Trivial Vanishing of Tor and the graded Tachikawa Conjecture

joint w/ J. Lyle and S. Sather-Wagstaff

(R, m, k) CM local ring, or, $R = \bigoplus_{i \geq 0} R_i$ with $R_0 = k$.

ω can. module.

Auslander, Ding, Solberg

Auslander - Reiten conjecture (1952) [ARC]

If $\operatorname{Ext}^i_R(M, M \oplus R) = 0$ for $i > 0 \Rightarrow M$ is free

ARC holds for: $c = \operatorname{edim}(R) - \dim(R)$ "codimension"
$e = e(R)$ "multiplicity"

1) R is a c.i. (Avramov - Buchweitz, '00)
2) R is Golod (Jorgensen, '99)
3) R is Gorenstein and $c \leq 4$ (Sega, '00)
4) $m^3 = 0$ (Huneke - Sega - Vraciu, '04)
5) $\frac{\operatorname{type}(R)}{e} < 2$ (Dao - Veliche, '08)

other results by Huneke - Leuschke, Nasseh - Sather-Wagstaff, Sega, Christensen - Holm, Lindo, Araya, ...

Theorem (Lyle – M, '19)

ARC holds if
1) $c \leq 3$
2) $e \leq \frac{7}{4}c + 1$
3) $e \leq c + 6$ and R is Gorenstein

In particular, ARC holds if $e \leq 8$, or, $e \leq 11$ and R is Gor.

Tachikawa – Conjecture (1973) [TC] Avramov – Buchweitz – Sega

If $\mathrm{Ext}^i_R(\omega, R) = 0$, $i > 0 \Rightarrow R$ is Gorenstein.

Rmk: $\mathrm{ARC} \overset{M=W}{\Longrightarrow} \mathrm{TC}$

TC also holds for:
1) $\mathrm{Type}(R) \leq 2$ (Huneke – Sega – Vraciu, '04)
2) R is generically Gor. (ABS, '05)
3) R is licci (ABS, '05)

Theorem (Lyle – M – Sather Wagstaff) "Graded TC."

If R is graded and char $k \neq 2$, then TC holds.

proof (sketch):
1) Reduce to Artinian Case.
2) If $\mathrm{Tor}^R_{i>0}(M, M) = 0$, we are able to compute $\lambda(\Lambda^2 M)$
3) The condition in TC can be stated as $\mathrm{Tor}^R_{i>0}(\omega, \omega) = 0$.

4) with $M = W$ our formula shows $\hat{\wedge}^2 W = 0 \Rightarrow W$ is cyclic. ∎

T.V. "final vanishing" $\operatorname{Tor}^R_{i \gg 0}(M,N) = 0 \overset{\checkmark}{\Leftarrow\Rightarrow}$ projdim $M < \infty$, projdim $N < \infty$

Not always.

EX: $R = \dfrac{k[\![x,y]\!]}{(x^2, y^2)}$, $M = R/xR$, $N = R/yR$

T.P. "Tor persistence" $\operatorname{Tor}^R_{i \gg 0}(M,M) = 0 \Leftarrow\Rightarrow$ projdim $M < \infty$.

open Question: Does every ring satisfy T.P.?

TP holds for:

1) $m^4 = 0$ and $\exists a \in m$ with $(0 :_R a)$ principal

(Sega, '11)

2) If $\hat{R} \cong Q/f$, f is Q-regular

and certain conditions on Q.

(Avramov – Iyengar – Nasseh – Sather-Wagstaff, '20).

Ex: Ring not covered by known results above

$$R = \frac{k[[x,y]]}{(x,y)^2} \otimes_k \frac{k[[u,v]]}{(u,v)^2}$$

Theorem (Lyle - M - Sather-Wagstaff)

If $m^3 = 0$, T.P. holds.