Lech-Mumford constant
(work in progress with Ilya Smirnov)

Throughout, all rings are commutative, Noetherian with multiplicative identity 1.

Lech's inequality (1960) (R,m) = local $I \subseteq R$ m-primary

Then $e(I) \leq d!\, e(R)\, \ell(R/I)$ $(*)$

Here $d = \dim R$ $e(I) := \lim_{n\to\infty} \frac{\ell(R/I^n)\, d!}{n^d}$ $e(R) := e(m)$

Example: ① Take $I = m$ in $(*)$

$LHS = e(R)$

$RHS = d!\, e(R)$

$(*)$ is sharp when $d \leq 1$

② Take $I = J^n$ $n \gg 0$ in $(*)$

$LHS = n^d e(J)$

$RHS \sim e(R)\, n^d e(J)$

$(*)$ is sharp when R = regular

Definition (R,m) = local $\quad C_{LM}(R) := \sup_{\sqrt{I}=m} \left\{ \dfrac{e(I)}{d!\, \ell(R/I)} \right\}$

$(d = \dim R)$ is the **Lech-Mumford constant** of R

Remark: ⓪ Lech's inequality $\Longleftrightarrow$ $C_{LM}(R) \leq e(R)$

① $C_{LM}(R) = e(R)$ if $d \leq 1$

② $C_{LM}(R) \geq 1$ by considering $I = J^n$ $n \gg 0$

③ $C_{LM}(R) = C_{LM}(\hat{R})$

Mumford use $e_0(R)$ to denote $C_{LM}(R)$

called the 0-th "flat multiplicity" of R

further defined $e_i(R) = e_0(R[[t_1 \cdots t_i]])$

$= C_{LM}(R[[t_1 \cdots t_i]])$

Proposition (Mumford 1977) (R,m) = local Then

$$e(R) \geq C_{LM}(R) \geq \underline{C_{LM}(R[[t_1]])} \geq C_{LM}(R[[t_1,t_2]]) \geq \cdots \geq 1$$

Definition (Mumford 1977) (R, m) = local is called

- **semi-stable** if $e_M(R[[t]]) = 1$
- **stable** if semistable and the sup in the definition of $e_M(R[[t]])$ is not attained

Theorem (Mumford 1977) Suppose X = projective scheme L = ample on X, If (X, L) is **asymptotically Chow semi-stable**. Then $\mathcal{O}_{X,x}$ is semistable for all $x \in X$

Chow point $\Phi_n(X)$ correspond to (X, L^n)

char 0

X = normal. proj. $\mathbb{Q}$-Gorenstein

Theorem (Odaka 12') (X, L) = asymptotically Chow semistable. Then $\mathcal{O}_{X,x}$ = log canonical $\forall x \in X$

log canonical: $Y \to \mathrm{Spec}(R) = X$ log resolution

coeff of exeptional in $K_{Y/X}$ is ≥ -1

Theorem (Ma – Smirnov 20) (R, m) = normal.

$\mathbb{Q}$-Gorenstein. char 0 (essentially finite type)

Then ① If R = semistable, then R = log canonical

② $C_{LM}(R) = 1$ and R = isolated singularity

then R = canonical

Examples: ⓪ (R,m) = regular $\Rightarrow$ $\underline{c_{LM}(R[[t_1,\dots,t_n]])}=1$

$\Rightarrow$ R is semistable (in fact, stable by Lech 1960')

① $(R,m)=\dfrac{k[[x_0,\dots,x_d]]}{f}$ hypersurface of dim d

- R = semistable $\Rightarrow$ $\deg f \leq d+1$
- $c_{LM}(R)=1$ $\Rightarrow$ $\deg f \leq d$

[MS]

② $\dim R=1$ CM R = semistable $\Longleftrightarrow$

[Mumford] R = regular or $R\cong \dfrac{k[x,y]}{xy}$

③ $\dim R=2$ CM $k=\bar{k}$ char 0 or $p>5$

[MS]: $c_{LM}(R)=1$ $\Longleftrightarrow$ R = regular, $\underline{\text{ADE}}$ or $\underline{A_\infty, D_\infty}$

R = semistable $\Rightarrow$ $\underbrace{\text{“ . . . . . . ”}}_{\text{incomplete list of candidates}}$

[Mumford, Shah, MS]

$\dfrac{k[x,y,z]}{xy}$

$\dfrac{k[x,y,z]}{x^2+y^2z}$