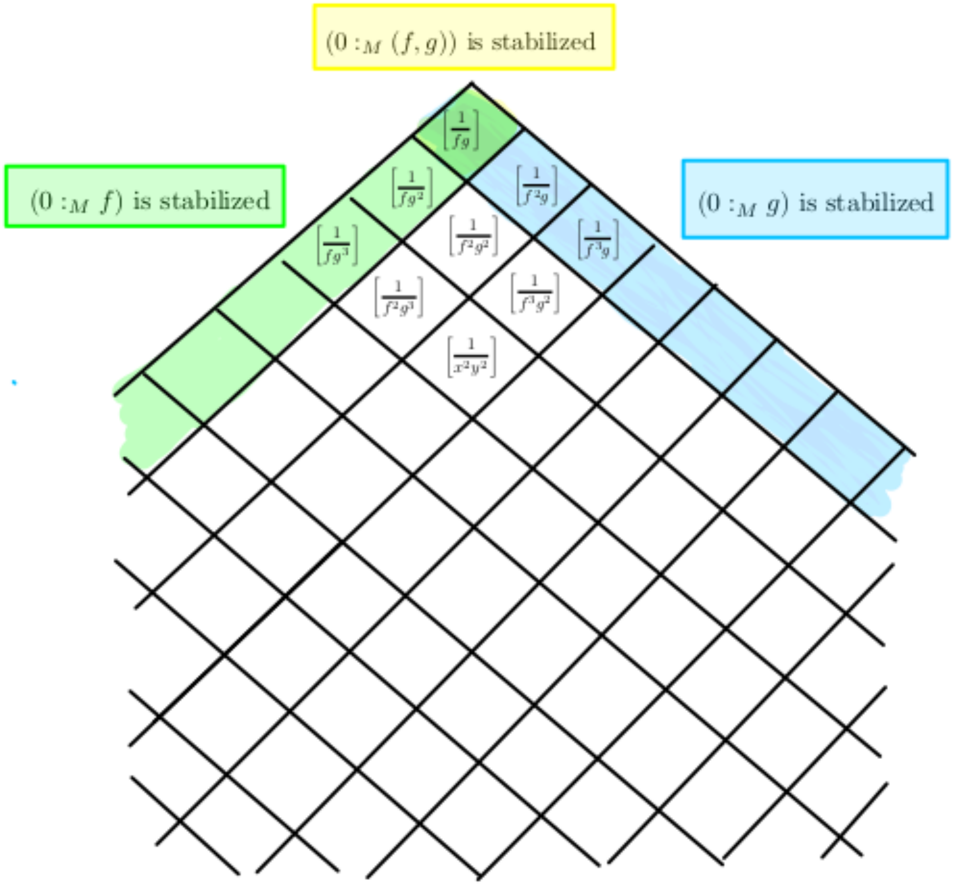

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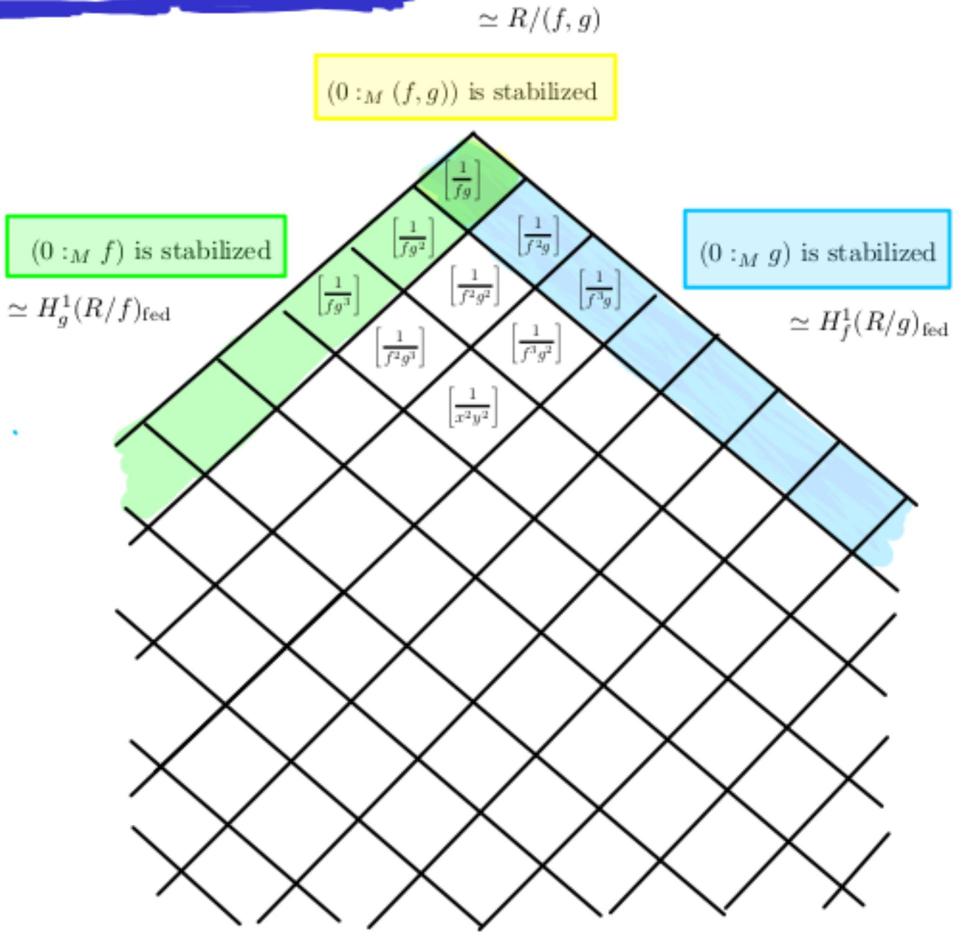

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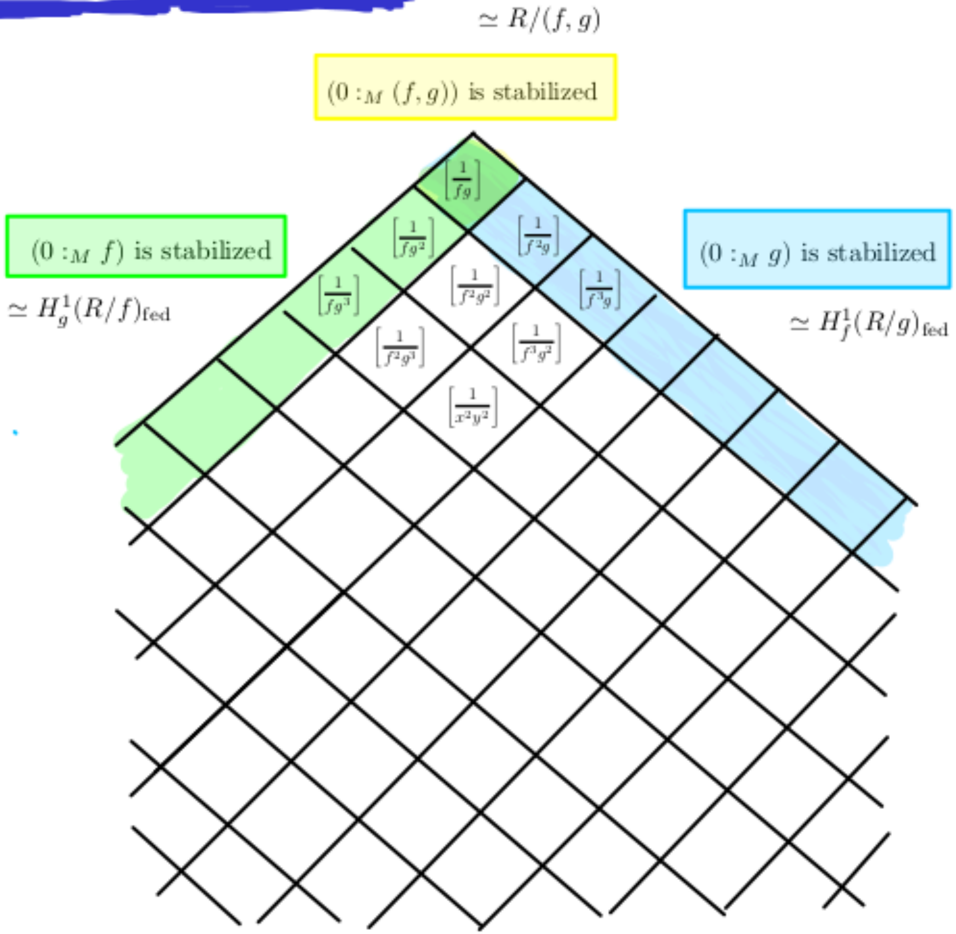

Recall: The structure morphism of this action has a nontrivial kernel...

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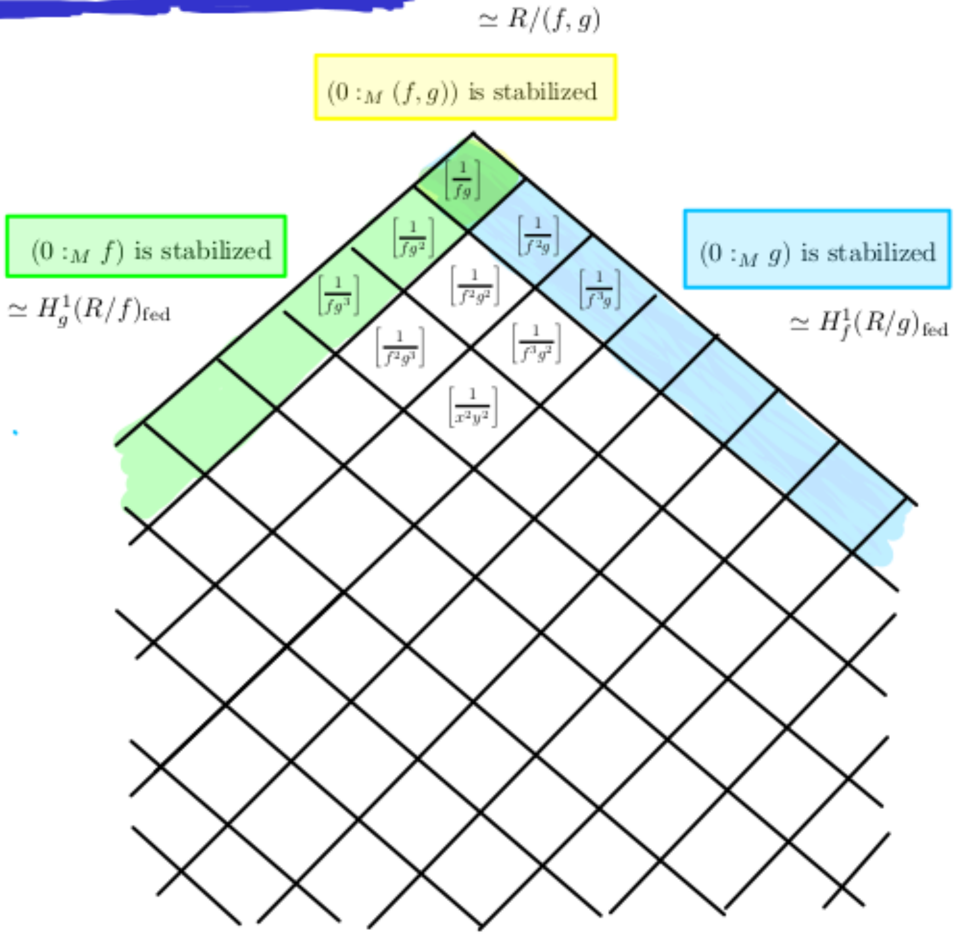

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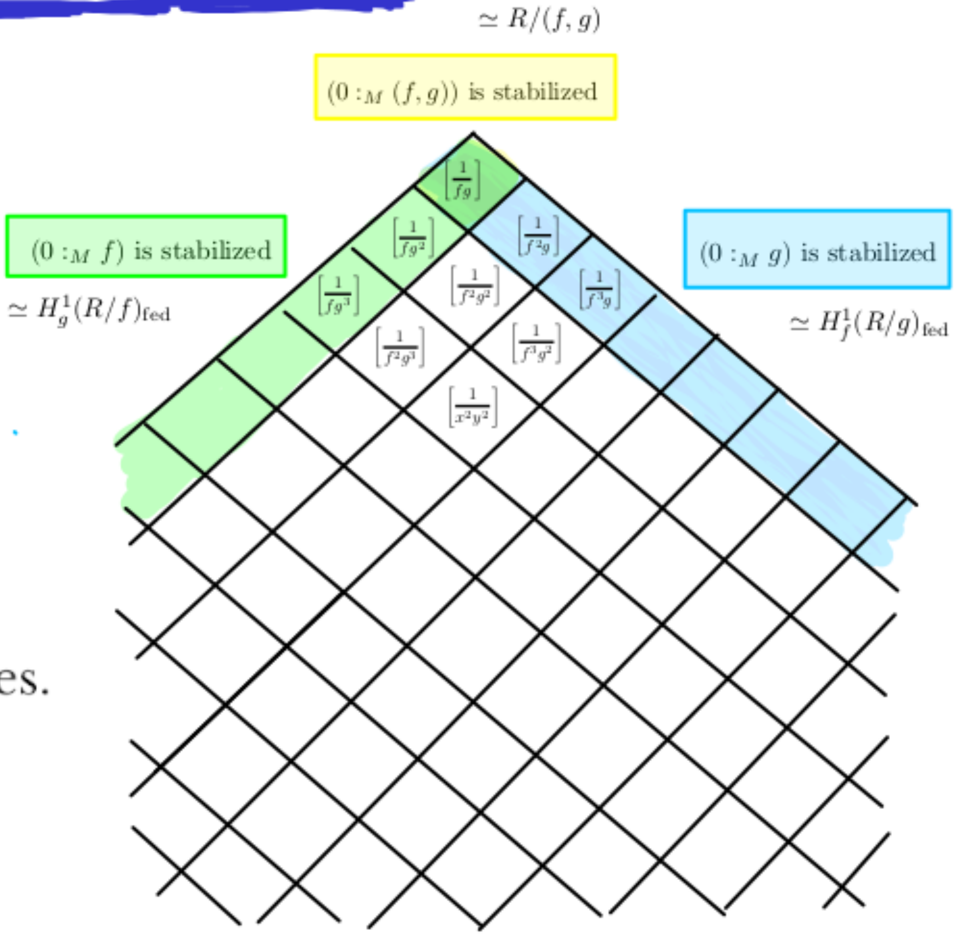

We extra structure
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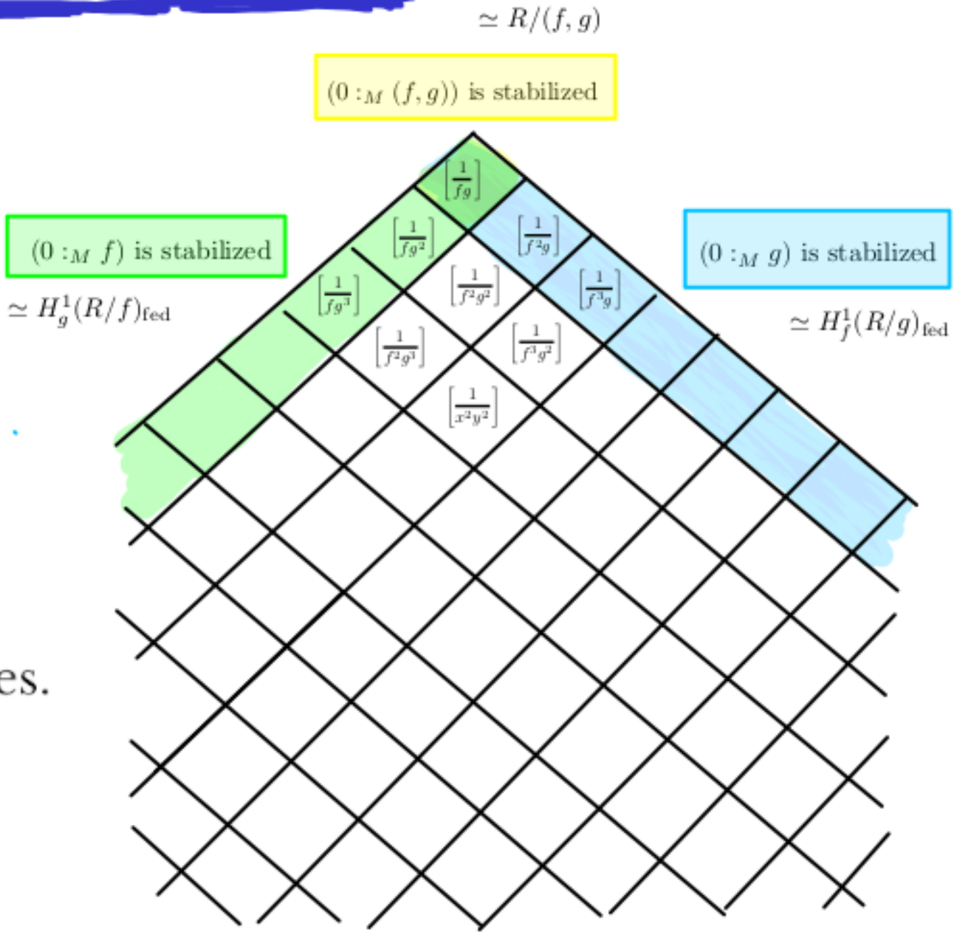

We extra structure from these F-submodules. Let's see how the pieces fit together....

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Example in codimension 2

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Have a complex of R<F>-modules with R<F>-linear maps

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$$\underbrace{\frac{R}{(f,g)}}_{\text{codim 2 C.I.}}$$

Example in codimension 2

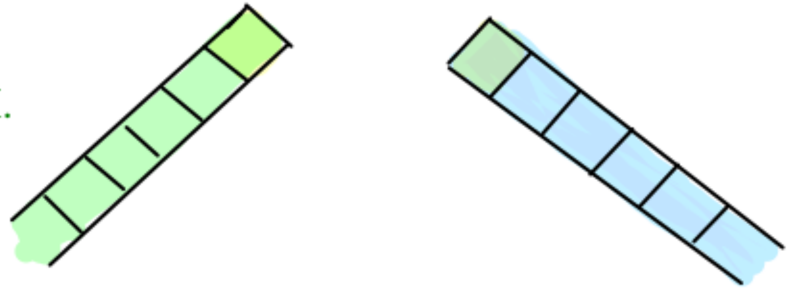
$M = H^2_{(f,g)}(R)_{\text{fed}}$

Have a complex of R<F>-modules with R<F>-linear maps

hypersurface terms

$$0 \to \frac{R}{(f,g)} \to H^1_g(R/f)_{\text{fed}} \oplus H^1_f(R/g)_{\text{fed}}$$

codim 2 C.I.



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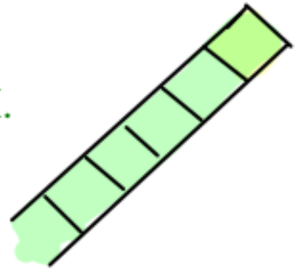
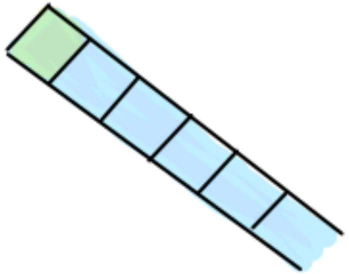
regular term





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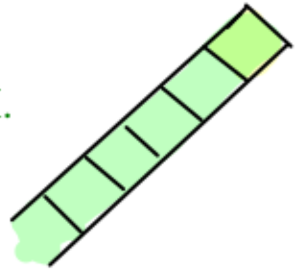

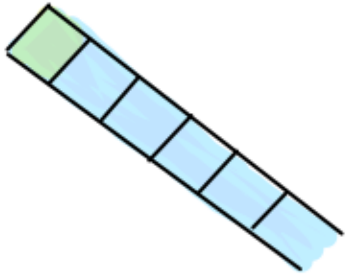

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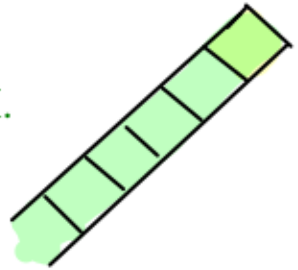

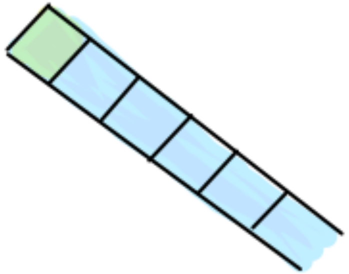

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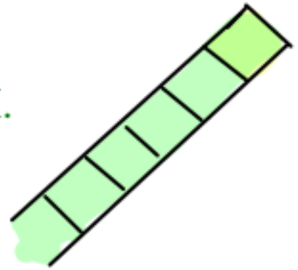

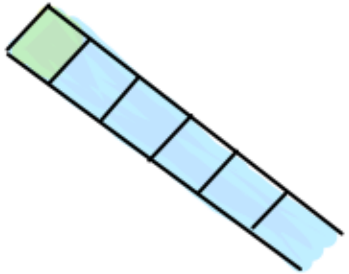

Example in codimension 2

Can check: this complex is exact

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Have a complex of R<F>-modules with R<F>-linear maps

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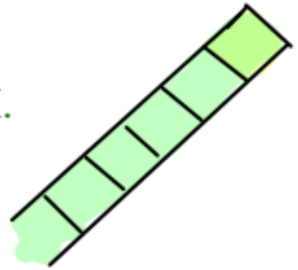
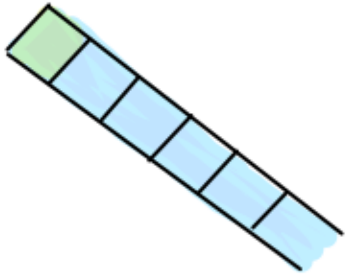
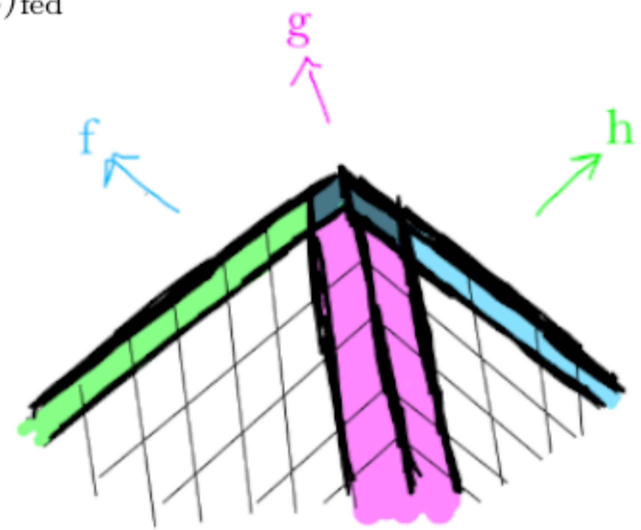




Similar picture in codimension 3

$M = H^3_{(f,g,h)}(R)_{\text{fed}}$



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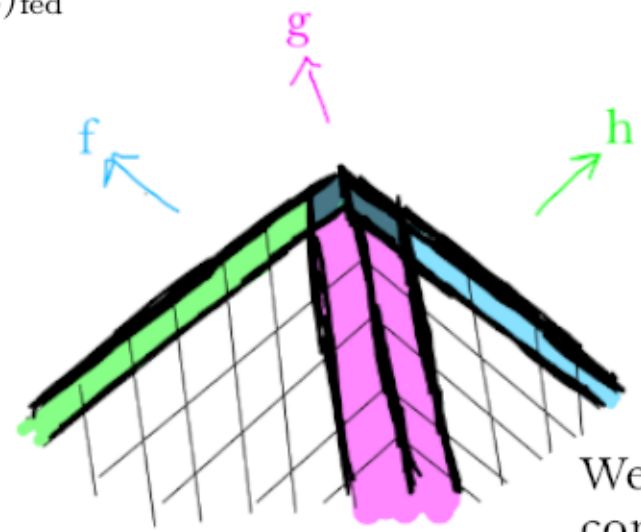

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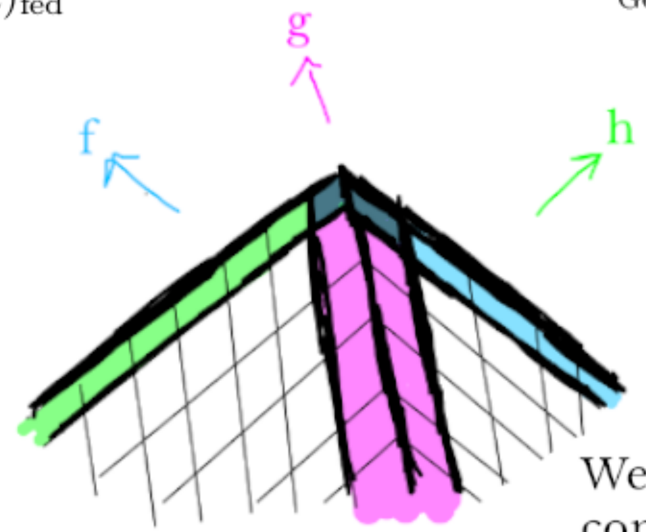

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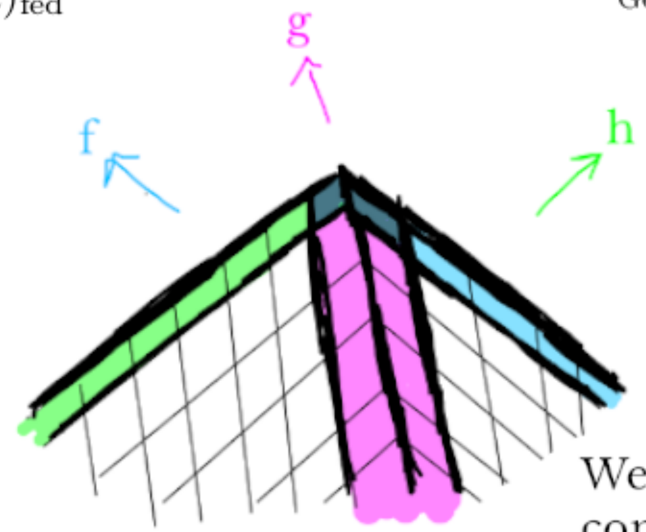

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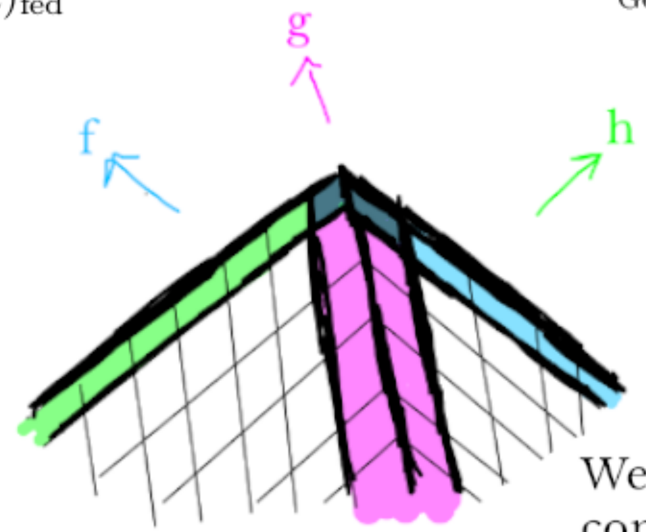

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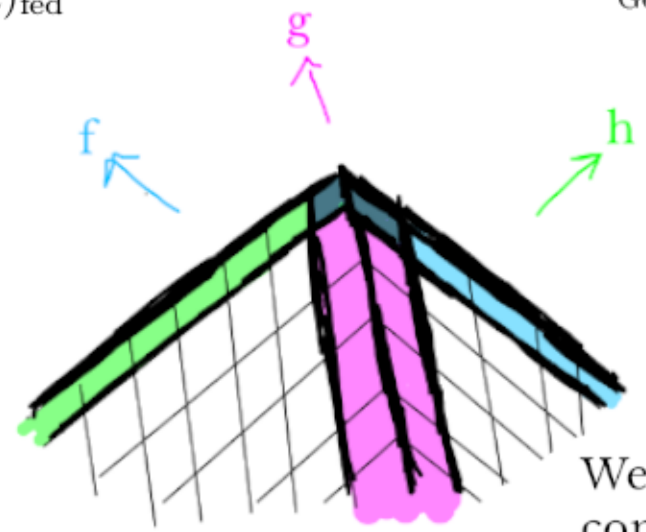

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and the augmentation is $H^c(\Delta^\bullet_{f_1,\cdots,f_c}(R)) \simeq H^c_{(f_1,\cdots,f_c)}(R)_{\text{nat}}$

Part 4

Applications

Local cohomlogy "at the height + k" level

$$\vdots$$

$$H_I^{\mathrm{ht}(I)+3}(R)$$

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$$H_I^{\mathrm{ht}(I)}(R)$$

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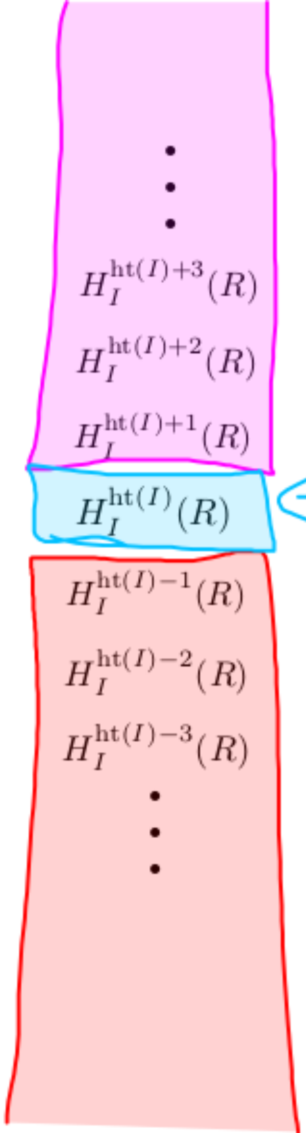

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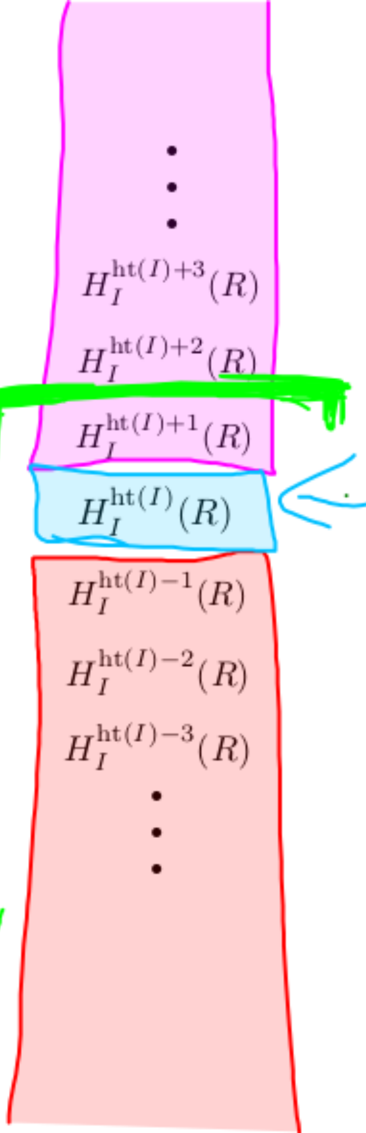

The height + 1 and lower cases are fully general
see [Hellus; 2000] (R is C-M local)
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that is, for each i, there is a possibly larger ideal I' depending on i, such that $H^i_I(R) \simeq H^i_{I'}(R)$ with $i \leq \mathrm{ht}(I') + 1$

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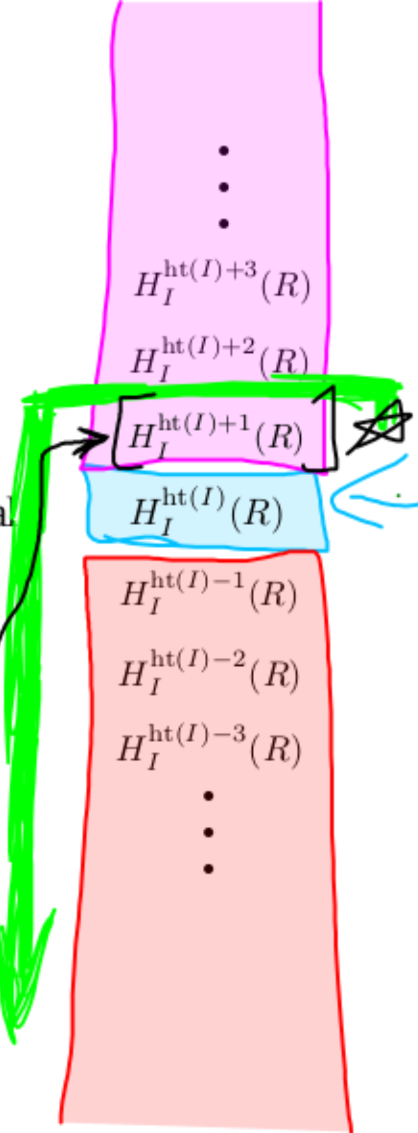

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Local cohomlogy "at the height + k" level

height + 2 and higher cases are not necessarily fully general. These can all be brought to the form height + 1 or lower, but may possess special properties that don't hold for all height + 1 modules.

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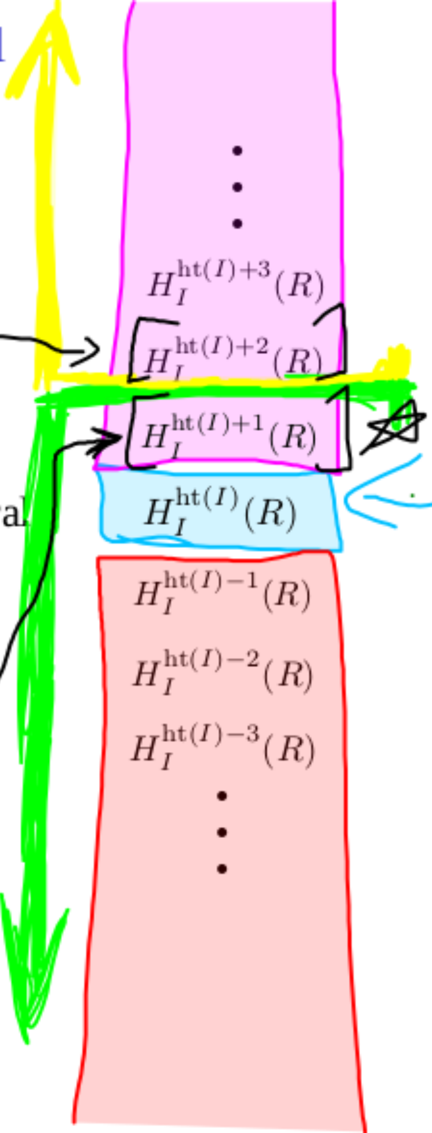

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Will sketch the argument for c=3...

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codimension 2 terms

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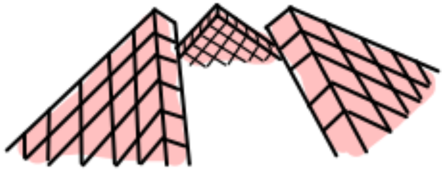

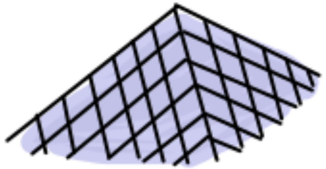

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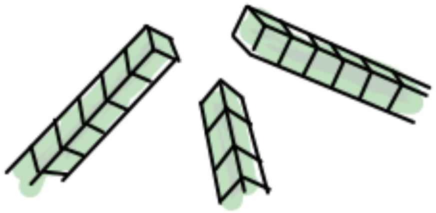

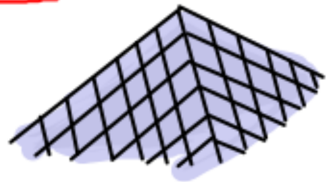

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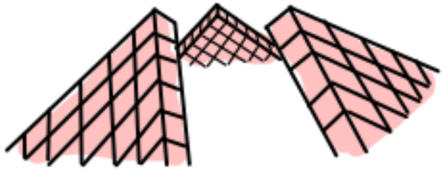

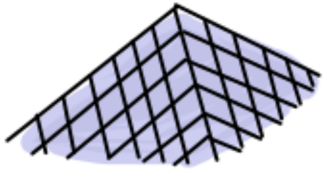

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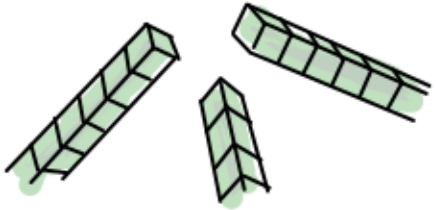

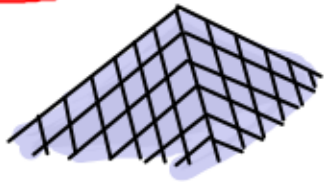

Get three important short exact sequences onto which we can apply $\Gamma_I(-)$.

Three key short exact sequences

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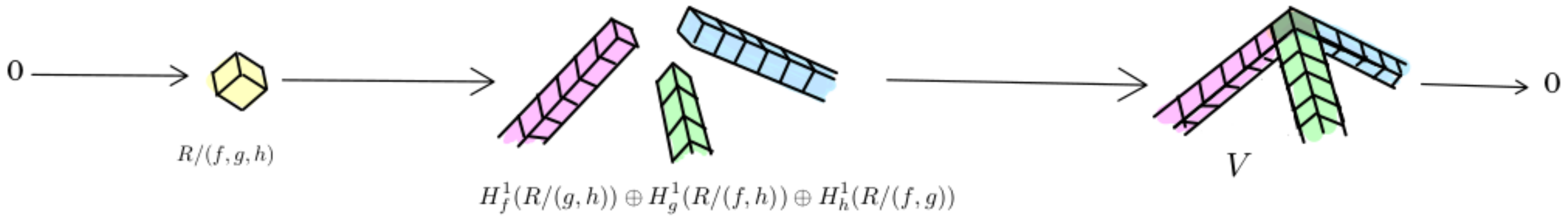

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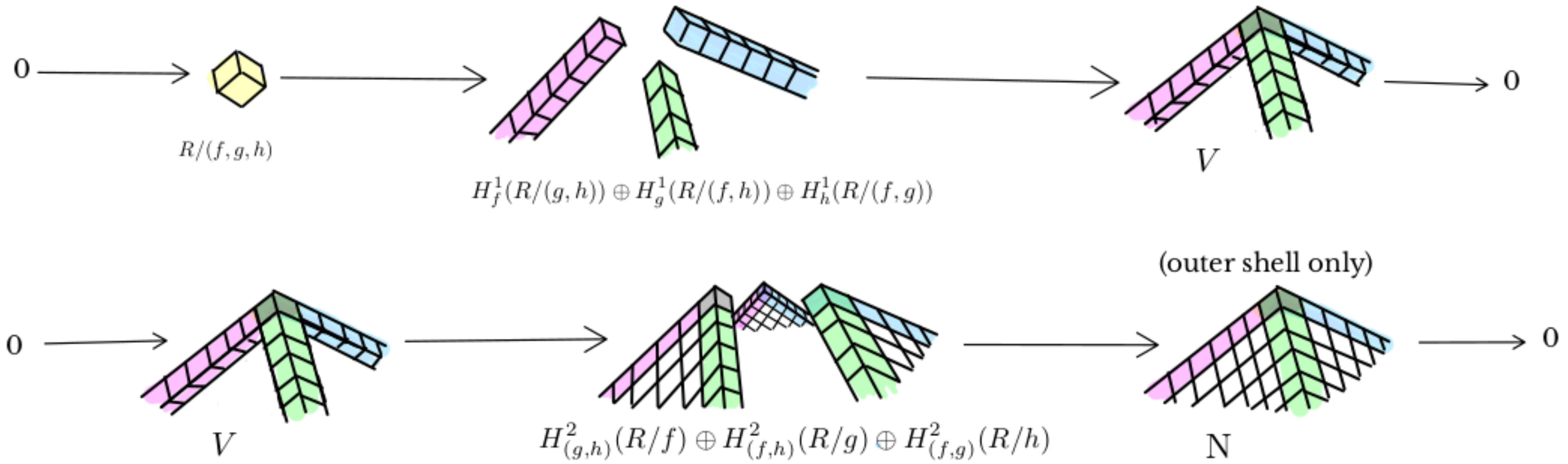

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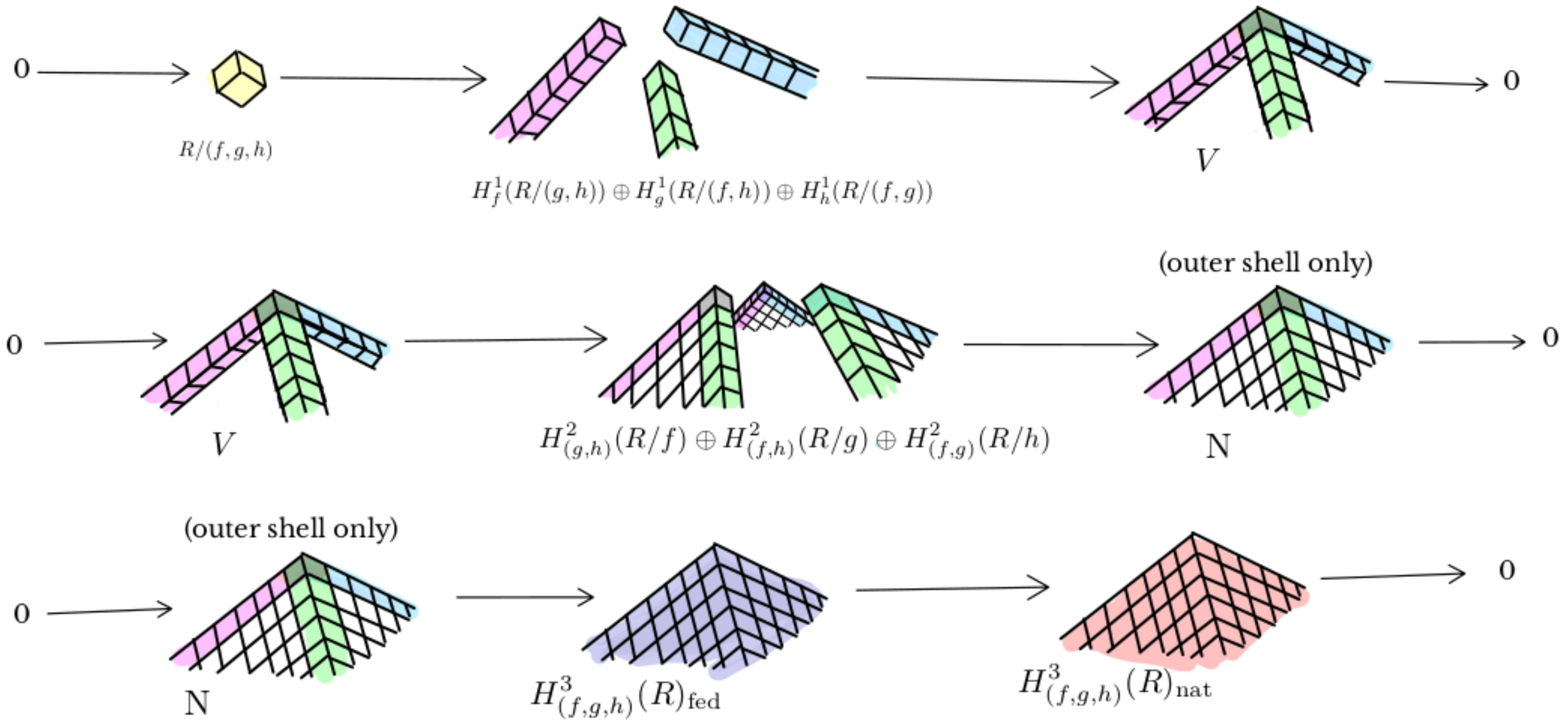

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- Base case is codimension 0 (LC of R itself), where we get key vanishing due to Peskine-Szpiro.
- End result: $\operatorname{Supp}\left(H_I^{\mathrm{ht}(I)+3}\left(\frac{R}{(f,g,h)}\right)\right)$ is closed!

Thank you!