Strongly F-regular Rings & Their Divisor Class Groups

<u>Conjecture</u>: Let (R, m, k) be a F-finite and strongly F-regular of prime char $p > 0$. Then the divisor class group $cl(R)$ is a f.g. group.

$R = (R, m)$ of prime char $p > 0$

$\forall e \in \mathbb{N}$ $\quad F^e: R \longrightarrow R$

$\qquad\qquad r \longmapsto r^{p^e}$

- For each $M \in Mod(M)$ let $F_*^e M = R$-mod. M obtained via restriction of scalars $/ F^e$.

<u>Example</u>: $R = \mathbb{F}_p[x_1, \ldots, x_d]$

$F_*^e R \cong R^{\oplus p^{ed}}$ with R-basis

$\{F_*^e x_1^{i_1} \cdots x_d^{i_d} \mid 0 \le i_j < p^e\}$.

- R is <u>F-finite</u>: $M \in mod(R) = \{$f.g. R-mods$\}$ $\implies F_*^e M$ remains f.g.

- R is strongly F-regular (SFR) if $\forall\, 0 \neq r \in R$ $\exists e \in \mathbb{N}$ and $\varphi \in \mathrm{Hom}_R(F^e_* R, R)$: $\varphi(F^e_* r) = 1$.

Examples: ① Regular Rings. ② Normal affine toric rings ③ Determinantal Rings ④ Direct summands of regular rings.

Remark: Every SFR R is a normal C-M domain.

Proposition: (R, m) SFR and M a torsion-free f.g. R-mod. Then $\exists e \in \mathbb{N}$ such that

$$F^e_* M \cong R \oplus -$$

Remark: If M is C-M $\Rightarrow$ M is torsion-free.

Pf $0 \neq h \in M$. M is t.f. $\Rightarrow$
$\exists\, \psi : M \to R$: $\psi(h) = r \neq 0$.
By SFR $\exists e \in \mathbb{N}$ and $\varphi : F^e_* R \to R$: $\varphi(F^e_* r) = 1$

$$F^e_* M \xrightarrow{F^e_* \psi} F^e_* R \xrightarrow{\varphi} R \qquad \Rightarrow F^e_* M \cong R \oplus -$$

$$F_*^e M \longrightarrow F_*^e R \xrightarrow{T} K \qquad \Rightarrow \quad F_*^e M \cong R \oplus -$$
$$F_*^e \eta \longmapsto F_*^e r \longmapsto 1 \qquad\qquad\qquad \blacksquare$$

<u>Theorem</u> (—) Let (R, m) SFR. $\exists e_0 \in \mathbb{N}$ such that $\forall$ MCM f.g. R-modules M

$$F_*^{e_0} M \cong R \oplus - .$$

<u>Divisor Class Groups</u>

- A <u>divisor</u> is a sum$/\mathbb{Z}$ of height 1 primes of R.

$$D = n_1 P_1 + \cdots + n_\ell P_\ell, \qquad n_i \in \mathbb{Z},$$
$$P_i \text{ ht } 1 \text{ Prime.}$$

- D a divisor let $R(D)$ be the corresponding fractional ideal.

- If $D = -n_1 P_1 - \cdots - n_\ell P_\ell$, $n_i \geq 0$ then $R(D) = P_1^{(n_1)} \cap \cdots \cap P_\ell^{(n_\ell)}$

- The divisor class group of R is

$$Cl(R) \cong \text{Divisors of } R / \sim$$

$$D_1 \sim D_2 \iff R(D_1) \cong R(D_2).$$

• $D \sim 0 \iff R(D) \cong R$

<u>Corollary</u>: (R, m) SFR & F-finite.
Then the torsion subgroup of $Cl(R)$, $T(Cl(R))$, is finite.

$\Rightarrow$ $Cl(R) \cong G \oplus T(Cl(R))$ where G is a torsion-free Abelian group.

We expect $Cl(R) \cong \mathbb{Z}^{\oplus N} \oplus T(Cl(R))$.

<u>Sketch Proof</u>: If D is torsion $\Rightarrow$ $R(D)$ is C-M.
(Patakfalvi–Schwede, Das–Se)

By Theorem if D is torsion then

$$F_*^{e_c} R(D) \cong R \oplus -.$$

In particular ☆ $F_*^{e_c} R(-p^{e_c} D) \otimes_R R(D) \cong R \oplus -$

Apply $- \otimes_R R(D)$ & $\mathrm{Hom}_R(\mathrm{Hom}_R(-, R), R)$ to ☆.

$$\text{LHS} \cong F_*^{e_c} R(-p^{e_c} D + p^{e_c} D) \cong F_*^{e_c} R$$

$$\text{RHS} \cong R(D) \oplus -$$

If D is torsion $\Rightarrow$ $F_*^e R \cong R(D) \oplus -$

If $D_1, \dots, D_t$ are distinct torsion divisors then by Krull-Schmidt

$$\Rightarrow F_*^e R \cong R(D_1) \oplus \cdots \oplus R(D_t) \oplus -$$

By rank considerations on $F_*^e R$ $\quad |T(Cl(R))| < \infty$

$\blacksquare$

$R(D) \subseteq ff(R)$

$\|$

Rank 1 & t.f.

D torsion $\Rightarrow$ $F_*^e R \cong R(D) \oplus -$

K_X = canonical $\quad F_*^e R \cong R(K_X) \oplus -$