MA595AGI: Algebraic Geometry I

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Contents

List of Symbols

Symbol	Description
$\rightsquigarrow$	specialization, 132
$\langle U, \varphi_U \rangle$	representative of a rational map $\varphi \colon X \dashrightarrow Y$, 50
$\langle U, f \rangle$	representative of a germ of a regular or rational function f, or a section f of a sheaf, 33, 116
$\sqrt{\mathfrak{a}}$	radical of an ideal $\mathfrak{a}$, 10
Δ	diagonal in $\mathbf{P}^n_k \times \mathbf{P}^n_k$, 50
Δf	difference function, 92
Δ_Y	diagonal in $Y \times Y$, 50
$\Gamma(U, \mathscr{F})$	sections of a sheaf $\mathscr{F}$ on an open set U, 115
Ω^p_X	sheaf of differential p-forms, 127
$\lambda(z_1, z_2, z_3, z_4)$	cross ratio, 86
$\mu_{\mathfrak{p}}(M)$	multiplicity of a graded module M at $\mathfrak{p}$, 94
φ_M	Hilbert function of a graded module M, 95
1	identity element in a ring R, xiii
$\mathbf{A}^n_k$	affine n-space over k, 6
$\mathbf{C}$	complex numbers
$\mathbf{N}$	natural numbers $\{0, 1, 2, \ldots\}$
$\mathbf{P}^n_k$	projective n-space over k, 17
$\mathbf{P}(V)$	projective space associated to a vector space V, 104
$\mathbf{Q}$	rational numbers
$\mathbf{R}$	real numbers
$\mathsf{Ab}(X)$	category of Abelian sheaves on a topological space X, 117
LRS	category of locally ringed spaces, 138
$\mathsf{Mod}(\mathcal{O}_X)$	category of sheaves of $\mathcal{O}_X$-modules, 126
$\mathsf{PAb}(X)$	category of Abelian presheaves on a topological space X, 117
$\mathsf{PSh}(X)$	category of presheaves of sets on a topological space X, 117
RS	category of ringed spaces, 138
Sch	category of schemes, 141
Sets	category of sets, 143
$\mathsf{Sh}(X)$	category of sheaves of sets on a topological space X, 117
$\mathsf{Top}(X)$	category of open subsets of a topological space X, 114
Var_k	category of varieties over k, 37
Vect_k	category of vector spaces over a field k, 145
$\mathfrak{m}_P$	maximal ideal of the local ring of a locally ringed space at a point P, 33, 138

Symbol	Description
$M(l)$	l-th twist of a graded module M, 93
$\tilde{M}$	sheaf associated to a module on $\mathrm{Spec}(A)$, 134
$\mathrm{PGL}_n(k)$	projective general linear group of degree n over a field k, 85
$S_{(f)}$	$(S_f)_0$ for a homogeneous element f in a graded ring S, 42
$S_{(\mathfrak{p})}$	$((S^h - \mathfrak{p})^{-1}S)_0$ for a homogeneous prime ideal $\mathfrak{p}$ in a graded ring S, 42
$S(Y)$	homogeneous coordinate ring of an algebraic set $Y \subseteq \mathbf{P}^n_k$, 20
S^h	homogeneous elements of a graded ring S, 21
$\mathrm{Sing}(Y)$	singular locus of a quasi-projective variety Y, 71
s_P	germ of a section of a sheaf at P, 116
$\mathrm{sp}(X)$	underlying topological space of a ringed space or scheme, 141
$\mathrm{Spé}(\mathscr{F})$	the espace étalé of a presheaf $\mathscr{F}$, 118
$T \star S$	composition of two natural transformations T and S, 145
$T_P(X)$	Zariski tangent space at P, 67
U_i	affine chart $\mathbf{P}^n_k - H_i \subseteq \mathbf{P}^n_k$, 20
$V(I)$	closed set in $\mathrm{Spec}(A)$ defined by an ideal $I \subseteq A$, 131
V^*	dual of a vector space V, 145
$X \times Y$	product of quasi-projective varieties X and Y, 47
$Z(T)$	affine algebraic set defined by a set of polynomials T, 7
$Z_+(T)$	projective algebraic set defined by a set of homogeneous polynomials T, 19

Conventions

(1) Let A and B be subsets of a set X. The *set difference* is denoted

$$A - B \coloneqq \{x \in X \mid x \in A \text{ and } x \notin B\}.$$

(2) All rings R will be assumed to be commutative with an identity element 1, unless stated otherwise. We may sometimes denote 1 by 1_R for clarity.

(3) All ring maps $\varphi\colon R \to S$ will be assumed to respect the identity element, i.e., $\varphi(1_R) = 1_S$.

Preface

These are notes for a graduate course on algebraic geometry (MA595AGI) taught at Purdue University in Fall 2024 and Fall 2025. The official course text is [Har77]. The section numbering for §§1.1–1.7 mirrors the numbering in [Har77]. We also suggest [Har92; $\mathrm{Sha13}_1$] as additional references for algebraic varieties and [EGAI; $\mathrm{EGAI}_{\mathrm{new}}$; EGAII; EGAIII_1; EGAIII_2; EGAIV_1; EGAIV_2; EGAIV_3; EGAIV_4] as additional references for schemes. The notes in the margins point to where in (some of) these texts one can find the material written down in these notes.

These notes will be continually updated throughout the semester. We have made an effort to reference the course text [Hoc17] for MA557 (Commutative Algebra) when using results from commutative algebra.

I would like to thank Farrah Yhee for innumerable helpful conversations.

CHAPTER 1

Varieties

1.0. Introduction

Instead of starting immediately with definitions, we start with some background and motivation from Euclidean geometry. There are many other sources of motivation: number theory, cryptography, architecture, robotics, etc. that we will not discuss. The material in this section will not be tested on homework or the exams. 8/25

1.0.1. Classical problems of antiquity. Algebraic geometry is a very old subject. We will begin with the following problems, which at first seem unrelated to what we ordinarily consider to be algebraic geometry. We will soon see that trying to solve these problems give rise naturally to algebro-geometric objects. [BK86, p. 3]

Problems 1.0.1 (Classical problems of antiquity [BK86, (1.2)]). The classical problems of antiquity were:

(1) Trisection of an arbitrary angle.
(2) Squaring the circle (before 1500 B.C.E.?): Given a circle with area A, can one construct a square with the same area A?
(3) Doubling the cube (Delian problem, 5th century B.C.E.?): Given a cube of volume V, can one construct a cube of volume $2V$? [BK86, p. 4]

All three problems are known to be impossible to solve using just a straightedge and compass. We will not provide full proofs here, but here are brief explanations of these impossibility results.

(1) [Wan1837] (see [Art07, §7.1] for this argument) The triple-angle formula says
$$\sin(3\beta) = 3\sin(\beta) - \sin^3(\beta).$$
Setting $\beta = \alpha/3$, $x = \sin(\beta)$, and $c = \sin(\alpha)$, we want to solve
$$x^3 - 3x + c = 0.$$
By Galois theory, a solution for such a cubic equation cannot always be found using a straightedge and compass for arbitrary c.
(2) [Lin1882] Squaring the circle is impossible since π is a transcendental number.
(3) [Wan1837] (see [Art07, §7.1] for this argument) Doubling the cube requires solving the equation $x^3 - 2 = 0$. If $\sqrt[3]{2}$ can be constructed using a straightedge and compass, then the field extension $\mathbf{Q}(\sqrt[3]{2})$ must be of degree equal to a power of 2. However, the field extension generated by $\sqrt[3]{2}$ is of degree 3 over $\mathbf{Q}$.

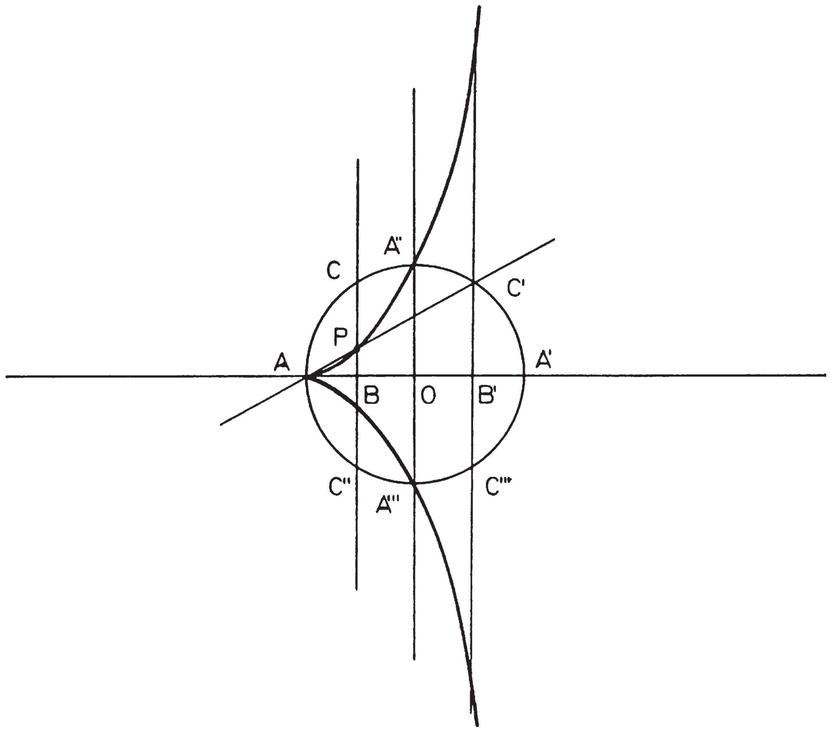


FIGURE 1.1. The cissoid of Diocles. From [BK86, p. 9].

What if we allowed ourselves to use more than a straightedge and compass? The ancient Greeks actually *were* able to trisect angles and double cubes, except they needed to use more complicated curves.

[BK86, pp. 9–12]

1.0.2. The cissoid of Diocles and doubling the cube. We start with the *cissoid of Diocles*, which we will use to double the cube.

In class, I drew the picture, pointed out x and y which satisfy $x^3 = 2y^3$, and wrote down (1.0.3).

CONSTRUCTION 1.0.2 (The cissoid of Diocles, c. 180 B.C.E.). We first give a pointwise construction. See Figure 1.1. Let $\overline{AA'}$ and $\overline{A''A'''}$ be two perpendicular lines with intersection O. We draw a circle around O. We draw two lines parallel and equidistant to $\overline{A''A'''}$, meeting $\overline{AA'}$ at points B and B', respectively. We denote by C, C', C'', C''' their intersections with the circle. The line through one of these points and A, for example $\overline{AC'}$, cuts the other parallel $\overline{CC''}$ at a point P. The collection of all such points P is the *cissoid of Diocles*. The point A is an example of a *cusp*, which is a type of singularity.

Diocles doubled the cube using the construction in Figure 1.2. Let M be the midpoint of $\overline{OA''}$ and let P be the intersection of $\overline{A'M}$ with the cissoid with $\overline{CB}$ and $\overline{C'B'}$ as above. Let $x = A'B$, $y = BC$, and $z = AB$. Then,

$$\frac{A'B}{PB} = \frac{A'O}{MO} = 2$$
$$\frac{A'B}{BC} = \frac{CB}{AB} = \frac{C'B'}{B'A'} = \frac{AB'}{B'C'} = \frac{AB}{BP}$$

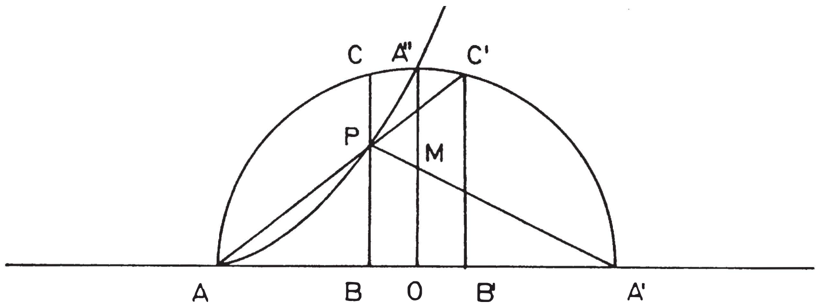


FIGURE 1.2. Constructing $\sqrt[3]{2}$. From [BK86, p. 10].

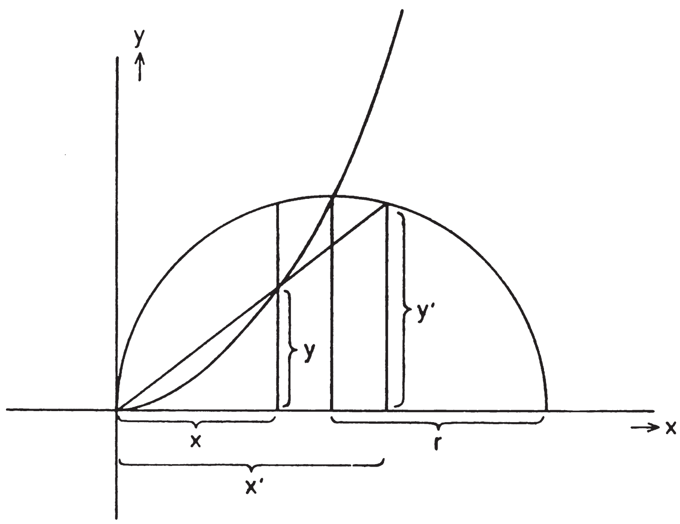


FIGURE 1.3. Notation for the cissoid. From [BK86, p. 74].

and hence

$$\frac{x}{y} = \frac{y}{z} = \frac{z}{\frac{1}{2}x}.$$

This shows that $x^3 = 2y^3$.

Finally, to be a true solution to the "doubling the cube" problem, we need to construct the cissoid of Diocles "organically." One such construction was found by Newton; see [BK86, p. 12].

From our point of view, the most relevant description is of the cissoid as an *algebraic curve*. With notation as in Figure 1.3, the points (x, y) on the cissoid satisfy [BK86, pp. 74–75]

$$\frac{y}{x} = \frac{y'}{x'}$$
$$x' = 2r - x$$
$$y' = \sqrt{r^2 - (r-x)^2}.$$

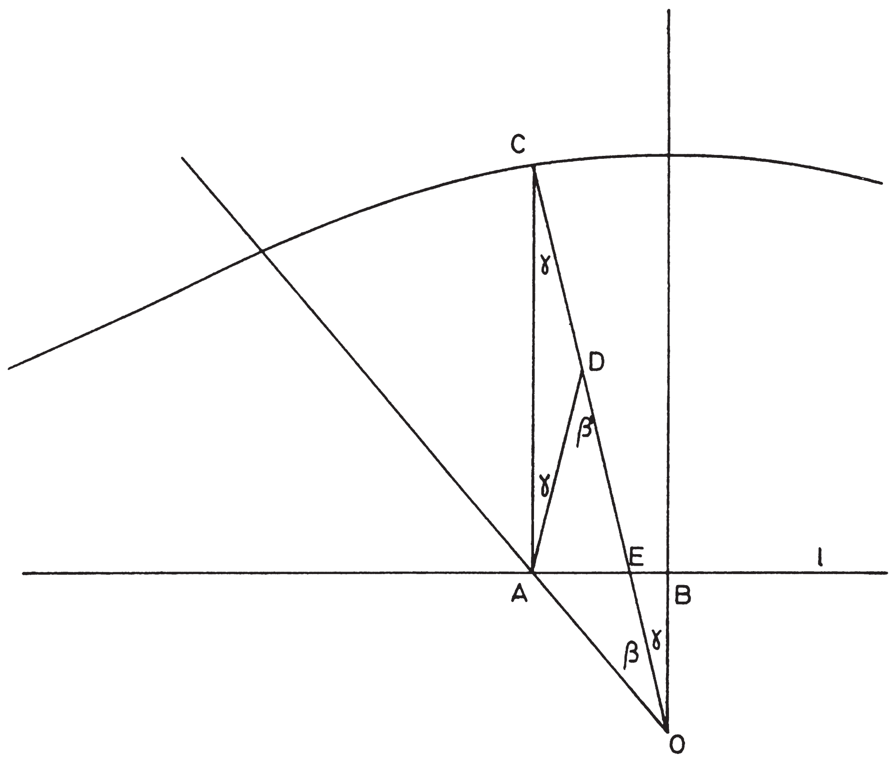


FIGURE 1.4. Trisecting the angle. From [BK86, p. 16].

Squaring both sides of the first equation and substituting x'^2 and y'^2 using the other two equations yields the following equation for the cissoid of Diocles:

$$y^2(2r - x) - x^3 = 0. \tag{1.0.3}$$

We therefore see that if we use curves defined by *cubic* polynomials, we can double the cube!

[BK86, pp. 13–16]

1.0.3. The conchoid of Nicomedes and trisecting the angle. Next, we describe the *conchoid of Nicomedes*, which we will use to trisect the angle.

In class, I drew the picture, pointed out α and γ which satisfy $\gamma = \frac{\alpha}{3}$, and wrote down (1.0.5).

CONSTRUCTION 1.0.4 (The conchoid of Nicomedes, c. 180 B.C.E.). Given a line ℓ, a point O at distance d from ℓ, and a segment k, let A be an arbitrary point on ℓ and P, P' the points on the line $\overline{OA}$ at distance k from A. The collection of all such points P, P' is a *conchoid of Nicomedes*. The form of the conchoid depends on the relationship between d and k as shown in Figure 1.5.

The conchoid of Nicomedes can be used to trisect an acute angle α as follows. Let $\alpha = \angle AOB$, where B is the foot of the perpendicular ℓ from A onto $\overline{OB}$ (see Figure 1.4). We then draw the conchoid for ℓ and O with $k = 2\,\overline{OA}$. The parallel to $\overline{OB}$ through A cuts the conchoid on the side away from O at a point C. The angle $\gamma = \angle BOC$ satisfies $\gamma = \frac{\alpha}{3}$. See [BK86, p. 16] for the proof.

[BK86, p. 75]

We also write down the equation for the conchoid of Nicomedes. With notation

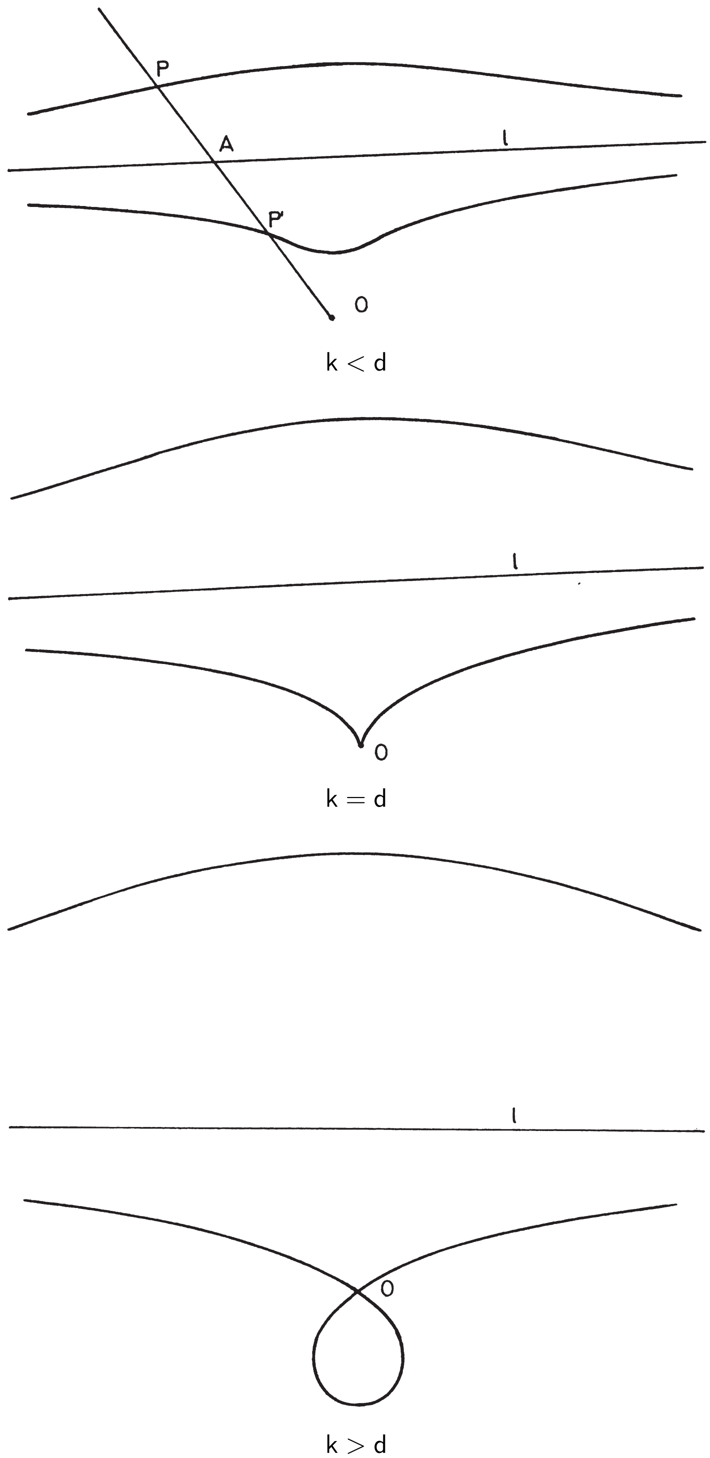


FIGURE 1.5. The conchoid of Nicomedes. From [BK86, pp. 14–15].

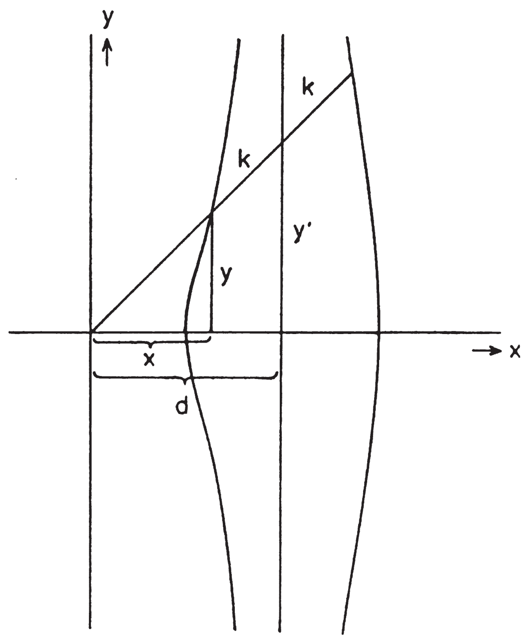


FIGURE 1.6. Notation for the conchoid. From [BK86, p. 75].

as in Figure 1.6, the points (x, y) on the conchoid satisfy

$$\frac{y}{x} = \frac{y'}{d}$$
$$(y' - y)^2 + (d - x)^2 = k^2.$$

After some calculation, we get the equation of the conchoid:

$$(y^2 + x^2)(d - x)^2 - k^2x^2 = 0. \tag{1.0.5}$$

Now that we see how polynomials of degree > 2 can be useful in geometric questions, we will start the official course material.

1.1. Affine Varieties

From now on, unless otherwise stated, the material in these notes can be tested on homework and the exams.

[Sha13$_1$, p. 23] Throughout this section, k denotes an algebraically closed field, which we call the *ground field*. We will explain why algebraically closed fields are nicer to work with by the end of this section.

[Har77, p. 1] **1.1.1. Affine space.** We start with the fundamental building block for all varieties: the affine n-space $\mathbf{A}_k^n$.

[Har92, p. 3] DEFINITION 1.1.1 (Affine n-space $\mathbf{A}_k^n$). Let $n \geq 0$ be an integer. The *affine n-space over k* is the set

$$\mathbf{A}_k^n := \{(a_1, a_2, \ldots, a_n) \mid a_i \in k \text{ for all } i\}.$$

An element $P \in \mathbf{A}^n_k$ is called a *point*. If $P = (a_1, a_2, \ldots, a_n)$ is a point in $\mathbf{A}^n_k$, the a_i are the *coordinates* of P.

We often drop the subscript on $\mathbf{A}^n_k$ if the ground field k is clear from context.

REMARK 1.1.2. The difference between the affine n-space $\mathbf{A}^n_k$ and the vector space k^n is that k^n has the additional structure of a vector space over k. For example, k^n has an additive structure and an action by multiplication by scalars in k. When we talk about morphisms and isomorphisms of varieties, the automorphisms (that is, the self-isomorphisms) of $\mathbf{A}^n_k$ will be allowed to translate the origin $(0, 0, \ldots, 0)$ to another point. [Har92, p. 3]

Analogous to how one glues together copies of $\mathbf{R}^n$ to form an n-dimensional manifold, we will eventually use affine n-spaces to build other spaces. However, to ensure the class of objects we obtain includes spaces with singularities like the cusp on the cissoid of Diocles, we will actually glue together subsets of affine n-spaces defined using polynomial equations.

To make this precise, we start by explaining how polynomials over k give rise to functions on $\mathbf{A}^n_k$.

EXAMPLE 1.1.3 (Regular functions on $\mathbf{A}^n_k$). Let [Har77, p. 2]

$$A = k[x_1, x_2, \ldots, x_n]$$

be the polynomial ring in n variables over the ground field k. We interpret the elements $f \in A$ as functions $f\colon \mathbf{A}^n_k \to k$ where

$$\begin{aligned} \mathbf{A}^n_k &\xrightarrow{f} k \\ P = (a_1, a_2, \ldots, a_n) &\longmapsto f(a_1, a_2, \ldots, a_n). \end{aligned}$$

Functions that arise from polynomials in this manner are examples of *regular functions on* $\mathbf{A}^n_k$.

DEFINITION 1.1.4 (Algebraic sets in $\mathbf{A}^n_k$). If $T \subseteq A = k[x_1, x_2, \ldots, x_n]$ is a set of polynomials, the *zero set of* T is

$$Z(T) \coloneqq \bigl\{ P \in \mathbf{A}^n_k \bigm| f(P) = 0 \text{ for all } f \in T \bigr\}.$$

If $\mathfrak{a} = (T)$ is the ideal generated by T in A, then $Z(T) = Z(\mathfrak{a})$. Since A is Noetherian by the Hilbert basis theorem (Theorem 1.1.30 below), any ideal $\mathfrak{a} \subseteq A$ can be generated by finitely many elements $f_1, f_2, \ldots, f_r$. Thus, for any subset $T \subseteq A$, we have

This is the definition in [Sha13$_1$, p. 23].

$$Z(T) = Z(f_1, f_2, \ldots, f_r) \tag{1.1.5}$$

for a finite set of elements $\{f_1, f_2, \ldots, f_r\} \subseteq A$. In (1.1.5), we omit the braces (or parentheses) around $f_1, f_2, \ldots, f_r$ to simplify notation.

A subset $Y \subseteq \mathbf{A}^n_k$ is an *algebraic set* if there exists a subset $T \subseteq A$ such that

$$Y = Z(T).$$

EXAMPLES 1.1.6. The equation for the cissoid of Diocles defines the algebraic set

$$Z\bigl(y^2(2r - x) - x^3\bigr) \subseteq \mathbf{A}^2_k$$

for every $r > 0$. The equation for the conchoid of Nicomedes defines the algebraic set

$$Z\bigl((y^2 + x^2)(d - x)^2 - k^2x^2\bigr) \subseteq \mathbf{A}^2_k$$

for all $k, d > 0$.

We prove that algebraic sets give rise to a natural topology on $\mathbf{A}_k^n$.

[Har77, Prop. I.1.1]
[Sha13₁, p. 24]

PROPOSITION 1.1.7 (Algebraic sets define a topology on $\mathbf{A}_k^n$).

(a) *Let $T_1, T_2 \subseteq A$. Then, $Z(T_1) \cup Z(T_2) = Z(T_1T_2)$, where*

$$T_1T_2 \coloneqq \{ fg \mid f \in T_1,\ g \in T_2 \}.$$

(b) *Let $\{T_\alpha\}$ be a family of subsets of A. Then,*

$$\bigcap_\alpha Z(T_\alpha) = Z\left(\bigcup_\alpha T_\alpha\right).$$

(c) *The empty set $\emptyset$ and the whole space $\mathbf{A}_k^n$ are algebraic sets.*

Proof. (a). For $\subseteq$, suppose $P \in Z(T_1) \cup Z(T_2)$. For every element $fg \in T_1T_2$, either $f(P) = 0$ or $g(P) = 0$. Thus, $P \in Z(T_1T_2)$. For $\supseteq$, we prove the contrapositive. Suppose that $P \notin Z(T_1) \cup Z(T_2)$. Then, there exist $f \in T_1$ and $g \in T_2$ such that $f(P) \neq 0$ and $g(P) \neq 0$. This implies that $(fg)(P) \neq 0$, and hence $P \notin Z(T_1T_2)$.

(b). We have

$$\begin{aligned}\bigcap_\alpha Z(T_\alpha) &= \bigcap_\alpha \{ P \in \mathbf{A}_k^n \mid f(P) = 0 \text{ for all } f \in T_\alpha \} \\ &= \left\{ P \in \mathbf{A}_k^n \;\middle|\; f(P) = 0 \text{ for all } f \in \bigcup_\alpha T_\alpha \right\} \\ &= Z\left(\bigcup_\alpha T_\alpha\right).\end{aligned}$$

(c). Note that $\emptyset = Z(1)$ and that $\mathbf{A}_k^n = Z(0)$. □

Recall (for example from [Mun00, p. 75]) that a *topology* on a set X consists of a collection of subsets of X called *open sets* that is stable under finite intersections and arbitrary unions and contains $\emptyset$ and X. Complements of open sets are called *closed sets*.

Proposition 1.1.7 allows us to make the following definition.

[Har77, p. 2]

DEFINITION 1.1.8 (The Zariski topology on $\mathbf{A}_k^n$). The *Zariski topology on* $\mathbf{A}_k^n$ is the topology whose closed sets are algebraic sets.

[Har77, Ex. I.1.1.1]

EXAMPLE 1.1.9. We describe the Zariski topology on the *affine line* $\mathbf{A}_k^1$ explicitly. Let $Z(T) \subseteq \mathbf{A}_k^1$ be an algebraic set. Since $k[x]$ is a PID, the ideal generated by T is a principal ideal (f). Since k is algebraically closed, we can write

$$f(x) = c\,(x - a_1)(x - a_2)\cdots(x - a_n)$$

where $c, a_1, a_2, \ldots, a_n \in k$. We see that

$$Z(T) = Z(f) = \{a_1, a_2, \ldots, a_n\}.$$

Thus, the algebraic sets in $\mathbf{A}_k^1$ are the finite subsets (including $\emptyset$) and $\mathbf{A}_k^1$ itself. The open sets in $\mathbf{A}_k^1$ are the empty set and complements of finite subsets (the *finite complement topology* [Mun00, p. 77, Example 3]). Note that $\mathbf{A}_k^1$ is therefore not Hausdorff.

1.1.2. Affine varieties. Before defining affine varieties, we first define the notion of irreducibility.

[Har77, p. 3]
[Sha13_1, p. 34]

DEFINITION 1.1.10. Let X be a topological space. We say that X is *irreducible* if it is nonempty and if, for every pair of proper closed subsets $X_1, X_2 \subsetneq X$, we have $X_1 \cup X_2 \subsetneq X$. Otherwise, we say that X is *reducible*.

When we say a subset $Y \subseteq X$ is irreducible, we will mean that Y is irreducible with the subspace topology.

REMARK 1.1.11 (Why the empty set is reducible). The condition that X is nonempty is part of Definition 1.1.10. Following [BouCA, p. 94], one can alternatively define X to be irreducible if for every finite collection of proper closed subsets $X_i \subsetneq X$, we have $\bigcup_i X_i \subsetneq X$. The empty set is reducible under this definition by using the empty collection of proper closed subsets in $\emptyset$.

We give a few examples.

EXAMPLE 1.1.12. The affine line $\mathbf{A}_k^1$ is irreducible because $\mathbf{A}_k^1$ is an infinite set (since k is algebraically closed) and its only proper closed subsets are finite (see Example 1.1.9). [Har77, Ex. I.1.1.2]

EXAMPLE 1.1.13. Consider the algebraic set

$$Z(xy) \subseteq \mathbf{A}_k^2.$$

Since $Z(xy) = Z(x) \cup Z(y)$, we see that $Z(xy)$ is reducible. We can visualize this algebraic set as the union of the y- and x-axes in $\mathbf{A}_k^2$.

EXAMPLE 1.1.14. Any nonempty open subset U of an irreducible space X is irreducible and dense. 8/27

[Har77, Ex. I.1.1.3, Exer. I.1.6]
[BouCA, p. 95, Prop. 3(i)]

Proof. We first show that U is dense. We can write

$$X = (X - U) \cup \bar{U}.$$

Since X is irreducible and U is nonempty, we see that $X - U \subsetneq X$ and hence $X = \bar{U}$. We now show that U is irreducible. If $U = U_1 \cup U_2$, then taking closures, we have

$$X = \bar{U} = \bar{U}_1 \cup \bar{U}_2.$$

Since X is irreducible, we have either $\bar{U} = \bar{U}_1$ or $\bar{U} = \bar{U}_2$. Taking intersections with U, either $U = U_1$ or $U = U_2$. □

EXAMPLE 1.1.15. If Y is an irreducible subset of X, then its closure $\bar{Y}$ in X is also irreducible.

[Har77, Ex. I.1.1.4, Exer. I.1.6]
[BouCA, p. 95, Prop. 2]

Proof. Suppose $\bar{Y} = A \cup B$ for A, B closed in $\bar{Y}$. Then,

$$Y = (Y \cap A) \cup (Y \cap B),$$

and since Y is irreducible, we have either $Y = Y \cap A$ or $Y = Y \cap B$. In each case, we have $\bar{Y} = \overline{Y \cap A} = A$ and $\bar{Y} = \overline{Y \cap B} = B$, respectively, and hence $\bar{Y}$ is irreducible. □

We now define affine varieties.

DEFINITION 1.1.16. An *affine variety* is an irreducible closed subset of $\mathbf{A}_k^n$ with the subspace topology. A *quasi-affine variety* is an open subset of an affine variety. [Har77, p. 3]

1.1.3. The algebra $\leftrightarrow$ geometry dictionary for affine space. Our next step is to understand how subsets of $\mathbf{A}^n_k$ and ideals in $k[x_1, x_2, \ldots, x_n]$ are related. To do so, we make the following definition:

[Har77, p. 3] DEFINITION 1.1.17. Let $Y \subseteq \mathbf{A}^n_k$ be a subset. The *ideal of* Y in A is

$$I(Y) \coloneqq \bigl\{f \in A \bigm| f(P) = 0 \text{ for all } P \in Y\bigr\}.$$

We therefore obtain a function Z mapping subsets $T \subseteq A$ to algebraic sets $Z(T)$ and a function I mapping subsets $Y \subseteq \mathbf{A}^n_k$ to ideals $I(Y) \subseteq A$. These functions satisfy the following properties.

[Har77, Prop. I.1.2] PROPOSITION 1.1.18.

(*a*) *If $T_1 \subseteq T_2$ are subsets of A, then $Z(T_1) \supseteq Z(T_2)$.*
(*b*) *If $Y_1 \subseteq Y_2$ are subsets of $\mathbf{A}^n_k$, then $I(Y_1) \supseteq I(Y_2)$.*
(*c*) *For any two subsets $Y_1, Y_2 \subseteq \mathbf{A}^n_k$, we have $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.*
(*d*) *For any ideal $\mathfrak{a} \subseteq A$, we have $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$, where $\sqrt{\mathfrak{a}}$ denotes the radical of $\mathfrak{a}$.*
(*e*) *For any subset $Y \subseteq \mathbf{A}^n_k$, we have $Z(I(Y)) = \bar{Y}$, where $\bar{Y}$ denotes the closure of Y in $\mathbf{A}^n_k$.*

Proof. (*a*). If $P \in Z(T_2)$, then $f(P) = 0$ for all $f \in T_2$. In particular, $f(P) = 0$ for all $f \in T_1$. Thus, $P \in Z(T_1)$.

(*b*). If $f \in I(Y_2)$, then $f(P) = 0$ for all $P \in Y_2$. In particular, $f(P) = 0$ for all $P \in Y_1$. Thus, $f \in I(Y_1)$.

(*c*). We have $f \in I(Y_1 \cup Y_2)$ if and only if $f(P) = 0$ for all $P \in Y_1 \cup Y_2$. This holds if and only if $f(P) = 0$ for all $P \in Y_1$ and for all $P \in Y_2$, which is equivalent to $f \in I(Y_1) \cap I(Y_2)$.

(*d*). Recall that the *radical* of an ideal $\mathfrak{a} \subseteq A$ is

$$\sqrt{\mathfrak{a}} \coloneqq \bigl\{f \in A \bigm| f^r \in \mathfrak{a} \text{ for some } r > 0\bigr\}.$$

If $f \in \sqrt{\mathfrak{a}}$, then $(f^r)(P) = 0$ for every $P \in Z(\mathfrak{a})$. Thus, $f(P) = 0$ for every $P \in Z(\mathfrak{a})$, and hence $f \in I(Z(\mathfrak{a}))$. The converse holds by Hilbert's Nullstellensatz (Theorem 1.1.19).

(*e*). We have $Y \subseteq Z(I(Y))$. Since the right-hand side is closed, taking closures, we obtain $\bar{Y} \subseteq Z(I(Y))$. Conversely, suppose that W is a closed set containing Y. Then, $W = Z(\mathfrak{a})$ for some ideal $\mathfrak{a} \subseteq A$, and hence $Z(\mathfrak{a}) \supseteq Y$. Since $\mathfrak{a} \subseteq I(Z(\mathfrak{a}))$ by (*d*) (or its proof), (*a*) implies $W = Z(\mathfrak{a}) \supseteq Z(I(Y))$. We therefore see that $Z(I(Y)) = \bar{Y}$. □

[Har77, Thm. I.1.3A]
[Hoc17, p. 52]
[AK21, (15.7)]
THEOREM 1.1.19 (Hilbert's Nullstellensatz). *Let k be an algebraically closed field, let $\mathfrak{a}$ be an ideal in $A = k[x_1, x_2, \ldots, x_n]$, and let $f \in A$ be a polynomial which vanishes at all points of $Z(\mathfrak{a})$. Then, $f^r \in \mathfrak{a}$ for some integer $r > 0$.*

Proposition 1.1.18 implies that there is a one-to-one correspondence between algebraic sets in $\mathbf{A}^n_k$ and radical ideals in A. This is one of the first instances where we have a nice translation between geometric properties and algebraic properties. This will be an ongoing theme throughout this course.

[Har77, Cor. I.1.4] COROLLARY 1.1.20. *There is a one-to-one inclusion-reversing correspondence*

$$\left\{\begin{matrix}\textit{algebraic sets}\\ \textit{in } \mathbf{A}_k^n\end{matrix}\right\} \overset{\text{1-1}}{\longleftrightarrow} \left\{\begin{matrix}\textit{radical ideals}\\ \textit{in } k[x_1, x_2, \ldots, x_n]\end{matrix}\right\}$$

$$Y \longmapsto I(Y)$$

$$Z(\mathfrak{a}) \longleftarrow\!\shortmid\; \mathfrak{a}.$$

Furthermore, an algebraic set is irreducible if and only if its ideal is a prime ideal.

Proof. The one-to-one correspondence holds by Proposition 1.1.18(*d*),(*e*) and is inclusion-reversing by Proposition 1.1.18(*a*),(*b*).

It remains to show the "furthermore" statement. $\Rightarrow$. If $fg \in I(Y)$, then

$$Y \subseteq Z(fg) = Z(f) \cup Z(g)$$

by Proposition 1.1.18(*c*). Thus, we have

$$Y = \big(Y \cap Z(f)\big) \cup \big(Y \cap Z(g)\big).$$

Since Y is irreducible, we have either $Y = Y \cap Z(f)$, in which case $Y \subseteq Z(f)$, or $Y = Y \cap Z(g)$, in which case $Y \subseteq Z(g)$. This shows that $f \in I(Y)$ and $g \in I(Y)$ in each respective situation.

$\Leftarrow$. Let $\mathfrak{p}$ be a prime ideal and suppose that $Z(\mathfrak{p}) = Y_1 \cup Y_2$. Then, $\mathfrak{p} = I(Y_1) \cap I(Y_2)$ by Proposition 1.1.18(*c*). Since $\mathfrak{p}$ is prime, either $\mathfrak{p} = I(Y_1)$ or $\mathfrak{p} = I(Y_2)$. Thus, $Z(\mathfrak{p}) = Y_1$ or $Z(\mathfrak{p}) = Y_2$, and hence $Z(\mathfrak{p})$ is irreducible. $\square$

Using Corollary 1.1.20, we can construct many examples of algebraic varieties.

Example 1.1.21. $\mathbf{A}_k^n$ is irreducible since $\mathbf{A}_k^n = Z(0)$, and $(0) \subseteq k[x_1, x_2, \ldots, x_n]$ is a prime ideal. [Har77, Ex. I.1.4.1]

Example 1.1.22. Let $f \in A = k[x, y]$ be an irreducible polynomial. Then, $(f) \subseteq A$ is a prime ideal since A is a UFD, and hence the zero set $Y = Z(f)$ is irreducible by Corollary 1.1.20. The zero set Y is called the *affine curve* defined by the equation $f(x, y) = 0$. If f has degree d, we say that Y is a curve of *degree* d. [Har77, Ex. I.1.4.2]

Example 1.1.23. Now let $f \in A = k[x_1, x_2, \ldots, x_n]$ be an irreducible polynomial. The affine variety $Y = Z(f)$ is called a *surface* if $n = 3$, or a *hypersurface* if $n > 3$. [Har77, Ex. I.1.4.3]

Example 1.1.24. A maximal ideal $\mathfrak{m} \subseteq A = k[x_1, x_2, \ldots, x_n]$ corresponds to a minimal irreducible closed subset of $\mathbf{A}_k^n$, which must be a point $P = (a_1, a_2, \ldots, a_n)$. We therefore see that every maximal ideal $\mathfrak{m} \subseteq A$ is of the form [Har77, Ex. I.1.4.4]

$$\mathfrak{m} = (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$$

for some $a_1, a_2, \ldots, a_n \in k$.

Example 1.1.25 (What happens if $k \neq \bar{k}$?). If k is not algebraically closed, then Proposition 1.1.18(*d*) is false. For example, if $k = \mathbf{R}$, then the curve [Har77, Ex. I.1.4.5, Exer. I.1.12]

$$\big\{x^2 + y^2 + 1 = 0\big\} \subseteq \mathbf{A}_{\mathbf{R}}^2$$

has no points, and hence is not irreducible. However, $x^2 + y^2 + 1$ is irreducible, and hence $(x^2 + y^2 + 1)$ is prime.

Now that we know how algebraic sets correspond to radical ideals (Corollary 1.1.20), we can attach a ring to any algebraic set.

DEFINITION 1.1.26. Let $Y \subseteq \mathbf{A}_k^n$ be an affine algebraic set. The *affine coordinate ring* of Y is [Har77, p. 4]

$$A(Y) := \frac{A}{I(Y)}.$$

[Har77, Rem. I.1.4.6, Exer. I.1.5]

REMARK 1.1.27. If Y is an affine variety (resp. affine algebraic set), then $A(Y)$ is a domain (resp. a reduced ring) that is finitely generated as a k-algebra. Conversely, any finitely generated k-algebra B which is a domain (resp. reduced) is the affine coordinate ring of an affine variety (resp. of an affine algebraic set): We can write B as a quotient of a polynomial ring $A = k[x_1, x_2, \ldots, x_n]$ by an ideal $\mathfrak{a} \subseteq A$, in which case $Y = Z(\mathfrak{a})$.

1.1.4. Noetherianity and irreducible decompositions. The topology of affine varieties has the following very important property.

[Har77, p. 5]

DEFINITION 1.1.28. A topological space X is *Noetherian* if it satisfies the *descending chain condition* for closed subsets: For any sequence

$$Y_1 \supseteq Y_2 \supseteq \cdots$$

of closed subsets in X, there is an integer r such that

$$Y_r = Y_{r+1} = \cdots .$$

The fact that affine varieties are Noetherian boils down to the following fact from commutative algebra. We will need the following definition.

DEFINITION 1.1.29. Let R be a ring. We say that R is *Noetherian* if it satisfies the *ascending chain condition* for ideals: For any sequence

$$I_1 \subseteq I_2 \subseteq \cdots$$

of ideals in R, there is an integer r such that

$$I_r = I_{r+1} = \cdots .$$

One can show (try it!) that R is Noetherian if and only if every ideal $I \subseteq R$ is finitely generated.

[Hoc17, p. 64]
[Rei95, (3.6)]
[AK21, (16.12)]

THEOREM 1.1.30 (The Hilbert basis theorem). *Let R be a Noetherian ring. Then, every finitely generated R-algebra is Noetherian.*

The case when $R = k$ is what Hilbert showed in [Hil1890].

[Har77, Ex. I.1.4.7]

EXAMPLE 1.1.31. $\mathbf{A}_k^n$ is a Noetherian topological space: A descending chain

$$Y_1 \supseteq Y_2 \supseteq \cdots$$

of closed subsets in $\mathbf{A}_k^n$ corresponds (via Corollary 1.1.20) to an ascending chain of ideals

$$I(Y_1) \subseteq I(Y_2) \subseteq \cdots$$

in $A = k[x_1, x_2, \ldots, x_n]$. Since A is Noetherian by the Hilbert basis theorem (Theorem 1.1.30), this chain stabilizes. Since $Y_i = Z(I(Y_i))$ for every i, the original chain of Y_i's also stabilizes.

9/3

We have seen that irreducible algebraic sets are nice because there is no way to break them apart into (finitely many) smaller ones. The following result says that reducible algebraic sets can always be decomposed into a finite union of algebraic varieties.

[Har77, Prop. I.1.5] PROPOSITION 1.1.32. *In a Noetherian topological space X, every nonempty closed subset Y can be expressed as a finite union*

$$Y = Y_1 \cup Y_2 \cup \cdots \cup Y_r \tag{1.1.33}$$

of irreducible closed subsets Y_i. If we require that $Y_i \not\supseteq Y_j$ for $i \neq j$, then the Y_i are uniquely determined. The Y_i are called the irreducible components *of Y, and the decomposition* (1.1.33) *is called the* irreducible decomposition *of Y.*

The following proof is an example of *Noetherian induction*.

Proof. We first show the existence of such a decomposition. Consider the set

$$\Sigma = \left\{ Z \subseteq X \text{ nonempty, closed} \;\middle|\; \begin{array}{c} Z \textit{ cannot} \text{ be written as a finite} \\ \text{union of irreducible closed subsets of } X \end{array} \right\}$$

partially ordered by inclusion. If $\Sigma \neq \emptyset$, then since X is Noetherian, Σ must contain an element Y minimal with respect to inclusion. Then, Y is not irreducible by construction of Σ, and hence we can write $Y = Y' \cup Y''$ where Y', Y'' are proper closed subsets of Y. By the minimality of Y, both Y' and Y'' can be written as a finite union of closed irreducible subsets, and hence Y can also be written as a finite union of closed irreducible subsets. This is a contradiction, and hence we conclude that every Y can be expressed as a finite union of irreducible closed subsets Y_i. By throwing out some of the Y_i as necessary, we may assume that $Y_i \not\supseteq Y_j$ for $i \neq j$.

We now show that the decomposition is unique. Suppose

$$Y = Y_1' \cup Y_2' \cup \cdots \cup Y_s'$$

is another decomposition. We induce on r. If $r = 1$, then Y is irreducible, and hence $Y_1 = Y = Y_1'$. Now suppose $r > 1$. We have

$$Y_1' = \bigcup_i (Y_1' \cap Y_i).$$

Since Y_1' is irreducible, we know that $Y_1' = Y_1' \cap Y_i$ for some i; after possibly rearranging the Y_i, we may assume that $i = 1$, and hence $Y_1' \subseteq Y_1$. Similarly, we have $Y_1 \subseteq Y_j'$ for some j. Then, we have $Y_1' \subseteq Y_j'$, and the condition that $Y_j' \not\supseteq Y_1'$ implies $j = 1$, and hence $Y_1 = Y_1'$. Now let $Z = \overline{(Y - Y_1)}$. Then,

$$Z = Y_2 \cup Y_3 \cup \cdots \cup Y_r = Y_2' \cup Y_3' \cup \cdots \cup Y_s'.$$

By inductive hypothesis, we obtain the uniqueness of the irreducible decomposition (1.1.33). □

Since $\mathbf{A}_k^n$ is Noetherian (Example 1.1.31), we obtain:

COROLLARY 1.1.34. *Every algebraic set in $\mathbf{A}_k^n$ can be expressed uniquely as a union of affine varieties, no one containing another.* [Har77, Cor. I.1.6]

This condition may look familiar to you: On Homework 1, I asked you to compute an irreducible decomposition!

1.1.5. Dimension. The last topic we want to discuss from this section is dimension. In algebraic geometry, we define dimension combinatorially by considering chains of irreducible closed subsets. By the Nullstellensatz (Theorem 1.1.19), this is suggested by the definition of dimension for rings.

DEFINITION 1.1.35. Let X be a topological space. The *dimension* $\dim(X)$ of X is the supremum of all integers n for which there exists a chain [Har77, p. 5] [EGAIV$_1$, (14.1.2)]

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

of distinct irreducible closed subsets of X. The *dimension* of an affine or quasi-affine variety is its dimension as a topological space.

[Har77, Ex. I.1.6.1] EXAMPLE 1.1.36. $\dim(\mathbf{A}^1_k) = 1$ since the irreducible closed subsets of $\mathbf{A}^1_k$ are the whole space and closed points (see Example 1.1.9).

EXAMPLE 1.1.37. The notion of dimension defined in Definition 1.1.35 is *very far* from what works for manifolds. If X is a Hausdorff space, then the only irreducible closed subsets are singletons. Thus, $\dim(X) = 0$ using the definition in Definition 1.1.35.

We now compare the dimension of an affine algebraic set with the dimension of its coordinate ring.

[Har77, p. 6] DEFINITION 1.1.38. Let A be a ring. The *height* of a prime ideal $\mathfrak{p}$ is the supremum of all integers $n \geq 0$ for which there exists a chain

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$$

of distinct prime ideals in $\mathfrak{p}$. The *(Krull) dimension* $\dim(A)$ of A is the supremum of the heights of all prime ideals.

REMARK 1.1.39. It is tempting to think that all Noetherian rings have finite Krull dimension. However, this is incorrect! Nagata constructed the first examples in [Nag75, §A1, Example 1]. The point is that the maximum length of a chain of prime ideals may depend on the maximal element in that chain. This will not happen for affine varieties since all coordinate rings are quotients of polynomial rings in finitely many variables over a field, and hence are finite-dimensional.

We can compare dimensions defined geometrically and algebraically using Corollary 1.1.20.

[Har77, Prop. I.1.7] PROPOSITION 1.1.40. *If Y is an affine algebraic set, then* $\dim(Y) = \dim(A(Y))$.

Proof. Suppose $Y \subseteq \mathbf{A}^n_k$. By Corollary 1.1.20, closed irreducible subsets of Y correspond to prime ideals of $k[x_1, x_2, \ldots, x_n]$ that contain $I(Y)$. In turn, these correspond to prime ideals of $A(Y)$ by [AM69, Proposition 1.1]. □

We now want to discuss different ways of computing dimensions. The following result allows us to use many tools from commutative algebra to compute dimension.

[Har77, Thm. I.1.8A] THEOREM 1.1.41. *Let k be a field and let B be an integral domain which is a finitely generated k-algebra. Then:*

(*a*) $\dim(B) = \operatorname{trdeg}_k(\operatorname{Frac}(B))$.

(*b*) (Dimension formula) *For any prime ideal* $\mathfrak{p} \subseteq B$, *we have*

$$\operatorname{ht}(\mathfrak{p}) + \dim(B/\mathfrak{p}) = \dim(B). \tag{1.1.42}$$

Proof. See [Mat89, Theorems 5.6, 15.5, and 15.6] or [Hoc17, p. 59]. □

Theorem 1.1.41 allows us to compute dimensions.

PROPOSITION 1.1.43. $\dim(\mathbf{A}^n_k) = n$. [Har77, Prop. I.1.9]

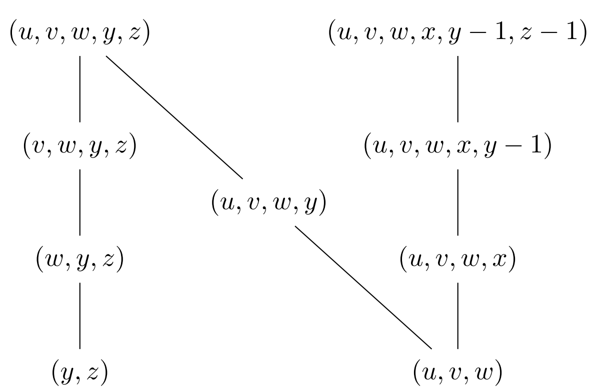
Proof. Combine Proposition 1.1.40 and Theorem 1.1.41(*a*). □

REMARK 1.1.44. (1.1.42) is often called the *dimension formula*. The dimension formula does not hold without the hypotheses in Theorem 1.1.41! For example:

(*a*) Suppose that (V, π) is a DVR with uniformizer π (e.g. the p-adic integers $\mathbf{Z}_p$ or a formal power series ring $k[\![X]\!]$) and set $B = V[T]$. Then,
$$\frac{V[T]}{(\pi T - 1)} \cong \operatorname{Frac}(V)$$
is a field (if $V = \mathbf{Z}_p$, then $\operatorname{Frac}(V) = \mathbf{Q}_p$), and hence $\mathfrak{p} = (\pi T - 1)$ is prime. However,
$$\operatorname{ht}(\mathfrak{p}) + \dim(B/\mathfrak{p}) = 1 + 0 < 2 = \dim(B).$$
Note that the issue in this example is that maximal ideals can have different heights.

(*b*) We can even construct an example where B is a localization of a finitely generated k-algebra and the maximal ideals in B all have the same height. We write down the example from [Hei17, Example 4.2]. Consider the ring [Hei17, Ex. 4.2]
$$B = \frac{k[u, v, w, x, y, z]}{(uy, uz, vy, vz, wy, wz)}.$$
Since
$$(uy, uz, vy, vz, wy, wz) = (y, z) \cap (u, v, w),$$
we can think of $\operatorname{MaxSpec}(B)$ geometrically as the union of k^4 and k^3 in k^6 along the x-axis, that is, the line $Z(u, v, w, y, z)$.

We now consider the localization
$$A = \Big(B - \big((u, v, w, y, z) \cup (u, v, w, x, y - 1, z - 1)\big)\Big)^{-1} B.$$
The ring is semi-local with maximal ideals corresponding to (u, v, w, y, z) and $(u, v, w, x, y - 1, z - 1)$, which are both of height 3, and hence $\dim(A) = 3$ with both maximal ideals of height 3. We then have the following diagram of inclusions of prime ideals in A:



This ring still satisfies the conclusion of does not satisfy the dimension formula (1.1.42) since setting $P = (u, v, w, y)$, we have

$$\mathrm{ht}(P) + \dim(A/P) = 1 + 1 = 2 < 3 = \dim(A).$$

[Har77, Prop. I.1.10]

PROPOSITION 1.1.45. *If Y is a quasi-affine variety, then $\dim(Y) = \dim(\bar{Y})$.*

Proof. Suppose we have a maximal chain of distinct irreducible closed subsets

$$\{P\} = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

in Y, where maximality forces $\{P\} = Z_0$. Taking closures and applying what we know about closures and irreducibility (Examples 1.1.14 and 1.1.15), we have a distinct irreducible closed subsets

$$\{P\} = \bar{Z}_0 \subsetneq \bar{Z}_1 \subsetneq \cdots \subsetneq \bar{Z}_n \tag{1.1.46}$$

that is also maximal. Thus, $\dim(Y) \le \dim(\bar{Y})$.

Next, we apply the dimension formula (1.1.42) to (1.1.46). By the dimension formula (1.1.42), we know that

$$\dim(Y) = n = \mathrm{ht}(\mathfrak{m}_P) + \dim(k) = \dim\bigl(A(\bar{Y})\bigr) = \dim(\bar{Y}).$$ □

REMARK 1.1.47. Note that the maximality of (1.1.46) makes it seem like we are done showing $\dim(Y) = \dim(\bar{Y})$. However, you could imagine that in $\bar{Y}$, different maximal chains have different lengths. This is what is prevented by the dimension formula (1.1.42)!

We end this section with a few more facts from commutative algebra. The following result is useful for bounding dimensions of hypersurfaces and intersections of hypersurfaces.

[Har77, Thm. I.1.11A]
[Hoc17, p. 124]

THEOREM 1.1.48 (Krull's height theorem [Kru38, Satz 7* on p. 220]). *Let A be a Noetherian ring and let $f \in A$ be an element which is neither a zero divisor nor a unit. Then, every minimal prime ideal $\mathfrak{p}$ containing f has height 1.*

Proof. This is proved in [AM69, Corollary 11.17]. However, the proof there is rather involved. For a simpler proof, see [Hoc17, p. 124] or [Nag75, (9.2)]. □

[Har77, Prop. I.1.12A]
[Hoc17, pp. 140–141]

PROPOSITION 1.1.49. *A Noetherian integral domain A is a UFD if and only if every prime ideal of height 1 is principal.*

Proof. See [Hoc17, pp. 140–141]. For references to the published literature, see [Mat89, Theorem 20.1] and [BouCA, Chapter VII, §3, no. 2, Theorem 1]. □

As a consequence, we have:

[Har77, Prop. I.1.13]

PROPOSITION 1.1.50. *A variety $Y \subseteq \mathbf{A}^n_k$ has dimension $n-1$ if and only if it is the zero set $Z(f)$ of a single nonconstant irreducible polynomial in $A = k[x_1, x_2, \ldots, x_n]$.*

Proof. If f is irreducible, then $Z(f)$ is a variety by Example 1.1.23. Moreover, (f) is a prime ideal of height 1 by Krull's height theorem (Theorem 1.1.48) and hence $\dim(Y) = n-1$ by the dimension formula (1.1.42). Conversely, suppose that $\dim(Y) = n-1$. Then, $I(Y) \subseteq A$ is a prime ideal of height 1. Since A is a UFD, we see that $\mathfrak{p} = (f)$ for an irreducible polynomial $f \in A$ by Proposition 1.1.49. □

[Har77, Rem. I.1.13.1]

REMARK 1.1.51. The analogue of Proposition 1.1.50 does *not* hold for prime ideals of height > 1. For example, there exist prime ideals of height 2 that cannot be generated by two elements. See [Har77, Exercise I.1.11] (which will be on Homework 2).

1.2. Projective Varieties

We continue to denote by k an algebraically closed field. 9/5

1.2.1. Motivation. Our next goal is to define projective varieties. Projective varieties will have a few advantages compared to affine varieties. From a topological point of view, $\mathbf{CP}^n$ is nice because it is compact in the Euclidean topology. From the point of view of algebraic geometry, working in projective space simplifies classification problems. You will see in [Har77, Exercise I.3.1(c)] that all curves defined by a quadratic equation in $\mathbf{P}^2_k$ are isomorphic to $\mathbf{P}^1_k$, in contrast with what you saw on Homework 1 for affine curves defined by a quadratic equation [Har77, Exercise I.1.1(c)].

For cubics, the difference between working in affine space and projective space is even more stark. Cubic curves were first investigated by Newton. This work was published in an appendix to *Opticks* (1704). See [BK86, pp. 93–98] for copies of some of the pages from this work. Newton was able to identify 72 different classes of cubic curves—but he was missing six classes in his published classification! (Later, Whiteside discovered that Newton knew about these cubics, since they appear in Newton's mathematical papers. See [Gui09, p. 111, n8].) [BK86, (2.5)]

Nevertheless, one of Newton's important ideas is that if you consider curves to be in the same class if they are related to each other by projecting away from points, then the classification problem becomes much more manageable. See Figure 1.7 for an illustration of how projections of a circle yield ellipses, parabolas, and hyperbolas. See Figure 1.8 for an illustration of how a cubical parabola are related to a semicubical parabola via projection.

As the name suggests, considering such projections leads one naturally to the field of *projective geometry*. Working in the projective plane, for example, forms the mathematical basis of one-point perspective in art.

1.2.2. Projective space, graded rings, homogeneous elements and ideals. We now start defining things formally.

DEFINITION 1.2.1. The *projective n-space* over k is [Har77, pp. 8–9]

$$\mathbf{P}^n_k := \frac{\mathbf{A}^{n+1}_k - \{(0,\ldots,0)\}}{(a_0,\ldots,a_n) \sim (\lambda a_0,\ldots,\lambda a_n) \text{ for all } \lambda \in k - \{0\}}.$$

We can think of the equivalence relation as identifying points lying on the same line through the origin.

An element $P \in \mathbf{P}^n_k$ is called a *point*. When using coordinates, we will write

$$P = [a_0 : a_1 : \cdots : a_n] \in \mathbf{P}^n_k$$

to differentiate points in $\mathbf{P}^n_k$ from points in $\mathbf{A}^n_k$. The a_i are the *homogeneous coordinates* of P.

We often drop the subscript k if the ground field k is clear from context.

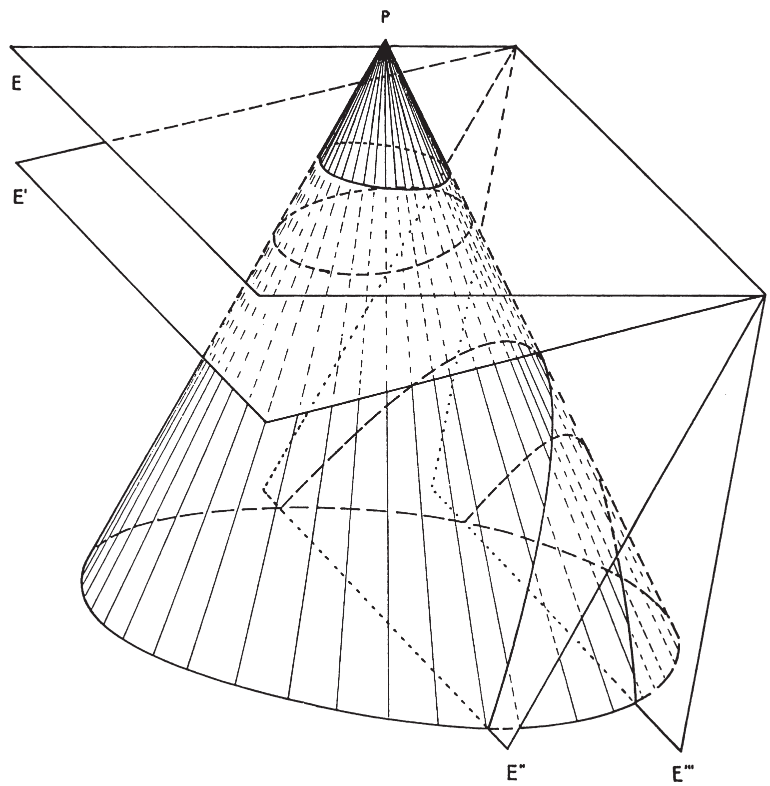


FIGURE 1.7. Projecting a circle in a plane. From [BK86, p. 100].

Remember from the previous section the key feature of algebraic geometry: We can translate questions back and forth between geometry and algebra in affine space. To setup our dictionary between geometry and algebra in projective space, we will need to consider the polynomial ring $k[x_0, x_1, \ldots, x_n]$ as a graded ring.

[Har77, p. 9] DEFINITION 1.2.2. A(n $\mathbf{N}$-)*graded ring* is a ring S together with a decomposition

$$S = \bigoplus_{d \geq 0} S_d$$

of S into a direct sum of Abelian groups S_d such that $S_d \cdot S_e \subseteq S_{d+e}$ for all $d, e \geq 0$. An element of S_d is called a *homogeneous element of degree d*. An ideal $\mathfrak{a} \subseteq S$ is a *homogeneous ideal* if

$$\mathfrak{a} = \bigoplus_{d \geq 0} (\mathfrak{a} \cap S_d).$$

Note that any element of S can be written uniquely as a (finite) sum of homogeneous elements by definition of direct sums of Abelian groups.

REMARK 1.2.3. There are different conventions for the degree of the 0 element. From the point of view of graded rings, one convention that makes sense is to say that 0 has all possible degrees.

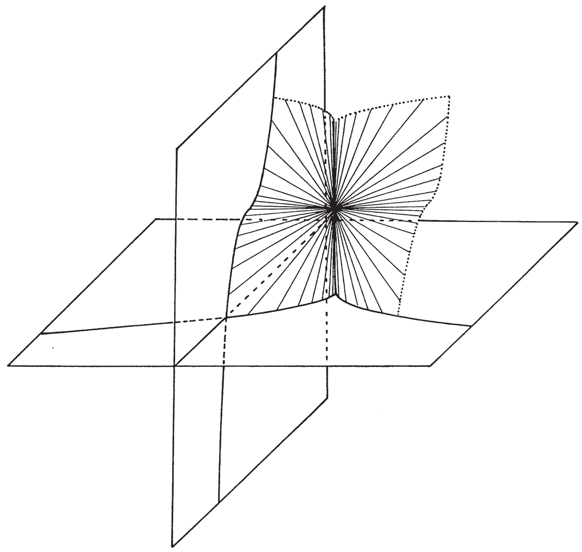


FIGURE 1.8. Cubical parabolas and semicubical parabolas are related via projection. From [BK86, p. 101].

We will need some facts about graded rings. Many of these were discussed in the homework for Commutative Algebra (MA557) and will be discussed on Worksheet 2. See also [Mat89, §13] and [$ZS75_2$, Chapter VII, §2] for proofs of (some of) these results.

FACTS 1.2.4. *Let S be a graded ring and let $\mathfrak{a} \subseteq S$ be an ideal.* [Har77, p. 9]

(1) *$\mathfrak{a}$ is homogeneous if and only if $\mathfrak{a}$ can be generated by homogeneous elements.*
(2) *Sums, products, intersections, and radicals of homogeneous ideals are homogeneous.*
(3) *If $\mathfrak{a}$ is homogeneous, to test whether $\mathfrak{a}$ is prime, it suffices to show that for all* homoeneous *elements $f, g \in S$, having $fg \in \mathfrak{a}$ implies $f \in \mathfrak{a}$ or $g \in \mathfrak{a}$.*

EXAMPLE 1.2.5. The main example of a graded ring we will consider is the polynomial ring [Har77, p. 9]

$$S = k[x_0, x_1, \dots, x_n],$$

where S_d is the set of all k-linear combinations of monomials of total weight d in $x_0, x_1, \dots, x_n$.

Homogeneous polynomials in S play an important role in working with projective space because while we cannot evaluate arbitrary polynomials by plugging in the homogeneous coordinates for a point $P \in \mathbf{P}^n_k$, we can ask whether a *homogeneous* polynomial $f \in S$ vanishes at P because

$$f(\lambda a_0, \lambda a_1, \dots, \lambda a_n) = \lambda^d f(a_0, a_1, \dots, a_n).$$

In other words, the property of a homogeneous polynomial being zero or not at P depends only on the equivalence class of $P = [a_0 : a_1 : \cdots : a_n]$.

DEFINITION 1.2.6. Let $T \subseteq S$ be a set of homogeneous elements of S. The *zero set* of T is [Har77, p. 9]

$$Z_+(T) \coloneqq \bigl\{P \in \mathbf{P}^n_k \bigm| f(P) = 0 \text{ for all } f \in T\bigr\}.$$

If $\mathfrak{a} \subseteq S$ is a homogeneous ideal, we define $Z_+(\mathfrak{a}) \coloneqq Z_+(T)$, where T is the set of all homogeneous elements in $\mathfrak{a}$. Since S is Noetherian, any set of homogeneous elements T has a finite subset $f_1, f_2, \ldots, f_r$ such that $Z_+(T) = Z_+(f_1, f_2, \ldots, f_r)$.

A subset $Y \subseteq \mathbf{P}^n_k$ is a *(projective) algebraic set* if there exists a set $T \subseteq S$ of homogeneous elements such that $Y = Z_+(T)$.

Algebraic sets define a topology.

[Har77, Prop. I.2.1] PROPOSITION 1.2.7 (Projective algebraic sets define a topology). *The class of algebraic sets is stable under finite unions and arbitrary intersections. The empty set and the whole space $\mathbf{P}^n_k$ are algebraic sets.*

Proof. The proof of Proposition 1.1.7 can be adapted to the projective setting. □

[Har77, p. 10] DEFINITION 1.2.8. The *Zariski topology* on $\mathbf{P}^n_k$ is the topology whose closed sets are the (projective) algebraic sets.

Since the property of being irreducible is purely topological, we can define projective varieties as follows.

[Har77, p. 10] DEFINITION 1.2.9. A *projective variety* is an irreducible algebraic set in $\mathbf{P}^n_k$ with the induced topology. An open subset of a projective variety is a *quasi-projective variety*. The *dimension* of a projective or quasi-projective variety is its dimension as a topological space.

Now that we know how to take zero sets of sets of homogeneous polynomials, we want to go in the other direction, i.e., from geometry to algebra.

[Har77, p. 10] DEFINITION 1.2.10. Let $Y \subseteq \mathbf{P}^n_k$ be an arbitrary subset. The *homogeneous ideal* of Y in S is the ideal

$$I(Y) \coloneqq \bigl\{ f \in S \bigm| f \text{ is homogeneous and } f(P) = 0 \text{ for all } P \in Y \bigr\} \cdot S.$$

If Y is an algebraic set, the *homogeneous coordinate ring* of Y is

$$S(Y) \coloneqq \frac{S}{I(Y)}.$$

1.2.3. Covering projective varieties by affine varieties. Our next goal is to show that projective n-space has an open covering by affine n-spaces. As a consequence, every projective (resp. quasi-projective) variety has an open covering by affine (resp. quasi-affine) varieties.

We start with some terminology.

[Har77, p. 10] DEFINITION 1.2.11. Let $f \in S$ be a homogeneous element. If f is linear, then $Z(f)$ is called a *hyperplane*. In particular, we denote

$$H_i \coloneqq Z_+(x_i)$$

for each $i \in \{0, 1, \ldots, n\}$, and set

$$U_i \coloneqq \mathbf{P}^n_k - H_i.$$

We see that

$$\mathbf{P}^n_k = \bigcup_{i=0}^{n} U_i$$

since for any $P = [a_0 : a_1 : \cdots : a_n] \in \mathbf{P}^n_k$, at least one of the coordinates a_i is nonzero.

We define a mapping

$$\begin{aligned} U_i &\xrightarrow{\quad\varphi_i\quad} \mathbf{A}^n_k \\ [a_0 : a_1 : \cdots : a_n] &\longmapsto \left(\frac{a_0}{a_i}, \frac{a_1}{a_i}, \ldots, \widehat{\frac{a_i}{a_i}}, \ldots, \frac{a_n}{a_i}\right) \end{aligned}$$

with a_i/a_i omitted in the right-hand side. Note that φ_i is well-defined since the ratios a_j/a_i are independent of the choice of homogeneous coordinates.

PROPOSITION 1.2.12. *The map φ_i is a homeomorphism of U_i with its induced topology to $\mathbf{A}^n_k$ with the Zariski topology.* [Har77, Prop. I.2.2]

Proof. First, we note that φ_i is bijective with inverse defined by

$$\varphi^{-1}(b_1, b_2, \ldots, b_n) = [b_1 : b_2 : \cdots : 1 : \cdots : b_n]$$

where 1 is in the i-th coordinate on the right-hand side. It therefore suffices to show that the closed subsets of U_i are identified with the closed subsets of $\mathbf{A}^n_k$ under φ_i. After rearranging coordinates, we may assume that $i = 0$, in which case we write U for U_0 and $\varphi\colon U \to \mathbf{A}^n_k$ for φ_0.

Let $A = k[y_1, y_2, \ldots, y_n]$ and denote by S^h the set of homogeneous elements of S. We define the maps

$$\begin{aligned} \alpha\colon S^h &\longrightarrow A \\ f &\longmapsto f(1, y_1, \ldots, y_n) \\ \beta\colon A &\longrightarrow S^h \\ g &\longmapsto x_0^{\deg(g)}\, g\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right). \end{aligned}$$

If $Y \subseteq U$ is a closed subset with closure $\bar{Y} \subseteq \mathbf{P}^n_k$, then $\bar{Y} = Z_+(T)$ for a subset $T \subseteq S^h$.

Set $T' = \alpha(T)$. We claim that $\varphi(Y) = Z(T')$. Consider a point

$$y = [b_0 : b_1 : b_2 : \cdots : b_n] \in U.$$

For every $f \in S$, we have

$$\alpha(f)\big(\varphi(y)\big) = f\left(1, \frac{b_1}{b_0}, \ldots, \frac{b_n}{b_0}\right) = b_0^{-\deg(f)}\, f(b_0, b_1, \ldots, b_n),$$

and hence $f(y) = 0$ if and only if $\alpha(f)(\varphi(y)) = 0$. Thus, we have the equivalences

$$\begin{aligned} {}_+y \in Y &\iff y \in Z_+(T) \cap U \\ &\iff f(y) = 0 \text{ for every } f \in T \\ &\iff \alpha(f)\big(\varphi(y)\big) = 0 \text{ for every } f \in T \\ &\iff g\big(\varphi(y)\big) = 0 \text{ for every } g \in \alpha(T). \end{aligned}$$

This shows that $\varphi(Y) = Z(T')$, and hence φ is closed.

Conversely, suppose that $W \subseteq \mathbf{A}^n_k$ is a closed subset, and write $W = Z(T')$ for a subset $T' \subseteq A$. We claim that $\varphi^{-1}(W) = Z_+(\beta(T')) \cap U$. Consider a point

$$y = [b_0 : b_1 : b_2 : \cdots : b_n] \in U.$$

For every $g \in A$, we have

$$\beta(g)(y) = b_0^{\deg(g)}\, g\left(\frac{b_1}{b_0}, \ldots, \frac{b_n}{b_0}\right) = b_0^{\deg(g)}\, g\bigl(\varphi(y)\bigr),$$

and hence $\beta(g)(y) = 0$ if and only if $g(\varphi(y)) = 0$. Thus, we have the equivalences

$$\begin{aligned} \varphi(y) \in W &\iff \varphi(y) \in Z(T') \\ &\iff g\bigl(\varphi(y)\bigr) = 0 \text{ for every } g \in T' \\ &\iff \beta(g)(y) = 0 \text{ for every } g \in T' \\ &\iff f(y) = 0 \text{ for every } f \in \beta(T'). \end{aligned}$$

This shows that $\varphi^{-1}(W) = Z_+(\beta(T')) \cap U$, and hence φ is continuous. □

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[Har77, Cor. I.2.3]

Corollary 1.2.13. *If Y is a projective (resp. quasi-projective) variety, then Y is covered by the open sets $Y \cap U_i$ for $i \in \{0, 1, \ldots, n\}$. These open sets are homeomorphic to affine (resp. quasi-affine) varieties via the map φ_i defined above.*

1.2.4. The algebra ↔ geometry dictionary for projective space. A key tool we will be using is the following, which again is on Worksheet 2. I encourage you to work it out on your own if you have not done so already. (If you need to use the result on homework, please submit a proof of what you need.) The idea of the proof is to use [Har77, Exercise I.2.3]. This latter exercise can be solved directly—reducing to the affine case would require understanding the quotient map

$$\mathbf{A}_k^n - \bigl\{(0, 0, \ldots, 0)\bigr\} \longrightarrow \mathbf{P}_k^n$$

better.

[Har77, Exer. I.2.4]
[ZS75$_2$, p. 172]

Proposition 1.2.14. *There is a one-to-one inclusion-reversing correspondence*

$$\left\{\begin{matrix} \textit{algebraic sets} \\ \textit{in } \mathbf{P}_k^n \end{matrix}\right\} \xleftrightarrow{\text{1-1}} \left\{\begin{matrix} \textit{homogeneous radical ideals} \\ \textit{in } k[x_0, x_1, x_2, \ldots, x_n] \\ \textit{not equal to } S_+ \end{matrix}\right\}$$

$$\begin{aligned} Y &\longmapsto I(Y) \\ Z_+(\mathfrak{a}) &\longleftarrow\!\!\shortmid\ \mathfrak{a}. \end{aligned}$$

Furthermore, an algebraic set is irreducible if and only if its ideal is a prime ideal.

Remark 1.2.15. Since S_+ does not appear in the right-hand side of this correspondence, it is sometimes called the *irrelevant* maximal ideal of S.

A key feature of $\mathbf{A}_k^n$ is that it is Noetherian and that algebraic sets admit irreducible decompositions. The same facts hold for $\mathbf{P}_k^n$.

[Har77, Exer. I.2.5]

Proposition 1.2.16.

(*a*) *$\mathbf{P}_k^n$ is a Noetherian topological space.*
(*b*) *Every algebraic set in $\mathbf{P}_k^n$ can be written uniquely as a finite union of irreducible algebraic sets, no one containing another. These are called its* irreducible components.

Proof. For (*a*), let $Z_0 \supseteq Z_1 \supseteq \cdots$ be a descending chain. This corresponds via Proposition 1.2.14 to an ascending chain of homogeneous radical ideals in $k[x_0, x_1, x_2, \ldots, x_n]$, and hence stabilizes by the Hilbert basis Theorem 1.1.30. For (*b*), we can apply Proposition 1.1.32 (which is a general statement about closed subsets of Noetherian topological spaces) since $\mathbf{P}_k^n$ is Noetherian by (*a*). □

1.2.5. An affine variety that is not a local complete intersection. Now that we understand how to work with projective varieties a bit more, I want to revisit a question from Homework 2. The theme for the rest of today is that determining the defining equations or the ideal of definition of a variety is surprisingly difficult.

EXAMPLE 1.2.17 (An affine variety that is not a local complete intersection). Let $Y \subseteq \mathbf{A}^n_k$ be an affine variety of dimension r. By Krull's height Theorem 1.1.48, we know that $I(Y)$ must have at least $n - \dim(Y)$ generators. While one might be optimistic and hope that there are always $n - \dim(Y)$ generators, this is not true as you saw/will see on Homework 2. [Har77, Exer. I.1.11]

Consider the curve $Y \subseteq \mathbf{A}^3_k$ defined by the kernel of the ring map

$$\begin{aligned} \varphi\colon k[x,y,z] &\longrightarrow k[t] \\ x &\longmapsto t^3 \\ y &\longmapsto t^4 \\ z &\longmapsto t^5. \end{aligned}$$

Since the codomain is a domain, we know that $\ker(\varphi)$ is prime, and hence $I(Y) = \ker(\varphi)$. Moroever, considering $k[x,y,z]$ as a graded ring where $\deg(x) = 3$, $\deg(y) = 4$, and $\deg(z) = 5$, we saw that $I(Y)$ is a *homogeneous* prime ideal. By matching the exponents on t on the images of the variables, we have

$$(xz - y^2, x^3 - yz, x^2y - z^2) \subseteq I(Y). \tag{1.2.18}$$

For the homework, I suggest using these three generators to prove that $I(Y)$ cannot be generated by 2 elements. You can use the dimension formula (1.1.42) to prove that $I(Y)$ is of height 2.

There are more questions we can ask about this example.

(1) It is even true that $I(Y)$ is not generated by 2 elements after localizing at $\mathfrak{m} = (x,y,z)$! A hint for showing this: first show that the localization map
$$\frac{I(Y)}{\mathfrak{m}\cdot I(Y)} \xrightarrow{\sim} \frac{I(Y)_\mathfrak{m}}{\mathfrak{m}\cdot I(Y)_\mathfrak{m}}$$
is an isomorphism where $I(Y)_\mathfrak{m} = I(Y)\cdot k[x,y,z]_\mathfrak{m}$. We now look at the images of the three elements in (1.2.18) modulo $\mathfrak{m}\cdot I(Y)$:
$$\overline{xz - y^2},\ \overline{x^3 - yz},\ \overline{x^2y - z^2} \in \frac{I(Y)}{\mathfrak{m}\cdot I(Y)}.$$
This is a graded module (by Worksheet 2), and is even a graded k-vector space where we make the identification $k \cong k[x,y,z]/\mathfrak{m}$. By the graded NAK lemma (from Homework 2 in MA557), to show that we need ≥ 3 generators, it suffices to show that these three elements are nonzero and k-linearly independent in $I(Y)/(\mathfrak{m}\cdot I(Y))$.

We say that Y is *not a local complete intersection* at the origin $(0,0,0) \in \mathbf{A}^3_k$. (An affine variety $Y \subseteq \mathbf{A}^n_k$ is a *local complete intersection* if $I(Y)_\mathfrak{m}$ is generated by $n - \dim(Y)$ elements for every maximal ideal $\mathfrak{m} \subseteq k[x_1, x_2, \ldots, x_n]$.)

(2) A lingering question remains: What *are* the actual generators of $I(Y)$? This question (and its generalization to other curves constructed in this way, where the weights $3, 4, 5$ are allowed to change) was the topic of Herzog's thesis (published as [Her70]). [Her70] [Kun13, p. 137ff]

1.2.6. The twisted cubic curve and bad behavior of generators of $I(Y)$ under projective closure. Next, I want to show you how the affine $\leftrightsquigarrow$ projective "translation" from last time (Proposition 1.2.12) does not always behave well with respect to the complete intersection property.

[Har77, Exer. I.1.2]

Example 1.2.19 (The twisted cubic curve). On Worksheet 1, you saw the example

$$Y = \bigl\{(t,t^2,t^3) \bigm| t \in k\bigr\} \subseteq \mathbf{A}^3_k.$$

This is an affine variety with

$$I(Y) = (x^2 - y, xy - z).$$

The inclusion $\supseteq$ holds by evaluating the generators at the points (t,t^2,t^3). To show the inclusion $\subseteq$, we first note that

$$\frac{k[x,y,z]}{(x^2-y,xy-z)} \cong k[x],$$

and hence $(x^2 - y, xy - z)$ is prime of height 2. On the other hand, the dimension formula (1.1.42) implies $\operatorname{ht}(I(Y)) = 2$. Thus, the inclusion $I(Y) \supseteq (x^2 - y, xy - z)$ is an inclusion of prime ideals of the same height, which must be an equality.

[Har77, Exer. I.2.9]

On Homework 3, I will ask you to consider the *projective closure* of Y in $\mathbf{P}^3_k$ obtained by embedding Y in $\mathbf{P}^3_k$ via the homeomorphism $\mathbf{A}^3_k \approx U_0 \subseteq \mathbf{P}^3_k$ and taking closures. The resulting projective variety is the *twisted cubic curve*

$$\bar{Y} \subseteq \mathbf{P}^3_k.$$

Note that $\bar{Y}$ is irreducible by Example 1.1.15.

Since we had nice equations defining the affine version of the twisted cubic curve on U_0, one might hope that homogenizing the generators yields generators for $I(\bar{Y})$. That is, one may hope that

[Har77, Exer. I.2.16(a)]

$$\beta(x^2 - y) = x^2 - yw \qquad \text{and} \qquad \beta(xy - z) = xy - zw$$

generate $I(\bar{Y})$. Set

$$W \coloneqq Z_+(x^2 - yw, xy - zw).$$

By Proposition 1.2.12, we know that $W \cap U_0$ is irreducible. The only possibility is that the behavior along the "hyperplane at infinity" $Z_+(w)$ is causing issues. To give us some intuition, we note that $W \cap Z_+(w) = Z_+(x,w)$ is a line that does not come from $\bar{Y}$.

We claim that the line $Z_+(x,w)$ is an irreducible component of W. To do this, consider the colon ideal

$$\begin{aligned}&\bigl((x^2 - yw, xy - zw) : (x,w)\bigr)\\ &\qquad \coloneqq \Bigl\{ f \in k[w,x,y,z] \Bigm| (x,w)\cdot f \subseteq (x^2 - yw, xy - zw)\Bigr\}.\end{aligned}$$

Note that

$$\begin{aligned} x(y^2 - xz) &= -z(x^2 - yw) + y(xy - zw)\\ w(y^2 - xz) &= -y(x^2 - yw) + x(xy - zw)\end{aligned}$$

and hence

$$\bigl((x^2 - yw, xy - zw) : (x,w)\bigr) \ni y^2 - xz.$$

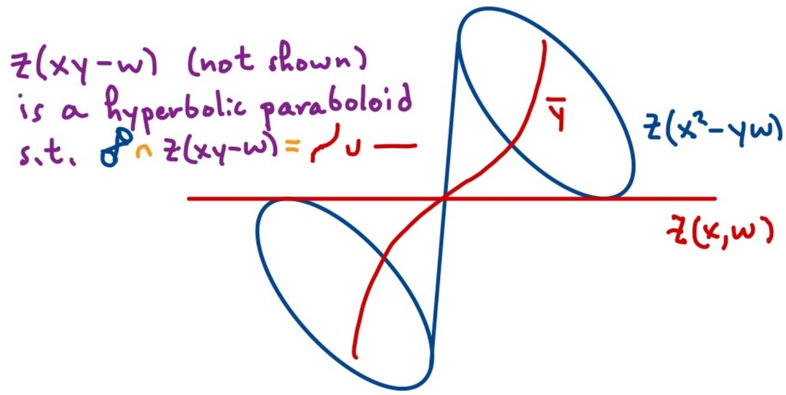

FIGURE 1.9. The twisted cubic curve is an irreducible component of an intersection of two quadrics. What is shown is the affine chart $z \neq 0$.

By definition of the colon ideal, we know that

$$\begin{aligned}&(x^2 - yw, xy - zw)\\ &\qquad \supseteq \big((x^2 - yw, xy - zw) : (x, w)\big) \cdot (x, w)\\ &\qquad \supseteq (y^2 - xz) \cdot (x, w).\end{aligned}$$

Taking zero sets, we see that

$$\begin{aligned}&Z_+(y^2 - xz) \cup Z_+(x, w)\\ &\qquad \supseteq Z_+\big((x^2 - yw, xy - zw) : (x, w)\big) \cup Z_+(x, w)\\ &\qquad \supseteq W\\ &\qquad \supseteq Z_+(x, w).\end{aligned}$$

To show that $Z_+(x, w)$ is an irreducible component of W, write $W = \bar{Y} \cup W'$ as a union of proper closed subsets. Such a decomposition exists by the existence of irreducible decompositions (Proposition 1.1.32). Taking intersections with $Z_+(w)$ in this chain of inclusions, we obtain

$$\begin{aligned}&Z_+(y^2 - xz, w) \cup Z_+(x, w)\\ &\qquad \supseteq \big(\bar{Y} \cap Z_+(w)\big) \cup \big(W' \cap Z_+(w)\big)\\ &\qquad \supseteq Z_+(x, w).\end{aligned}$$

Since

$$\bar{Y} = \big\{[s^3 : s^2t : st^2 : t^3] \bigm| s, t \in k\big\} \subseteq \mathbf{P}^3_k,$$

we have

$$\bar{Y} \cap Z_+(w) = \big\{[0 : 0 : 0 : 1]\big\}.$$

Since $Z_+(y^2 - xz, w)$ does not contain $Z_+(x, w)$, the implication above implies

$$Z_+(x, w) \supseteq W' \cap Z_+(w) \supseteq Z_+(x, w).$$

Thus, equality holds throughout and $W' \supseteq Z_+(x, w)$. It remains to show that $W' = Z_+(x, w)$. If $W' \supsetneq Z_+(x, w)$, then W' must be a surface containing $Z_+(x, w)$.

Since $W \cap U_0 = Y$ is a curve, this means that W' is a surface contained in $Z_+(w)$, and hence we must have $W' = Z_+(w)$. This contradicts the fact that

$$W' \cap Z_+(w) = Z_+(x, w).$$

The story continues on Homework 3, where for example, you will be asked to compute $I(\bar{Y})$. In this case, you can find the generators by hand. If you are working over a specific field and have specific generators for $I(Y)$, then you can compute $I(\bar{Y})$ using Gröbner bases. See [CLO15, Chapter 8, §4].

[Har77, Exer. I.2.16(a)]

Remark 1.2.20. We have shown that

$$Z_+(x^2 - yw, xy - zw) = \bar{Y} \cup Z_+(x, w)$$

is a union of the twisted cubic curve and a line (see Figure 1.9). Since both $Z_+(x^2 - yw)$ and $Z_+(xy - zw)$ are varieties (both polynomials are irreducible), we have therefore constructed an example of two varieties whose intersection is not a variety.

1.2.7. Cones over projective varieties. As we saw with [Har77, Exercises I.2.1–I.2.7] and in particular, with Proposition 1.2.14 stated above, one can prove various properties of algebraic sets in $\mathbf{P}^n_k$ and their homogeneous ideals by reducing somehow to the affine case (although sometimes, as in [Har77, Exercise I.2.3], I would recommend proving things directly). The affine case is often convenient because we can use familiar tools from commutative algebra. By reducing to the affine case, we were able to expand our algebra $\leftrightarrow$ geometry dictionary to the projective setting.

The results we showed last time (Proposition 1.2.12) give one way to reduce to the affine case:

(I) By Proposition 1.2.12, we can cover $\mathbf{P}^n_k$ (resp. a closed subset $Y \subseteq \mathbf{P}^n_k$) by affine open subsets U_i (resp. $Y \cap U_i$), which are homeomorphic to $\mathbf{A}^n_k$ (resp. to closed subsets in $\mathbf{A}^n_k$).

Moving back and forth using this homeomorphism can sometimes cause issues, however, as we saw with the twisted cubic in Example 1.2.19. Instead, we can mimic our intuition from [Har77, Exercises I.2.1–I.2.7]:

(II) Analyze everything by pulling back to the space $\mathbf{A}^n_k$ that surjects onto $\mathbf{P}^n_k$.

This is secretly what you were doing when for example, you deduced the homogeneous Nullstellensatz [Har77, Exercise I.2.1] from the affine version.

[Har77, Exer. I.2.10]

Definition 1.2.21. Let $Y \subseteq \mathbf{P}^n_k$ be a nonempty algebraic set, and consider the map

$$\begin{aligned}\theta\colon \mathbf{A}^{n+1}_k - \{(0,0,\ldots,0)\} &\longrightarrow \mathbf{P}^n_k\\ (a_0, a_1, \ldots, a_n) &\longmapsto [a_0 : a_1 : \cdots : a_n].\end{aligned}$$

The *affine cone* over Y is

$$C(Y) = \theta^{-1}(Y) \cup \{(0,0,\ldots,0)\} \subseteq \mathbf{A}^{n+1}_k.$$

The *projective cone* over Y is the projective closure

$$\overline{C(Y)} \subseteq \mathbf{P}^{n+1}_k.$$

See Figure 1.10 for an illustration.

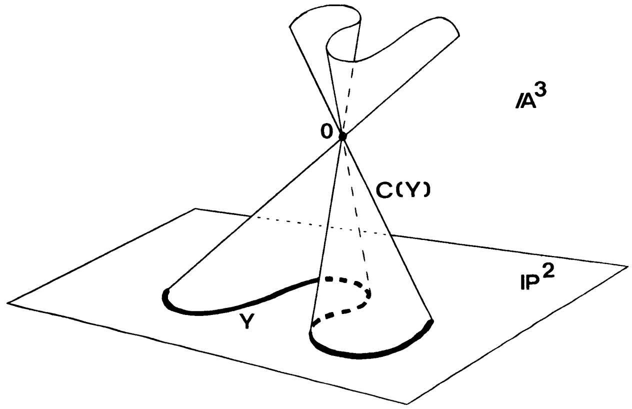

FIGURE 1.10. The cone over a curve in $\mathbf{P}^2_k$. From [Har77, p. 12].

Part of Homework 3 works out the basic properties of the affine cone. It turns out that $C(Y)$ is an algebraic set in $\mathbf{A}^{n+1}_k$ with defining ideal $I(Y) \subseteq k[x_0, x_1, \ldots, x_n]$ (this justifies the fact that $I(Y)$ could have two different meanings corresponding to the affine and projective cases, respectively). This also shows that the map

$$C(Y) - \{(0, 0, \ldots, 0)\} \longrightarrow Y$$

is a quotient map in the sense of topology, answering one of your questions from class.

Taking cones is nice because taking the cone over a complete intersection still gives a complete intersection. Similarly, if we have a set of r defining equations for Y as a projective variety, then there are r defining equations for $C(Y)$ as an affine variety. Surprisingly, as far as I know, the following question is open:

OPEN PROBLEM 1.2.22 [Lyu89, Problem 0.1]. *Let $Y \subseteq \mathbf{P}^n_k$ be a projective variety. If $C(Y)$ can be defined by r equations, can Y always be defined by r equations?*

1.2.8. The Segre embedding. For theoretical purposes, it is very useful to be able to take products of quasi-projective varieties. Given two affine spaces $\mathbf{A}^r_k$ and $\mathbf{A}^s_k$ of different dimensions, we know that we can consider $\mathbf{A}^r_k \times \mathbf{A}^s_k$ as the affine space $\mathbf{A}^{r+s}_k$. With projective spaces, however, there is no obvious way in which $\mathbf{P}^r_k \times \mathbf{P}^s_k$ can be realized as a projective variety. We therefore need to answer:

QUESTION 1.2.23. *How can we give the set $\mathbf{P}^r_k \times \mathbf{P}^s_k$ the structure of a projective variety?*

The key construction that will allow us to do this is called the *Segre embedding*. Hartshorne assigns this construction as an exercise. To help us get used to working in projective space, we work out the Segre embedding here.

EXAMPLE 1.2.24 (The Segre embedding). Fix integers $r, s > 0$. Consider the map (of sets) [Har77, Exer. I.2.14]

$$\psi\colon \mathbf{P}^r_k \times \mathbf{P}^s_k \longrightarrow \mathbf{P}^N_k$$
$$\big([a_0 : a_1 : \cdots : a_r], [b_0 : b_1 : \cdots : b_s]\big) \longmapsto [\cdots : a_i b_j : \cdots]$$

where $N = (r+1)(s+1) - 1 = rs + r + s$ and the coordinates $a_i b_j$ on the right-hand side are listed lexicographically.

Consider the ring map

$$\begin{aligned} \theta\colon k\Big[\{z_{ij}\}_{\substack{0\le i\le r\\ 0\le j\le s}}\Big] &\longrightarrow k[x_0,\ldots,x_r,y_0,\ldots,y_s] \\ z_{ij} &\longmapsto x_i y_j \end{aligned}$$

and set $\mathfrak{a} \coloneqq \ker(\psi)$. Note that $\mathfrak{a}$ is prime since $\operatorname{im}(\theta) \cong k[\{z_{ij}\}]/\mathfrak{a}$ is a subring of a domain.

We will show that ψ is well-defined, injective, and that $\operatorname{im}(\psi) = Z_+(\mathfrak{a})$. As a result, identifying $\mathbf{P}^r_k \times \mathbf{P}^s_k$ with its image $Z_+(\mathfrak{a})$ under ψ, we can give $\mathbf{P}^r_k \times \mathbf{P}^s_k$ the structure of a projective variety.

Step 1. ψ is well-defined.

First, the coordinates of $[\cdots : a_i b_j : \cdots]$ are not all 0 since there exist i_0, j_0 such that $a_{i_0} \neq 0$ and $b_{j_0} \neq 0$, and hence $a_{i_0} b_{j_0} \neq 0$.

Now let $\lambda, \mu \in k - \{0\}$. Then, we have

$$\begin{aligned} \psi\big(\lambda \cdot [a_0 : a_1 : \cdots : a_r], \mu \cdot [b_0 : b_1 : \cdots : b_s]\big) \\ &= [\cdots : \lambda\mu a_i b_j : \cdots] \\ &= \lambda\mu \cdot [\cdots : a_i b_j : \cdots] \\ &= [\cdots : a_i b_j : \cdots] \\ &= \psi\big([a_0 : a_1 : \cdots : a_r], [b_0 : b_1 : \cdots : b_s]\big) \end{aligned}$$

which shows that ψ is well-defined.

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Step 2. ψ is injective.

Suppose that $\psi(a,b) = \psi(c,d)$ for $a = [a_0 : \cdots : a_r], c = [c_0 : \cdots : c_r] \in \mathbf{P}^r_k$ and $b = [b_0 : \cdots : b_s], d = [d_0 : \cdots : d_s] \in \mathbf{P}^s_k$. Then, we have

$$[\cdots : a_i b_j : \cdots] = [\cdots : c_i d_j : \cdots].$$

Thus, there exists $\lambda \in k - \{0\}$ such that $a_i b_j = \lambda c_i d_j$ for all i, j. Choose indices i_0, j_0 such that $a_{i_0} \neq 0$ and $b_{j_0} \neq 0$. Since $a_{i_0} b_{j_0} = \lambda c_{i_0} d_{j_0}$, this implies $c_{i_0} \neq 0$ and $d_{j_0} \neq 0$ as well. Scaling a, b, c, d by $a_{i_0}^{-1}, b_{j_0}^{-1}, c_{i_0}^{-1}, d_{j_0}^{-1}$ respectively, we may assume that $a_{i_0} = b_{j_0} = c_{i_0} = d_{j_0} = 1$, in which case $\lambda = 1$. We then see that

$$\begin{aligned} a_i &= [\cdots : a_i b_j : \cdots]_{ij_0} = [\cdots : c_i d_j : \cdots]_{ij_0} = c_i \\ b_j &= [\cdots : a_i b_j : \cdots]_{i_0 j} = [\cdots : c_i d_j : \cdots]_{i_0 j} = d_j \end{aligned}$$

which shows that $(a,b) = (c,d)$, i.e., ψ is injective.

Step 3. $\operatorname{im}(\psi) = Z_+(\mathfrak{a})$.

$\subseteq$. Let $P = [\cdots : a_i b_j : \cdots] \in \operatorname{im}(\psi)$. We need to show that $f(P) = 0$ for all $f \in \mathfrak{a}$. We have

$$f(P) = \big(\theta(f)\big)(a,b) = 0$$

and hence $P \in Z(\mathfrak{a})$.

$\supseteq$. Suppose $Q = [\cdots : c_{ij} : \cdots] \in Z_+(\mathfrak{a})$. Fix i_0, j_0 such that $c_{i_0 j_0} \neq 0$. We claim that letting $a_i = c_{ij_0}$ and $b_j = c_{i_0 j}$ for all i, j, we have $\psi(a,b) = Q$. First, note that

$$\theta(z_{ij} z_{kl} - z_{kj} z_{il}) = x_i y_j x_k y_l - x_k y_j x_i y_l = 0$$

and hence $z_{ij}z_{kl} - z_{kj}z_{il} \in \mathfrak{a}$ for all $i, k \in \{0, 1, \ldots, r\}$ and $j, l \in \{0, 1, \ldots, s\}$. Thus,

$$a_i b_j = c_{ij_0} c_{i_0 j} = c_{ij} c_{i_0 j_0}$$

for all i, j. Thus,

$$\psi(a, b) = [\cdots : c_{ij} c_{i_0 j_0} : \cdots] = c_{i_0 j_0} \cdot [\cdots : c_{ij} : \cdots] = [\cdots : c_{ij} : \cdots].$$

This shows that $\operatorname{im}(\psi) = Z_+(\mathfrak{a})$.

Finally, I wanted to make the following cautionary remark.

REMARK 1.2.25. If X and Y are affine varieties, then it is true that

$$I(X) + I(Y) \subseteq I(X \cap Y),$$

but equality does not necessarily hold. For example, consider the parabola $X = Z(y - x^2)$ and the line $Y = Z(y)$. Then,

$$I(X) + I(Y) = (y - x^2, y) = (x^2, y) \neq (x, y) = I(X \cap Y).$$

However, it is true that $\sqrt{I(X) + I(Y)} = I(X \cap Y)$ by the Nullstellensatz (Theorem 1.1.19).

1.3. Morphisms

We continue to denote by k an algebraically closed field.

Now that we have defined and started working with (quasi-)affine varieties and (quasi-)projective varieties, we want to start discussing morphisms between them. This will give us a language for saying when two quasi-projective varieties are isomorphic, and allows us to discuss the *category* of (quasi-)affine varieties or (quasi-)projective varieties.

1.3.1. Regular functions on affine and quasi-affine varieties. In analogy with the case of smooth manifolds, what we want to do is to define a morphism of varieties to be some sort of map for which "algebraic" functions pull back. For manifolds, you do the same thing for smooth functions. The analogue of smooth functions that works for varieties is the class of *regular* functions.

Instead of proceeding like in [Har77], we will divide up our treatment of regular functions into three cases: the affine case, the quasi-affine case, and the quasi-projective case.

DEFINITION 1.3.1. Let $Y \subseteq \mathbf{A}^n_k$ be an affine variety. A function $f\colon Y \to k$ is a *regular function* if it is the restriction of a polynomial function on $\mathbf{A}^n_k$, or in other words, if f is the image of an element in $k[x_1, x_2, \ldots, x_n]$. We denote by $\mathcal{O}(Y)$ the ring of regular function on Y.

[Har77, Thm. I.3.2(a)] [Sha13$_1$, p. 25]

With this definition, we see that a regular function on Y is just an element in $A(Y)$: The elements in $I(Y)$ induce the 0 function on Y, and any other function is not the 0 function (by definition of $I(Y)$). We therefore have an isomorphism

$$A(Y) \coloneqq \frac{k[x_1, x_2, \ldots, x_n]}{I(Y)} \xrightarrow{\sim} \mathcal{O}(Y).$$

For a quasi-affine variety, this does not give the right notion.

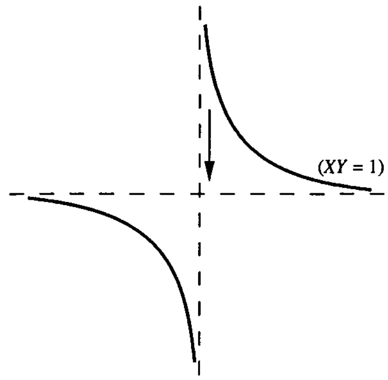

FIGURE 1.11. Projecting the hyperbola onto the x-axis has image $\mathbf{A}^1_k - \{0\}$. From [Rei95, Figure 4.8].

EXAMPLE 1.3.2. On $\mathbf{A}^1_k - \{0\}$, the function $1/x$ should be considered regular since it is a function defined algebraically and it still defines a function on $\mathbf{A}^1_k - \{0\}$. This is because eventually, we will show that $\mathbf{A}^1_k - \{0\}$ is isomorphic to $Z(xy-1) \subseteq \mathbf{A}^2_k$, which has affine coordinate ring isomorphic to $k[x, 1/x]$. The isomorphism is a very natural one: It comes from projecting the hyperbola onto the x-axis. See Figure 1.11.

The definition for quasi-affine varieties is therefore more complicated.

DEFINITION 1.3.3. Let $Y \subseteq \mathbf{A}^n_k$ be a quasi-affine variety. A function $f\colon Y \to k$ is *regular at a point* $P \in Y$ if there is an open neighborhood U with $P \in U \subseteq Y$ and polynomials

$$g, h \in A = k[x_1, x_2, \ldots, x_n]$$

such that h is nowhere 0 on U and $f = g/h$ on U. (Here, we interpret the polynomials as functions on $\mathbf{A}^n_k$ and therefore on Y via restriction.) We say that f is *regular on* Y if it is regular at every point of Y.

REMARK 1.3.4. There are two differences compared to Definition 1.3.1. First, Definition 1.3.3 involves rational functions of the form g/h. This fixes the issue raised about $\mathbf{A}^1_k - \{0\}$ before. Second, Definition 1.3.3 involves a *local* condition with a point P and a neighborhood U of P.

You may therefore be wondering:

QUESTION 1.3.5. *Is it necessary to allow for such a local condition, or can we always find* one *rational representation $f = g/h$ of a regular function that works for every point on a quasi-affine variety?*

We first give an example where even though one rational representation works, there are two different representations that do not look the same at first glance.

EXAMPLE 1.3.6 (Stereographic projection). Let $X = Z(x^2+y^2-1)$ be the circle and let $Y = X - \{(0,1)\}$. We can then define the regular function $Y \to k$ where

$$\begin{aligned} f\colon Y &\longrightarrow k \\ (x,y) &\longmapsto \frac{x}{1-y} \end{aligned}$$

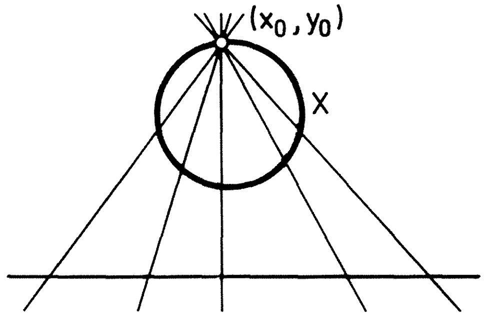

FIGURE 1.12. Stereographic projection of a circle. From [Sha13$_1$, Figure 4].

This is the stereographic projection of a circle onto the x-axis. See Figure 1.12. The function f is well defined since we have excluded the point $y = 1$. On the other hand, when $x \neq 0$, we have

$$\frac{x}{1-y} = \frac{x^2}{x(1-y)} = \frac{1-y^2}{x(1-y)} = \frac{y+1}{x}$$

and hence on the open set $\{x \neq 0\}$, the regular function f has two representations that do not look the same.

This example suggests that the answer is more subtle than we might expect. We now give another example answering Question 1.3.5 in the affirmative: There are regular functions in the sense of Definition 1.3.3 that do not arise as the restriction of a rational function on an affine variety containing Y as an open subset.

EXAMPLE 1.3.7 (Projecting away from a point on the cuspidal cubic). Consider the affine variety [Rös22]

$$X = Z(x^3 - y^2) \subseteq \mathbf{A}^2_k,$$

the *cuspidal cubic*. Let $Y = X - \{(1, -1)\}$. Consider the regular function

$$f\colon Y \longrightarrow k$$
$$(x, y) \longmapsto \begin{cases} \dfrac{1-y}{1-x} & \text{if } x \neq 1, \\[2ex] \dfrac{1+x+x^2}{1+y} & \text{if } y \neq -1. \end{cases}$$

This map takes a point on Y to the slope of the secant line connecting $(1,1)$ to (x,y) for $(x,y) \neq (1,1)$, and $f(1,1) = 3/2$. This function is well-defined since

$$\begin{aligned}\frac{1-y}{1-x} &= \frac{(1-y)(1+x+x^2)}{(1-x)(1+x+x^2)}\\ &= \frac{(1-y)(1+x+x^2)}{1-x^3}\\ &= \frac{(1-y)(1+x+x^2)}{1-y^2}\\ &= \frac{1+x+x^2}{1+y}\end{aligned}$$

when $x \neq 1$ and $y \neq 1$. On the other hand, we cannot write $f = g/h$ globally on Y since

$$\frac{g}{h} = \frac{1-y}{1-x} \implies (1-x)g = (1-y)h \implies (1-x) \mid h$$

and hence the expression g/h cannot be evaluated at $(1,1)$.

Note that we now have two definitions for regular functions on affine varieties. Our goal for most of today will be to show that the two notions coincide for affine varieties.

Before we start working towards this goal, we prove the following:

[Har77, Lem. I.3.1, Rem. I.3.1.1]

LEMMA 1.3.8. *A regular function on an affine variety or a quasi-affine variety is continuous when k is identified with $\mathbf{A}^1_k$ with its Zariski topology. Moreover, if f and g are regular functions on an affine or quasi-affine variety Y and $f = g$ on some nonempty open subset $U \subseteq Y$, then $f = g$ everywhere.*

Proof. Let $f\colon Y \to k$ be a regular function. It suffices to show that inverse images of closed sets are closed. Since the closed sets in $\mathbf{A}^1_k$ are finite sets of points, it suffices to show that

$$f^{-1}(a) = \bigl\{P \in Y \bigm| f(P) = a\bigr\}$$

is closed for every $a \in k$. The affine case holds since we have $f^{-1}(a) = Z(f-a)$ in this situation.

For the quasi-affine case, it suffices to show that $f^{-1}(a)$ is closed after restricting to an open cover of Y. Let $U \subseteq Y$ be an open subset on which $f = g/h$ for $g, h \in A$ and h is nowhere 0 on U. Then,

$$\begin{aligned}f^{-1}(a) \cap U &= \left\{P \in U \;\middle|\; \frac{g(P)}{h(P)} = a\right\}\\ &= \bigl\{P \in U \bigm| g(P) = ah(P)\bigr\}\\ &= Z(g - ah) \cap U\end{aligned}$$

which is closed.

The "moreover" statement follows by consider the regular function $f - g$. The set of points where $f - g = 0$ is closed and dense, and hence is equal to Y. ☐

1.3.2. Regular functions vs. affine coordinate rings. To prove that our two definitions of regular functions coincide for affine varieties, we need the following definitions.

[Har77, p. 16]
We switched Y, P in the subscript of $\mathcal{O}_{P,Y}$ to match notation used later for sheaves.

Definition 1.3.9 (The local ring at a point). Let Y be an affine or quasi-affine variety. We denote by $\mathcal{O}(Y)$ the ring of all regular functions on Y. If P is a point on Y, the *local ring of* Y *at* P is the ring

$$\mathcal{O}_{Y,P} \coloneqq \varinjlim_{U \ni P} \mathcal{O}(U)$$

(sometimes denoted $\mathcal{O}_P$) of germs of regular functions on Y at P, where the direct limit ranges over all open neighborhoods U of P. In other words, an element of $\mathcal{O}_{Y,P}$ is a pair $\langle U, f\rangle$ where U is an open subset of Y containing P and f is a regular function on U, and where we identify two such pairs $\langle U, f\rangle$ and $\langle V, g\rangle$ if $f = g$ on $U \cap V$. This forms an equivalence relation using the "moreover" statement in Lemma 1.3.8.

The ring $\mathcal{O}_{Y,P}$ is a local ring with maximal ideal $\mathfrak{m}_P$ consisting of all germs of regular functions that vanish at P. This is because if $f(P) \neq 0$, then $1/f$ is regular in a neighborhood of P. The evaluation map $\mathcal{O}_{Y,P} \to k$ has kernel $\mathfrak{m}_P$ and is surjective. Thus, the residue field $\mathcal{O}_{Y,P}/\mathfrak{m}_P$ is isomorphic to k.

Definition 1.3.10 (The function field or the field of rational function). Let Y be an affine or quasi-affine variety. The *function field* or the *field of rational functions* of Y is the field [Har77, p. 16]

$$K(Y) \coloneqq \varinjlim_{U} \mathcal{O}(U).$$

In other words, an element of $K(Y)$ is an equivalence class of pairs $\langle U, f\rangle$ where U is a nonempty open subset of Y, f is a regular function on U, and where we identify two pairs $\langle U, f\rangle$ and $\langle V, g\rangle$ if $f = g$ on $U \cap V$. The elements of $K(Y)$ are called *rational functions* on Y.

Note that $K(Y)$ is a field: Since Y is irreducible, any two nonempty open sets have nonempty intersection. We can therefore define addition and multiplication in $K(Y)$. If $\langle U, f\rangle \in K(Y)$ with $f \neq 0$, then we can restrict f to the open set

$$V = U - \bigl(U \cap Z(f)\bigr),$$

in which case $1/f$ is regular on V, and $\langle V, 1/f\rangle$ is an inverse for $\langle U, f\rangle$.

We have natural injective maps 9/15

$$\mathcal{O}(Y) \hookrightarrow \mathcal{O}_{Y,P} \hookrightarrow K(Y) \tag{1.3.11}$$

where the injectivity holds by the "moreover" statement in Lemma 1.3.8.

We are now ready to prove that the two definitions of regular functions coincide for affine varieties. We will in fact show more about how $\mathcal{O}(Y)$, $\mathcal{O}_{Y,P}$, and $K(Y)$ relate to different geometric properties of Y.

Theorem 1.3.12. *Let $Y \subseteq \mathbf{A}^n_k$ be an affine variety with affine coordinate ring $A(Y)$. We have the following:* [Har77, Thm. I.3.2]

(a) *The two definitions of regular functions on Y coincide: The "evaluation map"*

$$\alpha\colon A(Y) \xrightarrow{\sim} \mathcal{O}(Y)$$

is an isomorphism, where $\mathcal{O}(Y)$ is defined as for quasi-affine varieties.

(*b*) *For each point $P \in Y$, let $\mathfrak{m}_P \subseteq A(Y)$ be the ideal of functions vanishing at P. We have a 1-1 correspondence*

$$\begin{aligned} Y &\longrightarrow \operatorname{MaxSpec}\big(A(Y)\big) \\ P &\longmapsto \mathfrak{m}_P. \end{aligned}$$

(*c*) *For each $P \in Y$, the isomorphism α induces an isomorphism*

$$A(Y)_{\mathfrak{m}_P} \xrightarrow{\sim} \mathcal{O}_{Y,P}$$

and $\dim(\mathcal{O}_{Y,P}) = \dim(Y)$*.*

(*d*) *The isomorphism α induces an isomorphism*

$$\operatorname{Frac}\big(A(Y)\big) \xrightarrow{\sim} K(Y).$$

Thus, $K(Y)$ is a finitely generated extension field of k of transcendence degree $\dim(Y)$*.*

Proof. We proceed in a sequence of steps. Let $A = k[x_1, x_2, \ldots, x_n]$. We then consider the commutative diagram

$$\begin{array}{ccc} A & \twoheadrightarrow & A(Y) \\ & \searrow & \downarrow {\scriptstyle \exists!\,\alpha} \\ & & \mathcal{O}(Y). \end{array}$$

Since the kernel of $A \to \mathcal{O}(Y)$ is $I(Y)$, we obtain the injective map α by the universal property of quotient rings.

Step 1. Proof of (*b*).

By Corollary 1.1.20, there is a 1-1 correspondence between points of Y (which are minimal algebraic subsets of Y) and maximal ideal of A containing $I(Y)$. By [AM69, Proposition 1.1], these maximal ideals correspond to the maximal ideals of $A(Y)$. Using α to identify elements of $A(Y)$ with regular functions on Y, the maximal ideal corresponding to P is

$$\mathfrak{m}_P := \big\{ f \in A(Y) \bigm| f(P) = 0 \big\}.$$

Step 2. Proof of (*c*).

For every $P \in Y$, the universal property of localization implies there is a unique map β fitting in as the bottom horizontal map in the commutative diagram below:

$$\begin{array}{ccc} A(Y) & \xhookrightarrow{\ \alpha\ } & \mathcal{O}(Y) \\ \downarrow & & \downarrow \\ A(Y)_{\mathfrak{m}_P} & \xrightarrow[\exists!]{\beta} & \mathcal{O}_{Y,P}. \end{array}$$

The map β is injective because it factors as

$$A(Y)_{\mathfrak{m}_P} \xhookrightarrow{\alpha_{\mathfrak{m}_P}} \big(A(Y) - \mathfrak{m}_P\big)^{-1}\mathcal{O}(Y) \longhookrightarrow \mathcal{O}_{Y,P}.$$

The first map is injective since it is obtained from the inclusion $A(Y) \hookrightarrow \mathcal{O}(Y)$ by localizing and localization is exact. The second map is injective since it is just an inclusion map as subrings of $K(Y)$. The map β is surjective by the definition of

$\mathcal{O}_{Y,P}$ and the definition of a regular function. We conclude that β is an isomorphism, and moreover, we have

$$\dim(\mathcal{O}_{Y,P}) = \operatorname{ht}(\mathfrak{m}_P).$$

Since $A(Y)/\mathfrak{m}_P \cong k$, we see by Proposition 1.1.40 and the dimension formula (1.1.42) that $\dim(\mathcal{O}_P) = \dim(Y)$.

STEP 3. Proof of (*d*).

By (*c*), we know that α induces an isomorphism

$$\operatorname{Frac}\bigl(A(Y)\bigr) \xrightarrow{\sim} \operatorname{Frac}\bigl(\mathcal{O}_{Y,P}\bigr)$$

for every $P \in Y$. The right-hand side is equal to $K(Y)$ since every rational function is in some $\mathcal{O}_{Y,P}$, where we use the identification of both rings in $K(Y)$ as in (1.3.11). Next, $A(Y)$ is a finitely generated k-algebra, and hence $K(Y)$ is a finitely generated field extension of k. We can apply Proposition 1.1.40 and the dimension formula (1.1.42) to see that

$$\operatorname{trdeg}_k\bigl(K(Y)\bigr) = \dim(Y).$$

STEP 4. Proof of (*a*).

We first note that

$$\mathcal{O}(Y) \subseteq \bigcap_{P \in Y} \mathcal{O}_{Y,P}$$

as subrings of $K(Y)$. By (*b*) and (*c*), we therefore have

$$A(Y) \subseteq \mathcal{O}(Y) \subseteq \bigcap_{\mathfrak{m}} A(Y)_{\mathfrak{m}}$$

again as subrings of $K(Y)$, where $\mathfrak{m}$ runs over all maximal ideals of $A(Y)$. This inclusion is an equality since if B is an integral domain, then B is equal to the intersection (in $\operatorname{Frac}(B)$) of its localizations at all maximal ideals in B. $\square$

Before moving on, we want to emphasize one aspect of algebraic geometry that is so convenient and allows us to move back and forth between the global picture (i.e., what happens on all of X) and the local picture (i.e., what happens in a neighborhood of points on X). We said in (1.3.11) that the first map in

$$\mathcal{O}(Y) \hookrightarrow \mathcal{O}_{Y,P} \hookrightarrow K(Y)$$

is injective. The reason why is that if $f \in \mathcal{O}(Y)$ maps to $0 \in \mathcal{O}_{Y,P}$, then there is a neighborhood U of P where $f|_U = 0$. Since Y is a variety, it is irreducible(!), and hence Lemma 1.3.8 implies $f = 0$ on Y.

1.3.3. Regular functions on quasi-projective varieties. Morphisms. We now define regular functions on quasi-projective varieties.

DEFINITION 1.3.13. Let $Y \subseteq \mathbf{P}^n_k$ be a quasi-projective variety. A function $f\colon Y \to k$ is *regular at a point* $P \in Y$ if there is an open neighborhood $U \subseteq Y$ of P and homogeneous polynomials [Har77, p. 15]

$$g, h \in S = k[x_0, x_1, \ldots, x_n]$$

of the same degree such that h is nowhere 0 on U and $f = g/h$ on U. (Even though g and h are not functions on $\mathbf{P}^n_k$, their quotient is a well-defined function whenever $h \neq 0$ since they are homogeneous of the same degree.) We say that f is *regular on* Y if it is regular at every point.

We can define local rings and function fields/fields of rational functions of quasi-projective varieties in the same way as we did for affine and quasi-affine varieties (Definitions 1.3.9 and 1.3.10).

[Har77, Rem. I.3.1.1]

Lemma 1.3.14. *A regular function on a quasi-projective variety is continuous when k is identified with $\mathbf{A}^1_k$ with its Zariski topology. Moreover, if f and g are regular functions on a quasi-projective variety Y and $f = g$ on some nonempty open subset $U \subseteq Y$, then $f = g$ everywhere.*

Proof. The proof is similar to Lemma 1.3.8. Let $f\colon Y \to k$ be a regular function. It suffices to show that inverse images of closed sets are closed. Since the closed sets in $\mathbf{A}^1_k$ are finite sets of points, it suffices to show that

$$f^{-1}(a) = \{P \in Y \mid f(P) = a\}$$

is closed for every $a \in k$.

It suffices to show that $f^{-1}(a)$ is closed after restricting to an open cover of Y. Let $U \subseteq Y$ be an open subset on which $f = g/h$ for $g, h \in S$ homogeneous of the same degree and h is nowhere 0 on U. Then,

$$\begin{aligned} f^{-1}(a) \cap U &= \left\{ P \in U \;\middle|\; \frac{g(P)}{h(P)} = a \right\} \\ &= \{P \in U \mid g(P) = ah(P)\} \\ &= Z_+(g - ah) \cap U \end{aligned}$$

which is closed.

The "moreover" statement follows by consider the regular function $f - g$. The set of points where $f - g = 0$ is closed and dense, and hence is equal to Y. ☐

We can now define the category of varieties. We recall the definition of a category from Definition A.1.1:

[AK21, (6.1)]
[Hoc17, p. 8]

Definition 1.3.15. A *category* $\mathscr{C}$ consists of the following data:

(1) A class of *objects*.
(2) For every pair of objects A and B in $\mathscr{C}$, a set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ of *maps* or *morphisms*, such that $\mathrm{Hom}_{\mathscr{C}}(A, B)$ and $\mathrm{Hom}_{\mathscr{C}}(A', B')$ are disjoint unless $A = A'$ and $B = B'$. We write $f\colon A \to B$ to mean that $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$.
(3) For every triple of objects A, B, and C in $\mathscr{C}$, a *composition law*

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(A, B) \times \mathrm{Hom}_{\mathscr{C}}(B, C) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(A, C) \\ (f, g) &\longmapsto g \circ f \end{aligned}$$

satisfying the following axioms:

(*a*) For every object B, there is a distinguished *identity* morphism $\mathrm{id}_B\colon B \to B$ such that for every morphism $f\colon A \to B$, we have $\mathrm{id}_A \circ f = f$, and for every morphism $g\colon B \to C$, we have $g \circ \mathrm{id}_B = g$.
(*b*) Composition is associative: if $f\colon A \to B$, $g\colon B \to C$, and $g\colon C \to D$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

We say that $f\colon A \to B$ is a *isomorphism* with inverse $g\colon B \to A$ if $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

[Har77, p. 15]

Definition 1.3.16. Let k be a fixed algebraically closed field. A *variety over k* or simply a *variety* if k is clear from context, is any affine, quasi-affine, projective, or quasi-projective variety as defined above. We will often say *quasi-projective variety* for clarity because there are more general definitions for varieties in the literature.[1]

A *morphism* $\varphi\colon X \to Y$ is a continuous map such that for every open set $V \subseteq Y$ and every regular function $f\colon V \to k$, the function

$$f \circ \varphi\colon \varphi^{-1}(V) \longrightarrow k$$

is regular.

The composition of two morphisms is a morphism, and hence we obtain a category Var_k. An *isomorphism* $\varphi\colon X \to Y$ of two varieties is a morphism which admits an inverse morphism $\psi\colon Y \to X$ with $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$.

As an example of an isomorphism, we prove that the homeomorphism $\mathbf{A}^n_k \approx U_i \subseteq \mathbf{P}^n_k$ is an isomorphism of varieties. This implies that even though affine and quasi-affine varieties have multiple definitions for regular functions (you can consider them both as quasi-projective varieties), the different definitions coincide.

Proposition 1.3.17. *Let $U_i \subseteq \mathbf{P}^n_k$ be the open set defined by the equation $x_i \neq 0$. Then, the map $\varphi\colon U_i \to \mathbf{A}^n_k$ from Proposition 1.2.12 is an isomorphism.* [Har77, Prop. I.3.3]

Proof. The map φ is a homeomorphism by Proposition 1.2.12. It therefore suffices to show that the regular functions on open sets of U_i and $\mathbf{A}^n_k$ coincide. This follows since for a regular function f on an open subset V, the two maps

$$\begin{aligned}
\varphi^*\colon \mathcal{O}(V) &\longrightarrow \mathcal{O}\big(\varphi^{-1}(V)\big)\\
\frac{g(\ldots,x_{i-1},x_{i+1},\ldots)}{h(\ldots,x_{i-1},x_{i+1},\ldots)} &\longmapsto \frac{x_i^{\max\{\deg(g),\deg(h)\}}g(\ldots,x_{i-1}/x_i,x_{i+1}/x_i,\ldots)}{x_i^{\max\{\deg(g),\deg(h)\}}h(\ldots,x_{i-1}/x_i,x_{i+1}/x_i,\ldots)}\\
\frac{g(\ldots,x_{i-1},1,x_{i+1},\ldots)}{h(\ldots,x_{i-1},1,x_{i+1},\ldots)} &\longleftarrow\!\shortmid \frac{g(x_0,\ldots,x_n)}{h(x_0,\ldots,x_n)}
\end{aligned}$$

defined using α, β from Proposition 1.2.12 are inverses of each other. Here, the assignments in either direction are made for every *locally* defined representation $f = g/h$ as a rational function with h nowhere 0. You can check that the map φ^* is well-defined, and compose to the identity in either direction. □

We also give a local criterion for being an isomorphism.

Proposition 1.3.18. *Let $\varphi\colon X \to Y$ be a morphism of quasi-projective varieties.* [Har77, Exer. I.3.3]

(*a*) *For every $P \in X$, the morphism φ induces a homomorphism of local rings*

$$\varphi_P^*\colon \mathcal{O}_{Y,\varphi(P)} \longrightarrow \mathcal{O}_{X,P}.$$

(*b*) *The morphism φ is an isomorphism if and only if φ is a homeomorphism and the induced map φ_P^* on local rings is an isomorphism for all $P \in X$.*

Proof. (*a*). Recall that a homomorphism of local rings $\phi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ is a ring homomorphism such that $\phi(\mathfrak{m}) \subseteq \mathfrak{n}$. We claim the assignment

$$\begin{aligned}
\varphi_P^*\colon \mathcal{O}_{Y,\varphi(P)} &\longrightarrow \mathcal{O}_{X,P}\\
\langle V, g\rangle &\longmapsto \big\langle \varphi^{-1}(V), g\circ\varphi\big\rangle
\end{aligned}$$

[1]See [Wei62, p. 68; FAC, p. 226], although nowadays we often say "variety" to mean a special type of scheme (see [Har77, p. 105]).

is a well defined local map of local k-algebras. Suppose $\langle V', g'\rangle$ is another representative for $\langle V, g\rangle$. Then, we have

$$\begin{aligned}\big\langle \varphi^{-1}(V'), g'\circ\varphi\big\rangle &= \big\langle \varphi^{-1}(V\cap V'), g'\circ\varphi\big\rangle\\ &= \big\langle \varphi^{-1}(V\cap V'), g\circ\varphi\big\rangle\\ &= \big\langle \varphi^{-1}(V), g\circ\varphi\big\rangle\end{aligned}$$

by the fact that $g|_{V\cap V'} = g'|_{V\cap V'}$. The map φ_P^* is a map of k-algebras since

$$\begin{aligned}&\varphi_P^*\big(\langle V, f\rangle\cdot\langle V, g\rangle + \langle V, h\rangle\big)\\ &\quad= \varphi_P^*\big(\langle V, fg+h\rangle\big)\\ &\quad= \big\langle \varphi^{-1}(V), (fg+h)\circ\varphi\big\rangle\\ &\quad= \big\langle \varphi^{-1}(V), f\circ\varphi\big\rangle\cdot\big\langle \varphi^{-1}(V), g\circ\varphi\big\rangle + \big\langle \varphi^{-1}(V), h\circ\varphi\big\rangle\\ &\quad= \varphi_P^*\big(\langle V, f\rangle\big)\cdot\varphi_P^*\big(\langle V, g\rangle\big) + \varphi_P^*\big(\langle V, h\rangle\big).\end{aligned}$$

The map φ_P^* is local since germs of functions on Y that vanish at $\varphi(P)$ pull back to germs of functions on X that vanish at P, and hence

$$\varphi_P^*\big(\mathfrak{m}_{\varphi(P)}\big)\subseteq\mathfrak{m}_P.$$

9/17 (b). $\Rightarrow$. Since $\varphi\colon X\to Y$ is an isomorphism, there is an inverse morphism $\psi\colon Y\to X$ by definition. Since φ and ψ are continuous and are inverses of each other, we see that φ is a homeomorphism. To show that φ_P^* is an isomorphism, consider the maps ψ_Q^* for $Q\in Y$ defined as in (a). Then, we have

$$\begin{aligned}\big(\psi_{\varphi(P)}^*\circ\varphi_P^*\big)\big(\langle V, g\rangle\big) &= \psi_{\varphi(P)}^*\big(\big\langle\varphi^{-1}(V), g\circ\varphi\big\rangle\big)\\ &= \big\langle\psi^{-1}\big(\varphi^{-1}(V)\big), g\circ\varphi\circ\psi\big\rangle\\ &= \langle V, g\rangle\\ \big(\varphi_{\psi(Q)}^*\circ\psi_Q^*\big)\big(\langle U, f\rangle\big) &= \varphi_{\psi(Q)}^*\big(\big\langle\psi^{-1}(U), f\circ\psi\big\rangle\big)\\ &= \big\langle\varphi^{-1}\big(\psi^{-1}(U)\big), f\circ\psi\circ\varphi\big\rangle\\ &= \langle U, f\rangle\end{aligned}$$

and hence φ_P^* is an isomorphism.

$\Leftarrow$. We first claim that

$$\mathcal{O}(X) = \bigcap_{P\in X}\mathcal{O}_{X,P} \tag{1.3.19}$$

as subrings of $K(X)$. The inclusion $\subseteq$ holds by the injectivity of $\mathcal{O}(X)\hookrightarrow\mathcal{O}_{X,P}$ in Lemma 1.3.8. For the inclusion $\supseteq$, it suffices to note that an element in the right-hand side maps to a rational function on X that is regular at every point $P\in X$, which is a regular function on X.

We now return to proving $\Leftarrow$. For every open subset $V \subseteq Y$, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{O}(V) & = & \displaystyle\bigcap_{P \in \varphi^{-1}(V)} \mathcal{O}_{Y,\varphi(P)} \\ \Big\downarrow{\scriptstyle \varphi^*} & & \Big\downarrow{\scriptstyle \wr\, \bigcap_{P \in \varphi^{-1}(P)} \varphi_P^*} \\ \mathcal{O}\big(\varphi^{-1}(V)\big) & = & \displaystyle\bigcap_{P \in \varphi^{-1}(V)} \mathcal{O}_{X,P} \end{array}$$

where the right vertical map is an isomorphism by (1.3.19). □

1.3.4. Examples of morphisms. How can we write down morphisms? Proposition 1.3.17 shows us that sometimes we can just check the definition: We can upgrade a continuous map to be a morphism by checking the condition on regular functions. For morphisms $X \to Y$ where Y is affine, we have the following convenient result, which says that you can construct morphisms coordinate-wise.

Lemma 1.3.20. *Let X be a quasi-projective variety and let $Y \subseteq \mathbf{A}_k^n$ be an affine variety. A map of sets $\psi\colon X \to Y$ is a morphism if and only if $x_i \circ \psi$ is a regular function on X for every i, where $x_1, x_2, \ldots, x_n$ are the coordinate functions on $\mathbf{A}_k^n$.* [Har77, Lem. I.3.6]

Proof. Note that $x_i\colon \mathbf{A}_k^n \to k$ is a regular function for every i. This is where the affineness of Y is used.

$\Rightarrow$. If ψ is a morphism, then $x_i \circ \psi$ is a regular function on X by definition.

$\Leftarrow$. Suppose $x_i \circ \psi$ is regular. Then, for every polynomial $f = f(x_1, x_2, \ldots, x_n)$, the pullback $f \circ \psi$ is also regular on X. Since regular functions are continuous, we see that ψ^{-1} takes closed sets to closed sets, and hence ψ is continuous. It remains to check the condition on regular functions. Regular functions on open subsets of Y are locally quotients of polynomials. Thus, $g \circ \psi$ is regular for any regular function g on any open subset of Y by pulling back the numerator and denominator of a quotient of polynomials separately. We conclude that ψ is a morphism. □

We can now write down some interesting examples. An isomorphism of varieties is bijective and bicontinuous. Because of Proposition 1.3.17, you may be tempted to think that all bijective bicontinuous morphisms are isomorphisms, that is, that you can always "upgrade" homeomorphisms to actual isomorphisms of varieties. This is not always the case, as you will prove on Homework 4.

Example 1.3.21. The morphism [Har77, Exer. I.3.2]

$$\begin{aligned} \mathbf{A}_k^1 &\longrightarrow Z(x^3 - y^2) \subseteq \mathbf{A}_k^2 \\ t &\longmapsto (t^2, t^3) \end{aligned}$$

is a bijective bicontinuous morphism but is not an isomorphism. If $\operatorname{char}(k) = p > 0$, the *Frobenius morphism* is the map

$$\begin{aligned} \mathbf{A}_k^1 &\longrightarrow \mathbf{A}_k^1 \\ t &\longmapsto t^p \end{aligned}$$

is a bijective bicontinuous morphism but is not an isomorphism.

1.3.5. Mapping to affine vs. quasi-projective varieties. In order to write down non-affine examples, we prove the following powerful result about mapping to affine varieties. This result more or less says that the functor $X \mapsto \mathcal{O}(X)$ and $A(Y) \mapsto \operatorname{MaxSpec}(A(Y))$ are adjoint to each other. (I am not stating it like this since talking about adjoints of contravariant functors gets confusing!)

[Har77, Prop. I.3.5] PROPOSITION 1.3.22. *Let X be a quasi-projective variety and let Y be an affine variety. Then, the pull back map on regular functions induces a natural bijective mapping of sets*

$$\alpha\colon \operatorname{Hom}(X,Y) \xrightarrow{\sim} \operatorname{Hom}\bigl(A(Y),\mathcal{O}(X)\bigr)$$

where the left Hom *means morphisms of quasi-projective varieties and the right* Hom *means k-algebra homomorphisms.*

Proof. We first show that α is natural. Consider morphisms

$$W \xrightarrow{\varphi'} X \qquad \text{and} \qquad Y \xrightarrow{\varphi''} Z$$

where Z is affine. Then, the diagram

$$\begin{array}{ccc}
\operatorname{Hom}(W,Y) & \xrightarrow{\alpha_{WY}} & \operatorname{Hom}\bigl(A(Y),\mathcal{O}(W)\bigr) \\
{\scriptstyle -\circ\varphi'}\uparrow & & \uparrow{\scriptstyle -\circ\varphi'} \\
\operatorname{Hom}(X,Y) & \xrightarrow{\alpha_{XY}} & \operatorname{Hom}\bigl(A(Y),\mathcal{O}(X)\bigr) \\
{\scriptstyle \varphi''\circ -}\downarrow & & \downarrow{\scriptstyle \varphi''\circ -} \\
\operatorname{Hom}(X,Z) & \xrightarrow{\alpha_{XZ}} & \operatorname{Hom}\bigl(A(Z),\mathcal{O}(X)\bigr)
\end{array}$$

commutes by definition of a morphism as composition of functions. Here, we use Theorem 1.3.12 to make the identification $A(-) \cong \mathcal{O}(-)$ for affine varieties. Thus, the morphism α is natural in both X and Y.

It remains to show that α is a bijection. We define a map

$$\beta\colon \operatorname{Hom}\bigl(A(Y),\mathcal{O}(X)\bigr) \longrightarrow \operatorname{Hom}(X,Y)$$

as follows. Let $h\colon A(Y) \to \mathcal{O}(X)$ be a k-algebra homomorphism. Write $A(Y) = k[x_1,\ldots,x_n]/I(Y)$, and let $\bar{x}_i$ be the image of x_i in $A(Y)$. Consider the elements $\xi_i = h(\bar{x}_i) \in \mathcal{O}(X)$. These are global regular functions on X, and therefore define a mapping

$$\begin{aligned}
\psi\colon X &\longrightarrow \mathbf{A}^n_k \\
P &\longmapsto \bigl(\xi_1(P),\ldots,\xi_n(P)\bigr)
\end{aligned}$$

which is a morphism by Lemma 1.3.20. We show that the image of ψ is contained in Y. It suffices to show that for every $P \in X$ and $f \in I(Y)$, we have $f(\psi(P)) = 0$. But

$$f\bigl(\psi(P)\bigr) = f\bigl(\xi_1(P),\ldots,\xi_n(P)\bigr) = h\bigl(f(\bar{x}_1,\ldots,\bar{x}_n)\bigr)(P) = 0$$

since $f \in I(Y)$. Thus, ψ defines a map $X \to Y$.

We note that $\alpha \circ \beta = \mathrm{id}$ by the construction in the previous paragraph. Conversely, we have $\beta \circ \alpha = \mathrm{id}$ since the morphism $X \to Y$ is determined by each of its coordinates. See the proof of Lemma 1.3.20. ☐

Here are two important consequences of Proposition 1.3.22.

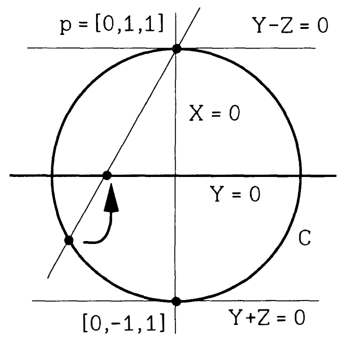

FIGURE 1.13. The projectivized stereographic projection. From [Har92, p. 21].

[Har77, Cor. I.3.7] COROLLARY 1.3.23. *Let X and Y be two affine varieties. Then, $X \cong Y$ if and only if $A(X) \cong A(Y)$.*

COROLLARY 1.3.24. *The functor $X \mapsto A(X)$ is an anti-equivalence of categories between the category of affine varieties over k and the category of finitely generated integral domains over k.* [Har77, Cor. I.3.8]

Proposition 1.3.22 tells us that for quasi-projective varieties Y, we can try to write down a morphism $\varphi\colon X \to Y$ by giving an $(n+1)$-tuple of homogeneous polynomials of the same degree. This procedure will define a morphism as long as they are not simultaneously 0 anywhere. This is because we can cover the codomain by affine varieties and check the condition for being a morphism locally. For example, the d-uple embedding [9/22] [Har92, p. 21]

$$\begin{aligned} \rho_d\colon \mathbf{P}^n_k &\longrightarrow \mathbf{P}^N_k \\ [a_0 : \cdots : a_n] &\longmapsto \bigl[\cdots : M_I(a_0,\ldots,a_n) : \cdots\bigr] \end{aligned}$$

is a morphism. (In fact, it is an isomorphism onto its image! This is on Homework 4.) Here, I ranges over multi-indices in $\mathbf{N}^{n+1}$ such that $|I| = d$, ordered lexicographically. However, not all morphisms can be described globally like this.

EXAMPLE 1.3.25 (Stereographic projection in $\mathbf{P}^2_k$). We revisit the stereographic projection of a circle (Example 1.3.6). Let [Har92, pp. 21–22]

$$X = Z_+(x^2 + y^2 - z^2)$$

be the (projective closure of the) circle. We claim that

$$\begin{aligned} \varphi_1\colon X - \bigl\{[0:1:1]\bigr\} &\longrightarrow \mathbf{P}^1_k \\ [x:y:z] &\longmapsto [x : z-y] \end{aligned}$$

extends to a morphism $X \to \mathbf{P}^1_k$. Consider the morphism

$$\begin{aligned} \varphi_2\colon X - \bigl\{[0:-1:1]\bigr\} &\longrightarrow \mathbf{P}^1_k \\ [x:y:z] &\longmapsto [y+z : x]. \end{aligned}$$

We see that the two morphisms glue to give a morphism on all of X. See Figure 1.13 for an illustration. This holds since on the open set where $x \neq 0$, $z - y \neq 0$, and $y + z \neq 0$, we have

$$\begin{aligned}[x : z - y] &= \bigl[x(y+z) : (z-y)(y+z)\bigr] \\ &= \bigl[x(y+z) : z^2 - y^2\bigr] \\ &= \bigl[x(y+z) : x^2\bigr] \\ &= [y+z : x].\end{aligned}$$

However, we cannot have a global description of φ in projective coordinates since

$$\begin{aligned}[g : h] = [x : z-y] &\implies (z-y)g = xh &&\implies x \mid g, \\ [g : h] = [y+z : x] &\implies xg = (y+z)h &&\implies x \mid h.\end{aligned}$$

1.3.6. Regular functions, local rings, and function fields of projective varieties. Our next goal is to prove an analogue of Theorem 1.3.12 for projective varieties. Our experience from complex analysis tells us that $\mathcal{O}(Y) = k$ would be reasonable.

[Kol87, Prop. 5.11]

Example 1.3.26. Let $X = \mathbf{P}^n_{\mathbf{C}}$. If we consider X in the Euclidean topology, then X is a compact complex manifold. Let $f \colon X \to \mathbf{C}$ be a regular function. Then, f is a holomorphic function on X. Since X is compact, the absolute value $|f|$ achieves a maximum at a point $P \in X$ by the extreme value theorem. This implies that f is constant by applying the maximum modulus principle (see, e.g., [SS03, Theorem 4.5] when $n = 1$ and [Hör90, p. 27] when $n \geq 2$) on a neighborhood of P.

To prove that this guess is correct, we need to replace analysis somehow. This is one place where we must use commutative algebra in an involved way to replace what feels like a simple analytic argument.

Before we begin, we need some more notation from the theory of graded rings.

[Har77, p. 18] [EGAII, (2.2.1), (2.2.7)]

Definition 1.3.27. Let S be a graded ring. Let $\mathfrak{p} \subseteq S$ be a homogeneous prime ideal. We set

$$S_{(\mathfrak{p})} \coloneqq \bigl(T^{-1}S\bigr)_0$$

where T is the set of homogeneous elements of S not in $\mathfrak{p}$. Note that $T^{-1}S$ has a natural $\mathbf{Z}$-grading where

$$\deg\left(\frac{f}{g}\right) = \deg(f) - \deg(g)$$

when f is a homogeneous element in S and $g \in T$. The ring $S_{(\mathfrak{p})}$ is a local ring with maximal ideal

$$\bigl(\mathfrak{p} \cdot T^{-1}S\bigr) \cap S_{(\mathfrak{p})}.$$

For example, if S is a domain, then setting $\mathfrak{p} = (0)$ yields a field $S_{((0))}$.

If $f \in S$ is a homogeneous element, we denote by $S_{(f)}$ the subring of elements of degree 0 in the localization S_f.

Remark 1.3.28. Please be careful with this notation. For example, when

$$S = k[x_0, x_1, \ldots, x_n],$$

the rings $S_{(x_i)}$ and $S_{((x_i))}$ are very different!

We now state our result about projective varieties.

[Har77, Thm. I.3.4] THEOREM 1.3.29. *Let $Y \subseteq \mathbf{P}^n_k$ be a projective variety with homogeneous coordinate ring $S(Y)$. We have the following:*

(a) $\mathcal{O}(Y) = k$.
(b) *Let $P \in Y$ be a point and let $\mathfrak{m}_P \subseteq S(Y)$ be the ideal generated by homogeneous $f \in S(Y)$ such that $f(P) = 0$. Then, $\mathcal{O}_{Y,P} \cong S(Y)_{(\mathfrak{m}_P)}$.*
(c) $K(Y) \cong S(Y)_{((0))}$.

The proof of (a) is the hardest and uses the notion of integral extensions from commutative algebra.

Proof. For every i, set
$$U_i \coloneqq \{x_i \neq 0\} \subseteq \mathbf{P}^n_k$$
and $Y_i \coloneqq Y \cap U_i$. Under the isomorphism $U_i \xrightarrow{\sim} \mathbf{A}^n_k$ from Proposition 1.3.17, we know that each Y_i is isomorphic to an affine variety. Recall that we have an isomorphism

$$\begin{aligned} \psi_i\colon S(Y)_{x_i} &\xrightarrow{\ \sim\ } A(Y_i)[x_i, x_i^{-1}] \\ x_j &\longmapsto \begin{cases} x_i y_j & \text{if } j \neq i, \\ x_i & \text{if } j = i \end{cases} \end{aligned} \tag{1.3.30}$$

from the proof of [Har77, Exercise I.2.6]. The map $(\psi_i)_0$ is inverse to the pullback map on regular functions under the isomorphism $Y \cap U_i \xrightarrow{\sim} Y_i$.

(b). Let $P \in Y$ be a point and choose i such that $P \in Y_i$. By Theorem 1.3.12(c), we have
$$\mathcal{O}_{Y_i,P} \xleftarrow{\sim} A(Y_i)_{\mathfrak{m}'_P}$$
where $\mathfrak{m}'_P$ is the maximal ideal of $A(Y_i)$ corresponding to P. We see that
$$(\psi_i)_0^{-1}\bigl(\mathfrak{m}'_P\bigr) = \mathfrak{m}_P \cdot S(Y)_{(x_i)}$$
by evaluating functions at P. We then see that
$$\mathcal{O}_{Y_i,P} \xleftarrow{\sim} A(Y_i)_{\mathfrak{m}'_P} \xleftarrow[\sim]{(\psi_i)_0} \Bigl(S(Y)_{(x_i)}\Bigr)_{\mathfrak{m}_P} = S(Y)_{(\mathfrak{m}_P)}$$
since localization is transitive and $x_i \notin \mathfrak{m}_P$.

(c). Choose i such that $Y_i \neq \emptyset$. We have
$$K(Y) = K(Y_i) = \operatorname{Frac}\bigl(A(Y_i)\bigr)$$
by Theorem 1.3.12(d). We see that
$$(\psi_i)_0^{-1}(0) = 0 \cdot S(Y)_{(x_i)}.$$
Thus, we have
$$K(Y_i) \cong A(Y_i)_{(0)} \xleftarrow[\sim]{(\psi_i)_0} \Bigl(S(Y)_{(x_i)}\Bigr)_0 = S(Y)_{((0))}.$$

(a). Let $f \in \mathcal{O}(Y)$ be a global regular function. Since Y is a variety, we can think of $\mathcal{O}(Y)$, $K(Y)$, and $S(Y)$ as subrings of $L \coloneqq \operatorname{Frac}(S(Y))$. For each i, we see that f is regular on Y_i, and hence
$$f \in A(Y_i) \cong S(Y)_{(x_i)}$$
by Theorem 1.3.12(a) and the isomorphism (1.3.30). The goal is to show that $f \in L$ is integral over k. Since k is algebraically closed, this would show that $f \in k$.

Write

$$f = \frac{g_i}{x_i^{N_i}} \in S(Y)_{(x_i)}$$

where $g_i \in S(Y)_{N_i}$. We then have $x_i^{N_i} f \in S(Y)_{N_i}$ for each i. Choose $N \geq \sum_i N_i$. By the pigeonhole principle, each monomial spanning $S(Y)_N$ as a k-vector space is divisible by $x_i^{N_i}$ for some i. We therefore have the inclusion

$$S(Y)_N \cdot f \subseteq S(Y)_N.$$

Iterating this inclusion, we have

$$S(Y)_N \cdot f^q \subseteq S(Y)_N$$

for every $q > 0$. In particular, fixing i such that $Y_i \neq \emptyset$, we see that

$$S(Y)[f] \subseteq x_i^{-N} \cdot S(Y)$$

as subrings of L. Since $S(Y)$ is Noetherian, we therefore see that $S(Y)[f]$ is a finitely generated $S(Y)$-module, and hence f is integral over $S(Y)$ by [AK21, Proposition 10.14]. This means there are elements $a_i \in S(Y)$ and an equation

$$f^m + a_1 f^{m-1} + \cdots + a_m = 0.$$

Since f has degree 0, we can take the 0-th degree homogeneous component to yield an algebraic equation with coefficients in $S(Y)_0 = k$ that is satisfied by f. Since k is algebraically closed, we conclude that $f \in k$. □

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This result is surprisingly powerful! We give two applications here.

[Har77, Exer. I.3.1(e)]

EXAMPLE 1.3.31. Let X be a projective variety that is isomorphic to an affine variety. We claim that X is one point. Let $\pi\colon X \to \{*\}$ be the unique map of sets. This is a morphism since it is continuous and the only regular functions on $\{*\}$ are constant functions, which pull back to constant regular functions on X.

Under the bijection in Proposition 1.3.22, the morphism π corresponds to the isomorphism

$$k \xrightarrow{\sim} \mathcal{O}(X) = k$$

using Theorem 1.3.29(a). Since X is affine, this shows that $X = \{*\}$.

More generally, every projective variety X that is *quasi-affine* is in fact a point. First, we have the morphisms

$$X \xrightarrow{\sim} X' \xhookrightarrow{\text{open}} \bar{X}' \xhookrightarrow{\text{closed}} \mathbf{A}_k^n$$

from the hypothesis that X is quasi-affine. We then have a factorization

$$k \hookrightarrow A(\bar{X}') \hookrightarrow \mathcal{O}(X') \xrightarrow{\sim} \mathcal{O}(X) = k$$

of the identity on k, where $A(\bar{X}') \hookrightarrow \mathcal{O}(X')$ is injective by the fact that both are naturally sub-k-algebras in $K(X')$. We therefore see that $k \hookrightarrow A(\bar{X}')$ must be bijective, and hence $A(\bar{X}') = k$. Thus, $\bar{X}'$ is a point.

[Har77, Exer. I.3.7]

EXAMPLE 1.3.32. Let $Y \subseteq \mathbf{P}_k^n$ be a projective variety of dimension ≥ 1 and let $H \subseteq \mathbf{P}_k^n$ be a hypersurface. We claim that $Y \cap H \neq \emptyset$. In [Har77, Exercise I.3.5] (Homework 4, Problem 4(b)), you will show that $\mathbf{P}_k^n - H$ is affine. If $Y \cap H = \emptyset$, then

$$Y = (\mathbf{P}_k^n - H) \cap Y$$

is both affine and projective, and hence is a point by Example 1.3.31. This contradicts the assumption that $\dim(Y) \geq 1$. As a special case, this shows that any two plane curves in $\mathbf{P}^2_k$ intersect.

1.3.7. Subvarieties. To end this section, we discuss three important technical aspects of varieties. We start with the notion of a subvariety.

Definition 1.3.33. A subset of a topological space is *locally closed* if it is an open subset of its closure, or equivalently, if it is the intersection of an open set and a closed set. [Har77, Exer. I.3.10]

If X is a quasi-affine (resp. quasi-projective) variety and Y is an irreducible locally closed subset of X, then Y is also a quasi-affine (resp. quasi-projective) variety in the same affine (resp. projective) space. We call this the *induced structure* on Y and we call Y a *subvariety* of X.

Now let Y be a subvariety of a quasi-projective variety X. The ring [Har77, Exer. I.3.13]

$$\mathcal{O}_{X,Y} \coloneqq \varinjlim_{\substack{U \subseteq X \\ U \cap Y \neq \emptyset}} \mathcal{O}(U)$$

is the *local ring of Y on X*. The elements of $\mathcal{O}_{X,Y}$ are germs $\langle U, f\rangle$ of regular functions on open subsets of X intersecting Y subject to the equivalence relation

$$\langle U, f\rangle = \langle V, g\rangle \iff f|_{U\cap V} = g|_{U\cap V}.$$

Proposition 1.3.34. *Let X be a quasi-projective variety.*

(*i*) *Let $\varphi\colon X \to Y$ be a morphism of quasi-projective varieties. Let $X' \subseteq X$ and $Y' \subseteq Y$ be subvarieties such that $\varphi(X') \subseteq Y'$. Then, $\varphi|_X\colon X' \to Y'$ is a morphism.* [Har77, Exer. I.3.10]

(*ii*) *Let $P \in X$ be a point. There is a 1-1 correspondence* [Har77, Exer. I.3.11]

$$\operatorname{Spec}(\mathcal{O}_{X,P}) \overset{\text{1-1}}{\longleftrightarrow} \left\{\begin{gathered}\textit{closed subvarieties of } X\\ \textit{containing } P\end{gathered}\right\}.$$

(*iii*) $\dim(\mathcal{O}_{X,P}) = \dim(X)$. [Har77, Exer. I.3.12]

(*iv*) *Let $Y \subseteq X$ be a subvariety. Then, $\mathcal{O}_{X,Y}$ is a local ring with residue field $K(Y)$ of dimension* $\dim(X) - \dim(Y)$. [Har77, Exer. I.3.13]

Proof Sketch. (*i*) follows since continuous maps restrict and since a regular function expressed as a rational function g/h on an open subset $V' \subseteq Y'$ can be lifted to some open subset $V \subseteq Y$, which pulls back correctly under φ. For (*ii*), you can reduce to the affine case by choosing some U_i containing P, in which case this reduces to Corollary 1.1.20. A similar strategy works to reduce (*iii*) to Theorem 1.3.12(*c*).

For (*iv*), the maximal ideal of $\mathcal{O}_{X,Y}$ is

$$\mathfrak{m}_{X,Y} \coloneqq \left\{\langle U, f\rangle \;\middle|\; f|_{U\cap Y} = 0\right\}$$

since the complement consists of units: If $\langle U, f\rangle$ is a germ such that $f \not\equiv 0$, then $\langle U - Z(f), 1/f\rangle$ is an inverse of $\langle U, f\rangle$. The fact that $\mathcal{O}_{X,Y}/\mathfrak{m}_{X,Y} \cong K(Y)$ follows from this description. One can calculate the dimension of $\mathcal{O}_{X,Y}$ by reducing to the affine case, showing that $A(X)_{I(Y)} \cong \mathcal{O}_{X,Y}$, and then using the dimension formula

(1.1.42) to show

$$\begin{aligned}\dim(\mathcal{O}_{X,Y}) &= \operatorname{ht}_{\mathcal{O}_{X,Y}}(\mathfrak{m}_{X,Y})\\ &= \operatorname{ht}_{A(X)}\bigl(I(Y)\bigr)\\ &= \dim\bigl(A(X)\bigr) - \dim\bigl(A(Y)\bigr)\\ &= \dim(X) - \dim(Y).\end{aligned}$$

□

[Har77, Exer. I.3.5, p. 25]

1.3.8. Affine open subsets form a basis for the Zariski topology. Recall that we say that a quasi-projective variety *is affine* if it is isomorphic to an affine variety. We saw before that every projective variety can be covered by affine open subsets obtained by taking complements of hyperplanes. We can do better: On any quasi-projective variety, the affine open subsets form a *basis* for the Zariski topology.

[Har77, Lem. I.4.2]

LEMMA 1.3.35. *Let $Y = Z(f) \subseteq \mathbf{A}^n_k$ be a hypersurface. Then, $\mathbf{A}^n_k - Y$ is isomorphic to*

$$H \coloneqq Z(x_{n+1}f - 1) \subseteq \mathbf{A}^{n+1}_k.$$

In particular, $\mathbf{A}^n_k - Y$ is affine with coordinate ring $k[x_1, x_2, \ldots, x_n]_f$.

Proof. The morphism

$$\begin{aligned} H &\longrightarrow \mathbf{A}^n_k - Y\\ (a_1,\ldots,a_n,a_{n+1}) &\longmapsto (a_1,\ldots,a_n)\end{aligned}$$

has inverse

$$\begin{aligned} \mathbf{A}^n_k - Y &\longrightarrow H\\ (a_1,\ldots,a_n) &\longmapsto \left(a_1,\ldots,a_n,\frac{1}{f(a_1,\ldots,a_n)}\right)\end{aligned}$$

and is therefore an isomorphism. □

[Har77, Prop. I.4.3]

PROPOSITION 1.3.36. *On any quasi-projective variety Y, there is a basis for the Zariski topology on Y consisting of affine open subsets.*

Proof. We want to show that for every point $P \in Y$ and every open neighborhood $U \ni P$, there exists an affine open neighborhood of P in U. Replacing Y by U, which is also a quasi-projective variety, we may assume that $U = Y$. Moreover, since any quasi-projective variety is covered by quasi-affine varieties by Corollary 1.2.13, we may assume that Y is quasi-affine in $\mathbf{A}^n_k$.

Let $Z = \bar{Y} - Y$, which is closed in $\mathbf{A}^n_k$. Since Z is closed in $\mathbf{A}^n_k$ and $P \notin Z$, we can find $f \in I(Z)$ such that $f(P) \neq 0$ (if $Z = \emptyset$, then $f = 1$ works). Set $H = Z(f)$. Then, $P \in Y - (Y \cap H)$, which is an open subset of Y. Finally, $Y - (Y \cap H)$ is a closed subset of $\mathbf{A}^n_k - H$, since $Z \subseteq H$ and hence

$$Y - (Y \cap H) = \bar{Y} - (Y \cap H).$$

We therefore see that $Y - (Y \cap H)$ is affine by Lemma 1.3.35. Thus, $Y - (Y \cap H)$ is the required affine open neighborhood of P. □

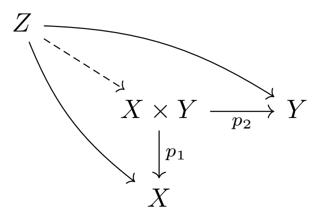
1.3.9. Products. We end with products. This give the first example where universal properties are used to construct morphisms.

[Har77, Exer. I.3.16, Thm. II.3.3] THEOREM 1.3.37. *Let $X \subseteq \mathbf{P}^n_k$ and $Y \subseteq \mathbf{P}^m_k$ be quasi-projective varieties.*

(*i*) *The product*
$$X \times Y \subseteq \mathbf{P}^n_k \times \mathbf{P}^m_k$$
is a quasi-projective variety after identifying $\mathbf{P}^n_k \times \mathbf{P}^m_k$ with its image under the Segre embedding (Example 1.2.24).

(*ii*) *The projection maps $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$ are morphisms of quasi-projective varieties.*

(*iii*) *The product variety $X \times Y$ together with the projection morphisms $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$ is a product of X and Y in the category Var_k. That is, $X \times Y$ and the two projections satisfy the following universal property: For every quasi-projective variety Z with morphisms $Z \to X$ and $Z \to Y$ making the solid diagram*



commute, there is a unique dashed morphism $Z \to X \times Y$ making the diagram commute.

Proof. We proceed in a few steps.

STEP 1. The case when X and Y are both affine. [Har77, Exer. I.3.15]

The goal is to reduce to a commutative-algebraic statement using Corollary 1.3.24. Let $X \subseteq \mathbf{A}^n_k$ and $Y \subseteq \mathbf{A}^m_k$. Since Corollary 1.3.24 is an anti-equivalence between affine *varieties* and *domains* of finite type over k, we need to first show:

SUBTEP 1.1. The set-theoretic product $X \times Y \subseteq \mathbf{A}^{n+m}_k$ is an affine variety. [Sha13_1, Thm. 1.6]

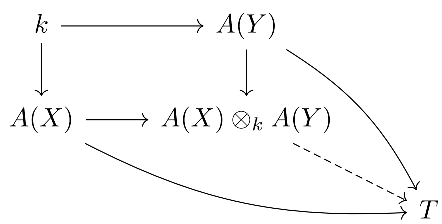
Note that $X \times Y$ is algebraic in $\mathbf{A}^{n+m}_k$ since
$$X \times Y = Z\bigl((I(X), I(Y)) \cdot A(\mathbf{A}^{n+m}_k)\bigr),$$
and hence we need to show that the set-theoretic product $X \times Y$ is irreducible in $\mathbf{A}^{n+m}_k$. Suppose that $X \times Y = Z_1 \cup Z_2$ for two closed subsets $Z_1, Z_2 \subseteq X \times Y$. Set
$$\begin{aligned} X_i &\coloneqq \bigl\{x \in X \bigm| \{x\} \times Y \subseteq Z_i\bigr\} \\ &= \bigl\{x \in X \bigm| (x,y) \subseteq Z_i \text{ for all } y \in Y\bigr\} \\ &= \bigcap_{y \in Y} \bigl\{x \in X \bigm| (x,y) \subseteq Z_i\bigr\} \end{aligned}$$
for $i \in \{1,2\}$. Since $\{x \in X \mid (x,y) \subseteq Z_i\} \cong (X \times \{y\}) \cap Z_i$ under the isomorphism $X \cong X \times \{y\}$, we see that each X_i is closed in X. Since
$$\{x\} \times Y = \bigl((\{x\} \times Y) \cap Z_1\bigr) \cup \bigl((\{x\} \times Y) \cap Z_2\bigr),$$
and $\{x\} \times Y \cong Y$ is irreducible, we see that $X = X_1 \cup X_2$. Since X is irreducible, this shows that $X = X_1$ or $X = X_2$, and hence $X \times Y = Z_1$ or $X \times Y = Z_2$.

9/26 SUBTEP 1.2. The set-theoretic product $X \times Y \subseteq \mathbf{A}_k^{n+m}$ is the categorical product of X and Y in the category of affine varieties over k.

By Corollary 1.3.24, it suffices to show the existence and universal property for the coproduct in the category of domains of finite type over k. The coproduct in this category is just the tensor product $A(X) \otimes_k A(Y)$ together with the k-algebra maps

$$\begin{aligned} A(X) &\longrightarrow A(X) \otimes_k A(Y) \\ a &\longmapsto a \otimes 1 \\ A(Y) &\longrightarrow A(X) \otimes_k A(Y) \\ b &\longmapsto 1 \otimes b. \end{aligned}$$

That is, $A(X) \otimes_k A(Y)$ together with these maps satisfy the following universal property [Hoc17, pp. 90–91; AK21, (8.17)][2]: For every k-algebra T with k-algebra maps $A(X) \to T$ and $A(Y) \to T$ making the solid diagram



conmute, there is a unique dashed map making the diagram commute. Since

$$A(X \times Y) = \frac{A(\mathbf{A}_k^{n+m})}{\big(I(X), I(Y)\big) \cdot A(\mathbf{A}_k^{n+m})}$$

satisfies the same universal property, we see that $A(X \times Y)$ is the coproduct of $A(X)$ and $A(Y)$ in the category of domains of finite type over k. Thus, the set-theoretic product $X \times Y$ is the product of X and Y in the category of affine varieties over k.

STEP 2. The case when X and Y are arbitrary quasi-projective varieties.

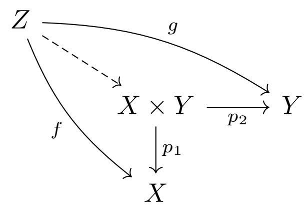
The goal is to apply the universal property for affine varieties X and Y shown in Step 1. Under the Segre embedding, the products $\mathbf{A}_k^n \times \mathbf{A}_k^m \subseteq \mathbf{P}_k^n \times \mathbf{P}_k^m$ map isomorphically to its image: On the open subset where $a_i b_j \neq 0$, we have the inverse map

$$\begin{aligned} \psi(\mathbf{P}_k^r \times \mathbf{P}_k^s) \cap \{a_i b_j \neq 0\} &\longrightarrow U_i \times V_j \\ [\cdots : a_i b_j : \cdots] &\longmapsto \left(\frac{a_1 b_j}{a_i b_j}, \ldots, \frac{a_n b_j}{a_i b_j}\right), \left(\frac{a_i b_1}{a_i b_j}, \ldots, \frac{a_i b_1}{a_i b_j}\right). \end{aligned}$$

Using this identification, we can construct morphisms $\mathbf{P}_k^n \times \mathbf{P}_k^m \to \mathbf{P}_k^n$ and $\mathbf{P}_k^n \times \mathbf{P}_k^m \to \mathbf{P}_k^m$ by working over each complement of a coordinate hyperplane. We can then restrict these morphisms using Proposition 1.3.34(i) to obtain morphisms $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$.

[2]Caution: The definition of the structure maps for tensor products of algebras in [AM69, pp. 30–31] is incorrect. See [Ele10].

By Proposition 1.3.36, we can cover X and Y by affine open subsets $\{U_i\}$ and $\{V_j\}$. Consider a diagram of the form



and choose an affine open covering $\{W_{ijk}\}$ of Z such that

$$W_{ijk} \subseteq f^{-1}(U_i) \cap g^{-1}(V_k)$$

for every i, j, k. By the universal property proved in Step 1, there are *unique* morphisms $W_{ijk} \to U_i \cap V_j$ for every i, j, k making the diagram commute. Because of this uniqueness, these morphisms are compatible with restriction, and hence glue together to form a *unique* morphism $Z \to X \times Y$. To spell this out, if morphisms $W_{ijk} \to U_i \times V_j$ and $W_{i'j'k'} \to U_{i'} \times V_{j'}$ are the morphisms induced by the universal property, then the diagram

$$\begin{array}{ccc} W_{ijk} & \longrightarrow & U_i \times V_j \\ \uparrow & & \uparrow \\ W_{ijk} \cap W_{i'j'k'} & \longrightarrow & (U_i \cap U_{i'}) \times (V_j \cap V_{j'}) \\ \downarrow & & \downarrow \\ W_{i'j'k'} & \longrightarrow & U_{i'} \times V_{j'} \end{array}$$

commutes since the top and bottom horizontal morphisms are induced by the universal property and the middle horizontal morphism is checked to be unique by covering the domain and codomain by affine opens and applying Step 1. The morphism $Z \to X \times Y$ constructed in this manner makes the diagram commute by construction since you can check commutativity locally on an open cover. □

1.4. Rational maps

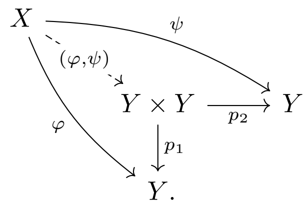
We continue to denote by k an algebraically closed field.

1.4.1. Rational maps and birational equivalence. Now that we understand the basic objects and morphisms between them in algebraic geometry, we can start discussing a part of algebraic geometry that is very special about algebraic geometry: Because everything is defined using polynomials or rational functions, varieties and morphisms between them are very "rigid."

We start with the following result, which says that if two morphisms coincide on a nonempty open set, then they are equal. This shows already how useful the product construction from last time is.

LEMMA 1.4.1. *Let X and Y be quasi-projective varieties. Let φ and ψ be two morphisms $X \to Y$, and suppose there exists a nonempty open subset $U \subseteq X$ such that $\varphi|_U = \psi|_U$. Then, $\varphi = \psi$.* [Har77, Lem. I.4.1]

Proof. By the universal property of products (Theorem 1.3.37(*iii*)), there is a commutative diagram



Now consider the product $Y \times Y$ as a locally closed subset of $\mathbf{P}^n_k \times \mathbf{P}^n_k$ considered as a quasi-projective variety using the Segre embedding:

$$Y \times Y \subseteq \mathbf{P}^n_k \times \mathbf{P}^n_k \subseteq \mathbf{P}^N_k.$$

Inside of $\mathbf{P}^n_k \times \mathbf{P}^n_k$, we have the *diagonal*

$$\Delta \coloneqq Z\Big(\big\{x_i y_j = x_j y_i \bigm| i, j \in \{0, 1, \ldots, n\}\big\}\Big) \subseteq \mathbf{P}^n_k \times \mathbf{P}^n_k.$$

Let $\Delta_Y \coloneqq \Delta \cap Y$. Since $\varphi|_U = \psi|_U$, we know that

$$(\varphi, \psi)(U) \subseteq \Delta_Y.$$

But since U is dense in X and Δ_Y is closed in $Y \times Y$, taking closures yields

$$(\varphi, \psi)(X) \subseteq \Delta_Y.$$

We therefore have $\varphi = \psi$ everywhere on X. □

Remark 1.4.2. In the proof above, the key observation is that Δ_Y is closed in $Y \times Y$. This is the analogue of being Hausdorff in algebraic geometry (cf. [BouGT, Chapter I, §8, no. 1, Proposition 1]). While this condition is automatic for quasi-projective varieties (as we saw in the proof above), it is not automatic if one wants to be able to glue varieties arbitrarily, or if one wants to work with schemes. For schemes, the condition is called being *separated*. See [FAC, p. 227] and [Har77, Chapter II, §4].

Using Lemma 1.4.1, we can define the following:

[Har77, p. 24] Definition 1.4.3. Let X, Y be quasi-projective varieties. A *rational map*

$$\varphi \colon X \dashrightarrow Y$$

is a *partially* defined morphism $\varphi_U \colon U \to Y$ where U is a nonempty open subset of X, subject to the equivalence relation

$$\langle U, \varphi_U \rangle \sim \langle V, \varphi_V \rangle \iff \varphi_U|_{U \cap V} = \varphi_V|_{U \cap V}.$$

A rational map φ is *dominant* if for some (and hence every, by Lemma 1.4.1) representative $\varphi_U \colon U \to Y$ of φ, the image of φ_U is dense in Y.

Remark 1.4.4. Hartshorne does not use the dashed arrow notation. It has become standard in birational geometry. There is some chance for confusion since I use dashed arrows in commutative diagrams for morphisms that I claim exist, so please ask if you are confused.

Lemma 1.4.1 shows that $\sim$ is indeed an equivalence relation. Since morphisms are continuous, we can compose rational maps. We can therefore consider the following category:

[Har77, p. 24] DEFINITION 1.4.5. We consider the category of quasi-projective varieties and dominant rational maps. An isomorphism in this category is called a *birational map*. In other words, a rational map

$$\varphi\colon X \dashrightarrow Y$$

is birational if there exists a rational map

$$\psi\colon Y \dashrightarrow X$$

such that $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$, where equality here means "equivalence as rational maps," If there is a birational map $X \dashrightarrow Y$, we say that X and Y are *birationally equivalent* or *birational*.

REMARK 1.4.6 (Birational equivalence vs. isomorphism). Classifying varieties up to birational equivalence is the focus of birational geometry, which is a subfield of algebraic geometry. Why is birational equivalence a good notion? The same rigidity that allowed us to show that regular functions that are equal on an open set are equal everywhere (Lemma 1.4.1) comes back to bite us here: there are not enough morphisms to make classifying varieties up to isomorphism a reasonable task, at least in higher dimensions. For curves (= Riemann surfaces), it is possible, but there is a stark contrast with the topological picture. Trying to classify curves up to isomorphism leads to the beautiful theory of moduli spaces, where there are still many open problems.

In higher dimensions, the philosophy is that we should instead classify varieties up to birational equivalence and then see what we can say about isomorphism classes. But studying varieties up to birational equivalence is already very interesting and powerful! I hope to convince you of this as the semester goes on. I will mention from time to time questions from birational geometry that we have the language to understand. But for now, our first goal is to justify why we can think of birational geometry as a fancy version of field theory: We are trying to understand fields from a geometric perspective.

1.4.2. Varieties vs. fields finitely generated over k. The following is the first main result in this section. The result says that the category of quasi-projective varieties with dominant rational maps is anti-equivalent to the category of finitely generated field extensions of k.

THEOREM 1.4.7. *Let X, Y be quasi-projective varieties. We then have a bijection* [Har77, Thm. I.4.4]

$$\left\{\begin{matrix}\text{dominant rational maps}\\ \varphi\colon X \dashrightarrow Y\end{matrix}\right\} \xleftrightarrow{\text{1-1}} \left\{\begin{matrix}k\text{-algebra maps}\\ K(Y) \to K(X)\end{matrix}\right\}$$
$$\big(\varphi\colon X \dashrightarrow Y\big) \longmapsto \Big(\langle V, f\rangle \mapsto \big\langle \varphi_U^{-1}(V), f \circ \varphi_U\big\rangle\Big). \tag{1.4.8}$$

This correspondence gives an anti-equivalence of categories between the category of quasi-projective varieties with dominant rational maps and the category of finitely generated field extensions of k.

Proof. We first make sure that $\langle \varphi_U^{-1}(V), f \circ \varphi_U\rangle$ defines a rational map on $\varphi_U^{-1}(V) \subseteq X$ where $V \subseteq Y$. Since $\varphi_U(U)$ is dense in Y, the inverse image $\varphi_U^{-1}(V)$ is a nonempty open subset of X, and $f \circ \varphi_U$ is a regular function on $\varphi_U^{-1}(V)$. This gives a rational function $\langle \varphi_U^{-1}(V), f \circ \varphi_U\rangle$ on X. This assignment yields a k-algebra map $K(Y) \to K(X)$ since addition and multiplication are defined pointwise, and adding or multiplying two rational functions can be done locally on open neighborhoods of points where the rational functions are regular.

To show the map (1.4.8) is a bijection, we construct an inverse. Let $\theta\colon K(Y) \to K(X)$ be a map of k-algebras. By Proposition 1.3.36, we know that Y is covered by affine varieties, so we may replace Y with an open subset to assume that Y is affine. We then find a commutative diagram

$$\begin{array}{ccc} K(Y) & \xrightarrow{\theta} & K(X) \\ \hookuparrow & & \hookuparrow \\ A(Y) & \dashrightarrow & \mathcal{O}(U) \end{array}$$

by choosing generators y_i for $A(Y)$ as a k-algebra and then choosing an open set $U \subseteq X$ where the $\theta(y_i)$ are regular functions on U. By Proposition 1.3.22, this map $A(Y) \to \mathcal{O}(U)$ corresponds to a morphism $U \to Y$ along which pulling back rational functions corresponds to θ.

It remains to show we have the claimed anti-equivalence of categories. We first show that $K(Y)$ is always finitely generated over k. If Y is a variety, then $K(Y) = K(U)$ for an open affine subset $U \subseteq Y$. Then, $K(U) = \operatorname{Frac}(A(U))$ is finitely generated over k by Theorem 1.3.12(*d*). Finally, we need to show that the functor $Y \mapsto K(Y)$ is essentially surjective. Let $y_1, y_2, \ldots, y_n \in K(Y)$ be generators as a field over k. Let I be the kernel of

$$\begin{aligned} k[x_1, x_2, \ldots, x_n] &\longrightarrow K(Y) \\ x_i &\longmapsto y_i. \end{aligned}$$

Then, $Z(I) \subseteq \mathbf{A}^n_k$ is an affine variety with affine coordinate ring

$$\frac{k[x_1, x_2, \ldots, x_n]}{I} \cong k[y_1, y_2, \ldots, y_n]$$

which has fraction field $K(Y)$. □

9/29 [Har77, Cor. I.4.5] Corollary 1.4.9. *Let X, Y be quasi-projective varieties. The following conditions are equivalent.*

(*i*) *X and Y are birational.*
(*ii*) *There are open subsets $U \subseteq X$ and $V \subseteq Y$ such that $U \cong V$.*
(*iii*) *$K(X) \cong K(Y)$ as k-algebras.*

Proof. (*i*) ⇒ (*ii*). Let $\varphi\colon X \dashrightarrow Y$ and $\psi\colon Y \dashrightarrow X$ be rational maps that are inverse to each other. After replacing Y by an open subset, we may assume that ψ is a morphism. Choose a representative $\langle U, \varphi\rangle$ for φ. Then, $\psi \circ \varphi = \mathrm{id}_U$ by Lemma 1.4.1. On the other hand, $\varphi \circ \psi$ is the identity on $\psi^{-1}(U)$.

(*ii*) ⇒ (*iii*) follows from the definition of function fields.

(*iii*) ⇒ (*i*) follows from Theorem 1.4.7. □

1.4.3. Every variety is birational to a hypersurface. Now that we have shown Theorem 1.4.7, we want to apply it to geometric questions. In other words, we want to use field theory to say something about varieties. I will mention later some applications in the opposite direction.

[Har77, Prop. I.4.9] Theorem 1.4.10. *Any quasi-projective variety X of dimension d is birational to a hypersurface Y in $\mathbf{P}^{d+1}_k$.*

Proof. We know that $K(X)$ is finitely generated as a field over k. Since k is algebraically closed, we know by [ZS75$_1$, Chapter II, Theorem 31 on p. 105] that $K(X)$ is *separably generated*, that is, there exists a transcendence basis $\{x_1, x_2, \ldots, x_d\}$ for $K(X)$ over k such that writing

$$k \subseteq k(x_1, x_2, \ldots, x_d) \subseteq K(X)$$

the second extension is a separable algebraic extension. By the theorem of the primitive element [ZS75$_1$, Chapter II, Theorem 19 on p. 84], there exists an element $y \in K(X)$ such that

$$k(x_1, x_2, \ldots, x_d, y) = K(X).$$

Since y is algebraic over $k(x_1, x_2, \ldots, x_d)$, it satisfies a polynomial equation

$$f(x_1, x_2, \ldots, x_d, y) = 0.$$

Clearing denominators and homogenizing, we obtain a homogeneous polynomial in $d+2$ variables defining the hypersurface Y in $\mathbf{P}^{d+1}_k$. □

REMARK 1.4.11. The way the result is proved may not be the most enlightening from a geometric perspective. On Homework 6, you will show that in fact, the birational map $X \dashrightarrow Y$ can be realized as an appropriately chosen linear projection. If this linear projection is chosen "generically" enough, you can do even better: Y is always *weakly normal*, a slightly weaker notion than normality [GT80; CM81; ZN84; RZN84; CGM87], and in low dimensions, we can say quite a few things about the singularities of Y [Rob71; Rob75; Doh08; DM24]. This allows us to reduce some questions about all quasi-projective varieties to just hypersurfaces. For example, see the proof of the Riemann–Roch theorem for curves in [ACGH85, Appendix A] and the proof of Noether's formula for surfaces in [GH78, Chapter 4, §6].

1.4.4. Rationality and unirationality. Last time, we saw that every variety is birational to a hypersurface. This is one way we can simplify varieties—if we have a problem that is invariant under birational maps, then we can try to replace our original variety with a hypersurface and solve the problem there, like I mentioned in Remark 1.4.11. It would be nice to be able to say what the result of this process is. Thus, an interesting question to ask is:

QUESTION 1.4.12. *When is a projective variety X birational to $\mathbf{P}^n_k$? In particular, when is a projective hypersurface X birational to $\mathbf{P}^n_k$?*

This question is very difficult, and has a deep history. There are many questions still open to this day. Before we move on, we give this property (and a related property) a name.

DEFINITION 1.4.13. Let X be a quasi-projective variety of dimension n.

(i) We say that X is *rational* if it is birational to $\mathbf{P}^n_k$. [Har77, Exer. I.4.4]

(ii) We say that X is *unirational* if there exists a dominant rational map $\mathbf{P}^n_k \dashrightarrow X$. [Kol96, (IV.3.1.2)]

Since birational maps are dominant, we see that rational ⇒ unirational. As we mentioned before, in birational geometry we often take projective closures to assume that the varieties we are studying are projective. By Theorem 1.4.7 and Corollary 1.4.9, we can think of rationality as saying that $K(X)$ is purely transcendental over k and unirationality as saying that $K(X)$ is contained in a purely transcendental extension over k of the same transcendence degree.

1.4.4.1. *The Lüroth problem.* Another way to think about unirationality is that X can be parametrized by an affine or projective space. You can imagine that it is sometimes easier to write down a parametrization of X that is not necessarily one-to-one. We therefore ask:

[Bea16]

QUESTION 1.4.14 (The Lüroth problem). *Let X be a unirational projective variety. Is X rational?*

See [Bea16] for a survey of the Lüroth problem. The answer to the Lüroth problem is "yes" in dimension 1.

THEOREM 1.4.15 (Lüroth's theorem [Lür1875]). *Let X be a quasi-projective curve. If X is unirational, then X is rational.*

Let us first see an analytic proof of this over $\mathbf{C}$, assuming that X is normal. (This is not going to tested, but I think it shows how useful algebraic topology and complex analysis can be in complex algebraic geometry!)

[Ele12]

Analytic proof of Lüroth's Theorem 1.4.15 over $\mathbf{C}$. We assume X is a complex projective normal curve. We will take it for granted that with the Euclidean topology, X is a compact complex manifold of dimension one. We will also take it for granted that the dominant rational map $\mathbf{P}^1_{\mathbf{C}} \dashrightarrow X$ extends to a surjective *morphism* $\mathbf{P}^1_{\mathbf{C}} \to X$ (we will prove this in [Har77, Chapter I, Proposition 6.8]). The complex structure gives the surface an orientation by multiplying by i.

We proceed by contradiction. Suppose that X is not rational. By the classification of surfaces from topology [Mun00, Chapter 12], we know that X is homeomorphic to S^2 (this is the case when $X \cong \mathbf{P}^1_{\mathbf{C}}$) or the connected sum $\mathbf{T}^g$ of g tori. Taking universal covers, since $\mathbf{P}^1_{\mathbf{C}}$ is simply connected, we obtain the following commutative diagram of topological spaces and continuous maps:

$$\begin{array}{ccc} \mathbf{P}^1_{\mathbf{C}} & \longrightarrow & \tilde{X} \\ \| & & \downarrow \\ \mathbf{P}^1_{\mathbf{C}} & \longrightarrow & X. \end{array}$$

By the classification of surfaces from topology, $\tilde{X}$ is an open unit disc or $\mathbf{C}$. Moreover, by the Riemann uniformization theorem, these maps can be upgraded to *holomorphic* maps. By the maximum modulus principle, the image of $\mathbf{P}^1_{\mathbf{C}}$ in $\tilde{X}$ is a point. This contradicts the surjectivity of $\mathbf{P}^1_{\mathbf{C}} \to X$. □

One drawback of this approach is that there is no way (at least, no way that I know of) to make this argument work over arbitrary algebraically closed fields. There are other geometric proofs of Lüroth's Theorem 1.4.15 that *do* work over arbitrary algebraically closed fields (see [Har77, Chapter IV, Example 2.5.5]), but they require more sophisticated tools from algebraic geometry. Nevertheless, Theorem 1.4.7 and Corollary 1.4.9 gives us one path forward: We can translate the statement of Lüroth's Theorem 1.4.15 into a field theory question!

[Oja90, Thm. 1.3]
[Mus17, Prop. 1.9]

Field-theoretic proof of Lüroth's Theorem 1.4.15. We show that if

$$k \subseteq K \subseteq k(t)$$

is a sequence of field extensions such that $\operatorname{trdeg}_k(K) = 1$, then K is of pure transcendence 1 over k (note that K may not be generated by t). Since $\operatorname{trdeg}_k(K) = 1$, it is enough to find some $\alpha \in K$ such that $K = k(\alpha)$.

First, the field extension $K \subseteq k(t)$ must be algebraic, and hence t is algebraic over K. Let

$$f(X) = X^n + a_1 X^{n-1} + \cdots + a_n \in K[X]$$

be the minimal polynomial of t over K. Since t is transcendental over k, we cannot have all $a_i \in k$. Thus, there is some i such that $a_i \in K - k$. We will show that in this case, $K = k(a_i)$. We know that the $a_i \in K \subseteq k(t)$, and hence we may write

$$a_i = \frac{u(t)}{v(t)}$$

where $u, v \in k[t]$ are relatively prime, and where at least one of them is of positive degree. Now consider the following polynomial:

$$F(X) = u(X) - a_i\, v(X) \in k(a_i)[X].$$

Since $F(t) = 0$, we know that $f(X) \mid F(X)$ in $K[X]$ by the minimality of $f(X)$. Thus,

$$F(X) = u(X) - a_i\, v(X) = f(X)\, g(X) \tag{1.4.16}$$

for some $g(X) \in K[X]$. We then claim the following:

Claim 1.4.17. $g(X) \in K$.

Showing the Claim would conclude the proof. We have a sequence of extensions

$$k \hookrightarrow k(a_i) \hookrightarrow K \hookrightarrow k(t).$$

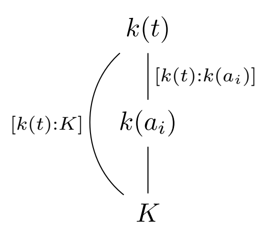
Since $g(X) \in K$, the polynomial

$$F(X) = u(X) - a_i\, v(X) \in k(a_i)[X]$$

is the minimal polynomial for t over both K and $k(a_i)$ simultaneously. Thus, we have

$$\big[k(t) : K\big] = \big[k(t) : k(a_i)\big].$$

Looking at the tower of fields



and counting degrees, we see that the inclusion $k(a_i) \subseteq K$ is an equality.

To prove Claim 1.4.17, the first step is to get rid of all denominators. By multiplying (1.4.16) by a suitable nonzero element of $c(t) \in k[t]$, we get a relation 10/1

$$c(t)\big(u(X)\, v(t) - v(X)\, u(t)\big) = f_1(t, X)\, g_1(t, X),$$

where $f_1(t, X), g_1(t, X) \in K[t, X]$ are obtained from f and g respectively by multiplication by an element in $k[t]$. Since $K[t, X]$ is a UFD, we can get rid of $c(t)$ by successively dividing by prime factors of $c(t)$ to get a relation

$$u(X)\, v(t) - v(X)\, u(t) = f_2(t, X)\, g_2(t, X), \tag{1.4.18}$$

where now $f_2(t,X), g_2(t,X) \in K[t,X]$ are obtained from f, g by multiplication by a nonzero element in $k[t]$.

We now look at the degrees in t on both sides. First,

$$\deg_t\bigl(u(X)\,v(t) - v(X)\,u(t)\bigr) \le \max\Bigl\{\deg\bigl(u(t)\bigr), \deg\bigl(v(t)\bigr)\Bigr\}.$$

Writing $f_2(t,X) = \gamma_0(t)\,X^n + \cdots + \gamma_n(t)$, we see that

$$\frac{\gamma_i(t)}{\gamma_0(t)} = a_i(t) = \frac{u(t)}{v(t)},$$

where $u(t), v(t)$ were relatively prime. This implies that in fact,

$$\deg_t\bigl(f_2(t,X)\bigr) \ge \max\Bigl\{\deg\bigl(u(t)\bigr), \deg\bigl(v(t)\bigr)\Bigr\}.$$

By looking at degrees in t on both sides of the relation (1.4.18), we have that $\deg_t(g_2(t,X)) = 0$, hence $g_2 \in K[X]$.

Now we claim that $g_2 \in K$. For sake of contradiction, suppose that $g_2 \notin K$. Then, there is a root $\gamma \in \bar{K}$ such that $g_2(\gamma) = 0$. Thus, we have

$$u(\gamma)\,v(t) - v(\gamma)\,u(t) = f_2(t,\gamma)\,g_2(\gamma) = 0,$$

and hence $u(\gamma)\,v(t) = v(\gamma)\,u(t)$. Since $u(t)$ and $v(t)$ are relatively prime and are not both constants, we must have $u(\gamma) = 0 = v(\gamma)$. This contradicts the choice of u, v as being relatively prime, and hence $g_2 \in K$. Finally, since $g \in K[X]$ and

$$g_2 = g \cdot \bigl(\text{element of } k[t]\bigr),$$

we have that $g \in K$. □

Remark 1.4.19. In higher dimensions, the surface case of the Lüroth problem (Question 1.4.14) also has an affirmative answer. This is due to Castelnuovo [Cas1896]. The original proof was geometric, which indicates how Theorem 1.4.7 and Corollary 1.4.9 can be used in the opposite direction to derive field-theoretic consequences from algebraic geometry. The case of dimensions ≥ 3 was open for a long time until it was resolved *in the negative* by Artin–Mumford [AM72], Clemens–Griffiths [CG72], and Iskovskih–Manin [IM71] around the same time.

1.4.4.2. *Hypersurfaces.* How about hypersurfaces? We give some examples of rational hypersurfaces.

[Har77, Exer. I.4.4]

Example 1.4.20 (Rational plane curves).

(*a*) Any conic in $\mathbf{P}^2_k$ is a rational curve. This is because any conic is *isomorphic* to $\mathbf{P}^1_k$ by [Har77, Chapter I, Exercise 3.1(c)], which was on your homework.

(*b*) The cuspidal cubic $y^2 = x^3$ is a rational curve. This is because $t \mapsto (t^2, t^3)$ is a bijection between the cuspidal cubic and $\mathbf{A}^1_k$ as you also saw on your homework [Har77, Chapter I, Exercise 3.2(a)].

(*c*) The nodal cubic

$$Y = \bigl\{y^2 z = x^2(x+z)\bigr\} \subseteq \mathbf{P}^2_k$$

is rational. To see this, we consider the projection φ away from the point $\mathbf{0} = [0:0:1] \in \mathbf{P}^2_k$ to the line $z = 0$. In coordinates, this is the morphism

$$\begin{aligned} \varphi\colon Y - \{\mathbf{0}\} &\longrightarrow \mathbf{P}^1_k \\ [x:y:z] &\longmapsto [x:y]. \end{aligned}$$

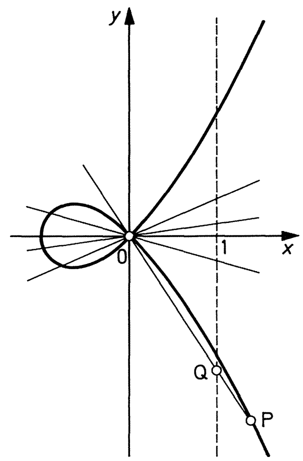


FIGURE 1.14. Projection of the nodal cubic. From [Sha13$_1$, Figure 3]

To give a more concrete geometric description, if we restrict to the affine charts $\{z \neq 0\}$ and $\{x \neq 0\}$, we obtain the map

$$\begin{aligned} Y \cap \{z \neq 0\} - \{\mathbf{0}\} &\longrightarrow \{x \neq 0\} \\ (x, y) &\longmapsto \frac{y}{x} \end{aligned}$$

sending a point $P \in Y \cap \{z \neq 0\}$ to the intersection point Q of the secant line $\overline{\mathbf{0}P}$ with the line $\{x = 1\}$.

We claim that the morphism

$$\begin{aligned} \psi \colon \mathbf{P}^1_k - \big\{[1:1], [1:-1]\big\} &\longrightarrow Y - \{P\} \\ [x:y] &\longmapsto \big[x(y^2 - x^2) : y(y^2 - x^2) : x^3\big] \end{aligned}$$

is an inverse for φ as rational maps. We have

$$\begin{aligned} (\varphi \circ \psi)[x:y] &= \varphi\big[x(y^2 - x^2) : y(y^2 - x^2) : x^3\big] \\ &= \big[x(y^2 - x^2) : y(y^2 - x^2)\big] \\ &= [x:y] \\ (\psi \circ \varphi)[x:y:z] &= \psi[x:y] \\ &= \big[x(y^2 - x^2) : y(y^2 - x^2) : x^3\big] \\ &= \big[x(y^2 - x^2) : y(y^2 - x^2) : z(y^2 - x^2)\big] \\ &= [x:y:z] \end{aligned}$$

as long as

$$[x:y] \neq \big\{[1:1], [1:-1]\big\}.$$

See Figure 1.14 for an illustration.

(*d*) We will see later [Har77, Chapter I, Exercise 6.2] that elliptic curves such as

$$\{y^2 = x^3 - x\} \subseteq \mathbf{A}^2_k$$

are not rational. The proof in [Har77, Chapter I, Exercise 6.2] is field-theoretic.

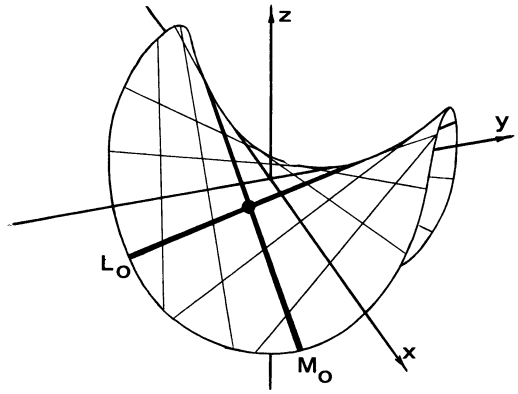


FIGURE 1.15. The quadric surface in $\mathbf{P}^3_k$. From [Har77, Figure 2].

FIGURE 1.16. Kobe Port Tower in the Kobe harbor (2006).

[Har77, Exer. I.4.5] EXAMPLE 1.4.21 (Quadric surfaces). The quadric surface $Q : xy = zw$ in $\mathbf{P}^3_k$ is birational to $\mathbf{P}^2_k$ since

$$Q \cong \mathbf{P}^1_k \times \mathbf{P}^1_k \supseteq U_0 \times U_0 \cong \mathbf{A}^2_k \subseteq \mathbf{P}^2_k,$$

where the first isomorphism is by [Har77, Chapter I, Exercise 2.15(a)]. On the other hand, Q is not isomorphic to $\mathbf{P}^2_k$: Q has lines that do not intersect each other [Har77, Chapter I, Exercise 2.15(b)] but every pair of curves in $\mathbf{P}^2_k$ intersect [Har77, Chapter I, Exercise 3.7(a)] (both of these were on your homework). See Figure 1.15 for an illustration and Figure 1.16 for a real-life example.

REMARK 1.4.22. In general, there are many results about rationality of hypersurfaces. See [LS24] for *very* recent progress. However, it has been a longtime open question whether there exists a cubic fourfold (i.e., a cubic hypersurface in $\mathbf{P}^5_k$) that is not rational. See [Has16] for a survey. See [KKPY25] for a recent proof that a very general cubic fourfold is not rational.

1.4.5. Blowing up. One of the most important examples of a birational map is the blowup of a variety at a point. Blowing up at points (and the more general notion of blowing up along ideals) is very important in part because of its role in resolutions of singularities—a topic with deep history here at Purdue. For some of the major work done here, see the work of Abhyankar [Abh65; Abh98], Lipman [Lip78], Włodarczyk [Wło05; ATW24] (the latter joint with Abramovich and Temkin, proved independently by McQuillan [McQ20]), Matsuki (joint with Kawanoue) [Kaw07; KM10; Mat20].

We saw in Theorem 1.4.10 that every variety is birational to a hypersurface. However, it is often the case that the hypersurface one gets is not a manifold (over $\mathbf{C}$) or (even) normal. Over the complex numbers, resolution of singularities asks: Is every complex projective variety birational to a complex projective manifold? This question can also be asked over other fields, but for now we point out that Hironaka [Hir64$_1$; Hir64$_2$] proved the existence of resolutions of singularities over $\mathbf{C}$.

The key construction used to resolve singularities is called *blowing up*. We now describe this construction.

CONSTRUCTION 1.4.23 (Blowing up $\mathbf{A}^n_k$ at the origin). Consider the product $\mathbf{A}^n_k \times \mathbf{P}^{n-1}_k$, which is a quasi-projective variety using the Segre embedding (Example 1.2.24) and our construction of products (Theorem 1.3.37). Let $x_1, x_2, \ldots, x_n$ be the coordinates on $\mathbf{A}^n_k$ and let $y_1, y_2, \ldots, y_n$ (note the numbering here is different from what we normally use!) be the coordinates on $\mathbf{P}^{n-1}_k$. Then, closed subsets of $\mathbf{A}^n_k \times \mathbf{P}^{n-1}$ are defined by polynomials in the x_i, y_j that are homogeneous with respect to the y_j since these are the polynomials that restrict well to the subsets $\mathbf{A}^n_k \times U_i \cong \mathbf{A}^{2n-1}_k$. [Har77, pp. 28–29]

The *blowup of $\mathbf{A}^n_k$ at the point* $\mathbf{0} = (0, 0, \ldots, 0)$ is

$$\operatorname{Bl}_{\mathbf{0}} \mathbf{A}^n_k \coloneqq \bigl\{ x_i y_j = x_j y_i \bigm| i, j = 1, 2, \ldots, n \bigr\}.$$

We denote

$$\begin{array}{ccc} \operatorname{Bl}_{\mathbf{0}} \mathbf{A}^n_k & \hookrightarrow & \mathbf{A}^n_k \times \mathbf{P}^{n-1}_k \\ & \underset{\varphi}{\searrow} & \downarrow \\ & & \mathbf{A}^n_k. \end{array}$$

The inverse image of $\mathbf{0}$ is called the *exceptional divisor* of the blowup.

We have the following properties of $\operatorname{Bl}_{\mathbf{0}} \mathbf{A}^n_k$:

(1) *φ gives an isomorphism*

$$\varphi|_{\operatorname{Bl}_{\mathbf{0}} \mathbf{A}^n_k - \varphi^{-1}(\mathbf{0})} \colon \operatorname{Bl}_{\mathbf{0}} \mathbf{A}^n_k - \varphi^{-1}(\mathbf{0}) \xrightarrow{\sim} \mathbf{A}^n_k - \{\mathbf{0}\}.$$

In particular, φ is birational, and if $P \in \mathbf{A}_k^n - \{\mathbf{0}\}$, then $\varphi^{-1}(P)$ consists of one point.

We claim the inverse for φ is given by

$$\begin{aligned}\psi\colon \mathbf{A}_k^n - \{\mathbf{0}\} &\longrightarrow \mathrm{Bl}_{\mathbf{0}}\,\mathbf{A}_k^n - \varphi^{-1}(\mathbf{0})\\ (a_1,a_2,\ldots,a_n) &\longmapsto \bigl((a_1,a_2,\ldots,a_n),[a_1:a_2:\cdots:a_n]\bigr).\end{aligned}$$

By definition, we have

$$(\varphi\circ\psi)(a_1,a_2,\ldots,a_n) = (a_1,a_2,\ldots,a_n)$$

and hence $\varphi\circ\psi = \mathrm{id}_{\mathbf{A}_k^n-\{\mathbf{0}\}}$. On the other hand, we want to show that

$$(\psi\circ\varphi)\bigl((a_1,a_2,\ldots,a_n),[b_1:b_2:\cdots:b_n]\bigr) = \bigl((a_1,a_2,\ldots,a_n),[a_1:a_2:\cdots:a_n]\bigr)$$

is equal to $((a_1,a_2,\ldots,a_n),[b_1:b_2:\cdots:b_n])$. Choose i such that $a_i \neq 0$. Then,

$$\begin{aligned}[b_1:b_2:\cdots:b_n] &= a_i\cdot[b_1:b_2:\cdots:b_n]\\ &= [a_ib_1:a_ib_2:\cdots:a_ib_n]\\ &= [a_1b_i:a_2b_i:\cdots:a_nb_i]\\ &= b_i\cdot[a_1:a_2:\cdots:a_n]\\ &= [a_1:a_2:\cdots:a_n]\end{aligned}$$

where the third equality follows from the relations $x_iy_j = x_jy_i$. For the fourth equation, we note that the third line implies $b_i \neq 0$: otherwise, the point $[a_1b_i:a_2b_i:\cdots:a_nb_i]$ would have all coordinates equal to 0, which does not define a point in $\mathbf{P}_k^{n-1}$.

(2) $\varphi^{-1}(\mathbf{0}) \cong \mathbf{P}_k^{n-1}$. Indeed,

$$\varphi^{-1}(\mathbf{0}) = \bigl\{(\mathbf{0},Q) \bigm| Q = [y_1:y_2:\cdots:y_n] \in \mathbf{P}_k^{n-1}\bigr\}$$

where the points Q have no restriction.

10/3 (3) *The points of $\varphi^{-1}(\mathbf{0})$ are in 1-1 correspondence with the set of lines through $\mathbf{0}$ in $\mathbf{A}_k^n$. In other words, the blowup replaces the point $\mathbf{0}$ with the set of tangent directions through $\mathbf{0}$.*

Let L be a line through $\mathbf{0}$ given parametrically by

$$\begin{cases} x_1 = a_1t\\ x_2 = a_2t\\ \quad\vdots\\ x_n = a_nt\end{cases}$$

where the a_i are not all 0 and $t \in \mathbf{A}_k^1$. We then see that

$$L' \coloneqq \varphi^{-1}\bigl(L - \{\mathbf{0}\}\bigr)$$

is given parametrically by $x_i = a_it$ and $y_i = a_it$. Since the y_i are homogeneous coordinates, we can cancel out t to see that L' is given parametrically by $x_i = a_it$ and $y_i = a_i$. These equations give the closure $\overline{L'}$ of L' in X parametrically. Finally, note that

$$\overline{L'} \cap \varphi^{-1}(\mathbf{0}) = \bigl\{[a_1:a_2:\cdots:a_n]\bigr\} \in \mathbf{P}_k^{n-1},$$

so sending L to Q gives a 1-1 correspondence between lines through $\mathbf{0} \in \mathbf{A}_k^n$ and points of $\varphi^{-1}(\mathbf{0})$.

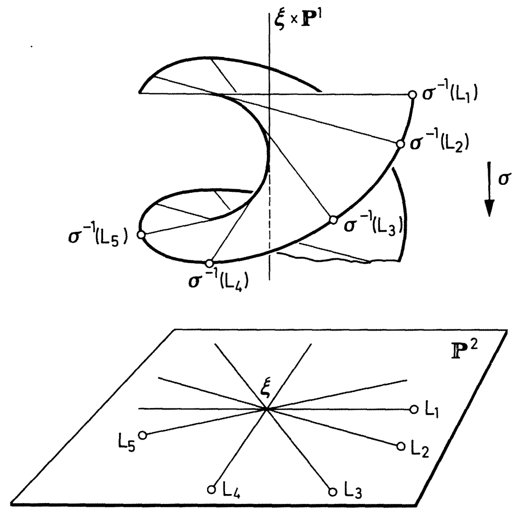


FIGURE 1.17. The blowup of $\mathbf{A}_k^2$ at the origin. From [Sha13$_1$, Figure 10].

(4) $\mathrm{Bl}_{\mathbf{0}}\,\mathbf{A}_k^n$ *is irreducible.*
Writing

$$\mathrm{Bl}_{\mathbf{0}}\,\mathbf{A}_k^n = \bigl(\mathrm{Bl}_{\mathbf{0}}\,\mathbf{A}_k^n - \varphi^{-1}(\mathbf{0})\bigr) \cup \varphi^{-1}(\mathbf{0}),$$

the first set is irreducible (it is isomorphic to $\mathbf{A}_k^n - \{\mathbf{0}\}$ by (1)), and every point in $\varphi^{-1}(\mathbf{0})$ is in the closure of a point in $\mathrm{Bl}_{\mathbf{0}}\,\mathbf{A}_k^n - \varphi^{-1}(\mathbf{0})$ by (3).

We now define blowups of affine varieties at points more generally.

DEFINITION 1.4.24. Let $Y \subseteq \mathbf{A}_k^n$ be a closed subvariety passing through $\mathbf{0}$. The *blowup of* Y *at a point* is [Har77, p. 29]

$$\mathrm{Bl}_{\mathbf{0}}\,Y \coloneqq \tilde{Y} \coloneqq \overline{\varphi^{-1}\bigl(Y - \{\mathbf{0}\}\bigr)},$$

where $\varphi\colon \mathrm{Bl}_{\mathbf{0}}\,\mathbf{A}_k^n \to \mathbf{A}_k^n$ is the blowup of $\mathbf{A}_k^n$ at the point $\mathbf{0}$ defined above. We denote $\varphi\colon \tilde{Y} \to Y$ the restriction of the morphism φ to $\tilde{Y}$.

To blowup at any other point $P \in \mathbf{A}_k^n$, we make a linear change of coordinates sending P to $\mathbf{0}$ to get the blowup $\mathrm{Bl}_P\,Y$ of Y at the point P.

REMARK 1.4.25. By (1) above, we see that $\varphi\colon \tilde{Y} \to Y$ induces an isomorphism away from $\mathbf{0}$. For now, the definition appears to depend on the embedding of Y in $\mathbf{A}_k^n$. We will give a more intrinsic description when we discuss blowups of schemes. [Har77, p. 29]

What does the blowup do geometrically? As we mentioned at the beginning of class, the basic idea is that blowing up "pulls apart" different directions of lines at O. This is easiest to see in $\mathbf{A}_k^2$.

EXAMPLE 1.4.26. Let us first look at the blowup of $\mathbf{A}_k^2$ itself at $\mathbf{0}$. We have

$$\mathrm{Bl}_{\mathbf{0}}\,\mathbf{A}_k^2 \coloneqq \{xu = yt\} \subseteq \mathbf{A}_k^2 \times \mathbf{P}_k^1$$

where the coordinates on $\mathbf{A}_k^2$ are (x, y) and the coordinates on $\mathbf{P}_k^1$ are $[t : u]$. We then see that when $t \neq 0$, we have

$$\frac{y}{x} = \frac{u}{t}$$

and when $u \neq 0$, we have

$$\frac{x}{y} = \frac{t}{u}.$$

The point $[t : u]$ corresponds to all possible slopes (including the undefined slope for a vertical line!) of lines through $\mathbf{0}$. Passing to open affine covers, we think of $\mathrm{Bl}_{\mathbf{0}} \mathbf{A}_k^2$ as being covered by an affine open that has a new coordinate corresponding to the slope $y/x = u/t$ of a line through the origin, and an affine open that has a new coordinate corresponding to the inverse of the slope $x/y = t/u$ of a line through the origin. See Figure 1.17.

Next, we see what happens when we blowup a curve inside of $\mathbf{A}_k^2$.

[Har77, Ex. I.4.9.1]

EXAMPLE 1.4.27. Consider the nodal cubic curve

$$Y \coloneqq \left\{y^2 = x^2(x+1)\right\} \subseteq \mathbf{A}_k^2.$$

Using the same notation as in the previous example, we call the exceptional divisor $E \coloneqq \varphi^{-1}(\mathbf{0})$ the *exceptional curve*.

We first compute the *total inverse image* $\varphi^{-1}(Y)$ of Y in $\mathrm{Bl}_{\mathbf{0}} \mathbf{A}_k^2$. This is the closed subset given by the equations $y^2 = x^2(x+1)$ and $xu = yt$ in $\mathbf{A}_k^2 \times \mathbf{P}_k^1$.

On the affine open subsets $t \neq 0$, we can set $t = 1$ and use u as our affine coordinate. On this open set, $\varphi^{-1}(Y)$ is given by the equations

$$y^2 = x^2(x+1)$$
$$y = xu$$

in $\mathbf{A}_k^3$ with coordinates x, y, u. Substituting, we get

$$x^2u^2 - x^2(x+1) = x^2\bigl(u^2 - (x+1)\bigr) = 0.$$

Thus, we have two irreducible components: one defined by $x = y = 0$ and u arbitrary, which is E, and the other defined by $u^2 = x + 1$ and $y = xu$. This is $\tilde{Y} \cap \{t \neq 0\}$, where $\tilde{Y}$ is the *strict transform* of Y. Note that on this open subset of $\mathrm{Bl}_{\mathbf{0}} \mathbf{A}_k^2$, $\tilde{Y} \cap E$ consists of two points $u = \pm 1$. These points correspond to the slopes of the two branches of Y at $\mathbf{0}$.

To finish the computation, we need to know what happens on the affine open subset $u \neq 0$. We set $u = 1$ and use t as our affine coordinate. On this open set, $\varphi^{-1}(Y)$ is given by the equations

$$y^2 = x^2(x+1)$$
$$x = yt$$

in $\mathbf{A}_k^3$ with coordinates x, y, u. Substituting, we get

$$y^2 - (yt)^2(yt+1) = y^2\bigl(1 - t^2(yt+1)\bigr) = 0.$$

Thus, we have two irreducible components: one defined by $x = y = 0$ and t arbitrary, which is E (this gives us no new information), and the other defined by $1 = t^2(yt+1)$, which is $\tilde{Y} \cap \{u \neq 0\}$. Note that the only points along E that was not contained in the open set $t \neq 0$ is when $t = 0$, but there are no such points that are also in $\tilde{Y}$.

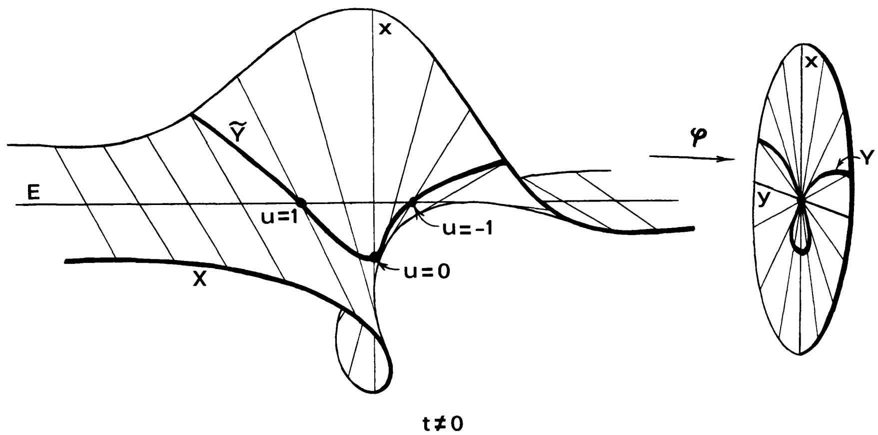


FIGURE 1.18. The blowup of the nodal cubic curve at the origin. From [Har77, Figure 3].

Thus, the affine open subset $u \neq 0$ gave us no new information we could not see already in the subset $t \neq 0$.

We therefore see that the blowup of Y at the origin separates the two branches of the nodal cubic curve at the origin to form the parabola

$$\{u^2 = x + 1\} \subseteq \mathbf{A}_k^2 \cong \{t \neq 0\} \subseteq \mathrm{Bl}_{\mathbf{0}} \mathbf{A}_k^2.$$

EXAMPLE 1.4.28. Let us take another curve, say the x-axis [Har77, Ex. I.4.9.1]

$$L = \{y = 0\} \subseteq \mathbf{A}_k^2.$$

We see that the total inverse image is $\varphi^{-1}(L) = \tilde{L} \cup E$ where $\tilde{L}$ and E intersect at the point $t = 1, u = 0$. Similarly, the y-axis L' has total inverse image $\varphi^{-1}(L') = \tilde{L}' \cup E$ where $\tilde{L}'$ and E intersect at the point $t = 0, u = 1$.

1.4.6. Cremona transformations. For arbitrary n, a birational self-map $\mathbf{P}_k^n \dashrightarrow \mathbf{P}_k^n$ is called a *Cremona transfomation*. The group of Cremona transformations is called the *Cremona group* $\mathrm{Bir}(\mathbf{P}_k^n)$. 10/6

We give a simple example:

EXAMPLE 1.4.29 (Plane Cremona transformations). The *quadratic transformation* is the rational map [Har77, Exer. I.4.6]

$$\begin{aligned} \varphi\colon \mathbf{P}_k^2 &\dashrightarrow \mathbf{P}_k^2 \\ [a_0 : a_1 : a_2] &\longmapsto [a_1a_2 : a_0a_2 : a_0a_1] \end{aligned}$$

that is defined when no two of a_0, a_1, a_2 are 0.

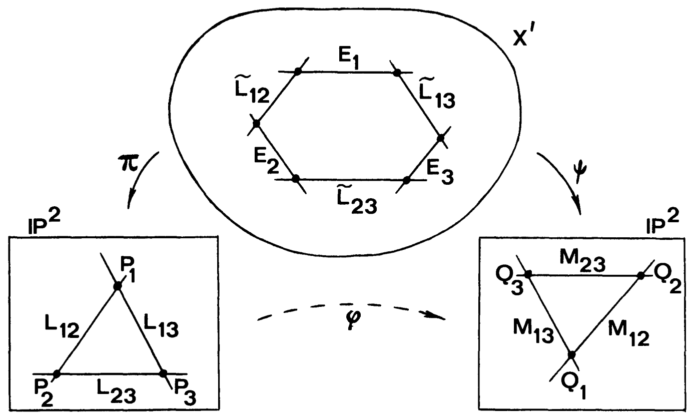


FIGURE 1.19. The quadratic transformation $\mathbf{P}^2_k \dashrightarrow \mathbf{P}^2_k$. From [Har77, Figure 21].

If we compose φ with itself on the open set $\{x_0x_1x_2 \neq 0\}$, we obtain the identity map

$$\begin{aligned}\varphi^2\big([a_0 : a_1 : a_2]\big) &= \varphi\big([a_1a_2 : a_0a_2 : a_0a_1]\big)\\ &= \varphi\left(\left[\frac{1}{a_0} : \frac{1}{a_1} : \frac{1}{a_2}\right]\right)\\ &= \left[\frac{1}{a_1a_2} : \frac{1}{a_0a_2} : \frac{1}{a_0a_3}\right]\\ &= [a_0 : a_1 : a_2]\end{aligned}$$

and hence φ is birational. In fact, φ induces an isomorphism

$$U \coloneqq \{x_0x_1x_2 \neq 0\} \xrightarrow{\sim} \{x_0x_1x_2 \neq 0\} =: V.$$

An open set on which φ is defined is $\mathbf{P}^2_k - \{[1:0:0],[0:1:0],[0:0:1]\}$; this ends up being the largest open subset where it is defined. The line $Z(x_0)$ maps to the point $P_1 = [1:0:0]$, the line $Z(x_1)$ maps to the point $P_2 = [0:1:0]$, and the line $Z(x_2)$ maps to the point $P_3 = [0:0:1]$.

[Har77, Ex. V.4.2.3] We can describe the quadratic transformation in terms of blowups as follows. We consider the variety

$$X' \coloneqq \{x_0y_0 = x_1y_1 = x_2y_2\} \subseteq \mathbf{P}^2_k \times \mathbf{P}^2_k.$$

The two projections $\pi\colon X' \to \mathbf{P}^2_k$ and $\psi\colon V \to \mathbf{P}^2_k$ can be obtained by blowing up 3 points in the respective copies of $\mathbf{P}^2_k$. Consider the open set $U_0 \subseteq \mathbf{P}^2_k$ given by $x_0 = 1$. Then, $U_0 \cong \mathbf{A}^2_k$, and

$$\pi^{-1}(U_0) = \{y_0 = x_1y_1 = x_2y_2\} \subseteq \mathbf{A}^2_k \times \mathbf{P}^2_k.$$

We can eliminate the variable y_0 in the second factor $\mathbf{P}^2_k$, and hence

$$\pi^{-1}(U_0) \cong \{x_1y_1 = x_2y_2\} \subseteq \mathbf{A}^2_k \times \mathbf{P}^1_k,$$

which is the equation of the blowup. By symmetry in $0,1,2$, we see that $\pi\colon V \to \mathbf{P}^2_k$ is indeed the blowup of $\mathbf{P}^2_k$ at the 3 points P_1, P_2, P_3. By symmetry in x, y, we see that $\psi\colon V \to \mathbf{P}^2_k$ is the blowup of $\mathbf{P}^2_k$ at the 3 points P_1, P_2, P_3 as well. So, $\psi \circ \pi^{-1}$ gives a birational map $\mathbf{P}^2_k \dashrightarrow \mathbf{P}^2_k$: it is obtained by blowup the points P_1, P_2, P_3

and then blowing down the strict transforms of the lines L_{23}, L_{13}, L_{12} connecting them. See Figure 1.19.

It remains to show that $\varphi = \psi \circ \pi^{-1}$. We claim that the open set where $x_0x_1x_2 \neq 0$, the preimage of a point $[a_0 : a_1 : a_2]$ in X' is the pair

$$\big([a_0 : a_1 : a_2], [a_1a_2 : a_0a_2 : a_0a_1]\big).$$

Over the open set U_0, we see that this is indeed the case by the previous paragraph, and the same applies to the other open affine charts by symmetry. Thus,

$$(\psi \circ \pi^{-1})\big([a_0 : a_1 : a_2]\big) = [a_1a_2 : a_0a_2 : a_0a_1] = \varphi\big([a_0 : a_1 : a_2]\big).$$

REMARK 1.4.30. It is known that the Cremona group $\mathrm{Bir}(\mathbf{P}^2_k)$ is generated by the projective linear group $\mathrm{PGL}_3(k)$ and the quadratic transformation. This is originally due to Max Noether although apparently there were some gaps in the proof. Guido Castelnuovo fixed some of these, but the proof was not completely fixed until the work of James Alexander [Ale1916]. According to [Dol25_1, Chapter 7, Historical Notes], Alexander's proof is considered to be correct.

A complete set of relations for the generators for $\mathrm{Bir}(\mathbf{P}^2_k)$ is known [Giz82], but the structure of $\mathrm{Bir}(\mathbf{P}^2_k)$ is apparently not well understood. Fairly recently, there have been more results, for example about the linear/topological simplicity and the non-simplicity of $\mathrm{Bir}(\mathbf{P}^n_k)$ [Bla10; CL13; BZ18]. In higher dimensions, much less is known, although the results in [BZ18] apply to higher dimensions as well.

REMARK 1.4.31. We have written the quadratic transformation as a composition of a blowup at three points point and the inverse of a blowup at three points. An open question is: Is every birational map of smooth projective varieties a sequence of blowups along smooth centers followed by a sequence of inverses of blowups along smooth centers? This is known as the *strong factorization conjecture*, and was posed by Hironaka [Hir64_1, Question (F′) on p. 149]. The *weak factorization conjecture* posed by Miyake and Oda [Oda78, p. 60] asks whether it is possible to factor a birational map by sequences of blowups and inverses of blowups along smooth centers. This last conjecture was solved by Włodarczyk [Wło03] and Abramovich–Karu–Matsuki–Włodarczyk [AKMW02].

1.5. Nonsingular varieties

So far, we have defined algebraic varieties and different ways to map between them (either via morphisms or rational maps). We now want to study algebraic varieties more closely. How can we tell them apart? What makes some varieties nicer than others? We saw in our blowup calculations that sometimes the curves we got were simpler just because they did not any self-intersections or cusps.

1.5.1. Nonsingular points. We want a precise way to determine whether our varieties are "simple" in this manner. Over the complex numbers, this corresponds to the subset of $\mathbf{C}^n$ having the structure of a complex manifold by the preimage theorem from differential topology (see [GP10, p. 21] for the preimage theorem for real manifolds). Translating the definition over to our setting, we make the following:

DEFINITION 1.5.1. Let $Y \subseteq \mathbf{A}^n_k$ be an affine variety, and let [Har77, p. 31]

$$f_1, f_2, \dots, f_t \in A = k[x_1, x_2, \dots, x_n]$$

be a set of generators for $I(Y)$. We say that Y is *nonsingular at a point* $P \in Y$ if the rank of the *Jacobian matrix*

$$\left(\frac{\partial f_i}{\partial x_j}(P)\right)$$

is $n-r$, where $r = \dim(Y)$. We say that Y is *nonsingular* if it is nonsingular at every point.

Note that this is only one possible definition of "smoothness." In [Zar47], Zariski discusses different possibilities for what smoothness should mean, which depending on the ground field k do not always coincide. However, even for this definition of smoothness, there are multiple equivalent formulations.

If you want to specify which definition you mean to someone not in the class, you can say you are defining smoothness via the *Jacobian criterion*. The usual rules for changes of coordinates shows that the rank of the Jacobian matrix does not depend on the choice of generators for $I(Y)$.

Remark 1.5.2. When we use partial derivatives in positive characteristic, we can get 0 often! For example,

$$\frac{\partial}{\partial x}x^p = p \cdot x^{p-1} = 0$$

in characteristic $p > 0$.

A fundamental realization of Zariski [Zar47] is that this definition does not depend on the way we embed Y into an affine space. This is because we can detect nonsingularity using properties of local rings. We start by defining this local property.

[Har77, p. 32]
[AM69, p. 123]
[Mat89, p. 105]

Definition 1.5.3. Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring. We say that A is a *regular local ring* if

$$\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim(A).$$

Note that by the NAK lemma [Hoc17, p. 101; AM69, Propsition 2.6; Mat89, Theorem 2.2], this condition says that $\mathfrak{m}$ can be generated by *exactly* $\dim(A)$ elements. Such a set of generators is called a *regular system of parameters*.

If $\mathcal{O}_{Y,P}$ is a local ring of nonsingular point on a variety Y, then you can think of this regular system of parameters as a local coordinate system.

1.5.2. Zariski's equivalence between smoothness and regularity over algebraically closed fields.

We now show Zariski's result:

[Har77, Thm. I.5.1]

Theorem 1.5.4 (Zariski [Zar47, Theorem 7]). *Let $Y \subseteq \mathbf{A}^n$ be an affine variety and let $P \in Y$ be a point. Then, Y is nonsingular at P if and only if the local ring $\mathcal{O}_{Y,P}$ is a regular local ring.*

Proof. Denote $A = k[x_1, x_2, \ldots, x_n]$. Let $P = (a_1, a_2, \ldots, a_n) \in \mathbf{A}_k^n$ and let

$$\mathfrak{a}_P = (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n) \subseteq A$$

be the corresponding maximal ideal. We define a linear map $\theta \colon A \to k^n$ by

$$\theta(f) = \left(\frac{\partial f}{\partial x_1}(P), \frac{\partial f}{\partial x_2}(P), \ldots, \frac{\partial f}{\partial x_n}(P)\right)$$

for $f \in A$. Then, $\{\theta(x_i - a_i)\}_{i=1}^n$ forms a basis for k^n, and $\theta(\mathfrak{a}_P^2) = 0$. Thus, we have an isomorphism

$$\theta' \colon \frac{\mathfrak{a}_P}{\mathfrak{a}_P^2} \xrightarrow{\sim} k^n.$$

We now let

$$\mathfrak{b} := I(Y) = (f_1, f_2, \ldots, f_t) \subseteq A.$$

Then, the rank of the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(P) & \frac{\partial f_1}{\partial x_2}(P) & \cdots & \frac{\partial f_1}{\partial x_n}(P) \\ \frac{\partial f_2}{\partial x_1}(P) & \frac{\partial f_2}{\partial x_2}(P) & \cdots & \frac{\partial f_2}{\partial x_n}(P) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_t}{\partial x_1}(P) & \frac{\partial f_t}{\partial x_2}(P) & \cdots & \frac{\partial f_t}{\partial x_n}(P) \end{pmatrix}_{t \times n}$$

is the dimension of $\theta(\mathfrak{b})$ as a subspace of k^n. Under the isomorphism θ',

$$\dim_k\bigl(\theta(\mathfrak{b})\bigr) = \dim_k\left(\frac{\mathfrak{b} + \mathfrak{a}_P^2}{\mathfrak{a}_P^2}\right).$$

On the other hand, $\mathcal{O}_{Y,P} = (A/\mathfrak{b})_{\mathfrak{a}_P}$, and hence if $\mathfrak{m}_P$ is the maximal ideal of $\mathcal{O}_{Y,P}$, we have

$$\frac{\mathfrak{m}_P}{\mathfrak{m}_P^2} \cong \frac{\mathfrak{a}_P}{\mathfrak{b} + \mathfrak{a}_P^2}.$$

By counting dimensions of k-vector spaces, we therefore see that

$$\dim_k\left(\frac{\mathfrak{m}_P}{\mathfrak{m}_P^2}\right) + \operatorname{rank}(J) = \dim_k\left(\frac{\mathfrak{a}_P}{\mathfrak{b} + \mathfrak{a}_P^2}\right) + \dim_k\left(\frac{\mathfrak{b} + \mathfrak{a}_P^2}{\mathfrak{a}_P^2}\right) = n. \tag{1.5.5}$$

Now let $r = \dim(Y)$. Then, $\mathcal{O}_{Y,P}$ is a local ring of dimension r by Theorem 1.3.12, and hence $\mathcal{O}_{Y,P}$ is a regular local ring if and only if $\dim_k(\mathfrak{m}_P/\mathfrak{m}_P^2) = r$. This is equivalent to $\operatorname{rank}(J) = n - r$, as claimed. □

DEFINITION 1.5.6. Let X be a quasi-projective variety over k and let $P \in X$. The *Zariski cotangent space at P* is the k-vector space [Har77, Exer. II.2.8]

$$\frac{\mathfrak{m}_{X,P}}{\mathfrak{m}_{X,P}^2}.$$

The *Zariski tangent space at P* is the k-vector space

$$T_P(X) := \left(\frac{\mathfrak{m}_{X,P}}{\mathfrak{m}_{X,P}^2}\right)^\vee$$

dual to the Zariski cotangent space.

1.5.3. Examples. Let us see some examples.

EXAMPLE 1.5.7. The nodal cubic $y^2 = x^2(x+1)$ and the cuspidal cubic $y^2 = x^3$ are both singular at the origin $\mathbf{0}$. To see this, we first write the curves as zero sets: the nodal cubic is $Z(y^2 - x^2(x+1))$ and the cuspidal cubic is $Z(y^2 - x^3)$. The Jacobians are

$$\begin{pmatrix} -2x(x+1) - x^2 & 2y \end{pmatrix} = \begin{pmatrix} -x(3x+2) & 2y \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} -3x^2 & 2y \end{pmatrix}.$$

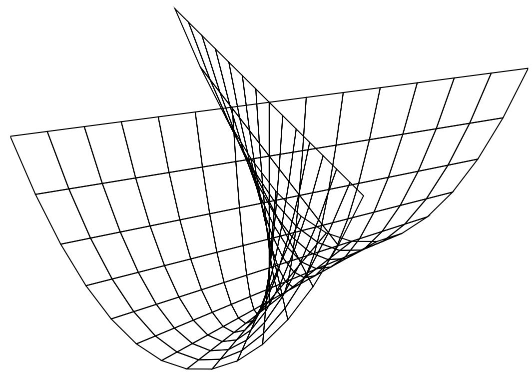


FIGURE 1.20. The Whitney umbrella.

For both curves, a necessary condition to have rank < 1 is that $y = 0$. The point $(-2/3, 0)$ is not on the nodal cubic, and hence the only singular points on these curves is the origin $\mathbf{0}$.

[Har77, Exer. I.5.2] On the other hand, the *Whitney umbrella* (also called a *pinch point* [Har77, Figure 5]) $xy^2 = z^2$ is singular along a whole line! The Jacobian for $xy^2 - z^2 = 0$ is

$$\begin{pmatrix} y^2 & 2xy & -2z \end{pmatrix}$$

which is 0 everywhere along the line $y = z = 0$, which is contained in the Whitney umbrella. See Figure 1.20.

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We saw last week that blowing up curve singularities can remove the singularities of the curve completely. This did not occur for the Whitney umbrella, which is a surface. You may be tempted to think that blowing up curve singularities resolves those singularities. This is not always the case.

[Har77, Exer. I.5.1(*a*)] EXAMPLE 1.5.8 (Tacnode). We consider the *tacnode*

$$Y := Z(x^2 - y^4 - x^4) \subseteq \mathbf{A}_k^2.$$

See Figure 1.21 for an illustration. The Jacobian of Y is

$$\begin{pmatrix} 2x - 4x^3 & -4y^3 \end{pmatrix}$$

which has rank < 1 only at the origin $\mathbf{0}$.

[Har77, Exer. I.5.6(*c*)] We compute the blowup of Y at $\mathbf{0}$. In U_t, we can dehomogenize the defining equations of $\varphi^{-1}(Y)$ by letting $t = 1$ to give

$$\begin{aligned} \varphi^{-1}(Y) \cap U_t &= Z(x^2 - y^4 - x^4, xu - y) \\ &= Z(x^2 - x^4u^4 - x^4, xu - y) \\ &= Z(x^2(1 - x^2u^4 - x^2), xu - y) \end{aligned}$$

Thus, the exceptional curve E in U_t is given by $x = y = 0$, u arbitrary, and the strict transform $\tilde{Y}$ in U_t is given by $1 - x^2u^4 - x^2 = 0$ and $xu = y$. Thus, $\tilde{Y}$ and E

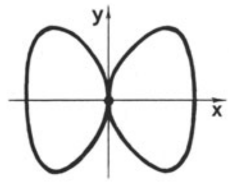

FIGURE 1.21. A tacnode. From [Har77, Figure 4].

do not intersect in U_t. Next, in U_u, we can dehomogenize by letting $u = 1$ to give

$$\begin{aligned}\varphi^{-1}(Y) \cap U_u &= Z(x^2 - y^4 - x^4, x - yt)\\ &= Z(y^2t^2 - y^4 - y^4t^4, x - yt)\\ &= Z(y^2(t^2 - y^2 - y^2t^4), x - yt)\end{aligned}$$

Thus, the exceptional curve E in U_u is given by $x = y = 0$, t arbitrary, and the strict transform $\tilde{Y}$ in U_u is given by $t^2 - y^2 - y^2t^4 = 0$ and $x = yt$. Thus, $\tilde{Y} \cap E \cap U_u$ consists of one point, $(0, 0) \times [0 : 1]$.

We want to identify the singularity $(0, 0) \times [0 : 1]$ with a plane curve singularity. So let $W = Z(s^2 - r^2 - r^2s^4) \subseteq \mathbf{A}^2_k$, and define

$$\begin{aligned}\tilde{Y} \cap U_u &\longrightarrow W & \qquad W &\longrightarrow \tilde{Y} \cap U_u\\ (x, y, t) &\longmapsto (y, t) & (r, s) &\longmapsto (rs, r, s)\end{aligned}$$

These maps are inverse to each other by definition.

This computation shows that $\tilde{Y}$ has a *node* (see [Har77, Exercise I.5.6(b)] for the definition) at $\varphi^{-1}(\mathbf{0})$ (where φ denotes the blowup of Y). In [Har77, Exercise I.5.6(b)], you will show that blowing up the node once resolves the singularity. Thus, we see that blowing up the tacnode twice resolves the singularity at $\mathbf{0}$.

Next, let's try to blowup a "higher-order" cusp.

EXAMPLE 1.5.9. Consider the plane curve [Har77, Exer. I.5.6(d)]

$$Y := Z(y^3 - x^5) \subseteq \mathbf{A}^2_k.$$

This is an example of a *triple point*. Now consider the inverse image $\varphi^{-1}(Y)$; in U_t, we can dehomogenize the defining equations of $\varphi^{-1}(Y)$ by letting $t = 1$ to give

$$\begin{aligned}\varphi^{-1}(Y) \cap U_t &= Z(y^3 - x^5, xu - y)\\ &= Z(x^3u^3 - x^5, xu - y)\\ &= Z(x^3(u^3 - x^2), xu - y)\end{aligned}$$

Thus, the exceptional curve E in U_t is given by $x = y = 0$, u arbitrary, and the strict transform $\tilde{Y}$ in U_t is given by $u^3 - x^2 = 0$ and $xu = y$. Thus, $\tilde{Y} \cap E \cap U_t$ consists of one point, $(0, 0) \times [1 : 0]$. On the other hand, in U_u dehomogenizing the defining equations of $\varphi^{-1}(Y)$ by letting $u = 1$ gives

$$\begin{aligned}\varphi^{-1}(Y) \cap U_u &= Z(y^3 - x^5, x - yt)\\ &= Z(y^3 - y^5t^5, x - yt)\\ &= Z(y^3(1 - y^2t^5), x - yt)\end{aligned}$$

and the exceptional curve E in U_u is given by $x = y = 0$, t arbitrary, while the strict transform $\tilde{Y}$ in U_u is given by $1 - y^2t^5 = 0$ and $x = yt$; these do not intersect. So it suffices to consider the point $(0,0) \times [1:0]$ in U_t. We claim it is a cusp (in the sense from before). For, let $W = Z(s^3 - r^2) \subseteq \mathbf{A}^2$, and define

$$\begin{aligned} \tilde{Y} \cap U_t &\longrightarrow W \\ (x, y, u) &\longmapsto (x, u) \end{aligned} \qquad\qquad \begin{aligned} W &\longrightarrow \tilde{Y} \cap U_t \\ (r, s) &\longmapsto (r, rs, s) \end{aligned}$$

These maps are clearly inverse to each other, and so we see that $\tilde{Y}$ indeed has a cusp at $\{(0,0) \times [1:0]\} = \varphi^{-1}(\mathbf{0})$ (where φ denotes the blowup of Y). Note this cusp is the same as that from [Har77, Exercise I.4.10]. We therefore see that blowing up this cusp again resolves the singularity at $(0,0) \times [1:0]$, and so blowing up twice resolves the singularity of our point $\mathbf{0} \in Y$.

1.5.4. Chevalley's example. We now give an example where Zariski's Theorem 1.5.4 from [Zar47] fails when k is not algebraically closed. This example shows how how can use the *proof* of Theorem 1.5.4 to actually compute whether explicit examples of local rings are regular local.

[Mur19, Ex. B.1.2] EXAMPLE 1.5.10 (Chevalley [Zar47, Example 3]). Let k be an imperfect field of characteristic $p > 2$, and let $a \in k - k^p$. For example, we can let $k = \mathbf{F}_p(t)$ and let $a = t$. Let $S = k[x, y]$ and $f = y^2 + x^p - a \in S$, and consider Chevalley's example [Zar47, Example 3]

$$R = \frac{S}{(f)} = \frac{k[x,y]}{y^2 + x^p - a}.$$

Note that R is smooth everywhere except at the maximal ideal $\mathfrak{m}_R \coloneqq (x^p - a, y)$, since the Jacobian for R is $(0\ 2y)$.

We claim that $R_{\mathfrak{m}_R}$ is a regular local ring, even though R fails the Jacobian criterion at this maximal ideal. We denote by $\mathfrak{m}_S$ the ideal $(x^p - a, y)S \subseteq S$. We have

$$\dim_{S/\mathfrak{m}_S}\left(\frac{\mathfrak{m}_S}{\mathfrak{m}_S^2}\right) = 2,$$

since S is regular. On the other hand, the defining equation $f = y^2 + x^p - a$ for R is nonzero modulo $\mathfrak{m}_S^2$, hence

$$\dim_{R/\mathfrak{m}_R}\left(\frac{\mathfrak{m}_R}{\mathfrak{m}_R^2}\right) = \dim_{S/\mathfrak{m}_S}\left(\frac{\mathfrak{m}_S}{\mathfrak{m}_S^2 + (f)}\right) = 1.$$

Thus, $R_{\mathfrak{m}_R}$ is regular. On the other hand, R fails the Jacobian criterion at this maximal ideal, in the sense that the Jacobian

$$\begin{pmatrix} 0 & 2y \end{pmatrix}$$

is equivalent to $(0\ 0)$ modulo the maximal ideal $(x^p - a, y)$.

REMARK 1.5.11 (Not tested). Chevalley's Example 1.5.10 is interesting because it shows that regularity does not behave well under field extensions. (In contrast, the definition of smoothness over non-algebraically closed fields will turn out to behave well.) Set

$$\begin{aligned} R' &\coloneqq R \otimes_k k(a^{1/p}) \\ &\cong \frac{k(a^{1/p})[x,y]}{y^2 + x^p - a} \\ &\cong \frac{k(a^{1/p})[x,y]}{y^2 + (x - a^{1/p})^p} \end{aligned}$$

and denote $\mathfrak{m}_{R'} = (x - a^{1/p}, y)R'$. We have that

$$\begin{aligned} y^2 + x^p - a &= y^2 + (x - a^{1/p})^2 \cdot (x - a^{1/p})^{2-p} \\ &\in (x - a^{1/p}, y)^2, \end{aligned}$$

and hence

$$\dim_{R'/\mathfrak{m}_{R'}}\left(\frac{\mathfrak{m}_{R'}}{\mathfrak{m}_{R'}^2}\right) = \dim_{S/\mathfrak{m}_S}\left(\frac{\mathfrak{m}_S}{\mathfrak{m}_S^2 + (f)}\right) = 2.$$

Thus, we see that R' is not regular at the maximal ideal $\mathfrak{m}_{R'}$.

1.5.5. The singular locus is closed. Our next goal is to show that *most* points on quasi-projective varieties are nonsingular. We start with an algebraic preliminary. Recall that denoting a local ring by $(A, \mathfrak{m}, k)$ is shorthand for "A is a local ring with maximal ideal $\mathfrak{m}$ and residue field k."

PROPOSITION 1.5.12. *Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring. We have* [Har77, Prop. I.5.2A] [AM69, Cor. 11.15] [Mat89, p. 105]

$$\operatorname{embdim}(A) \coloneqq \dim_k\left(\frac{\mathfrak{m}}{\mathfrak{m}^2}\right) \geq \dim(A).$$

The number $\operatorname{embdim}(A)$ is called the *embedding dimension* of A, although we will not use this terminology much.

Proof. By NAK [Hoc17, p. 101; AM69, Propsition 2.6; Mat89, Theorem 2.2], the embedding dimension $\operatorname{embdim}(A)$ equals the minimal number of generators of $\mathfrak{m}$. Now by Krull's Height Theorem 1.1.48, we know that

$$\dim(A) = \operatorname{ht}(\mathfrak{m}) \leq \operatorname{embdim}(A). \qquad \square$$

THEOREM 1.5.13. *Let Y be a quasi-projective variety. Then, the set* $\operatorname{Sing}(Y)$ *of singular points of Y is a proper closed subset of Y.* [Har77, Thm. I.5.3]

We call $\operatorname{Sing}(Y)$ the *singular locus* of Y.

Proof. We first show that $\operatorname{Sing}(Y)$ is closed. Let $Y \subseteq \mathbf{P}^n_k$ with an affine open cover $Y = \bigcup_i U_i$. It suffices to show that $\operatorname{Sing}(Y) \cap U_i$ is closed for every i. Replacing Y by the $Y \cap U_i$, we may therefore assume that Y is affine. By the proof of Theorem 1.5.4 (specifically (1.5.5)), we know that

$$\operatorname{embdim}(A) + \operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(P)\right) = n.$$

By Proposition 1.5.12, we therefore see that

$$\operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(P)\right) \leq n - r.$$

Thus,

$$\operatorname{Sing}(Y) = \left\{ P \in Y \,\middle|\, \operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(P)\right) < n - r \right\}$$
$$= Y \cap Z\left((n-r) \times (n-r) \text{ minors of } \left(\frac{\partial f_i}{\partial x_j}(P)\right)\right),$$

which is closed in Y.

It remains to show that $\operatorname{Sing}(Y) \subsetneq Y$. Since birational varieties have isomorphic open subsets, we may replace Y by a hypersurface birational to it (Theorem 1.4.10) to assume that $Y = Z(f) \subseteq \mathbf{A}_k^n$ is a hypersurface, where $f \in k[x_1, x_2, \ldots, x_n]$ is an irreducible polynomial. If $\operatorname{Sing}(Y) = Y$, then the functions $\partial f/\partial x_i$ are zero on Y, and hence

$$\frac{\partial f}{\partial x_i} \in I(Y)$$

for all i. But $I(Y)$ is the principal ideal generated by f, and hence

$$\deg\left(\frac{\partial f}{\partial x_i}\right) \le \deg(f) - 1$$

for all i. This means that we must have $\partial f/\partial x_i = 0$ for all i. This is impossible in characteristic 0, since if x_i occurs in f, then $\partial f/\partial x_i \neq 0$. If $\operatorname{char}(k) = p > 0$, then the fact that $\partial f/\partial x_i = 0$ for all i means that f is a polynomial in the p-th powers x_i^p of the variables x_i. Taking p-th roots of coefficients, we see that $f = g^p$ for a polynomial $g \in k[x_1, x_2, \ldots, x_n]$. This contradicts the assumption that f is irreducible. □

1.5.6. Classifying plane curve singularities: Multiplicity and Analytic isomorphism. A question we asked last time is: How do we classify plane curve singularities? Something you may have noticed is that it seems that the lowest degree terms have the most influence on a plane curve singularity, and at least gives a first measure for how "bad" a singularity can be. This gives us a first measure for how bad the singularities are.

[Har77, Exer. I.5.3] Definition 1.5.14 (Multiplicity of a plane curve). Let $Y = Z(f) \subseteq \mathbf{A}_k^2$ be a plane curve. Let $P = (a, b) \in \mathbf{A}_k^2$ be a point. Make a(n affine) linear change of coordinates so that P becomes the origin $(0, 0)$. Then, we can write

$$f = f_0 + f_1 + \cdots + f_d$$

where f_i is a homogeneous polynomial of degree i in x and y. The *multiplicity* of P on Y is

$$\mu_P(Y) := \min\{r \mid f_r \neq 0\}.$$

Note that $P \in Y$ if and only if $\mu_P(Y) > 0$. The linear factors of f_r are called the *tangent directions* at P.

Remark 1.5.15. A similar definition works well for hypersurfaces on nonsingular varieties. See [Har77, p. 388]. For a general quasi-projective variety (i.e., one that does not necessarily embed as a hypersurface on a nonsingular variety), one defines multiplicity using commutative-algebraic properties of the local ring $\mathcal{O}_{Y,P}$. The resulting definition is called *Hilbert–Samuel multiplicity*. See [Har77, Exercise V.3.4] for Hartshorne's discussion of this invariant. In general, there are various multiplicity functions that measure how bad singularities are.

We show that the multiplicity can detect nonsingular points.

LEMMA 1.5.16. *Let $Y \subseteq \mathbf{A}_k^2$ be a plane curve and let $P \in \mathbf{A}_k^2$ be a point. Then, $\mu_P(Y) = 1$ if and only if P is a nonsingular point of Y.* [Har77, Exer. I.5.3(a)]

Proof. By definition, $\mu_P(Y) = 1$ if and only if $f_1 \neq 0$. This is equivalent to one of $\partial f/\partial x$ or $\partial f/\partial y$ evaluated at $(0,0)$ being nonzero because higher order terms in f will not affect whether $\partial f/\partial x$ and $\partial f/\partial y$ vanish. But this is equivalent to the Jacobian evaluated at $(0,0)$ being nonzero, i.e., to Y being nonsingular at P. □

The multiplicity gives a nice invariant, but we want something finer. For example, we saw that the singularities of the node

$$y^2 = x^2(x+1) \tag{1.5.17}$$

or the cusp

$$y^2 = x^3$$

are resolved after one blowup, but the singularities of the higher order cusp

$$y^2 = x^5$$

are not. On the other hand, intuition from the Euclidean topology tells us that even though the nodal cubic (1.5.17) and the union of two axes

$$y^2 = x^2$$

look different algebraically, we might want to consider them to be "the same." So, we want a way to classify plane curve singularities that is finer than multiplicity but weaker than "the local rings are isomorphic": it is very hard to tell when two rings are isomorphic, and we will soon see that there are nonsingular points with non-isomorphic local rings. (This goes back to a question we had before: We have yet to construct an example of an irrational curve. But such an example would yield an example of two nonsingular points on curves that are not isomorphic.)

On the other hand, since a local ring on a variety over k is regular local if and only if its completion is isomorphic to a formal power series ring over k, we will try to make our equivalence relation the following: The local rings of the two curves at the origin have the same completions. This will be coarse enough to fix our issue with the fact that nonsingular points have non-isomorphic local rings, but fine enough to differentiate different order cusps. So, this new equivalence relation is a suitable notion to use to study singularities.

We recall the definition of the completion of a local ring.

DEFINITION 1.5.18. Let $(A, \mathfrak{m})$ be a Noetherian local ring. The $\mathfrak{m}$-*adic completion of* A is [Har77, p. 33] [Mat89, §8] [Hoc17, p. 156–157] [AM69, Ch. 10]

$$\begin{aligned}\hat{A} &:= \varprojlim_n A/\mathfrak{m}^n \\ &:= \left\{ (a_1, a_2, \ldots) \in \prod_{i=1}^n A/\mathfrak{m}^n \;\middle|\; a_j \equiv a_i \bmod \mathfrak{m}^i \text{ for all } j \geq i \right\}.\end{aligned}$$

An equivalent definition (in this case) is that $\hat{A}$ is formed by taking equivalence classes of Cauchy sequences in the $\mathfrak{m}$-adic topology on A.

When talking about Noetherian local rings, we will drop the prefix "$\mathfrak{m}$-adic." (One can define I-adic completions for other ideals $I \subseteq A$, but the case when $I = \mathfrak{m}$ is the most important one for now.)

Here are the important facts about completions we will use.

[Har77, Thm. I.5.4A]
[AM69, Ch. 10, 11]
[ZS75$_2$, Ch. VIII]

THEOREM 1.5.19. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with completion $\hat{A}$.*

(*a*) *$\hat{A}$ is a Noetherian local ring with maximal ideal $\hat{\mathfrak{m}} := \mathfrak{m}\hat{A}$, and there is a natural injective ring map $A \to \hat{A}$ (it is even faithfully flat).*
(*b*) *If M is a finitely generated A-module, then its completion $\hat{M}$ with respect to its $\mathfrak{m}$-adic topology is isomorphic to $M \otimes_A \hat{A}$.*
(*c*) $\dim(A) = \dim(\hat{A})$.
(*d*) *A is regular if and only if $\hat{A}$ is regular.*

The following is a very difficult theorem due to I. S. Cohen.

[ZS75$_2$, p. 307]
[Mat89, Thm. 29.4]

THEOREM 1.5.20 (Cohen structure theorem [Coh46]). *If $(A, \mathfrak{m}, k)$ is a complete regular local ring of dimension n containing some field, then*

$$A \cong k[\![x_1, x_2, \ldots, x_n]\!],$$

where $x_1, x_2, \ldots, x_n$ are the generators of $\mathfrak{m}$.

Using completions, we can define the following:

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DEFINITION 1.5.21. Let $P \in X$ and $Q \in Y$ be points on quasi-projective varieties. We say that P and Q are *analytically isomorphic* if there is an isomorphism

$$\hat{\mathcal{O}}_{X,P} \cong \hat{\mathcal{O}}_{Y,Q}$$

as k-algebras.

EXAMPLE 1.5.22. Let $P \in X$ and $Q \in Y$ be points on quasi-projective varieties.

(*i*) If P and Q are analytically isomorphic, then $\dim(X) = \dim(Y)$ by Theorem 1.5.19(*c*) and Theorem 1.3.12(*c*).
(*ii*) If P and Q are nonsingular points on quasi-projective varieties of the same dimension, then P and Q are analytically isomorphic. This follows from Theorem 1.5.19(*d*) and Theorem 1.5.20. This is the algebraic geometry analogue of the fact that two manifolds of the same dimension are locally isomorphic (in the topological, differential, or complex-analytic categories).

We now see that the origin on the nodal cubic and the union of two axes are analytically isomorphic.

EXAMPLE 1.5.23. Suppose $\mathrm{char}(k) \neq 2$. Let X be the nodal cubic curve

$$X := \{y^2 = x^2(x+1)\} \subseteq \mathbf{A}^2_k$$

and let Y be the algebraic set in $\mathbf{A}^2_k$ defined by the equation

$$Y := \{xy = 0\} \subseteq \mathbf{A}^2_k.$$

We claim that $\mathbf{0} \in X$ and $\mathbf{0} \in Y$ are analytically isomorphic.

For Y, we have not defined local rings for reducible algebraic sets. However, whatever the precise definition should be, the answer should be

$$\mathcal{O}_{Y,\mathbf{0}} = \left(\frac{k[x,y]}{(xy)}\right)_{(x,y)}.$$

We then see that

$$\hat{\mathcal{O}}_{Y,\mathbf{0}} = \frac{k[\![x,y]\!]}{(xy)}.$$

For X, we need to identify

$$\hat{\mathcal{O}}_{X,\mathbf{0}} = \frac{k[\![x,y]\!]}{\bigl(y^2 - x^2(x+1)\bigr)}.$$

Since $\operatorname{char}(k) \neq 2$, we have

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots$$

which is a unit in $k[\![x]\!]$. (This equation just means that the square of the right-hand side is equal to $1+x$.) Set g to be the formal power series on the right-hand side. Then, we have

$$\hat{\mathcal{O}}_{X,\mathbf{0}} = \frac{k[\![x,y]\!]}{\bigl(y^2 - (xg)^2\bigr)} \cong \frac{k[\![x,y]\!]}{(y^2 - x^2)} = \frac{k[\![x,y]\!]}{(y+x)(y-x)}$$

where the middle isomorphism follows from the fact that g is a unit. The k-algebra on the right-hand side is isomorphic to $\hat{\mathcal{O}}_{Y,\mathbf{0}}$.

You will see more on analyticly isomorphic singularities on Homework 7.

1.6. Nonsingular curves

Now that we have talked about the fundamental objects and morphisms we consider in algebraic geometry, we want to return to a question we had before: How do we classify quasi-projective varieties? This a general theme in subfields of mathematics. We first define some objects of interest. Then, we want to classify them, i.e., say when they are "different" or the "same" with respect to some equivalence relation. This becomes important when applying the theory to concrete problems: By applying a suitable classification, you can divide a proof up into cases, and treat them one case at a time. [Har77, p. 39]

In algebraic geometry, as we have discussed before, the basic theme of birational geometry is to classify quasi-projective varieties by proceeding in three steps. Recall that the goal of birational geometry is to classify varieties up to birational equivalence.

(*a*) Find a "nicest" representative in each birational equivalence class.
(*b*) Classify the representatives appearing in (*a*).

In general, these questions are very difficult! The classical approach is to interpret "nicest" to mean "nonsingular," but this has the drawback of making Question (*b*) more difficult. Another sometimes useful approach is to interpret "nicest" to mean "has the fewest number of defining equations," which we applied last time to show that the nonsingular locus is a nonempty open subset.

Progress in the last forty years has shown that interpreting "nicest" to mean "minimal" is very useful—this is the core philosophy of the minimal model program, where "minimal" roughly means that the variety does not have any extra exceptional divisors that can be blown down. With this interpretation, Question (*a*) is the core question in the minimal model program. There are still major open problems in the minimal model program in dimensions ≥ 4. Classifying the actual representatives (Question (*b*)) that appear is very difficult. (If you want to have more words to look up, Iskovskih, Shokurov, and Mori–Mukai classified Fano threefolds over the complex numbers [Isk77; Isk78; Šok79; Šok80; MM82; MM83; MM03]. In positive characteristic, there was progress due to Megyesi, Shepherd-Barron, and Saito [Meg98; She97; Sai03], but the full classification was only completed last year by Tanaka and Asai–Tanaka [Tan24_1; Tan24_2; AT25; Tan23]. I have heard that people have tried to prove analogous statements in dimension 4, but that not much progress has been made.) However, the story in dimension 2 is more or less complete—while we will not get to see the full story in this class (or even next semester), it is very interesting!

[Kol14]
[Sha13_2, p. 229ff]

In this section, we focus on the story in dimension 1. In this case, the story is well-understood, and goes back to the work of Riemann. Riemann's original motivations came from trying to compute elliptic integrals of the form

$$\int \frac{dx}{\sqrt{x^3 + ax^2 + bx + c}}$$

which can be traced back historically to the work of Euler, Abel, and Jacobi. Riemann's key insight was that instead of trying to deal with a multi-valued function like $\sqrt{x^3 + ax^2 + bx + c}$, it is better to look at the complex algebraic curve

$$\Gamma = \{y^2 = x^3 + ax^2 + bx + c\} \subseteq \mathbf{A}^2_{\mathbf{C}}.$$

The integral above then transforms into

$$\int_\Gamma \frac{dx}{y}.$$

This is (part of) the content of Riemann's existence theorem: Given a finite extension of $\mathbf{C}(t)$, how can we find a nonsingular complex algebraic curve with that function field?

We will approach this question from an algebraic perspective using valuation theory, which goes back at least to the work of Zariski [Zar40; Zar44]. From this point of view, what we will do is to define topological spaces using valuation-theoretically that we think of as "abstract" nonsingular curves. We will then show that these are all quasi-projective.

1.6.1. Valuations. We recall some definitions on valuations. You may have seen these in an algebraic number theory or commutative algebra course.

[Har77, pp. 39–40]

Definition 1.6.1. Let K be a field and let G be a totally ordered Abelian group (written additively). A *valuation of K with values in G* is a map $v\colon K - \{0\} \to G$ such that for all $x, y \in K - \{0\}$, we have

(1) $v(xy) = v(x) + v(y)$;
(2) $v(x + y) \geq \min\{v(x), v(y)\}$.

If v is a valuation, then the set

$$R = \bigl\{x \in K \bigm| v(x) \geq 0\bigr\} \cup \{0\}$$

is a subring of K, which we call the *valuation ring of v*. (Especially in algebraic number theory and arithmetic geometry, R is denoted by $\mathcal{O}_K$.) The valuation ring R is a local ring with maximal ideal

$$\mathfrak{m} = \bigl\{x \in K \bigm| v(x) > 0\bigr\} \cup \{0\}.$$

In general, a *valuation ring* is an integral domain which is the valuation ring of some valuation of its fraction field. If R is a valuation ring with quotient field K, we say that R is a *valuation ring of* K. If k is a subfield of K such that $v(x) = 0$ for all $x \in k - \{0\}$, we say that v is a *valuation of* K/k and that R is a *valuation ring* of K/k.

Note that valuation rings need not be Noetherian!

EXAMPLE 1.6.2. Let k be a field. Consider the non-Noetherian ring

[Har77, Exer. II.4.12(b)(3)]

$$k\left[x, y, \frac{x}{y}, \frac{x}{y^2}, \ldots\right]_{\mathfrak{m}} \subseteq k(x, y)$$

where $\mathfrak{m}$ is generated by all the generators on the left-hand side. This is the valuation ring of $k(x, y)$ with value group $\mathbf{Z}^2$ with the lexicographic order, where

$$v(x^i y^j) = (i, j).$$

In this section, we will focus on the case when $G = \mathbf{Z}$.

DEFINITION 1.6.3. A *non-trivial* valuation v is *discrete of rank 1* if its value group G is the integers. The corresponding valuation ring is called a *discrete rank 1 valuation ring*, abbreviated *DVR*.

[Har77, p. 40] We deviate from [Har77]. "Discrete valuation ring" can cause confusion because $\mathbf{Z}^2$ also seems like it should be discrete.

DVRs can be characterized in many ways.

THEOREM 1.6.4 [AM69, Proposition 9.2; Hoc17, p. 139]. *Let $(A, \mathfrak{m})$ be a Noetherian local domain of dimension 1. Then, the following conditions are equivalent.*

[Har77, Thm. I.6.2A]

(*i*) *A is a DVR.*
(*ii*) *A is normal (i.e. integrally closed in its fraction field).*
(*iii*) *A is a regular local ring.*
(*iv*) *$\mathfrak{m}$ is a principal ideal.*
(*v*) *A is a PID.*

(v) is not in [Har77] but follows from the fact that any element in $A - \mathfrak{m}$ is a unit.

DEFINITION 1.6.5. A *Dedekind domain* is a Noetherian integrally closed domain of dimension 1.

[Har77, p. 38]

Since integral closure and normality can be checked locally [AM69, Proposition 5.14], we can think of a Dedekind domain as a non-local version of a DVR. The following theorem tells us that taking integral closures in finite extensions of the fraction field do not take us outside the realm of Dedekind domains. Note that the key assertion in this theorem is that the integral closure of a Dedekind domain in a finite extension of its fraction field is Noetherian – this integral closure is an integrally closed domain (by definition) and is of dimension 1 (by Going Up).

THEOREM 1.6.6 (Corollary to the Krull–Akizuki theorem [Kru30; Aki35]). *The integral closure of a Dedekind domain in a finite extension of its fraction field is a Dedekind domain.*

[Har77, Thm. I.6.3A]

Proof. See [ZS75_1, Chapter V, §8, Theorem 19 on p. 281] for this statement. For the full Krull–Akizuki theorem, see [Mat89, Theorem 11.7]. □

REMARK 1.6.7. Caution: The analogue of the Krull–Akizuki theorem in higher dimensions is not true! Constructing an example of this is harder than the non-Noetherian valuation ring, so I will not do this in class.

1.6.2. Abstract nonsingular curves. We now return to our usual setting: k is our algebraically closed ground field. Recall that we want to understand function fields K of dimension 1 (i.e., finitely generated fields of transcendence degree 1) over k. We want to connect nonsingular curves Y with function field K and the set

$$C_K \coloneqq \{\text{DVRs of } K/k\}.$$

Note that this is reasonable: We have a map

$$\begin{aligned} Y &\longrightarrow C_K \\ P &\longmapsto \mathcal{O}_{Y,P} \end{aligned}$$

because if $P \in Y$ is a point on a nonsingular curve, then $\mathcal{O}_{Y,P}$ is a DVR by combining Zariski's Theorem 1.5.4 with Theorem 1.6.4. We have $\operatorname{Frac}(\mathcal{O}_{Y,P}) = K(Y) = K$ and $k \subseteq \mathcal{O}_{Y,P}$, and hence $\mathcal{O}_{Y,P}$ is a DVR of K/k.

We show this map is injective even without assuming that Y is of dimension 1. (For future context, this is another version of the statement that quasi-projective varieties are separated.)

[Har77, Lem. I.6.4] Lemma 1.6.8 (Uniqueness of centers of valuations). *Let Y be a quasi-projective variety. Let $P, Q \in Y$ and suppose that $\mathcal{O}_Q \subseteq \mathcal{O}_P$ as subrings of $K(Y)$. Then, $P = Q$.*

Proof. Write $Y \subseteq \mathbf{P}^n_k$. Replacing Y by $\bar{Y}$, we may assume that Y is projective. After a suitable linear change of coordinates, we may assume that neither P nor Q are in the hyperplane $H_0 \coloneqq Z(x_0)$. Then, $P, Q \in Y \cap (\mathbf{P}^n_k - H_0)$, which is affine, and hence we may assume that Y is affine.

Let $A = A(Y)$. Then, there are maximal ideals $\mathfrak{m}, \mathfrak{n} \subseteq A$ such that $\mathcal{O}_{Y,P} = A_\mathfrak{m}$ and $\mathcal{O}_{Y,Q} = A_\mathfrak{n}$. If $\mathcal{O}_Q \subseteq \mathcal{O}_P$, we must have $\mathfrak{m} \subseteq \mathfrak{n}$. But then, $\mathfrak{m}$ is a maximal ideal, and so $\mathfrak{m} = \mathfrak{n}$. This shows that $P = Q$ by Theorem 1.3.12(*b*). ☐

The trickier part is showing that $Y \to C_K$ is surjective. We will spend quite a while on this. The key algebraic input is the following.

Definition 1.6.9. If $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ are local rings contained in a field K, we say that B *dominates* A if $A \subseteq B$ and $\mathfrak{m}_B \cap A = \mathfrak{m}_A$.

[Har77, Thm. I.6.1A] Theorem 1.6.10. *Let K be a field. A local ring $R \subseteq K$ is a valuation ring if and only if it is a maximal element of the set of local rings contained in K partially ordered by domination. Every local ring contained in K is dominated by some valuation ring of K.*

Proof. See [BouCA, Chapter VI, §1, no. 2, Theorem 1] or [AM69, Chapter 5, p. 65 and Exercises, p. 72]. ☐

We now show:

[Har77, Lem. I.6.5] Lemma 1.6.11. *Let K be a function field of dimension 1 over k and let $x \in K$. Then,*

$$\{R \in C_K \mid x \notin R\}$$

is a finite set.

Proof. If R is a valuation ring, then $x \notin R$ if and only if $1/x \in \mathfrak{m}_R$. So, letting $y = 1/x$, we have to show that if $y \in K - \{0\}$, then

$$\{R \in C_K \mid y \in \mathfrak{m}_R\}$$

is finite. If $y \in k$, then this set is empty. We may therefore assume that $y \notin k$.

Consider the subring $k[y] \subseteq K$. Since k is algebraically closed, y is transcendental over k, and hence $k[y]$ is a polynomial ring. Moreover, since K is finitely generated and of transcendence degree 1 over k, we know that $k[y] \subseteq K$ is a finite field extension. Now let B be the integral closure of $k[y]$ in K. By the Krull–Akizuki Theorem 1.6.6, we know that B is a Dedekind domain. It is also a finitely generated k-algebra by Noether's theorem on the finiteness of integral closure [Har77, Theorem I.3.9A].

Now if y is contained in a DVR $R \in C_K$, then $k[y] \subseteq R$, and since R is integrally closed in K, we have $B \subseteq R$. Let $\mathfrak{n} = \mathfrak{m}_R \cap B$. Then, $\mathfrak{n}$ is a maximal ideal of B and B is dominated by R. But $B_\mathfrak{n}$ is also a DVR of K/k, and hence $B_\mathfrak{n} = R$ by the maximality of valuation rings (Theorem 1.6.10).

If furthermore $y \in \mathfrak{m}_R$, then $y \in \mathfrak{n}$. Now B is the affine coordinate ring of some affine variety Y (Remark 1.1.27). Since B is a Dedekind domain, $\dim(Y) = 1$ and Y is nonsingular. The statement that $y \in \mathfrak{n}$ is saying that y, as a regular function on Y, vanishes at the point of Y corresponding to $\mathfrak{n}$. Since $y \neq 0$, this can only happen at a finite number of points, which are in 1-1 correspondence with maximal ideals of B by Theorem 1.3.12, and $R = B_\mathfrak{n}$ is uniquely determined by the maximal ideal $\mathfrak{n}$. Thus, we conclude that $y \in \mathfrak{m}_R$ for only finitely many $R \in C_K$ as claimed. □

Corollary 1.6.12. *Any DVR of K/k is ismorphic to the local ring of a point on some nonsingular affine curve.* [Har77, Cor. I.6.6]

Proof. Given R, let $y \in R - k$. Then, the construction in the proof of Lemma 1.6.11 gives such a curve. □

We can now define abstract nonsingular curves. 10/22

Definition 1.6.13. Let K be a function field of dimension 1 over k, i.e., a finitely generated field of transcendence degree 1 over k. Let [Har77, p. 42]

$$C_K \coloneqq \bigl\{\text{DVRs of } K/k\bigr\}.$$

We sometimes call the elements $P \in C_K$ *points* and write $(R_P, \mathfrak{m}_P)$ for the corresponding DVR. Note that the set C_K is infinite, since letting B be the ring constructed in Lemma 1.6.11, we have an injection

$$\operatorname{MaxSpec}(B) \longhookrightarrow C_K$$

by Lemma 1.6.8, and $|\operatorname{MaxSpec}(B)| = |k|$ by [Har77, Exercise I.4.8]. We make C_K into a topological space by giving it the finite complement topology: The closed sets are finite sets and the whole space.

The ring of *regular functions* on an open subset $U \subseteq C_K$ is

$$\mathcal{O}(U) \coloneqq \bigcap_{P \in U} R_P.$$

An element $f \in \mathcal{O}(U)$ defines a function $U \to k$ by taking

$$f(P) \coloneqq \bigl(\text{the image of } f \text{ in } R_P/\mathfrak{m}_P \cong k\bigr),$$

where the identification with k holds by Corollary 1.6.12.

An *abstract nonsingular curve* is an open subset $U \subseteq C_K$, where K is a function field of dimension 1 over k. We consider U with the subspace topology and with the induced notion of regular functions on its open subsets.

REMARK 1.6.14. If $f, g \in \mathcal{O}(U)$ define the same function, then $f - g \in \mathfrak{m}_P$ for infinitely many $P \in C_K$, and hence $f = g$ by Lemma 1.6.11 and its proof. We can therefore identify elements of $\mathcal{O}(U)$ with a certain set of functions $U \to k$. Moreover, by the proof of Lemma 1.6.11 any $f \in K$ is a regular function on some open set U. By these two observations, the function field of C_K is K. [Har77, p. 42]

REMARK 1.6.15. The space C_K is an example of a Zariski–Riemann space. These are certain spaces of valuations that can be defined for function fields of arbitrary dimension, and in fact, much more generally. They first appeared in the work of Zariski [Zar40; Zar44] on resolution of singularities and local uniformization. Zariski called these spaces "Riemann manifolds." Nagata added Zariski's name to the terminology since "Riemann manifold" means something else in differential geometry [Nag62, Footnote on p. 2], and used them to prove the famous *Nagata compactification theorem*.

Zariski–Riemann spaces can also be thought of as non-Archimedean analytic spaces [FK06; Tem10]. They also appear naturally as limits of varieties or schemes, where the inverse system consists of all possible blowups of that variety or scheme [Nag63]. One reason they work well for the theory we are working out now is that in dimension 1, all blowups of nonsingular curves are trivial, i.e., they are the identity morphism. However, in dimensions ≥ 2, these spaces are much more complicated than their corresponding quasi-projective varieties. For example, for a surface S, the corresponding Zariski–Riemann space $\mathrm{ZR}(S)$ contains extra points corresponding to discrete rank 2 valuations like the one in Example 1.6.2. More recently, I used Zariski–Riemann spaces to prove vanishing theorems for schemes and a whole bunch of other geometric spaces [Mur25].

Our goal will be to show that C_K is isomorphic to a nonsingular projective curve and every abstract nonsingular curve is isomorphic to a nonsingular quasi-projective curve. However, we have not yet defined what a morphism of an abstract nonsingular curve is!

[Har77, p. 42] DEFINITION 1.6.16. A *morphism* $\varphi\colon X \to Y$ between abstract nonsingular curves or quasi-projective varieties is a continuous map such that for every open subset $V \subseteq Y$ and every regular function $f\colon V \to k$, the pullback $f \circ \varphi$ is a regular function on $\varphi^{-1}(V)$.

We can therefore consider the category whose objects are either abstract nonsingular curves or quasi-projective varieties, and whose morphisms are defined as above. Our goal will be to show that adjoining abstract nonsingular curves to our category of quasi-projective varieties essentially does not introduce any new objects. Here, "essentially" means "up to isomorphism."

We first show that we can think of every nonsingular quasi-projective curve as an abstract nonsingular curve.

[Har77, Prop. I.6.7] PROPOSITION 1.6.17. *Every nonsingular quasi-projective curve Y is isomorphic to an abstract nonsingular curve.*

Proof. Let $K = K(Y)$. Then, we have an injective map

$$\begin{aligned} Y &\hookrightarrow C_K \\ P &\longmapsto \mathcal{O}_P \end{aligned}$$

whose image is a subset $U \subseteq C_K$. Here, injectivity was shown in Lemma 1.6.8. We then get a bijection

$$\begin{aligned} \varphi\colon Y &\longleftrightarrow U \\ P &\longmapsto \mathcal{O}_P. \end{aligned}$$

We show that U is an open subset of C_K. First, since Y is infinite by [Har77, Exercise I.4.8], it suffices to show that U contains a nonempty open set (which would imply that $Y - U$ is finite). By replacing Y with an affine open subset, which exists by Proposition 1.3.36, we may assume that Y is affine.

Set $A = A(Y)$. Then, A is a finitely generated k-algebra. By Theorem 1.3.12, we know that $K = \operatorname{Frac}(A)$ and that

$$U = \{A_{\mathfrak{m}} \mid \mathfrak{m} \subseteq A \text{ maximal}\}.$$

Since all the $A_{\mathfrak{m}}$ are DVRs, Theorem 1.6.10 implies that U consists of all DVRs of K/k containing A. Now let $x_1, x_2, \ldots, x_n \in A$ be the set of generators of A over k. For every $P \in C_K$, we have $A \subseteq R_P$ if and only if $x_1, x_2, \ldots, x_n \in R_P$. Thus,

$$U = \bigcap_{i=1}^{n} \{P \in C_K \mid x_i \in R_P\}.$$

By Lemma 1.6.11, each of the sets in the intersection has finite complement, and is therefore open. Thus, U is also open.

It remains to show that φ is an isomorphism. It suffices to show that φ identifies regular functions. This follows from the definition of regular functions on U and the fact that for open subsests $V \subseteq Y$, we have

$$\mathcal{O}_Y(V) = \bigcap_{P \in V} \mathcal{O}_{Y,P}. \qquad \square$$

The "other" direction, namely, to construct a projective variety isomorphic to an abstract nonsingular curve, is harder. Somehow, we need to cook up a projective variety that contains an open subset isomorphic to C_K. The way we will do this is by first finding rational a map $C_K \dashrightarrow \mathbf{P}^n_k$ and then extending it to a morphism $C_K \to \mathbf{P}^n_k$. However, this morphism does not necessarily embed C_K into $\mathbf{P}^n_k$ as a closed subvariety! We will have to fix this issue momentarily, but for now, we prove that we can extend rational maps $C_K \dashrightarrow \mathbf{P}^n_k$ to *some* morphism. In scheme-theoretic language, the following is a version of the valuative criterion of properness [Har77, Theorem II.4.7]. Under this interpretation, Lemma 1.4.1 corresponds to the valuative criterion of separatedness [Har77, Theorem II.4.3].

Theorem 1.6.18 (Curve-to-projective extension theorem). *Let X be an abstract nonsingular curve, let $P \in X$, let Y be a projective variety, and let*

[Har77, Prop. I.6.8] The name for the theorem is from [Vak25, Thm. 15.3.1].

$$\varphi\colon X - \{P\} \longrightarrow Y$$

be a morphism. Then, there exists a unique morphism $\bar{\varphi}\colon X \to Y$ extending φ:

$$\begin{array}{ccc} X - \{P\} & \xrightarrow{\ \varphi\ } & Y \\ \big\downarrow & \overset{\exists!\ \bar{\varphi}}{\nearrow} & \\ X & & \end{array}$$

Proof. We first embed $Y \subseteq \mathbf{P}^n_k$. It then suffices to show that φ extends to a morphism $X \to \mathbf{P}^n_k$ since if it does, the image of this morphism would be contained in Y by taking closures. We may therefore assume that $Y = \mathbf{P}^n_k$.

We induce on n. If $n = 0$, then there is nothing to show: the morphism $X \to \mathbf{P}^0_k = \{*\}$ is just the constant map.

Now suppose $n \geq 1$. Let $x_0, x_1, \ldots, x_n$ be the homogeneous coordinates oon $\mathbf{P}^n_k$ and let U be the open subset $\{x_0 x_1 \cdots x_n \neq 0\} \subseteq \mathbf{P}^n_k$. If $\varphi(X - \{P\}) \cap U = \emptyset$, then $\varphi(X - \{P\}) \subseteq H_i \cong \mathbf{P}^{n-1}_k$ for some i, and we would be done by the inductive hypothesis. We may therefore assume that $\varphi(X - \{P\}) \cap U \neq \emptyset$.

For each i, j, x_i/x_j is a regular function on U. Pulling back these functions by φ, we obtain a regular function

$$f_{ij} = \varphi^*\left(\frac{x_i}{x_j}\right) \in K(X)$$

on an open subset of X for each i, j. Now let v be the valuation of K associated to the DVR $R_P \in C_K$. Set

$$r_i = v(f_{i0}) \in \mathbf{Z}$$

for each $i \in \{0, 1, \ldots, n\}$. Then, since $x_i/x_j = (x_i/x_0)/(x_j/x_0)$, we have

$$v(f_{ij}) = r_i - r_j$$

for each $i, j \in \{0, 1, \ldots, n\}$. Now let $\ell \in \{0, 1, \ldots, n\}$ be such that

$$r_\ell = \min\{r_0, r_1, \ldots, r_n\}.$$

Then, $v(f_{i\ell}) \geq 0$ for all i, and hence

$$f_{0\ell}, f_{1\ell}, \ldots, f_{n\ell} \in R_P.$$

We now set

$$\bar{\varphi}(Q) = \begin{cases} \bigl[f_{0\ell}(Q) : f_{1\ell}(Q) : \cdots : f_{n\ell}(Q)\bigr] & Q \in \varphi^{-1}(U_\ell) \cup \{P\}, \\ \varphi(Q) & Q \neq P. \end{cases}$$

We claim that $\bar{\varphi}$ is a morphism $X \to \mathbf{P}^n_k$ that extends φ, and that $\bar{\varphi}$ is unique. The uniqueness holds by Lemma 1.4.1. The definition we gave for $\bar{\varphi}$ would show it is a morphism as long as we can show that the two morphisms in the definition glue together correctly. (Hartshorne proceeds differently here, but what he does corresponds to checking that $Q \mapsto [f_{0\ell}(Q) : f_{1\ell}(Q) : \cdots : f_{n\ell}(Q)]$ defines a morphism in a neighborhood of P.) But this follows from construction: On U_ℓ, we can use the affine coordinates x_i/x_ℓ, and φ is determined by how the functions $f_{i\ell} = x_i/x_\ell$ pullback (see Lemma 1.3.20). ☐

We are now ready to prove what we wanted.

[Har77, Thm. I.6.9] Theorem 1.6.19. *Let K be a function field of dimension* 1 *oer k. Then, the abstract nonsingular curve C_K is isomorphic to a nonsingular projective curve Y.*

Proof. The idea is as follows. First, we will cover C_K with open subsets U_i that are isomorphic to nonsingular affine curves. For each i, we let Y_i be a projective closure of U_i. We can then use the curve-to-projective extension Theorem 1.6.18 to define a morphism $\varphi_i \colon C_K \to Y_i$. These morphisms do not necessarily identify C_K with a closed subvariety of a projective space: for example, the maps might not be injective, and may not induce isomorphisms on local rings. Instead, Hartshorne takes inspiration from the proof of Chow's lemma [Har77, Exercise II.4.10] and considers the morphism

$$C_K \longrightarrow \prod_i Y_i$$

induced by applying the universal property of products finitely many times. We will then set $Y = \overline{\operatorname{im}(\varphi)}$, which is a projective variety (note that $\prod_i Y_i$ is a subvariety of an iterated Segre product), and show that φ is an isomorphism of C_K onto Y.

STEP 1. Every point $P \in C_K$ has an open neighborhood that is isomorphic to an affine curve.

By Corollary 1.6.12, there exists an affine curve V and a point $Q \in V$ with $R_P \cong \mathcal{O}_Q$. It follows that $\operatorname{Frac}(V) = K$, and by Proposition 1.6.17, after possibly replacing V by an affine open subset containing Q, we know that V is isomorphic to an open subset of C_K containing P.

STEP 2. Construction of the map $\varphi \colon C_K \to Y$.

Since C_K is quasi-compact (the topology is the finite complement topology), we can find finitely many open subsets U_i, each of which are isomorphic to an affine variety V_i. Now write

$$V_i \subseteq \mathbf{A}_k^{n_i} \subseteq \mathbf{P}_k^{n_i}$$

and let $Y_i = \bar{V}_i$ be the projective closure. Then, Y_i is a projective variety, and we have a morphism $\varphi_i \colon U_i \to Y_i$ that is an isomorphism onto its image. By the curve-to-projective extension Theorem 1.6.18 applied to the finite set of points $C_K - U_i$, we can find a morphism $\bar{\varphi}_i \colon C_K \to Y_i$ extending φ_i. Let $\prod_i Y_i$ be the product of the projective varieties Y_i, which we recall is also projective (since it is a closed subvariety of an iterated Segre product). Consider the "diagonal" map

$$\begin{aligned} \varphi \colon C_K &\longrightarrow \prod_i Y_i \\ P &\longmapsto \prod_i \bar{\varphi}_i(P) \end{aligned}$$

induced by applying the universal property of products finitely many times, and let Y be the closure of the image of C_K. Then, Y is a projective variety, and $\varphi \colon C_K \to Y$ is a morphism whose image is dense in Y. By basic properties of closures and dimension, we know that Y is a curve.

STEP 3. φ is an isomorphism. 10/21

We apply the local criterion for isomorphisms (Proposition 1.3.18), which applies by working on one U_i at a time. We start with the condition on local rings. Let $P \in C_K$ be a point. We then have $P \in U_i$ for some i. Consider the commutative diagram

$$\begin{array}{ccc} C_K & \overset{\varphi}{\hookrightarrow} & Y \\ \uparrow & & \downarrow{\scriptstyle \pi} \\ U_i & \xrightarrow{\varphi_i} & Y_i \end{array}$$

of dominant morphisms, where π is the projection map onto the i-th factor. We then have the inclusions of local rings

$$\begin{array}{ccc} \mathcal{O}_{C_K,P} & \xhookleftarrow{\varphi_P^*} & \mathcal{O}_{Y,\varphi(P)} \\ \wr\big\downarrow & & \big\uparrow \pi^*_{\varphi_i(P)} \\ \mathcal{O}_{U_i,P} & \xleftarrow{\sim} & \mathcal{O}_{Y_i,\varphi_i(P)} \end{array}$$

in $K = K(Y_i)$ by [Har77, Exercise I.3.3]. Since the composition from the bottom right to the top left is an equality, we see that the top and right maps are isomorphisms. We therefore see that for every $P \in C_K$, the map

$$\varphi_P^* \colon \mathcal{O}_{Y,\varphi(P)} \longrightarrow \mathcal{O}_{C_K,P}$$

is an isomorphism.

It remains to show that φ is a bijection. Let $Q \in Y$. Then, $\mathcal{O}_Q$ is dominated by a DVR of K/k by Theorem 1.6.10, for example by taking the localization of the integral closure of $\mathcal{O}_Q$ in K at a maximal ideal lying over $\mathfrak{m}_Q$. But then, $R = R_P$ for some $P \in C_K$, and $\mathcal{O}_{Y,\varphi(P)} \cong R$, so by Lemma 1.6.8 we know that $Q = \varphi(P)$. Thus, φ is surjective. Moreover, φ is injective because distinct points of C_K correspond to distinct subrings of K. □

Here are some consequences of Theorem 1.6.18. You will see more on Homework 8.

[Har77, Cor. I.6.10] Corollary 1.6.20. *Every abstract nonsingular curve is isomorphic to a quasi-projective curve. Every nonsingular quasi-projective curve is isomorphic to an open subset of a nonsingular projective curve.*

Recall that a *curve* is a 1-dimensional quasi-projective variety.

[Har77, Cor. I.6.11] Corollary 1.6.21. *Every curve is birationally equivalent to a nonsingular projective curve.*

Proof. If Y is a curve with function field $K = K(Y)$, then Y is birationally equivalent to C_K, which is nonsingular and projective by Theorem 1.6.19. □

[Har77, Cor. I.6.12] Corollary 1.6.22. *We have the following equivalences of categories:*

$$\begin{array}{ccc} (i) & \left\{\begin{matrix}\textit{Nonsingular projective curves} \\ \textit{and dominant morphisms}\end{matrix}\right\} & \\ & \big\downarrow\wr & \\ (ii) & \left\{\begin{matrix}\textit{Quasi-projective curves} \\ \textit{and dominant rational maps}\end{matrix}\right\} & Y \\ & \big\downarrow\wr & \big\downarrow \\ (iii) & \left\{\begin{matrix}\textit{Function fields of dimension } 1 \textit{ over } k \\ \textit{and } k\textit{-algebra homomorphisms}\end{matrix}\right\}^{\mathrm{op}} & K(Y). \end{array}$$

Proof. The first functor is the faithful functor $(i) \hookrightarrow (ii)$ induced by the facts that all nonsingular projective curves are quasi-projective, and all dominant morphisms are dominant rational maps. The functor $(ii) \xrightarrow{\sim} (iii)$ is an equivalence of categories by Theorem 1.4.7.

It remains to show that $(i) \hookrightarrow (ii)$ is an equivalence of categories. The functor is essentially surjective since every quasi-projective curve Y is birational to the nonsingular projective curve $C_{K(Y)}$. The functor is full since if $X \dashrightarrow Y$ is a dominant rational map, then it extends to a dominant *morphism* by Theorem 1.6.18. □

1.6.3. The automorphism group of $\mathbf{P}^1_k$. As a concrete application, we compute the automorphism group of $\mathbf{P}^1_k$. This is one of the homework problems on Homework 8.

EXAMPLE 1.6.23. Think of $\mathbf{P}^1_k$ with coordinates x, y as $\{y \neq 0\} \cup \{\infty\}$. Recall from Homework 5 that a *fractional linear transformation* or a *Möbius transformation* of $\mathbf{P}^1_k$ is a map [Har77, Exer. I.6.7] [Ahl78, Ch. 3, §3.2]

$$z \longmapsto \frac{az+b}{cz+d} \tag{1.6.24}$$

for $a, b, c, d \in k$ such that $ad - bc \neq 0$. As you showed on Homework 5, the fractional linear transformations of $\mathbf{P}^1_k$ form a group, which we denote by $\mathrm{PGL}_2(k)$. We call this group the *projective general linear group* of degree 2 over k. In projective coordinates, the fractional linear transformation above is the endomorphism

$$\begin{aligned} \mathbf{P}^1_k &\longrightarrow \mathbf{P}^1_k \\ [s:t] &\longmapsto [as+bt : cs+dt]. \end{aligned}$$

We therefore have the surjective group homomorphism

$$\mathrm{GL}_2(k) \twoheadrightarrow \mathrm{PGL}_2(k),$$

which explains the notation $\mathrm{PGL}_2(k)$. Note the difference in notation compared to [Har77, Exercise I.6.6]: The notation PGL(1) in [Har77] is not standard.

We have an inclusion

$$\mathrm{PGL}_2(k) \hookrightarrow \mathrm{Aut}(\mathbf{P}^1_k)$$

which we claim is an isomorphism. By Corollary 1.6.22, we can identify the right-hand side with $\mathrm{Aut}_k(k(z))$, the automorphism group of $k(z)$ as a field over k. It therefore suffices to show that every field automorphism of $k(z)$ over k is of the form (1.6.24).

Let $\varphi\colon k(z) \to k(z)$ be an automorphism. Set

$$w = \varphi(z) = \frac{f(z)}{g(z)}$$

for $f(z), g(z) \in k[z]$ that are relatively prime. Then, we have

$$f(z) - w\,g(z) = 0,$$

and hence z satisfies the polynomial

$$f(T) - w\,g(T) \in k[w][T].$$

Now $f(T) - w\,g(T)$ is irreducible considered as an element in $k[w][T]$ since it is linear polynomial in $k(T)[w]$. By Gauss's lemma, we see that $f(T) - w\,g(T)$ is irreducible in $k(w)[T]$. We therefore see that $f(T) - w\,g(T)$ is a minimal polynomial for z over $k(w)$. Since φ is an isomorphism, we have

$$1 = \bigl[k(z) : k(w)\bigr] = \deg_T\bigl(f(T) - w\,g(T)\bigr),$$

and hence
$$\max\Big\{\deg\big(f(T)\big),\deg\big(g(T)\big)\Big\} \le 1.$$
Thus, we see that every automorphism of $k(z)$ is of the form (1.6.24). We note that $ad - bc \neq 0$ because if $ad - bc = 0$, then
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
is not invertible. This non-invertibility implies that $(a\ b)$ and $(c\ d)$ are scalar multiples of each other, and hence $(az+b)/(cz+d) \in k \cup \infty$, a contradiction.

We mention one consequence of the fact that $\mathrm{PGL}_2(k) \to \mathrm{Aut}(\mathbf{P}^1_k)$ is an isomorphism: This means that every automorphism of $\mathbf{P}^1_k$ is linear, and hence we can reduce questions about automorphisms of $\mathbf{P}^1_k$ to linear algebra! For example, the fact that any set of three distinct points in $\mathbf{P}^1_k$ can be mapped to $\{0, 1, \infty\}$ is a consequence of the linear algebra fact that any three lines through the origin in k^2 can mapped linearly to the diagonal, the x-axis, and the y-axis. If you have *four* points in $\mathbf{P}^1_k$, this corresponds to asking how you can map four lines with slopes z_1, z_2, z_3, z_4 under a linear automorphism of k^2. You showed on Homework 5 that you can map the line corresponding to z_2 to the diagonal, the line corresponding to z_3 to the x-axis, and the line corresponding to z_4 to the y-axis. Then, the line corresponding to z_1 maps to the line with slope given by the *cross ratio*
$$\lambda(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}.$$
This point of view using cross ratios may be helpful for you when solving [Har77, Exercise I.6.7] and [Har77, Exercise I.5.14(c)]. (While we are discussing [Har77, Exercise I.5.14], I wanted to mention that [Har77, Exercise I.5.14(d)] is particularly difficult – I uploaded a sketch of the uniqueness part of that exercise on Brightspace.)

1.6.4. Elliptic curves are irrational. Recall that a quasi-projective variety is *rational* if it is birational to a projective space $\mathbf{P}^n_k$. So far, we spent a lot of time talking about rationality of curves and proved Lüroth's Theorem 1.4.15, which says that unirational curves are rational. However, we have yet to see an example of an irrational variety!

Our goal is to construct such an irrational variety, following [Har77, Exercises I.6.1 and I.6.2]. We start with the following criterion for rationality.

[Har77, Exer. I.6.1] Proposition 1.6.25. *Let Y be a nonsingular rational curve which is not isomorphic to $\mathbf{P}^1_k$. We then have the following.*

(a) *Y is isomorphic to an open subset of $\mathbf{A}^1_k$.*
(b) *Y is affine.*
(c) *$A(Y)$ is a unique factorization domain.*

Proof. For (a), we know by Corollary 1.6.20 that Y is isomorphic to an open subset of a nonsingular projective curve $\bar{Y}$. By Corollary 1.6.22, there is an isomorphism
$$\varphi\colon \bar{Y} \xrightarrow{\sim} \mathbf{P}^1_k.$$
Since Y is not isomorphic to $\mathbf{P}^1_k$, we see that $\bar{Y} - Y \neq \emptyset$, and hence $\varphi(Y) \subseteq \mathbf{A}^1_k \subseteq \mathbf{P}^1_k$. Since $\mathbf{A}^1_k$ has the finite complement topology and φ is an isomorphism onto its

image, we see that Y is isomorphic to an open subset of $\mathbf{A}^1_k$. This also shows (*b*), since if $\varphi(Y) = \mathbf{A}^1_k - \{a_1, a_2, \dots, a_n\}$, then

$$Y \cong Z\left(t\prod_{i=1}^{n}(x - a_i) - 1\right) \subseteq \mathbf{A}^2_k.$$

Finally, $A(Y)$ is a UFD since the isomorphism above yields

$$A(Y) \cong k[x]_{\prod_{i=1}^{n}(x-a_i)},$$

and localizations of UFDs are UFDs. (One way to see this last fact is to use Proposition 1.1.49: A Noetherian domain is a UFD if and only if every height 1 prime is principal.) □

We can now give an example of an irrational curve by finding a nonsingular rational curve Y such that $A(Y)$ is not a UFD. One way we can think of this statement is that we are trying to construct a Dedekind domain of finite type over k with nontrivial divisor class group. One takeaway from this calculation is: Without good geometric invariants that can distinguish different curves or varieties, it is not easy to show that something is irrational.

EXAMPLE 1.6.26 (An elliptic curve). Assume $\operatorname{char}(k) \neq 2$. Consider the curve 11/3 [Har77, Exer. I.6.2]

$$Y := \left\{y^2 = x(x-1)(x-\lambda)\right\} \subseteq \mathbf{A}^2_k$$

for $\lambda \notin \{0, 1\}$. This is an example of an (affine) *elliptic curve*. In fact, all projective elliptic curves are the projective closure of an affine curve of the form above [Har77, Chapter IV, Proposition 4.6]. (The parameter λ appears when moving the four branch points of a $2:1$ morphism $Y \to \mathbf{P}^1_k$ to $\lambda, 0, 1, \infty$ via an automorphism of $\mathbf{P}^1_k$. We see that the cross ratio λ appears once again!)

We claim that Y is not a rational curve. We proceed in steps.

STEP 1. Y is nonsingular. Moreover, $A := A(Y)$ is normal.

The normality of $A(Y)$ is used to prove that $A(Y)$ is the integral closure of $k[x]$ in $K(Y)$. This is not used in the proof of irrationality.

We first show that Y is nonsingular. The Jacobian matrix for Y is

$$\big((x-1)(x-\lambda) + x(x-\lambda) + x(x-1) \quad -2y\big)\,.$$

If the Jacobian has rank less than $2 - 1 = 1$, then $y = 0$. By the equation for Y, this implies that $x \in \{0, 1, \lambda\}$. However, these three values for x are not zeroes for $(x-1)(x-\lambda) + x(x-\lambda) + x(x-1)$ since only two out of the three terms will vanish at $x \in \{0, 1, \lambda\}$. We conclude that Y is nonsingular.

To show that $A(Y)$ is normal, we know that by Zariski's Theorem 1.5.4, the local ring $\mathcal{O}_{Y,y}$ is a regular local ring for every $y \in Y$. By Theorem 1.6.4, this is the same thing as saying that $\mathcal{O}_{Y,y}$ is normal. Since normality is a local condition, this means that A is normal.

STEP 2. Consider the subring $k[x] \subseteq K := K(Y)$ generated by the image of x in A. Then, $k[x]$ is a polynomial ring and A is the integral closure of $k[x]$ in K.

Only the fact that $k[x]$ is a polynomial ring will be used in the proof of irrationality.

Since k is algebraically closed, x is transcendental over k, and hence $k[x]$ is a polynomial ring.

To compute the integral closure of $k[x]$ in K, we first note that this integral closure is contained in A because A is integrally closed by Step 1. Thus, to show that the integral closure of $k[x]$ is actually equal to A, it suffices to show that y is integral over $k[x]$. This holds since y satisfies the integral dependence relation

$$y^2 - x(x-1)(x-\lambda) = 0.$$

STEP 3. There is a k-algebra automorphism

$$\begin{aligned} \sigma\colon A &\longrightarrow A \\ x &\longmapsto x \\ y &\longmapsto -y. \end{aligned}$$

Note that σ defines a k-algebra automorphism of $k[x,y]$, which descends to A by the fact that σ fixes the equation $y^2 - x(x-1)(x-\lambda) = 0$.

STEP 4. There is a norm function

$$\begin{aligned} N\colon A &\longrightarrow A \\ a &\longmapsto a \cdot \sigma(a) \end{aligned}$$

such that $\operatorname{im}(N) \subseteq k[x]$. This norm function is multiplicative in the sense that $N(1) = 1$ and $N(ab) = N(a) \cdot N(b)$ for all $a, b \in A$.

Any $a(x,y) \in A$ is equal to $g(x) + c(x)\,y$ for polynomials $g(x), c(x) \in k[x]$. Then, we have

$$\begin{aligned} N(a) &= \bigl(g(x) + c(x)\,y\bigr)\bigl(g(x) - c(x)\,y\bigr) \\ &= g(x)^2 - c(x)^2\,y^2. \end{aligned}$$

Using the relation $y^2 = x(x-1)(x-\lambda)$, we obtain

$$N(a) = g(x)^2 - c(x)^2\,x(x-1)(x-\lambda) \in k[x]. \tag{1.6.27}$$

The fact that $N(1) = 1$ follows from definition since $\sigma(1) = 1$. To show that $N(ab) = N(a) \cdot N(b)$, we have

$$N(ab) = ab \cdot \sigma(ab) = \bigl(a \cdot \sigma(a)\bigr)\bigl(b \cdot \sigma(b)\bigr) = N(a) \cdot N(b)$$

because σ is a ring homomorphism.

STEP 5. The units in A are precisely $k^\times \coloneqq k - \{0\}$.

Suppose $ab = 1$ for $a, b \in A$. Then, we have

$$1 = N(1) = N(ab) = N(a) \cdot N(b) \in k[x].$$

Since $k[x]^\times = k^\times$, we have that $N(a), N(b) \in k^\times$. Writing $a = g(x) + c(x)\,y$ as before, (1.6.27) implies

$$N(a) = g(x)^2 - c(x)^2\,x(x-1)(x-\lambda)$$

lies in $k^\times$. Since we will repeatedly use the same argument in the rest of this example, we state the following:

CLAIM 1.6.28. *Suppose that*

$$N(a) = g(x)^2 - c(x)^2\,x(x-1)(x-\lambda)$$

has degree $\geq f$. *Then,* $\deg(c(x)) \leq f/2 - 1$.

Proof. We show the contrapositive Suppose that $\deg(c(x)) > f/2 - 1$, and in particular, $c(x) \neq 0$. Then, the highest degree term in

$$c(x)^2\, x(x-1)(x-\lambda)$$

is of odd degree $f' \geq f+1 > f$. For $N(a)$ to be of degree f, this implies $g(x)^2$ must have a term of degree f', and hence $g(x)$ is of degree $\geq f'/2$. This implies

$$\deg\bigl(N(a)\bigr) \geq f' > f. \qquad \square$$

Returning to the situation of Step 5, suppose that $N(a) \in k^\times$. By Claim 1.6.28, we see that $c(x) = 0$. If $N(a) \in k^\times$, then $g(x)$ must be constant to have $g(x)^2 \in k^\times$. Since all elements of $k^\times$ are also units in A, we see that $A^\times = k^\times$.

Step 6. A is not a UFD, and hence Y is not rational.

We first show that x and y are irreducible. Suppose $x = a \cdot b$ for nonunits a, b. Then, we have

$$x^2 = N(x) = N(a) \cdot N(b),$$

and hence either $N(a) = x^2$ and $N(b) \in k^\times$, or $N(a) = x$ and $N(b) = x$ (up to swapping and multiplication by units). In the first case, we have $b \in k^\times$ by the proof of Step 5, a contradiction. In the second case, write $a = g(x) + c(x)\,y$ as before. Claim 1.6.28 then implies $c(x) = 0$. We therefore have $a = g(x)$ and $N(a) = g(x)^2 \neq x$, a contradiction.

Similarly, suppose that $y = a \cdot b$ for nonunits a, b. Then,

$$-x(x-1)(x-\lambda) = N(y) = N(a) \cdot N(b),$$

and hence $N(a), N(b)$ have degree ≤ 3. Writing $a = g(x) + c(x)\,y$ and $b = h(x) + d(x)\,y$ as before, Claim 1.6.28 implies $c(x), d(x) \in k$. Moreover, we claim that $g(x), h(x)$ are linear. Otherwise, if $g(x)$ is of degree ≥ 2, then $\deg(N(a)) \geq 4$, a contradiction, and similarly for $N(b)$. This is a contradiction, since (1.6.27) implies two of the polynomials $x, x-1, x-\lambda$ divides either $g(x)$ or $h(x)$, which would force one of $g(x), h(x)$ to be of degree ≥ 2. Thus, both x, y are irreducible.

Finally, A is not a unique factorization domain since

$$y^2 = x(x-1)(x-\lambda).$$

This shows Y is not rational by Proposition 1.6.25(c)!

1.7. Intersections in projective space

Our goal now is to define intersection multiplicities for projective varieties in a projective space.

So far, we know that if Y and Z are projective varieties in $\mathbf{P}^n_k$, then $Y \cap Z$ may not be a variety (because it may not be irreducible). For example, the intersection of a conic and a line can be two points or one point, depending on how they intersect. A provisional definition for the *degree* of a curve is therefore the number of intersection points with a "sufficiently general" line. However, this definition is not very satisfactory because it is hard to work with and it is not precise (when is a line sufficiently general?).

Instead, we will rely on algebra. First, we use what we know about intersecting with hypersurfaces to give bounds for the dimension of $Y \cap Z$. Then, if $Y \cap Z$ is

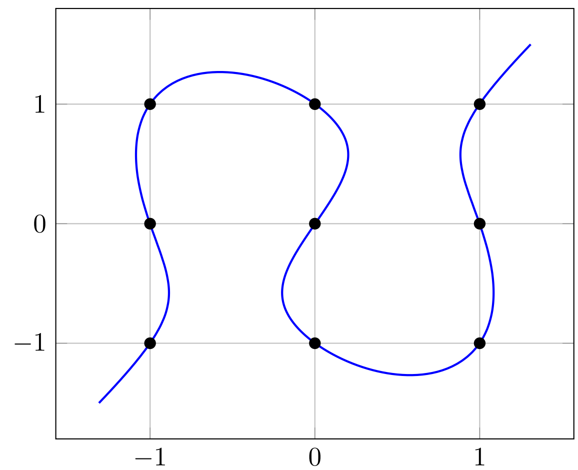

FIGURE 1.22. The nonsingular cubic curve $\{y^3 - y = 2(x^3 - x)\}$.

0-dimensional, we can ask how many points are in $Y \cap Z$. A concrete goal is to show the following:

THEOREM 1.7.1 (Bézout 1779 [Béz06]). *Let Y, Z be distinct curves in $\mathbf{P}^2_k$ of degrees d, e. Then, $Y \cap Z$ consists of de points* counted with multiplicity.

Our goal will be to understand what it means to count points "with multiplicity."

REMARK 1.7.2. We mention an interesting paradox that is related to Bézout's theorem. While known as *Cramer's paradox* or the *Cramer–Euler paradox*, this paradox is originally due to Maclaurin [Mac1720, p. 137]. See [BS11] for a wonderful account of this paradox and its history.

Suppose Y and Z are cubic curves in $\mathbf{P}^2_k$.

- By Bézout's Theorem 1.7.1, the intersection is expected to consist of nine points, and this expectation turns out to be correct if Y and Z are general in the moduli space of nonsingular cubic curves you constructed in [Har77, Exercise I.5.15(b)].
- On the other hand, since the moduli space of nonsingular cubic curves has dimension
$$\binom{2+3}{2} - 1 = 9$$
as you proved in [Har77, Exercise I.5.15(b)], and vanishing at a point in $\mathbf{P}^2_k$ gives a linear condition on the coefficients of the cubic, we expect that a set of nine points determines a unique cubic curve in $\mathbf{P}^2_k$ (this dimension count is known as Cramer's theorem, first proved in [Cra1750]).

For a concrete example, we consider an example due to Euler [Eul1750]. The two (singular!) cubics
$$Y = \{y^3 - y = 0\} \qquad \text{and} \qquad Z = \{x^3 - x = 0\}$$
in $\mathbf{A}^2_k$ intersect in a 3×3 grid of points in $\mathbf{A}^2_k$. Taking projective closures in $\mathbf{P}^2_k$, we obtain a 1-dimensional family of plane cubic curves
$$\left\{s(y^3 - yz^2) + t(x^3 - xz^2) = 0\right\} \subseteq \mathbf{P}^2_k$$

parametrized by $[s:t] \in \mathbf{P}^1_k$. This is an example of a *pencil* of curves. See [BS11] for an interactive visualization and see Figure 1.22 for a specific example.

1.7.1. Dimension of intersections. We start by computing bounds for the dimension of intersections of varieties.

PROPOSITION 1.7.3 (Affine dimension theorem). *Let Y, Z be closed subvarieties of dimension r, s in $\mathbf{A}^n_k$. Then, every irreducible component of $Y \cap Z$ has dimension $\geq r+s-n$.* [Har77, Prop. I.7.1] [Sha13$_1$, Thm. 1.24]

Proof. If Z is a hypersurface, then this is [Har77, Exercise I.1.8].

We now prove the general case. We know that

$$Y \times Z \subseteq \mathbf{A}^{2n}_k$$

has dimension $r+s$ since choosing Noether normalizations

$$k[x_1,x_2,\ldots,x_r] \subseteq A(Y)$$
$$k[y_1,y_2,\ldots,y_s] \subseteq A(Z)$$

the extension

$$k[x_1,x_2,\ldots,x_r,y_1,y_2,\ldots,y_s] \subseteq A(Y) \otimes_k A(Z)$$

is finite. (See also [Har77, Exercise I.3.15(d)].) We now consider the diagonal

$$\Delta = Z(x_1-y_1, x_2-y_2, \ldots, x_n-y_n) \subseteq \mathbf{A}^{2n}_k.$$

Since the diagonal morphism $\Delta\colon \mathbf{A}^n_k \hookrightarrow \mathbf{A}^{2n}_k$ is an isomorphism onto its image Δ, we know that

$$Y \cap Z \xrightarrow{\sim} (Y \times Z) \cap \Delta.$$

Thus, applying the hypersurface case n times, we see that every irreducible component of $Y \cap Z$ has dimension $\geq r+s-n$. ☐

THEOREM 1.7.4 (Projective dimension theorem). *Let Y, Z be closed subvarieties of dimension r, s in $\mathbf{P}^n_k$. Then, every irreducible component of $Y \cap Z$ has dimension $\geq r+s-n$. Furthermore, if $r+s-n \geq 0$, then $Y \cap Z$ is nonempty.* [Har77, Thm. I.7.2] [Sha13$_1$, Thm. 1.24]

Proof. The dimension bound holds by the affine dimension theorem (Proposition 1.7.3) since $\mathbf{P}^n_k$ is covered by affine n-spaces. To show that $Y \cap Z$ is nonempty, consider the affine cones $C(Y)$ and $C(Z)$ over Y and Z in $\mathbf{A}^{n+1}_k$. Then,

$$\dim\bigl(C(Y)\bigr) = r+1$$
$$\dim\bigl(C(Z)\bigr) = s+1$$

and $C(Y) \cap C(Z) \neq \emptyset$ since both cones contain the origin $(0,0,\ldots,0) \in \mathbf{A}^{n+1}_k$. By the affine dimension theorem (Proposition 1.7.3), we have

$$\dim\bigl(C(Y) \cap C(Z)\bigr) \geq (r+1)+(s+1)-(n+1) = r+s-n+1 > 0.$$

Thus, $C(Y) \cap C(Z)$ contains a point $Q \neq P$, and hence $Y \cap Z \neq \emptyset$. ☐

1.7.2. Numerical polynomials. To define intersection multiplicities, we will define the *Hilbert polynomial* associated to a projective variety $Y \subseteq \mathbf{P}^n_k$. The Hilbert polynomial is a numerical polynomial $P_Y \in \mathbf{Q}[z]$ that encodes various numerical invariants of Y. We will define P_Y using the homogeneous coordinate ring $S(Y)$. The definition will in fact work for any graded S-module, where $S = k[x_0, x_1, \ldots, x_n]$. 11/5

We start with some preliminaries on numerical polynomials.

[Har77, p. 49]
[Kle66, p. 295]

Definition 1.7.5. A *numerical polynomial* is a polynomial $P(z) \in \mathbf{Q}[z]$ for which there exists an integer $n_0 \in \mathbf{Z}$ such that $P(n) \in \mathbf{Z}$ for all $n \geq n_0$.

Convention 1.7.6. We will write "for all $n \gg 0$" for "there exists $n_0 \in \mathbf{Z}$ such that… for all $n \geq n_0$."

[Har77, Prop. I.7.3]
[Lan02, Lem. X.6.4]
[MT33, 2.12]
[Nag56b, App.]

Proposition 1.7.7.

(a) *If $P \in \mathbf{Q}[z]$ is a numerical polynomial, then there are integers $c_0, c_1, \ldots, c_r$ such that*
$$P(z) = c_0\binom{z}{r} + c_1\binom{z}{r-1} + \cdots + c_r$$
where
$$\binom{z}{r} = \frac{1}{r!}\, z(z-1)\cdots(z-r+1)$$
is the binomial coefficient function. In particular, $P(n) \in \mathbf{Z}$ for all $n \in \mathbf{Z}$.

(b) *If $f\colon \mathbf{Z} \to \mathbf{Z}$ is any function, and if there exists a numerical polynomial $Q(z)$ such that the difference function*
$$\Delta f := f(n+1) - f(n)$$
is equal to $Q(n)$ for all $n \gg 0$, then there exists a numerical polynomial $P(z)$ such that $f(n) = P(n)$ for all $n \gg 0$.

Proof. (a). We induce on the degree of P. The degree 0 case holds since P of degree 0 defines a function with a constant integer value. Next, since
$$\binom{z}{r} = \frac{z^r}{r!} + \cdots,$$
we can express any polynomial $P \in \mathbf{Q}[z]$ of degree r in the form stated with $c_0, c_1, \ldots, c_r \in \mathbf{Q}$. We have to show that $c_0, c_1, \ldots, c_r \in \mathbf{Z}$. For any polynomial P, we define the *difference polynomial*
$$\Delta P = P(z+1) - P(z).$$
Since
$$\Delta\binom{z}{r} = \binom{z+1}{r} - \binom{z}{r} = \binom{z}{r-1},$$
we have
$$\Delta P = c_0\binom{z}{r-1} + c_1\binom{z}{r-2} + \cdots + c_{r-1}.$$
By induction, $c_0, c_1, \ldots, c_{r-1} \in \mathbf{Z}$. Finally, $c_r \in \mathbf{Z}$ since $P(n) \in \mathbf{Z}$ for $n \gg 0$.

(b). Write
$$Q = c_0\binom{z}{r} + \cdots + c_r$$

where $c_0, c_1, \dots, c_r \in \mathbf{Z}$ using (a). Let

$$P = c_0\binom{z}{r+1} + \cdots + c_r\binom{z}{1}.$$

Then, $\Delta P = Q$, and $\Delta(f - P)(n) = 0$ for all $n \gg 0$. Thus, $(f - P)(n) = c_{r+1}$ for some constant $c_{r+1} \in \mathbf{Z}$ for all $n \gg 0$. We conclude that

$$f(n) = P(n) + c_{r+1}$$

for all $n \gg 0$. □

1.7.3. Dévissage for graded modules. Next, we need a version of prime cyclic filtrations for graded modules over graded rings. In algebraic geometry, this filtration result and the technique to prove statements using it is also called *dévissage*.

Definition 1.7.8. Let $S = \bigoplus_{d \geq 0} S_d$ be a graded ring. A *graded S-module* is an S-module M together with a decomposition [Har77, p. 50]

$$M = \bigoplus_{d \in \mathbf{Z}} M_d$$

as Abelian groups, such that $S_d \cdot M_e \subseteq M_{d+e}$ for every d, e. For any graded S-module M and any $l \in \mathbf{Z}$, the *l-th twist* $M(l)$ of M is the graded S-module with components

$$M(l)_d = M_{d+l}.$$

If M is a graded S-module, we define the *annihilator* of M is

$$\operatorname{Ann}_S(M) := \{s \in S \mid s \cdot M = 0\},$$

which is a homogeneous ideal in S because for homogeneous elements $m \in M$, an element $s \cdot m \in s \cdot M$ is zero if and only if $s_d \cdot m = 0$ for every homogeneous component s_d of s.

We now prove our result on prime cyclic filtrations.

Proposition 1.7.9 [BouCA, Chapter IV, §3, no. 1, Proposition 2]. *Let S be a Noetherian graded ring. Let M be a finitely generated graded S-module. Then, there exists a filtration* [Har77, Prop. I.7.4]

$$0 = M^0 \subseteq M^1 \subseteq \cdots \subseteq M^r = M$$

by graded S-submodules such that for every i, we have

$$\frac{M^i}{M^{i-1}} \cong \left(\frac{S}{\mathfrak{p}_i}\right)(l_i)$$

where $\mathfrak{p}_i$ is a homogeneous prime ideal of S and $l_i \in \mathbf{Z}$. The filtration is not unique, but for any such filtration we have:

(a) *If $\mathfrak{p}$ is a homogeneous prime ideal of S, then $\mathfrak{p} \supseteq \operatorname{Ann}_S(M)$ if and only if $\mathfrak{p} \supseteq \mathfrak{p}_i$ for some i. In particular, the minimal elements of the set $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r\}$ are the minimal primes of M, i.e., the minimal primes containing $\operatorname{Ann}_S(M)$.*

(b) *For every minimal prime of M, the number of times which $\mathfrak{p}$ occurs in the set $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r\}$ is equal to the length of $M_{\mathfrak{p}}$ over the local ring $S_{\mathfrak{p}}$, and hence is independent of the filtration.*

Proof. We proceed by Noetherian induction. Consider the set of graded submodules of M which admit such a filtration. This set is nonempty since it contains 0. Since M is Noetherian, there exists a maximal such submodule $M' \subseteq M$. Now consider $M'' = M/M'$. If $M'' = 0$, we are done. Otherwise, consider the set of ideals

$$\mathfrak{J} = \left\{ I_m = \operatorname{Ann}_S(m) \;\middle|\; \begin{matrix} m \in M'' \text{ is a nonzero} \\ \text{homogeneous element} \end{matrix} \right\}.$$

Each I_m is a homogeneous ideal and $I_m \neq S$. Since S is Noetherian, there exists an element $m \in M'' - \{0\}$ such that I_m is a maximal element of $\mathfrak{J}$. We claim that I_m is prime. Let $a, b \in S$ such that $ab \in I_m$ but $b \notin I_m$. Splitting into homogeneous components, we may assume a, b are homogeneous. We want to show that $a \in I_m$. Consider the element $bm \in M''$. Since $b \notin I_m$ we have $bm \neq 0$. We have $I_m \subseteq I_{bm}$, and by maximality of I_m, we have $I_m = I_{bm}$. But $ab \in I_m$, and hence $abm = 0$, which implies $a \in I_{bm} = I_m$ as required. We therefore see that I_m is a homogeneous prime ideal $\mathfrak{p} \subseteq S$. Letting $\deg(m) = l$, the graded S-module map

$$\begin{aligned} S(-l) &\longrightarrow M'' \\ 1 &\longmapsto m \end{aligned}$$

induces a graded S-module isomorphism

$$\left(\frac{S}{\mathfrak{p}}\right)(-l) \xrightarrow{\sim} S \cdot m \subseteq M''$$

since $\operatorname{Ann}_S(m) = \mathfrak{p}$. Letting $N' \subseteq M$ be the inverse image of $S \cdot m \subseteq M'$ in M, we have $M' \subseteq N'$ and $N'/M' \cong (S/\mathfrak{p})(-l)$. So N' has a filtration of the type required, contradicting the maximality of M'. We therefore see that $M' = M$, i.e., M has a filtration of the required form.

(a). Given a filtration for M, we have

$$\mathfrak{p} \supseteq \operatorname{Ann}_S(M) \iff \mathfrak{p} \supseteq \operatorname{Ann}_S\big(M^i/M^{i-1}\big) \text{ for some } i.$$

But $\operatorname{Ann}_S((S/\mathfrak{p}_i)(l_i)) = \mathfrak{p}_i$.

(b). We localize at a minimal prime $\mathfrak{p}$. Since $\mathfrak{p}$ is minimal in the set $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r\}$, we see that

$$\frac{M^i_{\mathfrak{p}}}{M^{i-1}_{\mathfrak{p}}} = \begin{cases} 0 & \text{if } \mathfrak{p}_i \neq \mathfrak{p} \\ k(\mathfrak{p}) & \text{if } \mathfrak{p}_i = \mathfrak{p} \end{cases}$$

where in the latter case, we forget the grading. We therefore see that

$$\operatorname{length}_{S_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \text{number of times } \mathfrak{p} \text{ occurs in } \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r\}. \qquad \square$$

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[Har77, p. 51]

Definition 1.7.10. Let S be a Noetherian graded ring and let M be a finitely generated graded S-module. If $\mathfrak{p}$ is a minimal prime of M, the *multiplicity* of M at $\mathfrak{p}$ is

$$\mu_{\mathfrak{p}}(M) \coloneqq \operatorname{length}_{S_{\mathfrak{p}}}(M_{\mathfrak{p}}).$$

Since it is often hard to compute lengths directly from the definition, we mention the following alternative descriptions in terms of vector space dimension.

Example 1.7.11. Let M be a module over a local ring $(R, \mathfrak{m})$.

[Mur24ca, Ex. 8.11.4]
[Hoc17, p. 122]

(i) Suppose that $\mathfrak{m} \cdot M$. Then, M is naturally a module over $R/\mathfrak{m}$, and

$$\operatorname{length}_R(M) = \dim_{R/\mathfrak{m}}(M).$$

[AK21, Exer. 19.19] *(ii)* Suppose that k is a field and that R is a k-algebra such that the composition

$$k \longrightarrow R \longrightarrow R/\mathfrak{m}$$

is bijective. Then,

$$\mathrm{length}_R(M) = \dim_k(M).$$

1.7.4. Hilbert polynomial. We can now define the Hilbert polynomial and Hilbert function of a graded module M over the polynomial ring $S = k[x_0, x_1, \dots, x_n]$.

Definition 1.7.12. Let M be a graded module over the ring $S = k[x_0, x_1, \dots, x_n]$. [Har77, p. 51] The *Hilbert function* is the function φ_M given by

$$\varphi_M(l) = \dim_k(M_l)$$

for every $l \in \mathbf{Z}$.

Theorem 1.7.13 (Hilbert–Serre [FAC, n° 80]). *Let M be a finitely generated graded $S = k[x_0, x_1, \dots, x_n]$-module. Then, there is a unique polynomial $P_M(z) \in \mathbf{Q}[z]$ such that $\varphi_M(l) = P_M(l)$ for all $l \gg 0$. Furthermore,* [Har77, Thm. I.7.5]

$$\deg\bigl(P_M(z)\bigr) = \dim\bigl(Z_+\bigl(\mathrm{Ann}_S(M)\bigr)\bigr).$$

Definition 1.7.14. The polynomial $P_M(z)$ of Theorem 1.7.13 is the *Hilbert polynomial* of M. [Har77, p. 52]

Proof of Theorem 1.7.13. We proceed by induction on

$$r = \dim\bigl(Z_+\bigl(\mathrm{Ann}_S(M)\bigr)\bigr).$$

Suppose $r = -1$, i.e., we have

$$(x_0, x_1, \dots, x_n) \subseteq \sqrt{\mathrm{Ann}_S(M)}.$$

We claim that $\varphi_M(l) = 0$ for all $l \gg 0$. If $M = 0$, this is true since $M_l = 0$ for all l. If $M \neq 0$, since M is finitely generated, there exists a degree d such that $M_{\le d}$ generates M. But then, every element in $M_{\le d}$ is annihilated by $(x_0, x_1, \dots, x_n)^s$ for some $s \ge 1$ (by the pigeonhole principle), and hence $M_l = 0$ for all $l \ge d + s$. Thus, $P_M(z) = 0$ is the Hilbert polynomial and

$$\deg\bigl(P_M(z)\bigr) = \dim\bigl(Z_+\bigl(\mathrm{Ann}_S(M)\bigr)\bigr) = -1.$$

For the inductive case $r \ge 0$, we proceed by dévissage. If

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence of graded S-modules, then $\varphi_M = \varphi_{M'} + \varphi_{M''}$ by the rank–nullity theorem and

$$Z_+\bigl(\mathrm{Ann}_S(M)\bigr) = Z_+\bigl(\mathrm{Ann}_S(M')\bigr) \cup Z_+\bigl(\mathrm{Ann}_S(M'')\bigr)$$

because

$$\sqrt{\mathrm{Ann}_S(M)} = \sqrt{\mathrm{Ann}_S(M') \cap \mathrm{Ann}_S(M'')}$$

(see the proof of [Mur24ca, Proposition 8.14.2]). Thus, if the theorem is true for M' and M'', then it is true for M. By Proposition 1.7.9, M has a filtration with quotients of the form $(S/\mathfrak{p})(l)$ where $\mathfrak{p}$ is a homogeneous prime ideal and $l \in \mathbf{Z}$. By induction on the length of the filtration, we may therefore reduce to the case when $M \cong (S/\mathfrak{p})(l)$ where we may assume that $r = \dim(Z_+(\mathfrak{p}))$ by the inductive hypothesis on r. The shift l corresponds to a change of variables $z \mapsto z + l$, and hence it suffices to consider the case when $M = S/\mathfrak{p}$ where $r = \dim(Z_+(\mathfrak{p}))$.

Since $\mathfrak{p} \neq (x_0, x_1, \dots, x_n)$, choose $x_i \notin \mathfrak{p}$, and consider the exact sequence

$$0 \longrightarrow \frac{S}{\mathfrak{p}}(-1) \xrightarrow{x_i \cdot -} \frac{S}{\mathfrak{p}} \longrightarrow \frac{S}{(\mathfrak{p}, x_i)} \longrightarrow 0.$$

Set $M'' \coloneqq S/(\mathfrak{p}, x_i)$. Then,

$$\varphi_{M''}(l) = \varphi_{S/\mathfrak{p}}(l) - \varphi_{S/\mathfrak{p}}(l-1) = (\Delta\varphi_{S/\mathfrak{p}})(l-1)$$

is eventually equal to a polynomial $P_{M''}(z)$ of degree $\dim(Z_+(\mathfrak{p}, x_i))$ by the inductive hypothesis. By Proposition 1.7.7 and its proof, we therefore see that φ_M is eventually equal to a polynomial $P_M(z)$ of degree $\dim(Z_+(\mathfrak{p}, x_i)) + 1$. On the other hand, $Z_+(\mathfrak{p}, x_i) = Z_+(\mathfrak{p}) \cap H$ where $H = Z_+(x_i)$ and $Z_+(\mathfrak{p}) \not\subseteq H$ by the choice of x_i. By the projective dimension theorem (Theorem 1.7.4), we have

$$\dim\bigl(Z_+(\mathfrak{p})\bigr) = \dim\bigl(Z_+(\mathfrak{p}, x_i)\bigr) + 1. \qquad \square$$

[Har77, p. 52] Definition 1.7.15. If $Y \subseteq \mathbf{P}^n_k$ is an algebraic set of dimension r, we define the *Hilbert polynomial of* Y $P_Y(z)$ to be the Hilbert polynomial $P_{S(Y)}(z)$ of its homogeneous coordinate ring $S(Y)$. (By Theorem 1.7.13, it is a polynomial of degree r.) The *degree* $\deg(Y)$ *of* Y is $r!$ times the leading coefficient of $P_Y(z)$:

$$P_Y(z) = \frac{\deg(Y)}{r!} z^r + \cdots .$$

[Har77, Prop. I.7.6] Proposition 1.7.16.

(*a*) *If* $Y \subseteq \mathbf{P}^n_k$ *is nonempty, then* $\deg(Y)$ *is a positive integer.*
(*b*) *Suppose* $Y = Y_1 \cup Y_2$ *where* $\dim(Y_1) = \dim(Y_2) = r$ *and* $\dim(Y_1 \cap Y_2) < r$. *Then,* $\deg(Y) = \deg(Y_1) + \deg(Y_2)$.
(*c*) *We have*

$$P_{\mathbf{P}^n_k}(z) = \binom{z+n}{n}$$

and $\deg(\mathbf{P}^n_k) = 1$.
(*d*) *Let* $f \in S$ *be a homogeneous polynomial of degree* d. *Then,*

$$\begin{aligned} P_{S/(f)}(z) &= \binom{z+n}{n} - \binom{z-d+n}{n} \\ &= \frac{d}{(n-1)!}\, z^{n-1} + \cdots . \end{aligned}$$

Suppose, moreover, that f *is a product of* distinct *irreducible factors. Then,* $H = Z_+(f)$ *satisfies* $\deg(H) = d$. *In other words, the new definition of degree is consistent with the definition of the degree of a hypersurface as defined in Example 1.1.22.*

Proof. (*a*). Since $Y \neq \emptyset$, we know that P_Y is a nonzero polynomial of degree $r = \dim(Y)$. By Proposition 1.7.7(*a*), $\deg(Y) = c_0 \in \mathbf{Z}$. It is a positive integer because for $l \gg 0$, we have

$$P_Y(l) = \varphi_{S/I}(l) \geq 0.$$

(*b*). Let I_1 and I_2 be the ideals for Y_1 and Y_2, respectively. Then, $I = I_1 \cap I_2$ is the ideal of Y. We then have the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I & \xrightarrow{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} & I_1 \oplus I_2 & \xrightarrow{[1\ -1]} & I_1 + I_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S & \xrightarrow{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} & S \oplus S & \xrightarrow{[1\ -1]} & S & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \dfrac{S}{I} & \xrightarrow{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} & \dfrac{S}{I_1} \oplus \dfrac{S}{I_2} & \xrightarrow{[1\ -1]} & \dfrac{S}{I_1 + I_2} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where the top two rows and all three columns are exact. By the snake lemma, the botttom row is also exact. Now $Z_+(I_1 + I_2) = Y_1 \cap Y_2$, which has smaller dimension. Thus,

$$\deg\big(P_{S/(I_1+I_2)}(z)\big) < r$$

and hence the leading coefficient of $P_{S/I}(z)$ is the sum of the leading coefficients of $P_{S/I_1}(z)$ and $P_{S/I_2}(z)$.

(*c*). The Hilbert polynomial of $\mathbf{P}^n_k$ is the polynomial $P_S(z)$. For $l > 0$, the "dots and dividers" proof from the Homework shows that

$$\varphi_S(l) = \binom{l+n}{n}.$$

Thus,

$$P_S(z) = \binom{z+n}{n}.$$

In particular, the leading coefficient of $P_S(z)$ is $1/n!$. Thus, $\deg(\mathbf{P}^n_k) = 1$.

(*d*). We have an exact sequence of graded S-modules

$$\begin{array}{c} 0 \longrightarrow S(-d) \xrightarrow{f\cdot -} S \longrightarrow \dfrac{S}{(f)} \longrightarrow 0 \\ 1 \longmapsto f. \end{array}$$

Thus,

$$\varphi_{S/(f)}(l) = \varphi_S(l) - \varphi_S(l-d)$$

and the Hilbert polynomial of $S/(f)$ is

$$\begin{aligned} P_{S/(f)}(z) &= \binom{z+n}{n} - \binom{z-d+n}{n} \\ &= \frac{d}{(n-1)!}\, z^{n-1} + \cdots . \end{aligned}$$

When f is a product of distinct irreducible factors and $H = Z_+(f)$, we therefore have $\deg(H) = d$. $\square$

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1.7.5. Bézout's theorem. We are now ready to prove Bézout's theorem. We start by defining the intersection multiplicity of a projective variety and a hypersurface.

[Har77, p. 53] DEFINITION 1.7.17. Let $Y \subseteq \mathbf{P}^n_k$ be a projective variety of dimension r. Let H be a hypersurface not containing Y. Then, by the Projective Dimension Theorem 1.7.4, we have

$$Y \cap H = Z_1 \cup Z_2 \cup \cdots \cup Z_s$$

where the Z_j are varieties of dimension $r-1$. Let $\mathfrak{p}_j$ be the homogeneous prime ideal of Z_j. The *intersection multiplicity of* Y *and* H *along* Z_j is

$$i(Y,H;Z_j) \coloneqq \mu_{\mathfrak{p}_j}\left(\frac{S}{I_Y + I_H}\right)$$

where $I_Y = I(Y)$ and $I_H = I(H)$.

We can now state our version of Bézout's theorem. Note that in this statement, Y can be a projective variety of arbitrary positive dimension. We will then state Bézout's theorem (the case when Y and Z are curves) as a corollary.

[Har77, Thm. I.7.7] THEOREM 1.7.18. *Let* Y *be a projective variety of dimension* ≥ 1 *in* $\mathbf{P}^n_k$. *Let* $f \notin I_Y$ *be a homogeneous polynomial of degree* d *that is a product of* distinct *irreducible factors and set* $H = Z_+(f)$. *Let* $Z_1, Z_2, \ldots, Z_s$ *be the irreducible components of* $Y \cap H$. *Then, we have*

$$\sum_{j=1}^{s}\big(i(Y,H;Z_j)\cdot \deg(Z_j)\big) = \big(\deg(Y)\big)\cdot\big(\deg(H)\big).$$

Proof. Consider the exact sequence of graded S-modules

$$\begin{array}{ccccccccc} 0 \longrightarrow & \dfrac{S}{I_Y}(-d) & \xrightarrow{f\cdot -} & \dfrac{S}{I_Y} & \longrightarrow M \longrightarrow 0 \\ & 1 & \longmapsto & f & \end{array}$$

where $M = S/(I_Y + I_H)$. Note that multiplication by f is injective on S/I_Y since the image of f in S/I_Y is nonzero. Taking Hilbert polynomials, we have

$$P_M(z) = P_Y(z) - P_Y(z-d). \tag{1.7.19}$$

The result will follow from comparing the leading coefficients of both sides of this equation, as follows. Let Y have dimension r and degree e. Then,

$$P_Y(z) = \frac{e}{r!}z^r + \cdots.$$

On the right-hand side of (1.7.19), we have

$$\left(\frac{e}{r!}z^r + \cdots\right) - \left(\frac{e}{r!}(z-d)^r + \cdots\right) = \frac{de}{(r-1)!}z^{r-1} + \cdots. \tag{1.7.20}$$

On the other hand, we compute the Hilbert polynomial for M by dévissage. By Proposition 1.7.9, M has a filtration

$$0 = M^0 \subseteq M^1 \subseteq \cdots \subseteq M^q = M$$

with quotients $M^i/M^{i-1} \cong (S/\mathfrak{q}_i)(l_i)$. We then have

$$P_M(z) = \sum_{i=1}^{q} P_i(z)$$

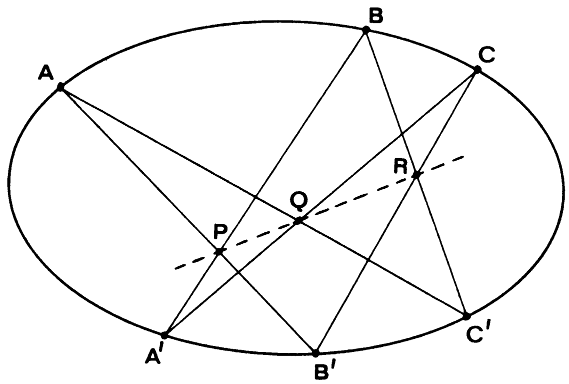

FIGURE 1.23. Pascal's "Hexagrammum Mysticum" theorem. From [Har77, Figure 22].

where letting $r_i = \dim(Z_+(\mathfrak{q}_i))$ and $f_i = \deg(Z_+(\mathfrak{q}_i))$, we have

$$P_i(z) = \frac{f_i}{r_i!} z^{r_i} + \cdots$$

since the shift l_i does not affect the leading coefficient of $P_i(z)$. Since we are only interested in the leading coefficient of $P_M(z)$, we may ignore the $P_i(z)$ of degree $< r-1$. We are therefore left with the $P_i(z)$ corresponding to the minimal primes of M, which are the primes $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_s$ corresponding to the Z_j. Each $\mathfrak{p}_j$ appears $\mu_{\mathfrak{p}_j}(M)$ times in the prime cyclic filtration for M, and hence the leading coefficient of $P_M(z)$ is

$$\frac{1}{(r-1)!}\left(\sum_{j=1}^{s}\bigl(i(Y,H;Z_j)\cdot\deg(Z_j)\bigr)\right).$$

Comparing this with (1.7.20), we are done. □

COROLLARY 1.7.21 (Bézout 1779 [Béz06]). *Let Y, Z be distinct curves in $\mathbf{P}^2_k$ of degrees d, e. Let $Y \cap Z = \{P_1, P_2, \ldots, P_s\}$. Then,* [Har77, Cor. I.7.8]

$$\sum_{j=1}^{s} i(Y,Z;P_j) = de.$$

The statement also holds if Y and Z are possibly reducible algebraic sets all of whose components are of dimension 1 such that Y and Z have no irreducible component in common.

Proof. This follows from Theorem 1.7.18 working one component of Y at a time after noting that a point has Hilbert polynomial 1, and hence degree 1. □

1.7.6. Pascal's theorem. As a fun application, we prove the following theorem due to Pascal, which is a generalization of Pappus's hexagon theorem [Pap86, 7.206 and 7.207]. Pappus's theorem is the special case of Pascal's theorem where Γ is the union of two lines.

THEOREM 1.7.22 (Hexagrammum Mysticum, Pascal 1640 [Smi29, pp. 326–330]). *Let A, B, C, A', B', C' be six distinct points on an irreducible plane curve $\Gamma \subseteq \mathbf{P}^2_k$ of* [Har77, Exer. V.4.5]

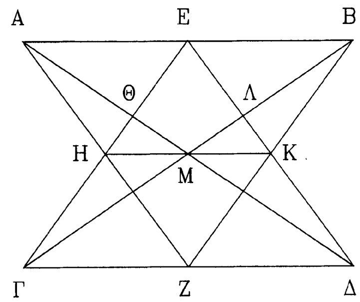

FIGURE 1.24. Pappus's hexagon theorem. From [Pap86, Text Figure 138].

degree 2*. Then, the three points* $P = \overline{AB'} \,.\, \overline{A'B}$*,* $Q = \overline{AC'} \,.\, \overline{A'C}$*, and* $R = \overline{BC'} \,.\, \overline{B'C}$ *are collinear.*

If Γ *is reducible, the same conclusion holds assuming that the six points have the following configuration: Writing* $\Gamma = \Gamma_1 \cup \Gamma_2$ *as a union of two lines, we have* $A, B, C \in \Gamma_1$*,* $A', B', C' \in \Gamma_2$*, and* $\Gamma_1 \,.\, \Gamma_2 \notin \{A, B, C, A', B', C'\}$*.*

See Figures 1.23 and 1.24 for illustrations.

Proof. Let $f = \ell_1\ell_2\ell_3$ be the homogeneous cubic polynomial vanishing on $\overline{AB'}$, $\overline{BC'}$, and $\overline{CA'}$. Let $g = m_1m_2m_3$ be the homogeneous cubic polynomial vanishing on $\overline{BA'}$, $\overline{CB'}$, and $\overline{AC'}$. We then consider the homogeneous cubic polynomial

$$h_\lambda = f + \lambda g$$

where $\lambda \in k$. We choose a point $x \in \Gamma$ as follows:

- When Γ is irreducible, choose
$$x \in \Gamma - \{A, B, C, A', B', C'\}.$$
- When Γ is reducible, choose
$$x = \Gamma_1 \,.\, \Gamma_2,$$

Now choose $\lambda \in k$ such that $h_\lambda(x) = 0$. Then, $Z_+(h_\lambda)$ is a possibly reducible curve of degree 3 intersecting Γ at at least 7 points. By Bézout's theorem (Corollary 1.7.21), we therefore see that $Z_+(h_\lambda)$ contains Γ (when Γ is reducible, this still holds since $Z_+(h_\lambda)$ intersects both Γ_1 and Γ_2 in at least 4 points by the choice of x). However, $Z_+(h_\lambda)$ contains the 9 points $A, B, C, A', B', C', P, Q, R$ as well, 6 of which are on Γ. Note that the points P, Q, R cannot lie on Γ because of the assumptions on the six points A, B, C, A', B', C'. Thus, writing

$$Z_+(h_\lambda) = \Gamma \cup L$$

for a line L, we know that L contains the other 3 points P, Q, R. □

1.8. There are exactly 27 lines on a smooth cubic surface

Our goal for the next few lectures is to prove the statement in the section title:

THEOREM 1.8.1 (Cayley [Cay1849] and Salmon [Sal1849]). *Let* $X \subseteq \mathbf{P}^3_k$ *be a nonsingular cubic surface in* $\mathbf{P}^3_k$*. Then, there are exactly 27 lines on* X*.*

Salmon in [Sal1915, Footnote on p. 183] explains how Cayley and Salmon proved this theorem in their correspondences. See also [Dol25_2, Chapter 9, Historical Notes] for the history of this result and other results on cubic surfaces. The proof we present is Cayley's proof from [Cay1849] in modern language. We draw from the accounts in [Bea96, Chapter IV], [Rei13, §7], [Sha13_1, Chapter 4, §2.5], and [Dol25_2, Theorem 9.1.13].

For a while, we will focus on proving that any cubic surface contains at least one line. This can be using elementary methods (see [Rei13, Proposition 7.2]). However, since we have developed many of the necessary tools to show the existence of a line more conceptually, we will proceed more along the lines of [Bea96, Lemma IV.14] and [Sha13_1, Theorem 1.28] and use this opportunity to introduce some fundamental theorems and constructions in algebraic geometry that you might not see until much later if we waited until we had the language of schemes. There are three ingredients we need to prove this existence result.

(1) The image of a morphism of projective varieties is closed (Theorem 1.8.2). This generalizes a problem from Homework.
(2) For a morphism $\pi\colon Y \to X$ of projective varieties, the set
$$\bigl\{x \in X \bigm| \dim\bigl(\pi^{-1}(x)\bigr) \geq k\bigr\}$$
is closed in X for every integer $k \geq 0$ (Theorem 1.8.5). This is called the *theorem on the dimension of fibers* in [Sha13_1].
(3) There is a projective variety that parametrizes d-planes through the origin in k^n. This variety is called the *Grassmannian* $G(d,n)$.

1.8.1. Images of projective varieties. We start with the following result, which we prove using elimination theory (in the form you saw on Homework 7 and 8 using resultants).

THEOREM 1.8.2. *Let X be a quasi-projective variety and let* [Har92, Thm. 3.12]
$$\pi\colon X \times \mathbf{P}^n_k \longrightarrow X$$
be the projection onto the first factor. For any closed subset $Y \subseteq X \times \mathbf{P}^n_k$, the image $\pi(Y)$ of Y is closed in X.

Proof. Working with each member of an affine open cover of X, we may assume that X is a closed subvariety of $\mathbf{A}^m_k$ with coordinates $x_1, x_2, \ldots, x_m$. Working one irreducible component of Y at a time, we may assume that Y is irreducible.

We proceed by induction on n. When $n = 0$, there is nothing to show. Now suppose that $n > 0$. Let J be the ideal defining Y in $X \times \mathbf{P}^n_k$, which is homogeneous with respect to the $\mathbf{N}$-grading on
$$A(X)[y_0, y_1, \ldots, y_n]$$
where $A(X)$ has degree 0 and each y_i has degree 1. We claim there exists a point $P \in \mathbf{P}^n_k$ such that $Y \not\subseteq X \times \{P\}$. Otherwise, we would have $Y \subseteq X \times \{P\}$ for every $P \in \mathbf{P}^n_k$. Since $X \times \{P\}$ and $X \times \{P'\}$ do not intersect for $P \neq P'$, this implies that Y is empty, contradicting the assumption that Y is irreducible.

We now consider the morphism
$$\pi_n\colon X \times \bigl(\mathbf{P}^n_k - \{P\}\bigr) \longrightarrow X \times \mathbf{P}^{n-1}_k$$

obtained from the universal property of products where the morphism $X \to X$ is the identity and the morphism $\mathbf{P}^n_k - \{P\} \to \mathbf{P}^{n-1}_k$ is the projection away from P. We claim that

$$\pi_n\bigl(Y - \bigl(X \times \{P\}\bigr)\bigr)$$

is closed. In fact, we will show that this set is equal to

$$Z\Bigl(\bigl\{\operatorname{Res}\bigl(f(\underline{x},\underline{y}), g(\underline{x},\underline{y})\bigr) \bigm| f, g \in J\bigr\}\Bigr)$$

where the resultant is formed by thinking of f and g as polynomials in y_n with coefficients in $A(X)[y_0, y_1, \ldots, y_{n-1}]$. It suffices to show this equality of sets after restricting to each point $Q \in X$. In this case, the resultant specializes to the resultant

$$\operatorname{Res}\bigl(f(Q,\underline{y}), g(Q,\underline{y})\bigr)$$

of f, g where the coefficients are evaluated at Q. We have

$$\begin{aligned}\pi_n\Bigl(Y - \bigl(X \times \{P\}\bigr)\Bigr) &\cap \bigl(\{Q\} \times \mathbf{P}^{n-1}_k\bigr)\\ &= (\pi_n)|_{\{Q\}\times\mathbf{P}^n_k}\Bigl(Y \cap \bigl(\{Q\} \times \mathbf{P}^n_k\bigr) - \{P\}\Bigr)\end{aligned}$$

and $(\pi_n)|_{\{Q\}\times\mathbf{P}^n_k}$ is the projection away from P. These sets therefore have the required description in terms of resultants by Homework 8, Problem 9.

We now consider the composition

$$\pi' \colon X \times \bigl(\mathbf{P}^n_k - \{P\}\bigr) \xrightarrow{\pi_n} X \times \mathbf{P}^{n-1}_k \longrightarrow X,$$

which coincides with the restriction of π to $X \times \mathbf{P}^n_k - \{P\}$. Applying the inductive hypothesis, the image $\pi'(Y - (X \times \{P\}))$ in X is a closed set W. Since the image of Y in X is contained in W, the set Y is closed in $X \times \mathbf{P}^n_k$, and

$$\overline{Y - \bigl(X \times \{P\}\bigr)} = Y,$$

we conclude that

$$W = \pi'\Bigl(Y - \bigl(X \times \{P\}\bigr)\Bigr) \subseteq \pi(Y) \subseteq \overline{\pi\Bigl(Y - \bigl(X \times \{P\}\bigr)\Bigr)} = W$$

is closed. □

As a consequence, we have:

[Har92, Thm. 3.13] Theorem 1.8.3. *Let Y be a projective variety and let $f\colon Y \to X$ be a morphism to a projective variety. Then, $f(Y)$ is a closed subvariety of X.*

Proof. We can factor f as

$$\begin{array}{ccccc} Y & \xrightarrow{(\mathrm{id}_Y, f)} & Y \times X & \hookrightarrow & \mathbf{P}^m_k \times X \\ & & & & \downarrow \pi \\ & \searrow^{f} & & & X. \end{array}$$

By Theorem 1.8.2, it suffices to note that

$$\Gamma_f \coloneqq (\mathrm{id}_Y, f)(Y) = (f, \mathrm{id}_X)^{-1}(\Delta_X)$$

is closed in $Y \times X$. Finally, $f(Y)$ is irreducible since it is the image of an irreducible set. □

This gives a new proof of Theorem 1.3.29(*a*).

[Har92, Cor. 3.14]

COROLLARY 1.8.4. *Let f be a regular function on a projective variety Y. Then, f is constant.*

Proof. Apply Theorem 1.8.3 to the composition $Y \to \mathbf{A}^1_k \hookrightarrow \mathbf{P}^1_k$. □

Another application of Theorem 1.8.3 is that it gives a new proof of Homework 8, Problem 8.

1.8.2. Dimension of fibers. The next ingredient we need is a result on the dimension of fibers of a morphism. The key ingredient here is Theorem 1.8.3 and the relative version of the Noether normalization theorem we proved in the commutative algebra course [Mur24ca]. Along the way, we will also need to use systems of parameters.

THEOREM 1.8.5 (The theorem on the dimension of fibers). *Let $\pi\colon Y \to X$ be a morphism of quasi-projective varieties. Set $\dim(X) = m$ and $\dim(Y) = n$.*

(*i*) *We have* [Sha13₁, Thm. 1.25]

$$\dim\bigl(\pi^{-1}(x)\bigr) \geq n - m$$

for all $x \in \pi(Y)$.

(*ii*) *If π is dominant, then there exists a nonempty open subset $U \subseteq X$ such that $\pi(Y) \supseteq U$ and*

$$\dim\bigl(\pi^{-1}(x)\bigr) = n - m$$

for all $x \in U$.

(*iii*) *Consider the projection morphism*

Cf. [Sha13₁, Cor. on p. 76]. This statement in [Sha13₁] is incorrect. See [Spe20].

$$X \times \mathbf{P}^n_k \longrightarrow X$$

and consider a closed subset $Y \subseteq X \times \mathbf{P}^n_k$. Denote by $\pi\colon Y \to X$ the composition. Then,

$$\bigl\{x \in X \bigm| \dim\bigl(\pi^{-1}(x)\bigr) \geq k\bigr\}$$

is closed in X.

Proof. For (*i*), let $f_1, f_2, \ldots, f_m$ be a system of parameters at x, which we recall is a sequence of $\dim(X)$ elements in the maximal ideal of $\mathcal{O}_{X,x}$ that generate the maximal ideal defining x up to radical [Hoc17, p. 128]. Such a sequence exists by the proof of Krull's height theorem [Hoc17, p. 125]. (Extracting the construction from the proof, one constructs such a sequence by finding $f_1 \in \mathfrak{m}$ avoiding the minimal primes of $\mathcal{O}_{X,x}$, taking the quotient by f_1, and repeating this process.) Replacing X by a suitable open neighborhood of x, we may assume that $f_1, f_2, \ldots, f_m$ are regular functions on X. Now $\pi^{-1}(x)$ is a union of irreducible components of

$$\pi^{-1}\bigl(Z(f_1, f_2, \ldots, f_m)\bigr) = Z(\pi^* f_1, \pi^* f_2, \ldots, \pi^* f_m),$$

which has dimension $\geq n - m$ by the affine dimension theorem (Proposition 1.7.3).

For (*ii*), we can replace X by an open subset to assume that X is affine with affine coordinate ring D. We can then cover Y by affine open subsets V_i with affine coordinate rings R_i containing D. By the relative Noether normalization theorem [Hoc17, p. 50], there exists an open subset

$$U_i = V_i - Z(c_i) \subseteq V_i$$

such that $(R_i)_{c_i}$ is module-finite over a polynomial ring $D_{c_i}[z_1, z_2, \ldots, z_{d_i}]$ over D_{c_i} for some integer d_i. Comparing the transcendence degree of fraction fields, we see that $d_i = n - m$. Thus, for any point $x \in U_i$ corresponding to a maximal ideal $\mathfrak{m} \subseteq D_{c_i}$, we see that

$$\dim\bigl(\pi^{-1}(x) \cap V_i\bigr) = \dim\bigl((D_{c_i}/\mathfrak{m})[z_1, z_2, \ldots, z_{n-m}]\bigr) = n - m.$$

Finally, we see that taking $U = \bigcap_i U_i$ works since dimension can be computed as the maximum of the dimension of each member of an open cover.

For (*iii*), we proceed by induction on $\dim(X)$. If $\dim(X) = 0$, there is nothing to show. Now let $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_r$ be the irreducible decomposition of Y. Denote by $\pi_i \colon Y_i \to X$ the restriction of π to each irreducible component. Since

$$\pi^{-1}(x) = \bigcup_{i=1}^{r} \pi_i^{-1}(x),$$

saying that $\dim(\pi^{-1}(x)) \geq k$ is equivalent to $\dim(\pi_i^{-1}(x)) \geq k$ for some i. Thus,

$$\bigl\{x \in X \bigm| \dim\bigl(\pi^{-1}(x)\bigr) \geq k\bigr\} = \bigcup_{i=1}^{r} \bigl\{x \in X \bigm| \dim\bigl(\pi_i^{-1}(x)\bigr) \geq k\bigr\}.$$

We may therefore assume that Y is irreducible. By Theorem 1.8.3, we know that $\pi(Y)$ is closed. Replacing X by $\pi(Y)$, we may assume that π is surjective. Now set $n = \dim(Y)$ and $m = \dim(X)$. If $k \leq n - m$, we are done by (*i*). Now suppose $k > n - m$. By (*ii*), there is an open subset of X over which $\dim(\pi^{-1}(x)) = n - m < k$ for all $x \in X$. Let $Z = X - U$. Then, $\dim(Z) < \dim(X)$. Applying induction to the restriction of π to each irreducible component of Z, we are done. □

11/1
[Har92, Lect. 6]
[Sha13$_1$, Ex. 1.24]

1.8.3. Grassmannians. We now introduce the last ingredient: Grassmannians. Let V be a k-vector space. The idea of the Grassmannian $G(d, V)$ is that we want a projective(!) variety whose points correspond to d-planes in V. To make the definition easier to state, we introduce a coordinate-free notation for projective space:

$$\mathbf{P}(V) \coloneqq \frac{V - \{0\}}{k^*}$$

where by k^* we mean that we quotient out by the usual action of k^* on $V - \{0\}$ by scalar multiplication. When $d = 1$ and $V = k^n$, this is just $\mathbf{P}^n_k$.

Definition 1.8.6 (Grassmannians). Let V be a k-vector space of dimension n and let $d \geq 1$ be an integer. As a set, the *Grassmannian* $G(d, V)$ is

$$G(d, V) \coloneqq \bigl\{d\text{-dimensional subspaces of } V\bigr\}.$$

If $V = k^n$ for some n, we also denote the Grassmannian by $G(d, n)$.

We give $G(d, V)$ the structure of a projective variety as follows. First, we consider the map

$$\begin{aligned} G(d, V) &\longrightarrow \mathbf{P}\Bigl(\bigwedge^d V\Bigr) \\ \operatorname{Span}_k\{v_1, v_2, \ldots, v_d\} &\longmapsto [v_1 \wedge v_2 \wedge \cdots \wedge v_d]. \end{aligned} \tag{1.8.7}$$

This map is well-defined because different choices of bases correspond to multiplying $v_1 \wedge v_2 \wedge \cdots \wedge v_d$ by the determinant of the change of coordinates matrix, which is a nonzero constant in k. The map (1.8.7) is injective because for a point

$$[v_1 \wedge v_2 \wedge \cdots \wedge v_d] \in \mathbf{P}\Bigl(\bigwedge^d V\Bigr),$$

the preimage of $[v_1 \wedge v_2 \wedge \cdots \wedge v_d]$ is the kernel W in the exact sequence

$$0 \longrightarrow W \longrightarrow V \xrightarrow{(v_1 \wedge \cdots \wedge v_d) \wedge -} \bigwedge^{d+1} V,$$

which is exactly d-dimensional.

Definition 1.8.8. The embedding (1.8.7) is called the *Plücker embedding*. Given a choice of basis on $V \cong k^n$, the $\binom{n}{d}$ coordinates p_I on $\bigwedge^d V$ for subsets $I \subseteq \{1, 2, \ldots, n\}$ of cardinality d are called *Plücker coordinates*. In Plücker coordinates, the Plücker embedding is the map sending $\operatorname{Span}_k\{v_1, v_2, \ldots, v_d\}$ to the point in $\mathbf{P}(\bigwedge^d V)$ whose I-th coordinate is

$$\text{determinant of minor of } \begin{pmatrix} - & v_1 & - \\ - & v_2 & - \\ & \vdots & \\ - & v_d & - \end{pmatrix} \text{ whose columns are indexed by } I.$$

Now that we have embedded $G(d, V)$ into $\mathbf{P}(\bigwedge^d V)$, we want to show that identifying $G(d, V)$ with its image under the Plücker embedding, the Grassmannian $G(d, V)$ is a projective variety.

Theorem 1.8.9. *The Grassmannian $G(d, V)$ is a closed subset of $\mathbf{P}(\bigwedge^d V)$.*

Proof of Theorem 1.8.9. Let

$$X := \left\{ ([w], [v]) \in \mathbf{P}\Big(\bigwedge^d V\Big) \times \mathbf{P}(V) \;\middle|\; w \wedge v = 0 \in \bigwedge^{d+1} V \right\}.$$

Looking at the short exact sequence

$$0 \longrightarrow K \longrightarrow \bigwedge^d V \otimes_k V \longrightarrow \bigwedge^{d+1} V \longrightarrow 0,$$

we see that $X = \mathbf{P}(K) \subseteq \mathbf{P}(\bigwedge^d V \otimes_k V)$ is a closed set.

Let $\pi\colon X \to \mathbf{P}(\bigwedge^d V)$ be the projection. We claim that

$$G(d, V) = \left\{ [w] \in \mathbf{P}\Big(\bigwedge^d V\Big) \;\middle|\; \dim\Big(\pi^{-1}\big([w]\big)\Big) \geq d - 1 \right\},$$

which is closed by the theorem on the dimension of fibers (Theorem 1.8.5(*iii*)). The inclusion $\subseteq$ holds since $[w] \in G(d, V)$ implies $([w], [v]) \in X$ for all $[v] \in \mathbf{P}(V)$. It remains to show the inclusion $\supseteq$. If $\dim(\pi^{-1}([w])) \geq d - 1$, then

$$\dim \left\{ v \in V \;\middle|\; w \wedge v = 0 \in \bigwedge^{d+1} V \right\} \geq d.$$

Choose a basis $v_1, v_2, \ldots, v_r$ for this vector space for $r \geq d$. Then, we can write

$$w = v_1 \wedge v_2 \wedge \cdots \wedge v_r \wedge \eta$$

for some $\eta \in \bigwedge^{d-r} V$ by the linear algebra result Lemma 1.8.10 below. In particular, this means that $d \geq r$. Thus, $d = r$ and $\eta \in k^*$. We conclude that

$$[w] = [v_1 \wedge v_2 \wedge \cdots \wedge v_r]. \qquad \square$$

LEMMA 1.8.10. *Let V be a finite dimensional vector space (over an arbitrary field). Let $w \in \bigwedge^d V$ and let $v_1, v_2, \ldots, v_r$ be linearly independent elements in V. Suppose that $v_i \wedge w = 0$ for all i. Then, there exists $\eta \in \bigwedge^{d-r} V$ such that*

$$w = v_1 \wedge v_2 \wedge \cdots \wedge v_r \wedge \eta.$$

Proof. Let $n = \dim V$ and let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V, such that the first r vectors v_i are $v_1, v_2, \ldots, v_r$. Write

$$w = \sum_{1 \le i_1 < i_2 < \cdots < i_d \le n} a_{(i_1, i_2, \ldots, i_d)} v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_d},$$

where the $v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_d}$ form a basis for $\bigwedge^d V$ by [Lan02, Chapter XIX, Proposition 1.1]. Then,

$$\begin{aligned} v_i \wedge w &= \sum_{1 \le i_1 < i_2 < \cdots < i_d \le n} a_{(i_1, i_2, \ldots, i_d)} v_i \wedge v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_d} \\ &= \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_d \le n \\ i \notin \{i_1, i_2, \ldots, i_d\}}} a_{(i_1, i_2, \ldots, i_d)} v_i \wedge v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_d} \\ &= 0 \end{aligned}$$

implies that $a_{(i_1, i_2, \ldots, i_d)} = 0$ for all $(i_1, i_2, \ldots, i_d)$ not containing i, since the

$$v_i \wedge v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_d}$$

are linearly independent in $\bigwedge^{d+1} V$ by [Lan02, Chapter XIX, Proposition 1.1].

Applying the above to each i, we have that $a_{(i_1, i_2, \ldots, i_d)} = 0$ for all $(i_1, i_2, \ldots, i_d)$ not containing $1 \le i \le r$. Thus, letting

$$b_{(i_{r+1}, i_{r+2}, \ldots, i_d)} \coloneqq a_{(1, 2, \ldots, r, i_{r+1}, i_{r+2}, \ldots, i_d)},$$

we have

$$\begin{aligned} \omega &= \sum_{1 \le i_{r+1} < i_{r+2} < \cdots < i_d \le n} b_{(i_{r+1}, i_{r+2}, \ldots, i_d)} v_1 \wedge \cdots \wedge v_r \wedge v_{i_{r+1}} \wedge \cdots \wedge v_{i_d} \\ &= v_1 \wedge \cdots \wedge v_r \wedge \left(\sum_{1 \le i_{r+1} < i_{r+2} < \cdots < i_d \le n} b_{(i_{r+1}, i_{r+2}, \ldots, i_d)} v_{i_{r+1}} \wedge \cdots \wedge v_{i_d} \right), \end{aligned}$$

Letting η be the parenthesized element above, which is in $\bigwedge^{d-r} V$, we are done. □

We want to compute the dimension of $G(d, V)$ and show it is in fact a projective variety. To do so, we use the following:

CONSTRUCTION 1.8.11. Choose a basis for V. For a subset $I \subseteq \{1, 2, \ldots, n\}$ of cardinality d, the intersection of $G(d, V)$ with the open open subset $\{p_I \neq 0\} \subseteq \mathbf{P}(\bigwedge^d V)$ corresponds to the set

$$\mathrm{GL}_d(k) \Big\backslash \left\{ \begin{matrix} d \times n \text{ matrices whose columns} \\ \text{indexed by } I \text{ are linearly independent} \end{matrix} \right\}$$

under the Plücker embedding. This set is in bijection with

$$\mathbf{A}^{d(n-d)} \cong \left\{ \begin{matrix} d \times n \text{ matrices whose columns} \\ \text{indexed by } I \text{ form } \mathrm{Id}_d \end{matrix} \right\}. \tag{1.8.12}$$

Finally, the map sending such a matrix to its set of minors is polynomial in its entries (by construction of the determinant). The sets (1.8.12) form an open cover of $G(d, V)$, and each one is irreducible.

REMARK 1.8.13. The equations that would come out of unraveling the proof of Theorem 1.8.9 are not optimal. In fact, the Grassmannian $G(d, V)$ can be written as the vanishing locus of quadratic relations in the Plücker coordinates, called the *Plücker relations*. For example, the Plücker relation defining $G(2, 4)$ in $\mathbf{P}(\bigwedge^2 k^4) \cong \mathbf{P}^5_k$ is

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0.$$

1.8.4. Every cubic surface contains a line. We are now ready to start the proof of Theorem 1.8.1.

THEOREM 1.8.14. *Every cubic surface $X \subseteq \mathbf{P}^3_k$ contains a line.* [Bea96, Lem. IV.14] [Sha13$_1$, Thm. 1.28] [Rei13, Prop. 7.2]

Proof. Let V be the 4-dimensional vector space whose projectivization is $\mathbf{P}^3_k$. Consider the vector space $C_3 = \operatorname{Sym}^3(V)$ corresponding to homogeneous degree 3 polynomials on V, which has dimension 20. The corresponding projective space

$$\mathbf{P}(C_3) \cong \mathbf{P}_k^{\binom{6}{3}-1} = \mathbf{P}^{19}_k$$

parametrizes (possibly reducible) cubic surfaces in $\mathbf{P}^3_k$. Consider the *incidence variety*

$$I \coloneqq \bigl\{(f, \ell) \in \mathbf{P}(C_3) \times G(2, V) \bigm| f|_\ell \equiv 0\bigr\} \subseteq \mathbf{P}(C_3) \times G(2, V).$$

We claim that I is closed in $\mathbf{P}(C_3) \times G(2, V)$. For a line $\ell \in G(2, V)$, we can choose coordinates w, x, y, z on V so that $\ell = Z(w, x)$. Then, the condition $f|_\ell \equiv 0$ says that the terms of f with y^3, y^2z, yz^2, z^3 have coefficient 0. This is indeed a polynomial condition on the coefficients of f.

We compute the dimension of I. Consider the projection $p_2 \colon I \to G(2, 4)$. The fibers of p_2 have dimension $19 - 4 = 15$ since as computed above, the condition that a cubic polynomial contains a line imposes 4 linear conditions on the coordinates on $\mathbf{P}(C_3)$. By Theorem 1.8.5(*ii*), we see that

$$\dim(I) = \dim\bigl(G(2, 4)\bigr) + 15 = 2(4 - 2) + 15 = 19$$

We now consider the projection $\pi \colon I \to \mathbf{P}(C_3)$. Both projective algebraic sets have dimension 19. Moreover, by Theorem 1.8.3, we know that $\pi(I)$ is closed in $\mathbf{P}(C_3)$. But there is at least one cubic surface $w^3 - xyz = 0$ with only finitely many lines: In the chart $w \neq 0$, a line parametrized as $t \mapsto (a_1t + b_1, a_2t + b_2, a_3t + b_3)$ cannot lie on the surface since

$$1 = (a_1t + b_1)(a_2t + b_2)(a_3t + b_3)$$

cannot hold for all t unless the a_i vanish. In the complement $w = 0$, there are only 3 lines: $0 = xyz$. Thus, Theorem 1.8.5(*ii*) shows that π is surjective, and hence every cubic surface contains a line. □

We also give a more instructive example.

EXAMPLE 1.8.15 (The Fermat cubic). Suppose $\operatorname{char}(k) \neq 3$. Consider the *Fermat cubic* $Z(c_{\text{Fermat}})$, where [Rei13, Exer. 7.6]

$$c_{\text{Fermat}} \coloneqq w^3 + x^3 + y^3 + z^3.$$

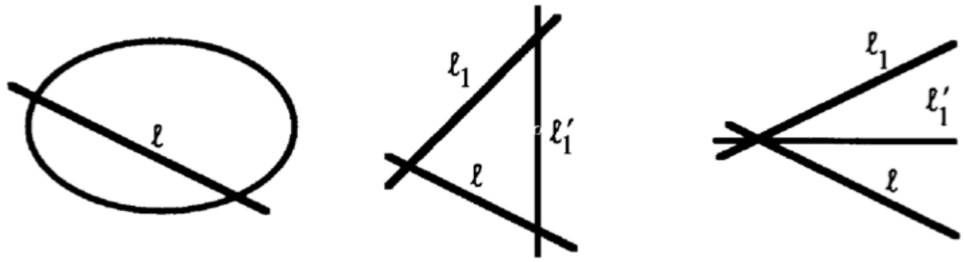

FIGURE 1.25. Line plus conic. From [Rei13, Figure 7.2].

Consider the lines $[s : as : t : bt]$ for a, b cube roots of -1 obtained by embedding

$$\begin{aligned} \mathbf{P}^1_k &\longrightarrow \mathbf{P}^3_k \\ [s : t] &\longmapsto [s : as : t : bt]. \end{aligned}$$

There are $3 \cdot 3 = 9$ choices of a, b, which give distinct embeddings of $\mathbf{P}^1_k$ by construction. One can show (see the Homework) that permuting coordinates gives us 3 times more the number of lines we had before, bringing the total up to 27, and that there are no more lines than these.

1.8.5. There are exactly 27 lines on a cubic surface. Given the existence of a line, we can now investigate the configuration of lines on a smooth cubic surface X. For another account of the proof, see [Sha13$_1$, Chapter 4, §2.5].

[Rei13, Prop. 7.1, Prop. 7.3] [Bea96, Lem. IV.15]

PROPOSITION 1.8.16. *Given a line $\ell \subseteq X$, there are exactly five pairs (ℓ_i, ℓ'_i) of lines on X meeting ℓ in such a way that*

(i) *For $i \in \{1, 2, \ldots, 5\}$, the lines ℓ, ℓ_i, ℓ'_i are coplanar.*
(ii) *For $i \neq j$, we have $(\ell_i \cup \ell'_i) \cap (\ell_j \cup \ell'_j) = \emptyset$.*

Proof. We first find all the lines that meet ℓ. Consider the 1-dimensional family (a "pencil") of planes $\{\Pi_{[\mu:\lambda]}\}_{[\mu:\lambda]\in\mathbf{P}^1_k}$ through ℓ. We then have

$$X \cap \Pi_{[\mu:\lambda]} = \ell \cup C_{[\mu:\lambda]}$$

where $C_{[\mu:\lambda]}$ is a conic.

We claim that this conic cannot be a double line and cannot contain ℓ. In other words, the possibilities we have are those in Figure 1.25. If the plane $L = 0$ intersects X along the line $M = 0$ and the line $N = 0$ counted twice, then X is defined by an equation

$$LQ + MN^2 = 0$$

where Q is a homogeneous quadratic polynomial and L, M, N are homogeneous linear polynomials. The partial derivaties for such a cubic surface are of the form

$$L_x Q + LQ_x + M_x N^2 + 2MNN_x$$

and similarly for w, y, z. These partial derivatives vanish at the two points where $L = N = Q = 0$, and hence X is singular at those two points. Thus, if $C_{[\mu:\lambda]}$ is singular, it is the union of two lines distinct from ℓ but meeting it, and these lines are coplanar with ℓ. The two possibilities are:
Since any line in X intersecting ℓ must be contained in some plane $\Pi_{[\mu:\lambda]}$, we know that the lines arising from singular $C_{[\mu:\lambda]}$ are the lines in X meeting ℓ.

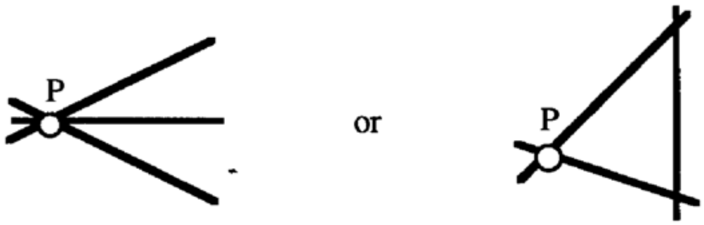

FIGURE 1.26. 3 concurrent lines or a triangle. From [Rei13, Figure 7.1].

We want to show that there are exactly 5 distinct planes Π_i where the $C_{[\mu:\lambda]}$ are singular. Choose coordinates on $\mathbf{P}^3_k$ so that $\ell = Z(w,z)$. Then, X is given by an equation of the form

$$f = Ax^2 + Bxy + Cy^2 + Dx + Ey + F \tag{1.8.17}$$

where $A, B, C, D, E, F \in k[w,z]$ are homogeneous of degrees 1,1,1,2,2,3, respectively. The plane $\Pi_{[\mu:\lambda]}$ is given by $\{\mu z = \lambda w\}$ for points $[\lambda : \mu] \in \mathbf{P}^1_k$. Whenever $\mu \neq 0$, we obtain a conic

$$C_{[1:\lambda]} : A(1,\lambda)x^2 + B(1,\lambda)xy + C(1,\lambda)y^2 + D(1,\lambda)x + E(1,\lambda)y + F(1,\lambda) = 0$$

obtained by substituting in $z = \lambda w$ and dividing f by w. Similarly, whenever $\lambda \neq 0$, we obtain a conic

$$C_{[\mu:1]} : A(\mu,1)x^2 + B(\mu,1)xy + C(\mu,1)y^2 + D(\mu,1)x + E(\mu,1)y + F(\mu,1) = 0.$$

These conics are singular if and only if

$$\Delta(w,z) = 4ACF + BDE - AE^2 - B^2F - CD^2 = 0$$

using what we know about when quadratic forms define a singular hypersurface from Homework 7, Problem 6. (In characteristic 2, you can redo the proof for the classification of conics from Homework 4, Problem 1. Note that $\Delta(w,z)$ is 4 times the usual discriminant. Cf. the proof of [Bea96, Lemma IV.15].) Note that $\Delta(w,z)$ is a homogeneous quintic polynomial. To show that there are 5 planes Π_i such that the $C_{[\mu:\lambda]}$ are singular, it suffices to show that $\Delta(w,z)$ does not have multiple roots.

Choose coordinates so that $z = 0$ is a root of $\Delta(w,z)$ and let $\Pi : z = 0$ be the corresponding plane. We want to show that $\Delta(w,z)$ is not divisible by z^2. By Figure 1.25, we know that $\Pi \cap X$ is a set of 3 lines. Depending on which situation we are in Figure 1.26, we choose coordinates so that

(1) (Triangle) $\ell : w = 0$, $\ell_1 : x = 0$, $\ell_1' : y = 0$.
(2) (Concurrent) $\ell : w = 0$, $\ell_1 : x = 0$, $\ell_1' : x = w$.

In case (1), we have

$$f = wxy + zg$$

where g is quadratic. Using the notation from (1.8.17), we then have $B = w + az$ and $z \mid A, C, D, E, F$. Thus, modulo terms divisible by z^2, we have

$$\Delta \equiv -w^2F \mod z^2.$$

Moreover, the point $[1:0:0:0] \in X$, and hence nonsingularity at this point implies that F must contain the term w^2z with nonzero coefficient. This shows that z^2

does not divide F. In case (2), we have

$$f = x(x-w)w + zg$$

and hence $A = w + aZ$ and $D = -w^2 + zL$ where L is linear. Then, $z \mid B, C, E, f$ and z does not divide D. We can then proceed similarly (exercise). $\square$

[Rei13, Cor. 7.4] [Bea96, Thm. IV.13, Lem. IV.15]

COROLLARY 1.8.18. *There exist two disjoint lines $\ell, m \subseteq X$. Also, X is rational.*

Proof Sketch. We know that $\ell_1 \cap \ell_2 = \emptyset$. We can then write the rational map

$$\varphi\colon X \dashrightarrow \ell \times m \qquad \psi\colon X \dashrightarrow \ell \times m$$

as follows. If $P \in \mathbf{P}^3_k - (\ell \cup m)$, then there exists a unique line n through P which meets both ℓ and n. Setting $\Phi(P) = (\ell \cap n, m \cap n) \in \ell \times m$, we have a morphism

$$\Phi\colon \mathbf{P}^3_k - (\ell \cup m) \longrightarrow \ell \times m$$

whose fiber above $(Q, R) \in \ell \times m$ is the line QR in $\mathbf{P}^3_k$. We then restrict Φ to X to get the rational map φ.

Conversely, for $(Q, R) \in \ell \times m$, we let $n = QR$ in $\mathbf{P}^3_k$. By Proposition 1.8.16, we know there are only finitely many lines on X meeting ℓ, so for almost all values of (Q, R), n intersects X in 3 points $\{P, Q, R\}$. We can therefore define ψ by $(Q, R) \mapsto P$. $\square$

REMARK 1.8.19. The proof is a sketch since I have not shown why φ and ψ are rational maps, i.e., defined using polynomials!

Now we have to find all the lines on X. We note the following useful fact:

LEMMA 1.8.20. *Any line $n \subseteq X$ must meet exactly one of ℓ_i, ℓ_i'.*

Proof. The line n meets the plane Π_i, and $\Pi_i \cap X = \ell \cup \ell_i \cup \ell_i'$. Moreover, n cannot meet both ℓ_i and ℓ_i' since then, $n \subseteq \pi_i$, contradicting Figure 1.26. $\square$

Using this, the remaining lines in X can be sorted out using the following:

[Rei13, Lem. 7.5]

LEMMA 1.8.21. *If $\ell_1, \ell_2, \ell_3, \ell_4 \subseteq \mathbf{P}^3_k$ are disjoint lines, then either:*

(*i*) *All 4 lines ℓ_i lie on a smooth quadric $Q \subseteq \mathbf{P}^3_k$ and they have an infinite number of common transversals (i.e., lines that intersect them).*

(*ii*) *The 4 lines ℓ_i do not lie on any quadric Q, in which case they have either 1 or 2 common transversals.*

Proof. Let Q be a smooth quadric containing ℓ_1, ℓ_2, ℓ_3. Note that such a quadric exists since ℓ_1, ℓ_2, ℓ_3 do not intersect. This is because quadrics form a $\binom{5}{2} - 1 = 9$ dimensional family, and so we can choose 3 points on each line for a quadric to pass through, which give linear relations on the coefficients. Each line must lie in the quadric by Bézout's theorem. The quadric is nonsingular because otherwise, if the quadric is a cone, the three lines intersect at the vertex, or if the quadric is a union of two planes (that possibly coincide), then two of the lines must lie on one of the planes and therefore intersect.

Since the three lines ℓ_1, ℓ_2, ℓ_3 do not intersect, we see they live in one of the rulings on the quadric surface Q by Homework 3, Problem 4. Now if ℓ_4 lies on Q then it must be part of the same ruling and we are in situation (*i*). If ℓ_4 is not on Q, then it must intersect Q in 1 or 2 points. The member(s) of the other ruling on Q containing these points give the transversals in case (*ii*). $\square$

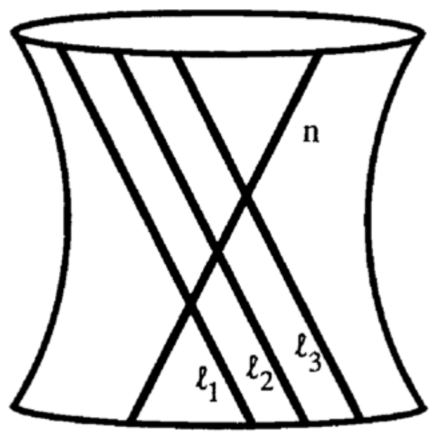

FIGURE 1.27. Quadric surface through 3 lines. From [Rei13, Figure 7.3].

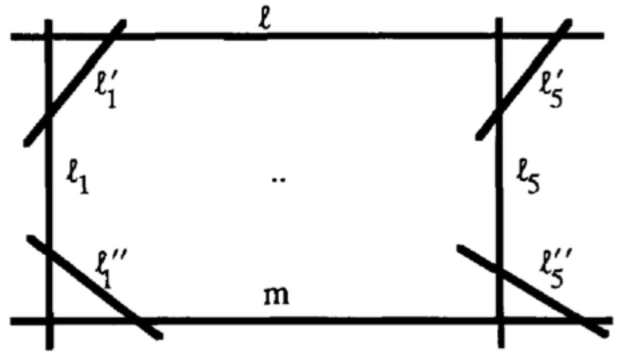

FIGURE 1.28. Configuration of lines on a smooth cubic surface. From [Rei13, Figure 7.4].

We are now ready to put everything to get our 27 lines. Let ℓ and m be disjoint lines As we saw already in Lemma 1.8.20, m meets exactly one out of each of the 5 pairs ℓ_i, ℓ_i' meeting ℓ. Renumbering the pairs, we may assume that m meets the ℓ_i. We then have the configuration of lines shown in Figure 1.28. By Proposition 1.8.16 applied to m, we get another set of lines ℓ_i'' for $i \in \{1,2,3,4,5\}$ and the 5 pairs of lines meeting m are (ℓ_i, ℓ_i''). By Lemma 1.8.20, we know that ℓ_i'' meets one of ℓ, ℓ_j, ℓ_j' for $i \neq j$. Since $i \neq j$, we know that the only possibility is that ℓ_i'' meets ℓ_j' for $i \neq j$.

CLAIM 1.8.22. [Rei13, Clm. 7.6]

(*i*) *If $n \subseteq S$ is any other line than these 17, then n meets exactly 3 out of the 5 lines $\ell_1, \dots, \ell_5$.*

(*ii*) *Given any choice $\{i, j, k\}$ of 3 elements of the set $\{1,2,3,4,5\}$, there is a unique line $\ell_{ijk} \subseteq X$ meeting ℓ_i, ℓ_j, ℓ_k.*

Proof. We first note that given four disjoint lines on X, they cannot all lie on a quadric Q, since otherwise $Q \subseteq X$ by Bézout's theorem, contradicting the irreducibility of X.

(i). If n meets ≥ 4 of the ℓ_i, then by Lemma 1.8.21, $n = \ell$ or $n = m$, which is a contradiction. If n meets ≤ 2 of the ℓ_i then it meets ≥ 3 of the ℓ_i' by Lemma 1.8.20. Thus, after possibly reordering, n meets $\ell_2', \ell_3', \ell_4', \ell_5'$ or $\ell_1, \ell_3', \ell_4', \ell_5'$. In either situation, ℓ and ℓ_1'' are common transversals of the 5 disjoint lines $\ell_1, \ell_2', \ell_3', \ell_4', \ell_5'$. By Lemma 1.8.21 again, if n meets ≥ 4 of these then $n = \ell$ or $n = \ell_1''$. This is a contradiction.

(ii). By Proposition 1.8.16, there are 10 lines meeting ℓ_1. So far, only 4 have been accounted for: $\ell, \ell_1', m, \ell_1''$. The 6 other lines must meet exactly 2 of the 4 remaining lines $\ell_2, \dots, \ell_5$ by (i). There are exactly $6 = \binom{4}{2}$ possibly choices, so they must all occur! $\square$

We conclude that the lines on X are:

$$\{\ell, m, \ell_i, \ell_i', \ell_i'', \ell_{ijk}\}$$

and there are

$$1 + 1 + 5 + 5 + 5 + \binom{5}{3} = 1 + 1 + 5 + 5 + 5 + 10 = 27$$

of them! This concludes the proof of Theorem 1.8.1. $\square$

CHAPTER 2

Schemes

2.0. Introduction

So far, we have restricted to studying quasi-projective varieties over algebraically closed fields. However, there are many reasons why we would want to enlarge our category to encompass more objects. 11/4 [Har77, §I.8]

2.0.1. Why schemes? We list some sample objects that we would like our category to encompass.

(1) (Non-algebraically closed fields) For applications to number theory, it becomes necessary to work over non-algebraically closed fields. For example, Fermat's "last theorem" [Dio1670, Arithmeticorum Liber II, Quæstio VIII, Observatio Domini Petri de Fermat] (now a theorem due to Wiles [Wil95; TW95]) asks whether the algebraic curve
$$\{x^n + y^n = z^n\} \subseteq \mathbf{P}^2_{\mathbf{Q}}$$
has any points over $\mathbf{Q}$ when $n > 2$. Over finite fields, the Weil conjectures [Wei49] (now theorems due to Dwork [Dwo60], Grothendieck [Gro68], and Deligne [Del74]), and were likely one of the motivations for Weil to develop his foundations for algebraic geometry [Wei62].

(2) (Non-quasi-projective varieties) Weil [Wei62] and Serre [FAC] both developed ways to discuss abstract varieties that are not necessarily quasi-projective varieties. This becomes necessary because sometimes it is easier to construct objects by gluing together affine varieties – for example Abelian varieties – in which case one must show that the resulting object is a quasi-projective variety.

(3) (Other ground rings) In commutative algebra, many theorems can be stated (and are easier to prove) in the general framework of Noetherian rings. In algebraic geometry, we want to be able to do something similar, i.e., to put all Noetherian rings (and possibly more) on the same footing. This allows one to take localizations and completions freely without leaving the category in question, which is essential (for example) in Hironaka's proof of resolutions of singularities [Hir64_1; Hir64_2] for varieties over fields of characteristic zero. (Hironaka's inductive proof in fact works with schemes of finite type over quasi-excellent local $\mathbf{Q}$-algebras.) The case of Dedekind domains of possibly mixed characteristic and objects of finite type over them (considered by Nagata [Nag56a] before Grothendieck) is one important example of this, since these schemes can interpolate between positive characteristic and characteristic zero.

(4) (Reducible and non-reduced) When working with intersections, we saw that oftentimes intersections of varieties could become reducible. Moreover,

in §1.7, we saw the importance of keeping track of *multiplicities* of varieties in intersections. These multiplicities keep track of the fact that sums of radical or prime ideals are not necessarily prime or radical.

Schemes will give us a common language that encompass all these features! Schemes were developed by Grothendieck and Dieudonné in [EGAI; EGAII; EGAIII$_1$; EGAIII$_2$; EGAIV$_1$; EGAIV$_2$; EGAIV$_3$; EGAIV$_4$; EGAI$_{\text{new}}$]. On the other hand, higher-dimensional analogues of abstract curves, called *Zariski–Riemann spaces*, are not schemes. So, in your algebraic geometry journey, you may need to go even further.

2.0.2. What do we need? To give scheme theory a firm foundation, we will introduce *sheaves*, which keep track of regular functions on open subsets in a way that gluing becomes easier. From a commutative-algebraic point of view, sheaves (more precisely, quasi-coherent sheaves) are what correspond to modules. Note that we have not discussed modules very much in this course! We will also introduce *sheaf cohomology*, which will be a good source of invariants we can use to differentiate different varieties or schemes. Sheaf cohomology will require developing some tools from homological algebra.

2.1. Sheaves

We introduce the notion of a sheaf. These provide a systematic way to keep track of local algebraic data on a topological space. For example, for a quasi-projective variety X, the structure sheaf $\mathcal{O}_X$ keeps track of the regular functions that are defined on an open subset U.

2.1.1. Presheaves. We start with presheaves.

[Har77, p. 61]
[God73, I.1.9]

Definition 2.1.1. Let X be a topological space. We can consider the category $\mathsf{Top}(X)$ whose objects are the open subsets of X, and whose morphisms are the inclusion maps. Let $\mathscr{C}$ be a category. A *presheaf $\mathscr{F}$ on X with values in $\mathscr{C}$* is a contravariant functor

$$\mathscr{F}\colon \mathsf{Top}(X)^{\mathrm{op}} \longrightarrow \mathscr{C}.$$

If $\mathscr{C}$ is the category of Abelian groups, rings, or sets, then we say that $\mathscr{F}$ is a presheaf of Abelian groups, rings, or sets. For Abelian groups, we sometimes say *Abelian presheaf*.

[TohokuI, p. 154]

Remark 2.1.2. In class, I will state everything for (pre)sheaves of sets and Abelian groups. The case for sets often implies things for other (pre)sheaves with values in a *concrete* category $\mathscr{C}$, which you can think of roughly as a category of objects whose morphisms are set maps that respect some extra structure. However, making this precise would require introduce more terminology from category theory, so we will not do so.

We spell out what a presheaf $\mathscr{F}$ of sets (resp. Abelian groups) is. A presheaf $\mathscr{F}$ of sets (resp. Abelian groups) consists of the following data:

(*a*) For every open subset $U \subseteq X$, a set (resp. an Abelian group) $\mathscr{F}(U)$.
(*b*) For every inclusion $V \subseteq U$ of open subsets of X, a function (resp. a group homomorphism) $\rho^U_V\colon \mathscr{F}(U) \to \mathscr{F}(V)$. We call these *restriction maps*.

These data are subject to the following conditions:

[Har77, p. 61] has the extra condition that $\mathscr{F}(\emptyset) = 0$. This is not standard.

(1) ρ^U_U is the identity map $\mathscr{F}(U) \to \mathscr{F}(U)$.

(2) If $W \subseteq V \subseteq U$ are three open subsets, then $\rho^U_W = \rho^V_W \circ \rho^U_V$.

DEFINITION 2.1.3. If $\mathscr{F}$ is a presheaf on X, we refer to $\mathscr{F}(U)$ as the *sections* over the open set U. We sometimes use the notation [Har77, p. 61] [God73, I.1.9]

$$\Gamma(U, \mathscr{F}) \coloneqq \mathscr{F}(U).$$

If $V \subseteq U$ is an inclusion of open sets, we sometimes write $s|_V$ instead of $\rho^U_V(s)$ for $s \in \mathscr{F}(U)$.

2.1.2. Sheaves. We can now define sheaves as a kind of presheaf that is determined by local data.

DEFINITION 2.1.4. Let X be a topological space and let $\mathscr{F}$ be a presheaf on X with values in a category $\mathscr{C}$. We say that $\mathscr{F}$ is a *sheaf with values in* $\mathscr{C}$ if it satisfies the following conditions: [Har77, p. 61] [God73, II.1.1]

(3) Let $\{U_i\}_{i\in I}$ be a family of open subsets of X with union U. Let $s', s'' \in \mathscr{F}(U)$. If $s'|_{U_i} = s''|_{U_i}$ for every i, then $s' = s''$.
(4) Let $\{V_i\}_{i\in I}$ be a family of open subsets of X with union V. Let $s_i \in \mathscr{F}(V_i)$ be elements such that for all $i, j \in I$, we have

$$s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}.$$

Then, there exists $s \in \mathscr{F}(V)$ such that $s|_{V_i} = s_i$ for all $i \in I$. (Note (3) implies that s is unique.)

REMARK 2.1.5 (The sheaf conditions in terms of an equalizer—not necessary). A convenient way (that you do not need to know!) to package the definition of a sheaf is that a presheaf $\mathscr{F}$ of sets is a sheaf if the diagram [EGAI$_{\text{new}}$, (**0**, 3.1.1)]

$$\mathscr{F}(U) \longrightarrow \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \cap U_j)$$

is an equalizer diagram for every family of open sets $\{U_i\}_{i\in I}$. See Definition A.4.1 for the definition of an equalizer. This means that the elements of $\prod_i \mathscr{F}(U_i)$ that map to the same element in $\prod_{i,j} \mathscr{F}(U_i \cap U_j)$ are *exactly* the elements in the image of $\mathscr{F}(U)$. An Abelian presheaf is a sheaf if the sequence

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\prod_i \rho^U_{U_i}} \prod_i \mathscr{F}(U_i) \xrightarrow{\prod_{i,j}(\rho^{U_i}_{U_i\cap U_j} - \rho^{U_j}_{U_i \cap U_j})} \prod_{i,j} \mathscr{F}(U_i \cap U_j)$$

is exact for every family of open sets $\{U_i\}_{i\in I}$. (This sequence probably looks familiar: We saw it on Homework 5, Problem 1 when computing $\mathcal{O}(\mathbf{A}^2 - \{(0,0)\})$!)

REMARK 2.1.6. What does a sheaf map the empty set to? Hartshorne [Har77, p. 61] includes $\mathscr{F}(\emptyset) = 0$ as part of the definition of a presheaf. This condition is actually a consequence of the definition of a sheaf: If one considers the *empty* family of open subsets $\{V_i\}_{i\in I}$ in (3), then the s_i are an element of the empty product in the category $\mathscr{C}$. The empty product is the final object in the category $\mathscr{C}$. For sheaves of sets, the final object in the category of sets is the singleton, and hence $\mathscr{F}(\emptyset) = \{*\}$. For Abelian sheaves, the final object in the category of Abelian groups is 0, and hence $\mathscr{F}(\emptyset) = 0$. [Stacks, Tag 006U]

EXAMPLE 2.1.7. Let X be a quasi-projective variety over an algebraically closed field k. For each open subset $U \subseteq X$, let $\mathcal{O}_X(U)$ be the ring of regular functions from U to k, and for each $V \subseteq U$, let [Har77, Ex. II.1.0.1]

$$\rho_V^U \colon \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V)$$

be the restriction map (in the usual sense). Then, $\mathcal{O}_X$ is a presheaf of rings on X by definition of a regular function, and is a sheaf on X since a function that is locally 0 is 0, and a function which is regular locally is regular. We call $\mathcal{O}_X$ the *sheaf of regular functions* on X.

[Har77, Ex. II.1.0.2] EXAMPLE 2.1.8. Similarly, one can define the sheaf of continuous real-valued functions on any topological space, the sheaf of differentiable functions on a differentiable manifold, the sheaf of holomorphic functions on a complex manifold, etc.

[Har77, Ex. II.1.0.3] [God73, II.1.4] We use this notation instead of $\mathscr{A}$ since $\underline{\mathbf{Z}}$ is a useful example.

EXAMPLE 2.1.9. Let X be a topological space and let A be an object in $\mathscr{C}$. The *constant sheaf* $\underline{A}_X$ on X determined by A is defined as follows. Consider A with the discrete topology. For any open set $U \subseteq X$, we set

$$\underline{A}_X(U) \coloneqq \left\{ \begin{matrix} \text{continuous maps} \\ U \to A \end{matrix} \right\}.$$

Together with the usual restriction maps, we obtain a sheaf $\underline{A}_X$. We sometimes drop the subscript X.

If U is an open set whose connected components are open (which is always true if X is locally connected), then $\underline{A}_X(U)$ is a direct product of copies of A, one for each connected component of U.

2.1.3. Stalks. The following definition gives the analogue of a germ of a function and the local rings of a quasi-projective variety for arbitrary sheaves.

[Har77, p. 62] [God73, II.1.2] DEFINITION 2.1.10. Let X be a topological space and let $\mathscr{F}$ be a presheaf on X with values in a cocomplete category (i.e., a category with arbitrary small colimits, in particular direct limits). Let $P \in X$ be a point. The *stalk* of $\mathscr{F}$ at P is

$$\mathscr{F}_P \coloneqq \varinjlim_{U \ni P} \mathscr{F}(U)$$

where the transition maps in the direct system are the restriction maps. The elements of $\mathscr{F}_P$ are called *germs* of sections of $\mathscr{F}$ at the point P. If $s \in \mathscr{F}(U)$ is a section, then the germ of s at $P \in U$ is s_P.

Spelling out the definition of a direct limit, the elements of $\mathscr{F}_P$ are represented by pairs $\langle U, s\rangle$ where U is an open neighborhood of P and $s \in \mathscr{F}(U)$, subject to the equivalence relation that

$$\langle U, s\rangle \sim \langle V, t\rangle \iff \begin{matrix} \text{there exists an open neighborhood} \\ W \subseteq U \cap V \text{ of } P \text{ such that } s|_W = t|_W. \end{matrix}$$

[Har77, p. 62] EXAMPLE 2.1.11. Let X be a quasi-projective variety over an algebraically closed field k. Then, the stalk $\mathcal{O}_{X,P}$ of the structure sheaf is the same thing as the local ring of P on X.

2.1.4. Morphisms.

We now define morphisms of (pre)sheaves.

[Har77, p. 62] [God73, I.1.9]

DEFINITION 2.1.12. Let X be a topological space. Let $\mathscr{F}, \mathscr{G}$ be two presheaves on X with values in $\mathscr{C}$. A *morphism* $\varphi\colon \mathscr{F} \to \mathscr{G}$ is a natural transformation $\mathscr{F} \Rightarrow \mathscr{G}$. More explicitly, φ consists of the data of a morphism $\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$ in $\mathscr{C}$ such that whenever $V \subseteq U$ is an inclusion, the diagram

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\varphi(U)} & \mathscr{G}(U) \\ {\scriptstyle \rho_V^U}\downarrow & & \downarrow{\scriptstyle \rho_V'^U} \\ \mathscr{F}(V) & \xrightarrow{\varphi(V)} & \mathscr{G}(V) \end{array}$$

commutes, where the vertical maps are the restriction maps for $\mathscr{F}$ and $\mathscr{G}$, respectively. We use the same definition for sheaves. An *isomorphism* is a morphism which has a two-sided inverse.

We therefore obtain the categories 11/6

$$\mathsf{Sh}(X) \hookrightarrow \mathsf{PSh}(X)$$

of sheaves and presheaves of sets on X, respectively, where the arrow denotes that $\mathsf{Sh}(X)$ forms a full subcategory of $\mathsf{PSh}(V)$. The analogues for Abelian (pre)sheaves are

$$\mathsf{Ab}(X) \hookrightarrow \mathsf{PAb}(X)$$

which are connected to the categories for sheaves via forgetful functors. Since these forgetful functors reflect isomorphisms, we will state results for sheaves of sets.

To prove Proposition 2.1.14 below, we define the following:

EXAMPLE 2.1.13. Let X be a topological space and let $\{A_P\}_{P\in X}$ be a family of sets indexed by X. Then, the assignment

[Stacks, Tag 006X] [God73, II.3.1] [Har77, Exer. II.1.16(e)]

$$U \longmapsto \prod_{P\in U} A_P$$

defines a sheaf Π.

By the definition of direct limits, a morphism $\varphi\colon \mathscr{F} \to \mathscr{G}$ of presheaves on X induces a morphism

$$\varphi_P\colon \mathscr{F}_P \longrightarrow \mathscr{G}_P$$

on stalks for any point $P \in X$. The following result illustrates the local nature of sheaves.

[Har77, Prop. II.1.1]

PROPOSITION 2.1.14. *Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves of sets on a topological space X. If φ is an isomorphism, then the induced maps $\varphi_P\colon \mathscr{F}_P \to \mathscr{G}_P$ on stalks are isomorphisms for every $P \in X$. Conversely:*

- (a) *If φ_P is injective for every $P \in X$ and if $\mathscr{F}$ satisfies the sheaf property (3), then $\varphi(U)$ is injective for every open set $U \subseteq X$.*
- (b) *If φ_P is an isomorphism for every $P \in X$, the presheaf $\mathscr{F}$ is a sheaf, and $\mathscr{G}$ satisfies the sheaf property (3), then $\varphi(U)$ is an isomorphism for every open set $U \subseteq X$.*

As a consequence, if $\mathscr{F}, \mathscr{G}$ are sheaves and the maps φ_P are isomorphisms for every P, then φ is an isomorphism.

Proof. The direction ⇒ follows from the construction of direct limits.

For the converse, denote by $\Pi(\mathscr{F})$ the sheaf in Example 2.1.13 obtained from the family $\{\mathscr{F}_P\}_{P\in X}$. We first show:

CLAIM 2.1.15. *If $\mathscr{F}$ satisfies the sheaf property (3), then the map*

$$\mathscr{F}(U) \longrightarrow \pi(\mathscr{F})(U)$$

is injective.

If $s, t \in \mathscr{F}(U)$ map to the same elements, then there is an open covering W_P of U for which $s|_{W_P} = t|_{W_P}$ for every P by the definition of germs. The sheaf property (3) then shows that $s = t$.

We now show (*a*). If φ_P is injective for every $\varphi(U)$, then the commutativity of the diagram

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\varphi(U)} & \mathscr{G}(U) \\ \big\downarrow & & \big\downarrow \\ \prod_{P\in U} \mathscr{F}_P & \xrightarrow{\prod_P \varphi_P} & \prod_{P\in U} \mathscr{G}_P. \end{array} \tag{2.1.16}$$

shows that the top horizontal map $\varphi(U)$ is injective. Here we use Claim 2.1.15 to show that the left vertical map is injective.

We now show (*b*). Suppose we have a section $t \in \mathscr{G}(U)$. Then, for every $P \in U$, there are an open neighborhood $V_P \ni P$ and a section $s(P) \in \mathscr{F}(V_P)$ such that

$$\varphi_P\big(s(P)_P\big) = t_P$$

for every $P \in U$. After possibly shrinking the V_P, we may assume that

$$\varphi\big(s(P)\big)|_{V_P} = t|_{V_P}$$

for every P. Now if P, Q are two points, then $s(P)|_{V_P\cap V_Q}$ and $s(Q)|_{V_P\cap V_Q}$ are two sections of $\mathscr{F}(V_P \cap V_Q)$, which are both sent by φ to $t|_{V_P\cap V_Q}$. By the injectivity shown in (*a*), this implies that these two sections $s(P)|_{V_P\cap V_Q}$ and $s(Q)|_{V_P\cap V_Q}$ are equal. By the sheaf property (4), we then see there is a section $s \in \mathscr{F}(U)$ such that $s|_{V_P} = s(P)$ for every P. Finally, we have $\varphi(U)(s) = t$ by the commutativity of (2.1.16) and Claim 2.1.15 applied to $\mathscr{G}$.

We can therefore define an inverse morphism $\varphi^{-1} : \mathscr{G} \to \mathscr{F}$, where compatibility with restriction maps holds by the fact that $\varphi^{-1}(U)$ is the inverse for $\varphi(U)$ on every open subset $U \subseteq X$. □

2.1.5. Espace étalé and sheafification. Our next goal is to (1) describe sheaves in terms of topological spaces, and (2) construct a functor called *sheafification* that turns a presheaf into a sheaf.

[Har77, Exer. II.1.13, Prop.-Def. II.1.2] [God73, II.1.2]

DEFINITION 2.1.17 (Espace étalé and sheafification). Let $\mathscr{F}$ be a presheaf of sets on X. We define a topological space $\text{Spé}(\mathscr{F})$ called the *espace étalé* of $\mathscr{F}$ as follows.

(1) As a set,

$$\text{Spé}(\mathscr{F}) \coloneqq \coprod_{P\in X} \mathscr{F}_P.$$

We define a projection map

$$\pi\colon \operatorname{Sp\acute{e}}(\mathscr{F}) \longrightarrow X$$
$$\mathscr{F}_P \ni s \longmapsto P.$$

For each open set $U \subseteq X$ and each section $s \in \mathscr{F}(U)$, we obtain a map

$$\bar{s}\colon U \longrightarrow \operatorname{Sp\acute{e}}(\mathscr{F})$$
$$P \longmapsto s_P$$

which is a section of π over U, i.e., $\pi \circ \bar{s} = \mathrm{id}_U$. Note that for every inclusion $V \subseteq U$, we have

$$\bar{s}|_V = \overline{s|_V}.$$

(2) The topology on $\operatorname{Sp\acute{e}}(\mathscr{F})$ is the topology generated by all subsets of $\operatorname{Sp\acute{e}}(\mathscr{F})$ of the form $\bar{s}(U)$ for all $U \subseteq X$ open and all $s \in \mathscr{F}(U)$. Note that π is a continuous map. [TohokuI, p. 154]

The espace étalé defines a *sheaf* $\mathscr{F}^\#$ where

$$\mathscr{F}^\#(U) \coloneqq \bigl\{\text{continuous sections } U \to \operatorname{Sp\acute{e}}(\mathscr{F})\bigr\}.$$

We obtain a morphism

$$\theta\colon \mathscr{F} \longrightarrow \mathscr{F}^\#$$

which we call the *sheafification* of $\mathscr{F}$.

The espace étalé description of a sheaf is from [SHC50/51, Exp. XIV] and predates the one given above. The notation for the sheafification in [Har77] is $\mathscr{F}^+$. We use the notation $\mathscr{F}^\#$ to match other sources, for example [Art62, §2.1].

We now have the following:

Proposition 2.1.18. *Let X be a topological space and let $\mathscr{F}$ be a presheaf of sets on X.*

(*i*) *θ_P is bijective for every $P \in X$.*
(*ii*) *If $\mathscr{F}$ satisfies the sheaf property (3), then $\theta(U)$ is injective for every open set $U \subseteq X$.*
(*iii*) *If $\mathscr{F}$ is a sheaf, then θ is an isomorphism.*
(*iv*) *The sheafification is the left adjoint of the forgetful functor $\mathsf{Sh}(X) \hookrightarrow \mathsf{PSh}(X)$ or $\mathsf{Ab}(X) \hookrightarrow \mathsf{PAb}(X)$, i.e., for every sheaf $\mathscr{G}$ on X, there is a bijection*

$$\theta^*\colon \operatorname{Hom}(\mathscr{F}^\#, \mathscr{G}) \xrightarrow{\sim} \operatorname{Hom}(\mathscr{F}, \mathscr{G})$$

induced by precomposition by θ that is functorial in both $\mathscr{F}$ and $\mathscr{G}$.

[Har77, Exer. II.1.13, Prop.-Def. II.1.2] [God73, II.1.2]

[Stacks, Tag 0080]

Proof. (*i*) holds by definition of the stalk $\mathscr{F}_P$. (*ii*) and (*iii*) follow from (*i*) and Proposition 2.1.14.

For (*iv*), naturality in $\mathscr{G}$ holds by definition as precomposition by θ, and naturaliy in $\mathscr{F}$ holds by the fact that if $\mathscr{F}_1 \to \mathscr{F}_2$ is a morphism of presheaves, we have a corresponding morphism $\mathscr{F}_1^\# \to \mathscr{F}_2^\#$ of sheaves.

We want to show that θ^* is bijective. This follows from the commutative diagram

$$\begin{array}{ccccc} \mathscr{F}(U) & \xrightarrow{\theta_{\mathscr{F}}(U)} & \mathscr{F}^\#(U) & \hookrightarrow & \prod_{P \in U} \mathscr{F}_P \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{G}(U) & \xrightarrow[\sim]{\theta_{\mathscr{G}}(U)} & \mathscr{G}^\#(U) & \hookrightarrow & \prod_{P \in U} \mathscr{G}_P \end{array}$$

where $\theta_{\mathscr{G}}(U)$ is an isomorphism by (*iii*): There exists an induced map $\mathscr{F}^\#(U) \to \mathscr{G}(U)$ using that $\theta_{\mathscr{G}}(U)$ is an isomorphism, and this map is unique since the horizontal maps on the right are injective by Claim 2.1.15. □

2.1.6. Properties of morphisms of sheaves of sets. We now define some properties of morphisms of sheaves.

[God73, II.1.6]
[Har77, p. 64]

DEFINITION 2.1.19. Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of sets on a topological space X. We say that φ is *injective* (resp. *surjective*) if φ_P is injective (resp. surjective) for every $P \in X$.

[God73, II.1.6]

EXAMPLE 2.1.20. Note that $\varphi\colon \mathscr{F} \to \mathscr{G}$ injective implies $\varphi(U)$ is injective for every open set $U \subseteq X$ by Proposition 2.1.14(a).

The analogue for surjectivity does not hold. For example, let $\mathcal{O}_{\mathbf{C}}$ denote the sheaf consisting of holomorphic functions $U \mapsto \mathbf{C}$ for all $U \subseteq \mathbf{C}$ under addition, and let $\mathcal{O}_{\mathbf{C}}^{\times}$ denote the sheaf consisting of holomorphic functions $U \mapsto \mathbf{C}^{\times}$ for all $U \subseteq \mathbf{C}$ under multiplication. These are sheaves since a function on U is holomorphic if and only if it is holomorphic on each set of any open cover of U. Now consider the exponential map

$$\begin{aligned} \exp\colon \mathcal{O}_{\mathbf{C}} &\longrightarrow \mathcal{O}_{\mathbf{C}}^{\times} \\ f &\longmapsto \exp(f). \end{aligned}$$

This is surjective since every non-vanishing holomorphic function on $\mathbf{C}$ *locally* has a logarithm on an open subset of $\mathbf{C}$ by choosing an appropriate branch of the logarithm function. However, if $U = \mathbf{C}$, then $\exp(U)$ is not surjective since there does not exist a global logarithm on $\mathbf{C}$.

[God73, II.1.8]
[Har77, p. 64]

DEFINITION 2.1.21. Let $\mathscr{F}$ be a sheaf of sets on a topological space X. A sheaf $\mathscr{G}$ is a *subsheaf* of $\mathscr{F}$ if, for every open subset $U \subseteq X$, the sets $\mathscr{G}(U)$ are subsets of $\mathscr{F}(U)$, and the restriction maps are the restriction maps for $\mathscr{F}$.

Now let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of sets on a topological space X. The *image* $\operatorname{im}(\varphi)$ of φ is the sheafification

$$\operatorname{im}(\varphi) \coloneqq \Bigl(U \longmapsto \operatorname{im}\bigl(\varphi(U)\bigr)\Bigr)^{\#}.$$

Equivalently (up to isomorphism), the image can be described as the sheaf of sections of the image of $\text{Spé}(\mathscr{F})$ in $\text{Spé}(\mathscr{G})$. The description in terms of the espace étalé shows that the image $\operatorname{im}(\varphi)$ can be identified with a subsheaf of $\mathscr{G}$.

[God73, II.1.9]
[Har77, p. 65]

DEFINITION 2.1.22. Let $\mathscr{F}$ be a sheaf of sets on a topological space X. Suppose that for every open $U \subseteq X$, we have an equivalence relation $R(U)$ on $\mathscr{F}(U)$. We say that the collection $\{R(U)\}_{U \subseteq X}$ is an equivalence relation on $\mathscr{F}$ if it satisfies the following property: For $s, t \in \mathscr{F}(U)$, we have $s \equiv t \bmod R(U)$ if and only if there exists an open cover $U = \bigcup_i U_i$ such that $s|_{U_i} \equiv t|_{U_i} \bmod R(U_i)$ for every i. We then define the *quotient sheaf* to be the sheafification

$$\mathscr{F}/R \coloneqq \bigl(U \longmapsto \mathscr{F}(U)/R(U)\bigr)^{\#}.$$

This presheaf satisfies the sheaf property (3), and hence injects into $\mathscr{F}/R$. The canonical morphism $\mathscr{F} \to \mathscr{F}/R$ is surjective.

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2.1.7. Functors associated to continuous maps. We now define some basic operations associated to continuous maps of topological spaces.

[Har77, p. 65]

DEFINITION 2.1.23. Let $f\colon X \to Y$ be a continuous map of topological spaces.

[God73, II.1.13]

(*i*) Let $\mathscr{F}$ be a sheaf of sets or Abelian groups on X. The *direct image sheaf* $f_*\mathscr{F}$ on Y is defined by

$$(f_*\mathscr{F})(V) := \mathscr{F}\big(f^{-1}(V)\big)$$

for any open subset $V \subseteq Y$. For morphisms $\varphi\colon \mathscr{F} \to \mathscr{G}$ and $\psi\colon \mathscr{G} \to \mathscr{H}$, we have the commutative diagram

$$\begin{array}{ccccc} \mathscr{F}\big(f^{-1}(V)\big) & \xrightarrow{\varphi(f^{-1}(V))} & \mathscr{G}\big(f^{-1}(V)\big) & \xrightarrow{\psi(f^{-1}(V))} & \mathscr{H}\big(f^{-1}(V)\big) \\ \rho^V_{V'}\downarrow & & \downarrow\rho'^V_{V'} & & \downarrow\rho''^V_{V'} \\ \mathscr{F}\big(f^{-1}(V')\big) & \xrightarrow{\varphi(f^{-1}(V'))} & \mathscr{G}\big(f^{-1}(V')\big) & \xrightarrow{\psi(f^{-1}(V'))} & \mathscr{H}\big(f^{-1}(V')\big) \end{array}$$

for all inclusions $V' \subseteq V$ of open sets in Y. Thus, the direct image is functorial in $\mathscr{F}$, and we obtain functors

$$f_*\colon \mathsf{Sh}(X) \longrightarrow \mathsf{Sh}(Y),$$
$$f_*\colon \mathsf{Ab}(X) \longrightarrow \mathsf{Ab}(Y).$$

(*ii*) Let $\mathscr{G}$ be a sheaf of sets or Abelian groups on Y. The *inverse image presheaf* $f_{\mathcal{P}}^{-1}\mathscr{G}$ on X is defined by [God73, II.1.12]

$$(f_{\mathcal{P}}^{-1}\mathscr{G})(U) := \varinjlim_{V \supseteq f(U)} \mathscr{G}(V)$$

for any open subset $U \subseteq X$, where the direct limit is taken over all open sets $V \subseteq Y$ containing $f(U)$. The *inverse image sheaf* $f^{-1}\mathscr{G}$ is the sheafification

$$f^{-1}\mathscr{G} := (f_{\mathcal{P}}^{-1}\mathscr{G})^{\#}.$$

We describe how $f_{\mathcal{P}}^{-1}$ acts on maps of presheaves. For morphisms $\varphi\colon \mathscr{F} \to \mathscr{G}$ and $\psi\colon \mathscr{G} \to \mathscr{H}$, we have the commutative diagram

$$\begin{array}{ccccc} \mathscr{F}(V) & \xrightarrow{\varphi(V)} & \mathscr{G}(V) & \xrightarrow{\psi(V)} & \mathscr{H}(V) \\ \rho^V_{V'}\downarrow & & \downarrow\rho^V_{V'} & & \downarrow\rho^V_{V'} \\ \mathscr{F}(V') & \xrightarrow{\varphi(V')} & \mathscr{G}(V') & \xrightarrow{\psi(V')} & \mathscr{H}(V') \end{array}$$

where the vertical morphisms are the restriction morphisms $\rho^V_{V'}$ running over $V \supseteq V' \supseteq f(U)$. Taking direct limits in each column, we obtain the morphisms

$$f_{\mathcal{P}}^{-1}\mathscr{F}(U) \xrightarrow{f_{\mathcal{P}}^{-1}\varphi(U)} f_{\mathcal{P}}^{-1}\mathscr{G}(U) \xrightarrow{f_{\mathcal{P}}^{-1}\psi(U)} f_{\mathcal{P}}^{-1}\mathscr{H}(U).$$

by the universal property for direct limits (see Definition A.5.4). Note by the universal property of direct limits applied to $\psi(V) \circ \varphi(V)$, we get that

$$f_{\mathcal{P}}^{-1}(\psi \circ \varphi)(U) = f_{\mathcal{P}}^{-1}(\psi)(U) \circ f_{\mathcal{P}}^{-1}(\varphi)(U).$$

Thus, $f_{\mathcal{P}}^{-1}$ defines a functor on presheaves. Taking sheafifications, we obtain the functors

$$f^{-1}\colon \mathsf{Sh}(Y) \longrightarrow \mathsf{Sh}(X),$$
$$f^{-1}\colon \mathsf{Ab}(Y) \longrightarrow \mathsf{Ab}(X).$$

As a special case, if f is an inclusion of a subset $X \subseteq Y$ with the induced topology, we call

$$\mathscr{G}|_X \coloneqq i^{-1}\mathscr{G}$$

[God73, II.1.5] the *restriction* of $\mathscr{G}$ to X. Note the stalk of $\mathscr{G}|_X$ at $P \in X$ is just $\mathscr{F}_P$.

[Har77, Exer. II.1.17] EXAMPLE 2.1.24 (Skyscraper sheaves). Let X be a topological space and let $P \in X$ be a point. Let A be a set or an Abelian group. The *skyscraper sheaf at P with value A* is the sheaf $i_P(A)$ defined by

$$i_P(A)(U) = \begin{cases} A & \text{if } P \in U \\ \{*\} & \text{otherwise} \end{cases}$$

for sets and

$$i_P(A)(U) = \begin{cases} A & \text{if } P \in U \\ 0 & \text{otherwise} \end{cases}$$

for Abelian groups. If $i_P \colon \{P\} \hookrightarrow X$ is the inclusion, we can also describe the skyscraper sheaf as $i_{P*}\underline{A}_{\{P\}}$.

An important result connecting these functors is the following.

[Har77, Exer. II.1.18]
[EGAI, (**0**, 3.5.4)]
[EGAI$_{\text{new}}$, (**0**, 3.5.4)]

PROPOSITION 2.1.25. *Let $f \colon X \to Y$ be a continuous map of topological spaces. For sheaves $\mathscr{F}$ and $\mathscr{G}$ of sets or Abelian groups, there is a bijection of sets*

$$\operatorname{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) \xrightarrow{\sim} \operatorname{Hom}_Y(\mathscr{G}, f_*\mathscr{F}) \tag{2.1.26}$$

natural in both $\mathscr{F}$ and $\mathscr{G}$. Here we say that f^{-1} is the left adjoint *of f_*, and that f_* is a* right adjoint *of f^{-1}.*

Proof. We proceed in steps.

STEP 1. It suffices to show the statement after replacing f^{-1} by the functor $f_{\mathcal{P}}^{-1}$ on presheaves, where

$$f_{\mathcal{P}}^{-1}\mathscr{F}(U) = \varinjlim_{V \supseteq f(U)} \mathscr{G}(U).$$

By Proposition 2.1.18(*iv*), we have the commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) & \longrightarrow & \operatorname{Hom}_Y(\mathscr{G}, f_*\mathscr{F}) \\ & \searrow^{\sim} \qquad \nearrow & \\ & \operatorname{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, \mathscr{F}) & \end{array}$$

natural in both $\mathscr{F}$ and $\mathscr{G}$. If we show that the right diagonal map is a bijection, then the commutativity of the diagram shows that the top horizontal map is a bijection.

STEP 2. Constructing a morphism

$$\varepsilon_{\mathcal{P}} \colon f_{\mathcal{P}}^{-1} f_*\mathscr{F} \longrightarrow \mathscr{F}$$

natural in $\mathscr{F}$.

First, note

$$\begin{aligned}(f_{\mathcal{P}}^{-1} f_* \mathscr{F})(U) &= \varinjlim_{V \supseteq f(U)} f_* \mathscr{F}(V) \\ &= \varinjlim_{V \supseteq f(U)} \mathscr{F}\big(f^{-1}(V)\big).\end{aligned}$$

Thus, using the commutative diagram

$$\begin{array}{ccc} f_*\mathscr{F}(V) & \xrightarrow{f_*\varphi(V)} & f_*\mathscr{F}'(V) \\ {\scriptstyle \rho^{f^{-1}(V)}_{f^{-1}(V')}}\big\downarrow & & \big\downarrow{\scriptstyle \rho'^{f^{-1}(V)}_{f^{-1}(V')}} \\ f_*\mathscr{F}(V') & \xrightarrow{f_*\varphi(V')} & f_*\mathscr{F}'(V') \\ {\scriptstyle \rho^{f^{-1}(V')}_{U}}\big\downarrow & & \big\downarrow{\scriptstyle \rho'^{f^{-1}(V')}_{U}} \\ \mathscr{F}(U) & \xrightarrow{\varphi(U)} & \mathscr{F}'(U) \end{array}$$

and the universal property of direct limits, we can define $\varepsilon_{\mathcal{P}} \colon f_{\mathcal{P}}^{-1} f_* \mathscr{F} \to \mathscr{F}$ using the dashed maps in the diagram below:

$$\begin{array}{ccc} f_*\mathscr{F}(V) & \xrightarrow{f_*\varphi(V)} & f_*\mathscr{F}'(V) \\ {\scriptstyle \rho^{f^{-1}(V)}_{f^{-1}(V')}}\big\downarrow & & \big\downarrow{\scriptstyle \rho'^{f^{-1}(V)}_{f^{-1}(V')}} \\ f_*\mathscr{F}(V') & \xrightarrow{f_*\varphi(V')} & f_*\mathscr{F}'(V') \\ {\scriptstyle \pi_{V'}}\big\downarrow & & \big\downarrow{\scriptstyle \pi'_{V'}} \\ f_{\mathcal{P}}^{-1} f_*\mathscr{F}(U) & \xrightarrow{f_{\mathcal{P}}^{-1} f_*\varphi(U)} & f_{\mathcal{P}}^{-1} f_*\mathscr{F}'(U) \\ {\scriptstyle (\varepsilon_{\mathcal{P}}(\mathscr{F}))(U)}\,\vdots\downarrow & & \vdots\downarrow{\scriptstyle (\varepsilon_{\mathcal{P}}(\mathscr{F}'))(U)} \\ \mathscr{F}(U) & \xrightarrow{\varphi(U)} & \mathscr{F}'(U) \end{array}$$

where V, V' range over all open subsets of X such that $V \supseteq V' \supseteq f(U)$, which implies

$$f^{-1}(V) \supseteq f^{-1}(V') \supseteq f^{-1}\big(f(U)\big) \supseteq U,$$

and the vertical maps to the third line are equal to π_V and π'_V. The naturality of $\varepsilon_{\mathcal{P}}$ (the bottom square) follows since if we have a map $\varphi \colon \mathscr{F} \to \mathscr{F}'$, the map $f_{\mathcal{P}}^{-1} f_* \mathscr{F}(U) \to \mathscr{F}'(U)$ is unique by the universal property for $f_{\mathcal{P}}^{-1} f_* \mathscr{F}(U)$ applied to $\rho_U^{f^{-1}(V)} \circ f_*\varphi(V)$ and $\rho_U^{f^{-1}(V')} \circ f_*\varphi(V')$.

Step 3. Constructing a morphism

$$\eta_{\mathcal{P}} \colon \mathscr{G} \longrightarrow f_* f^{-1} \mathscr{G}$$

natural in $\mathscr{G}$.

Noting that

$$f_* f_{\mathcal{P}}^{-1} \mathscr{G}\big(U\big) = f_{\mathcal{P}}^{-1} \mathscr{G}\big(f^{-1}(U)\big) = \varinjlim_{V \supseteq f(f^{-1}(U))} \mathscr{G}(V), \tag{2.1.27}$$

we can define $\eta_{\mathcal{P}}\colon \mathscr{G} \to f_* f_{\mathcal{P}}^{-1}\mathscr{G}$ by considering the commutative diagram

$$\begin{array}{ccc}
\mathscr{G}(U) & \xrightarrow{\psi(U)} & \mathscr{G}'(U) \\
{\scriptstyle \rho_V^U}\downarrow & & \downarrow{\scriptstyle \rho_V'^U} \\
\mathscr{G}(V) & \xrightarrow{\psi(V)} & \mathscr{G}'(V) \\
{\scriptstyle \rho_{V'}^V}\downarrow & & \downarrow{\scriptstyle \rho_{V'}'^V} \\
\mathscr{G}(V') & \xrightarrow{\psi(V')} & \mathscr{G}'(V')
\end{array}$$

where V, V' range over all open subsets of Y such that

$$U \supseteq V \supseteq V' \supseteq f\big(f^{-1}(U)\big),$$

and applying the universal property of the direct limit in the bottom square to obtain the commutative diagram

$$\begin{array}{ccc}
\mathscr{G}(U) & \xrightarrow{\qquad\psi(U)\qquad} & \mathscr{G}'(U) \\
{\scriptstyle (\eta_{\mathcal{P}}(\mathscr{G}))(U)}\downarrow & & \downarrow{\scriptstyle (\eta_{\mathcal{P}}(\mathscr{G}'))(U)} \\
f_* f_{\mathcal{P}}^{-1}\mathscr{G}(U) & \xrightarrow{f_* f_{\mathcal{P}}^{-1}\psi(U)} & f_* f_{\mathcal{P}}^{-1}\mathscr{G}'(U).
\end{array}$$

The naturality of $\eta_{\mathcal{P}}$ follows since if we have a map $\psi\colon \mathscr{G} \to \mathscr{G}'$, the map

$$f_* f_{\mathcal{P}}^{-1}\mathscr{G}(U) \longrightarrow f_* f_{\mathcal{P}}^{-1}\mathscr{G}'(U)$$

is the unique map induced by the universal property for $f_* f_{\mathcal{P}}^{-1}\mathscr{G}(U)$, that is, the universal property for the direct limit in (2.1.27), applied to $\pi'_V \circ \psi(V)$ and $\pi'_{V'} \circ \psi(V')$.

Step 4. The bijection (2.1.26).

Now we want to define the maps defining the bijection in (2.1.26). The map from right to left is defined as

$$H_1\colon \operatorname{Hom}_Y(\mathscr{G}, f_*\mathscr{F}) \xrightarrow{f_{\mathcal{P}}^{-1}} \operatorname{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, f_{\mathcal{P}}^{-1} f_*\mathscr{F}) \xrightarrow{\varepsilon_{\mathcal{P}}(\mathscr{F})\circ -} \operatorname{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, \mathscr{F}).$$

The map from left to right is defined as

$$H_2\colon \operatorname{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, \mathscr{F}) \xrightarrow{f_*} \operatorname{Hom}_Y(f_* f_{\mathcal{P}}^{-1}\mathscr{G}, f_*\mathscr{F}) \xrightarrow{-\circ\eta_{\mathcal{P}}(\mathscr{G})} \operatorname{Hom}_Y(\mathscr{G}, f_*\mathscr{F}).$$

We need to show they are inverse to each other. First, $H_1 \circ H_2 = \mathrm{id}$ since for all $V \supseteq V' \supseteq f(U)$ and $\varphi \in \operatorname{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, \mathscr{F})$, we have the diagram

$$\begin{array}{ccccc}
\mathscr{G}(V) & \xrightarrow{(f_*\varphi\circ\eta_{\mathcal{P}}(\mathscr{G}))(V)} & f_*\mathscr{F}(V) & \xrightarrow{\rho_U^{f^{-1}(V)}} & \mathscr{F}(U) \\
{\scriptstyle \pi_V}\downarrow & & \downarrow & & \\
f_{\mathcal{P}}^{-1}\mathscr{G}(U) & \xrightarrow{f_{\mathcal{P}}^{-1}(f_*\varphi\circ\eta_{\mathcal{P}}(\mathscr{G}))(U)} & f_{\mathcal{P}}^{-1} f_*\mathscr{F}(U) & \xrightarrow{\varepsilon_{\mathcal{P}}(\mathscr{F})(U)} & \mathscr{F}(U)
\end{array}$$

where the composition along the bottom row is $\varphi(U)$. Applying the universal property of $f_{\mathcal{P}}^{-1}\mathscr{G}(U)$ to the compositions $\mathscr{G}(V) \to \mathscr{F}(U)$, we have

$$\varphi(U) = \varepsilon_{\mathcal{P}}(\mathscr{F})(U) \circ f_{\mathcal{P}}^{-1}\big(f_*\varphi \circ \eta_{\mathcal{P}}(\mathscr{G})\big)(U) = (H_1 \circ H_2)(\varphi)(U)$$

by the uniqueness part of the universal property of $f_{\mathcal{P}}^{-1}\mathscr{G}(U)$.

Next, $H_2 \circ H_1 = \mathrm{id}$ since (2.1.27) combined with

$$\begin{aligned}\varinjlim_{V \supseteq f(f^{-1}(U))} f_*\mathscr{F}(V) &= \varinjlim_{V \supseteq f(f^{-1}(U))} \mathscr{F}\big(f^{-1}(V)\big)\\ &= \varinjlim_{f^{-1}(V) \supseteq f^{-1}(f(f^{-1}(U)))} \mathscr{F}\big(f^{-1}(V)\big)\\ &= \varinjlim_{f^{-1}(V) \supseteq f^{-1}(U)} \mathscr{F}\big(f^{-1}(V)\big)\\ &= f_*\mathscr{F}(U)\end{aligned}$$

gives that for all $U \supseteq V \supseteq V' \supseteq f(f^{-1}(U))$, we have the diagram

$$\begin{array}{ccccc} \mathscr{G}(U) & \xrightarrow{\rho^U_V} & \mathscr{G}(V) & \xrightarrow{\psi(V)} & f_*\mathscr{F}(V)\\ & \searrow^{\eta_{\mathcal{P}}(\mathscr{G})} & \downarrow & & \downarrow\\ & & f_*f_{\mathcal{P}}^{-1}\mathscr{G}(U) & \xrightarrow{f_*(\varepsilon_{\mathcal{P}}(\mathscr{F})\circ f_{\mathcal{P}}^{-1}\psi)(U)} & f_*\mathscr{F}(U)\end{array}$$

for every morphism $\psi \in \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$. In particular, letting $U = V$, we have

$$\psi(U) = \big(f_*(\varepsilon_{\mathcal{P}}(\mathscr{F}) \circ f_{\mathcal{P}}^{-1}\psi) \circ \eta_{\mathcal{P}}(\mathscr{G})\big)(U) = (H_2 \circ H_1)(\psi)(U).$$

STEP 5. Naturality of (2.1.26) in $\mathscr{F}$ and $\mathscr{G}$.

Now, naturality in $\mathscr{F}$ follows since if $\varphi\colon \mathscr{F} \to \mathscr{F}'$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F}) & \xrightarrow{\mathrm{Hom}_Y(\mathscr{G}, f_*\varphi)} & \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F}')\\ {\scriptstyle f_{\mathcal{P}}^{-1}}\downarrow & & \downarrow{\scriptstyle f_{\mathcal{P}}^{-1}}\\ \mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, f_{\mathcal{P}}^{-1}f_*\mathscr{F}) & \xrightarrow{\mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, f_{\mathcal{P}}^{-1}f_*\varphi)} & \mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, f_{\mathcal{P}}^{-1}f_*\mathscr{F}')\\ {\scriptstyle \mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, \varepsilon_{\mathcal{P}}(\mathscr{F}))}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, \varepsilon_{\mathcal{P}}(\mathscr{F}))}\\ \mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, \mathscr{F}) & \xrightarrow{\mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, \varphi)} & \mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, \mathscr{F}')\end{array}$$

commutes by functoriality of $f_{\mathcal{P}}^{-1}$ in the top square and naturality of $\varepsilon_{\mathcal{P}}$ in the bottom square, and then since the composition of the vertical maps gives H_1. Finally, naturality in $\mathscr{G}$ follows since if $\psi\colon \mathscr{G} \to \mathscr{G}'$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, \mathscr{F}) & \xleftarrow{\mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\psi, \mathscr{F})} & \mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}', \mathscr{F})\\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*}\\ \mathrm{Hom}_Y(f_*f_{\mathcal{P}}^{-1}\mathscr{G}, f_*\mathscr{F}) & \xleftarrow{\mathrm{Hom}_Y(f_*f_{\mathcal{P}}^{-1}\psi, f_*\mathscr{F})} & \mathrm{Hom}_Y(f_*f_{\mathcal{P}}^{-1}\mathscr{G}', f_*\mathscr{F})\\ {\scriptstyle \mathrm{Hom}_Y(\eta_{\mathcal{P}}(\mathscr{G}), f_*\mathscr{F})}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_Y(\eta_{\mathcal{P}}(\mathscr{G}), f_*\mathscr{F})}\\ \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F}) & \xleftarrow{\mathrm{Hom}_Y(\psi, f_*\mathscr{F})} & \mathrm{Hom}_Y(\mathscr{G}', f_*\mathscr{F})\end{array}$$

commutes by functoriality of f_* in the top square and naturality of $\eta_{\mathcal{P}}(\mathscr{G})$ in the bottom square, and then since the composition of the vertical maps gives H_2. □

2.1.8. Sheaf hom.

We also define a sheafy version of Hom.

DEFINITION 2.1.28 (Sheaf $\mathscr{H}om$). Let X be a topological space. Let $\mathscr{F}$, $\mathscr{G}$ be sheaves of sets or Abelian groups on X. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves of Abelian [Har77, Exer. II.1.15] [God73, II.1.7] groups, the set $\operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ of morphisms of the restricted shaves has a natural structure of Abelian group for every open set $U \subseteq X$. The presheaf

$$U \longmapsto \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$$

is a sheaf called the *sheaf of local morphisms* of $\mathscr{F}$ into $\mathscr{G}$, "sheaf hom" for short, and is denoted

$$\mathscr{H}om(\mathscr{F}, \mathscr{G})$$

Proof that $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ is a sheaf. To show that $U \mapsto \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is a sheaf, suppose $\{V_i\}$ is an open cover of U and $s, s' \in \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is such that

$$s|_{V_i} = s(V_i) = s'(V_i) = s'|_{V_i}$$

for all i. Let $f \in \mathscr{F}(V)$ be arbitrary for some $V \subset U$. Then,

$$s(V_i \cap V)(f|_{V_i \cap V}) = s'(V_i \cap V)(f|_{V_i \cap V}),$$

and so since $\mathscr{G}$ is a sheaf, $s(V)(f) = s'(V)(f)$. Thus, $s = s'$. Now suppose that we have elements $s_i \in \operatorname{Hom}(\mathscr{F}|_{V_i}, \mathscr{G}|_{V_i})$ such that for each i, j,

$$s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}.$$

If $f \in \mathscr{F}(V)$, then

$$s_i(V \cap V_i \cap V_j)(f|_{V \cap V_i \cap V_j}) = s_j(V \cap V_i \cap V_j)(f|_{V \cap V_i \cap V_j}) \in \mathscr{G}(V \cap V_i \cap V_j)$$

gives that there exists a well-defined image $s(V)(f) \in \mathscr{G}(V)$ since $\mathscr{G}$ is a sheaf. □

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[God73, II.2.1, II.2.2]
[Har77, pp. 64–65, p. 109]

2.1.9. Sheaves of modules and Abelian groups. We now discuss sheaves of modules. These include sheaves of Abelian groups as a special case.

DEFINITION 2.1.29. Let X be a topological space. Let $\mathcal{O}$ be a sheaf of rings on X. A *sheaf $\mathscr{F}$ of $\mathcal{O}$-modules*, or an *$\mathcal{O}$-module* for short, is a sheaf of sets such that every $\mathscr{F}(U)$ is a $\mathcal{O}(U)$-module and the restriction maps $\rho^U_V \colon \mathscr{F}(U) \to \mathscr{F}(V)$ are maps of modules compatible with the ring maps $\rho^U_V \colon \mathcal{O}(U) \to \mathcal{O}(V)$. A *morphism* $\varphi \colon \mathscr{F} \to \mathscr{G}$ of $\mathcal{O}$-modules is a morphism of sheaves of sets such that each $\mathscr{F}(U) \to \mathscr{G}(U)$ is a morphism of $\mathcal{O}(U)$-modules. This forms the category $\mathsf{Mod}(\mathcal{O})$.

A pair $(X, \mathcal{O})$ where $\mathcal{O}$ is a sheaf of rings on X is called a *ringed space*. In this situation, we often denote $\mathcal{O}$ by $\mathcal{O}_X$ and call it the *structure sheaf*.

EXAMPLE 2.1.30. The 0 sheaf is a sheaf of $\mathcal{O}$-modules. By properties of categories of modules, we know that there are always unique morphisms $0 \to \mathscr{F}$ and $\mathscr{F} \to 0$, i.e., the 0 sheaf is a *0 object* in the category $\mathsf{Mod}(\mathcal{O})$.

EXAMPLE 2.1.31. If $\mathcal{O} = \underline{\mathbf{Z}}_X$, then $\mathsf{Mod}(\underline{\mathbf{Z}}_X) = \mathsf{Ab}(X)$.

REMARK 2.1.32. It is often useful to think about sheaves of non-commutative rings, for example the sheaf of rings of differential operators $\mathscr{D}_X$ [EGAIV_4, Corollaire 16.8.10]. We will stick to commutative rings for simplicity.

One fact you will verify on the homework is that the category $\mathsf{Mod}(\mathcal{O}_X)$ of $\mathcal{O}_X$-modules – in particular, the category $\mathsf{Ab}(X)$ of Abelian sheaves – are Abelian categories. For now, we explain what exact sequences (and related notions) are in this category.

[Har77, pp. 64–65]
[God73, II.2.4]

DEFINITION 2.1.33. Let $(X, \mathcal{O}_X)$ be a ringed space.

(*i*) Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of $\mathcal{O}_X$-modules. The *kernel* $\ker(\varphi)$ of φ is

$$\ker(\varphi) := \Big(U \longmapsto \ker\big(\varphi(U)\big)\Big).$$

This is already a sheaf (by the sheaf condition (3)) and is a subsheaf of $\mathscr{F}$. Note that φ is injective on an open set U if and only if $\ker(\varphi)(U) = 0$ because by definition of stalks using direct limits, we have [Har77, Exer. II.1.2(*a*)]

$$\ker(\varphi)_P \xrightarrow{\sim} \ker(\varphi_P) \tag{2.1.34}$$

for all points $P \in X$. To show the isomorphism (2.1.34), consider the commutative diagram

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\varphi(U)} & \mathscr{G}(U) \\ {\scriptstyle \rho^U_V}\downarrow & & \downarrow{\scriptstyle \rho'^U_V} \\ \mathscr{F}(V) & \xrightarrow{\varphi(V)} & \mathscr{G}(V) \\ {\scriptstyle \pi_V}\downarrow & & \downarrow{\scriptstyle \pi'_V} \\ \mathscr{F}_P & \xrightarrow{\varphi_P} & \mathscr{G}_P \end{array}$$

for open subsets $V \subseteq U \subseteq X$ containing P. The commutativity of the diagram shows that $\ker(\varphi)_P \subseteq \ker(\varphi_P)$. Conversely, if $\langle U, f\rangle \in \ker(\varphi_P)$, then we know that $\langle U, \varphi(U)(f)\rangle = 0$. By the sheaf condition (3), this implies there exists an open subset $V \subseteq U$ such that

$$\varphi(U)(f)|_V = \varphi(V)\big(f|_V\big) = 0.$$

(*ii*) We say that a sequence of $\mathcal{O}_X$-modules [God73, II.2.5]

$$\cdots \longrightarrow \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^{i} \xrightarrow{\varphi^{i}} \mathscr{F}^{i+1} \longrightarrow \cdots$$

is *exact* if $\ker(\varphi^i) = \operatorname{im}(\varphi^{i-1})$ as subsheaves of $\mathscr{F}^i$ for every i.

(*iii*) Let $\mathscr{F}'$ be a sheaf of sub-$\mathcal{O}_X$-modules of $\mathscr{F}$, i.e., every $\mathscr{F}'(U)$ is a sub-$\mathcal{O}_X(U)$-module of $\mathscr{F}(U)$. The *quotient sheaf* $\mathscr{F}/\mathscr{F}'$ is the sheafification [God73, II.2.3]

$$\mathscr{F}/\mathscr{F}' := \big(U \longmapsto \mathscr{F}(U)/\mathscr{F}'(U)\big)^{\#}.$$

This is a special case of the construction for sets. The stalks are $(\mathscr{F}/\mathscr{F}')_P = \mathscr{F}_P/\mathscr{F}'_P$ by Proposition 2.1.18(*i*).

(*iv*) Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of $\mathcal{O}_X$-modules. The *cokernel* $\operatorname{coker}(\varphi)$ of φ is the sheafification [God73, II.2.4]

$$\operatorname{coker}(\varphi) := \Big(U \longmapsto \operatorname{coker}\big(\varphi(U)\big)\Big)^{\#}.$$

This is a quotient sheaf of $\mathscr{F}$.

We give an example of an exact sequence from toplogy.

EXAMPLE 2.1.35. Let X be a smooth manifold of dimension n. For every $p \geq 0$, consider the sheaf Ω^p_X of differential forms of degree p on X, which is defined by setting [God73, Ex. II.2.5.1]

$$\Omega^p_X(U) := \big\{\text{differential forms of degree } p \text{ on } U\big\}.$$

The exterior derivative

$$\Omega^p_X(U) \longrightarrow \Omega^{p+1}_X(U)$$

defines morphisms of sheaves

$$d^p \colon \Omega_X^p \longrightarrow \Omega_X^{p+1}$$

of sheaves of real vector spaces. We have an injection

$$j \colon \underline{\mathbf{R}}_X \hookrightarrow \Omega_X^0$$

of the constant sheaf with value $\mathbf{R}$ into Ω_X^0 by considering locally constant real-valued functions as smooth functions on X. The *Poincaré lemma* then implies that the sequence

$$0 \longrightarrow \underline{\mathbf{R}}_X \xrightarrow{j} \Omega_X^0 \xrightarrow{d^0} \Omega_X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \Omega_X^n \longrightarrow 0 \tag{2.1.36}$$

is exact: Since exactness is a local question, we can work locally to assume that $X = \mathbf{R}^n$, in which case this is the original Poincaré lemma. See [Lee13, Corollary 17.15].

Note that if we take sections in (2.1.36), we do not get an exact sequence! This is where de Rham cohomology comes from: the i-th de Rham cohomology group is defined as

$$H^i_{\mathrm{dR}}(X) \coloneqq \mathbf{h}^i\Big(0 \longrightarrow \Omega_X^0(X) \longrightarrow \Omega_X^1(X) \longrightarrow \cdots \longrightarrow \Omega_X^n(X) \longrightarrow 0\Big).$$

For example, letting $X = S^1$ and taking global sections, we have

$$H^1_{\mathrm{dR}}(S^1) \cong \mathbf{R}.$$

See [BT82, Example 2.6] or [Lee13, Theorem 17.21].

We will discuss cohomology groups coming from sheaves next semester. For now, we want to mention that even if taking sections is not an exact functor, it is still *left* exact.

[Har77, Exer. II.1.8]
[God73, p. 133]

Proposition 2.1.37. *Let X be a topological space and let $U \subseteq X$ be an open subset. The functor*

$$\begin{aligned} \mathsf{Mod}(\mathcal{O}_X) &\longrightarrow \mathsf{Mod}\big(\Gamma(U, \mathcal{O}_X)\big) \\ \mathscr{F} &\longmapsto \Gamma(U, \mathscr{F}) \end{aligned}$$

is left exact.

Proof. Consider a left exact sequence

$$0 \longrightarrow \mathscr{F}' \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}''.$$

We then have the sequence

$$0 \longrightarrow \Gamma(U, \mathscr{F}') \xrightarrow{\varphi(U)} \Gamma(U, \mathscr{F}) \xrightarrow{\psi(U)} \Gamma(U, \mathscr{F}'')$$

that we need to show is exact. The sequence is automatically exact at $\Gamma(U, \mathscr{F}')$ since a morphism of sheaves is injective if and only if it is injective on sections by (2.1.34) (the morphism $\ker(\varphi) \to 0$ is an isomorphism if and only if it is an isomorphism on stalks).

By functoriality, we know that the composition $\psi(U) \circ \varphi(U)$ is the 0 map (you can check the map is 0 on stalks since you can check that the image is 0 on stalks). We therefore have the inclusion

$$\mathrm{im}(\varphi)(U) \subseteq \ker(\psi)(U)$$

as sub-$\mathcal{O}_X(U)$-modules of $\Gamma(U, \mathscr{F})$. It remains to show the opposite inclusion $\operatorname{im}(\varphi)(U) = \ker(\psi)(U)$. We have the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(U,\mathscr{F}') & \xrightarrow{\varphi(U)} & \Gamma(U,\mathscr{F}) & \xrightarrow{\psi(U)} & \Gamma(U,\mathscr{F}'') \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathscr{F}'_P & \xrightarrow{\varphi_P} & \mathscr{F}_P & \xrightarrow{\psi_P} & \mathscr{F}''_P \end{array}$$

Suppose $t \in \ker(\psi)(U)$. Then, $\psi(U)(t) = 0$, and hence

$$\psi(U)(t)_P = \psi_P(t_P) = 0$$

for all $P \in U$. Since the bottom row is exact, there exists $s_P \in \mathscr{F}'_P$ such that $\varphi_P(s_P) = t_P$. We claim we can lift s_P to some $s \in \mathscr{F}'(U)$. For each P, pick an open set $V_P \ni P$ and $r_P \in \mathscr{F}'(V_P)$ such that $s_P = \langle V_P, r_P \rangle$. For each $W \coloneqq V_P \cap V_Q$, we have

$$\varphi(W)(r_P|_W) = \varphi(W)(r_Q|_W) = t|_W.$$

By injectivity of φ we have that

$$r_P|_W = r_Q|_W.$$

By the sheaf property (4), the r_P therefore glue to form a section $s \in \mathscr{F}(U)$. Thus, since $\varphi(U)(s) = t$ for all P, we have $\ker(\psi)(U) \subseteq \operatorname{im}(\varphi)(U)$. □

We end today by connecting sheaf theory to the theory of varieties we developed so far in the course.

Example 2.1.38. Let X be a quasi-projective variety over an algebraically closed field k, which we consider as a ringed space by letting $\mathcal{O}_X$ be the sheaf of regular functions on X. [Har77, Exer. II.1.21]

(*a*) (Sheaves of ideals) Let $Y \subseteq X$ be a closed subset. For each open set $U \subseteq X$, let

$$\mathscr{I}_Y(U) \coloneqq \{ f \in \mathcal{O}_X(U) \mid f(P) = 0 \text{ for all } P \in Y \cap U \}.$$

This forms a presheaf since the restriction maps on $\mathcal{O}_X$ restrict to restriction maps on $\mathscr{I}_Y$. We claim $\mathscr{I}_Y$ is in fact a sheaf. Since $\mathscr{I}_Y(U) \subseteq \mathcal{O}_X(U)$, we have that $\mathscr{I}_Y$ satisfies the sheaf property (3). If we have an open set $U = \bigcup V_i$ and we have a compatible family of regular functions $s_i \in \mathcal{O}_X(V_i)$ that vanish on $Y \cap V_i$ for all i, then they glue together to a regular function $s \in \mathcal{O}_X(U)$, that moreover vanishes on $Y \cap U$ since regular functions are defined locally. Thus, $\mathscr{I}_Y$ is a sheaf, called the *sheaf of ideals of Y* and is a subsheaf of $\mathcal{O}_X$.

(*b*) (The restriction short exact sequence) If Y is a subvariety, then the quotient sheaf $\mathcal{O}_X/\mathscr{I}_Y$ is isomorphic to $i_*(\mathcal{O}_Y)$, where $i \colon Y \to X$ is the inclusion, and $\mathcal{O}_Y$ is the sheaf of regular functions on Y.

To prove this, first note we have a sheaf morphism

$$\psi \colon \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y$$

given by restricting regular functions. The kernel of this morphism consists of exactly those regular functions that vanish on all points of $Y \cap U$, giving us the sequence

$$0 \longrightarrow \mathscr{I}_Y \longrightarrow \mathcal{O}_X \xrightarrow{\psi} i_*\mathcal{O}_Y \longrightarrow 0$$

which is exact at $\mathscr{I}_Y$ and $\mathcal{O}_X$ by (*a*), and at $i_*\mathcal{O}_Y$ by definition of a regular function locally on affines as a restriction of a polynomial function on $\mathbf{A}^n_k$. (Note: This requires some argument!)

We now look at the specific example of $X = \mathbf{P}^1_k$.

(*c*) (Taking global sections is not necessarily exact) Let $Y = \{P, Q\}$ for distinct points $P, Q \in \mathbf{P}^1_k$. We then have the exact sequence

$$0 \longrightarrow \mathscr{I}_Y \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_P \oplus i_*\mathcal{O}_Q \longrightarrow 0$$

by (*b*). However, the induced map of k-vector spaces

$$k = \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(X, i_*\mathcal{O}_P \oplus i_*\mathcal{O}_Q) \cong k^{\oplus 2}$$

of global sections is the map $a \mapsto (a, a)$, which is not surjective.

(*d*) (The sheaf of rational functions) Let $\mathscr{K}_X$ be the constant sheaf on X with value $K(X)$, the function field of X. We call $\mathscr{K}_X$ the *sheaf of rational functions on* X. We claim we have the sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathscr{K}_X \longrightarrow \bigoplus_{P \in X} i_P\left(\frac{K(X)}{\mathcal{O}_{X,P}}\right) \longrightarrow 0. \tag{2.1.39}$$

The second map is defined by

$$\begin{array}{ccc} \mathscr{K}_X & \longrightarrow & \prod_{P \in X} i_P\left(\frac{K(X)}{\mathcal{O}_{X,P}}\right) \\ & \searrow & \uparrow \\ & & \bigoplus_{P \in X} i_P\left(\frac{K(X)}{\mathcal{O}_{X,P}}\right) \end{array}$$

where the top horizontal map is the evaluation map, and the diagonal dashed map exists since a rational map on $\mathbf{P}^1_k$ is not regular at only finitely many points on $\mathbf{P}^1_k$. The sequence (2.1.39) is exact since after taking stalks, we have

$$0 \longrightarrow \mathcal{O}_{X,P} \longrightarrow K(X) \longrightarrow \frac{K(X)}{\mathcal{O}_{X,P}} \longrightarrow 0$$

which is exact. We can therefore identify

$$\mathscr{K}_X/\mathcal{O}_X \cong \bigoplus_{P \in X} i_P\left(\frac{K(X)}{\mathcal{O}_{X,P}}\right).$$

(*e*) (The first Cousin problem) While taking global sections is not exact in general, the sequence

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(X, \mathscr{K}_X) \longrightarrow \Gamma(X, \mathscr{K}_X/\mathcal{O}_X) \longrightarrow 0$$

is exact. In this case, you can compute this directly (Exercise). Algebraically, this follows from the fact that a certain sheaf cohomology group $H^1(X, \mathcal{O}_X)$ vanishes. Analytically, this is the first Cousin problem in several complex variables for the complex manifold $\mathbf{CP}^1$. For example, see [Hör90, Theorem 5.5.1] for the first Cousin problem over Stein manifolds and see [GR79, Chapter V, §2, Theorem 1] for the first Cousin problem over more general complex manifolds.

2.2. Schemes

We are now ready to define schemes. The idea is to associate a topological space to a commutative ring A, and use A to construct a sheaf of rings on it. The resulting object is the *spectrum* $\operatorname{Spec}(A)$. As a topological space, this is a familiar object from commutative algebra: $\operatorname{Spec}(A)$ is the set of prime ideals in A. This means we have more points than just maximal ideals, which correspond to points on the corresponding variety when A is of finite type over an algebraically closed field. 11/15

We will then glue these objects together to form a general scheme, in the same way that one glues together copies of $\mathbf{R}^n$ to build a manifold. An important class of examples is the class of schemes of the form $\operatorname{Proj}(S)$ where S is a graded ring. When S is a graded domain of finite type over an algebraically closed field k such that $S_0 = k$, these schemes will be the scheme-theoretic version of projective varieties.

2.2.1. The spectrum of a ring as a topological space. We start by defining the spectrum of a ring.

Definition 2.2.1 [EGAI, (1.1); EGAI_{new}, (1.1)]. Let A be a ring. As a set, the *spectrum* $\operatorname{Spec}(A)$ of A is the set of prime ideals in A. As a topological space, the closed sets in $\operatorname{Spec}(A)$ are sets of the form

[Har77, p. 70]

$$V(I) := \{P \in \operatorname{Spec}(A) \mid I \subseteq P\}$$

for every ideal $I \subseteq A$. This topology is called the *Zariski topology*.

When we think of $\operatorname{Spec}(A)$ geometrically, we sometimes denote its points by x, in which case the corresponding prime ideals are denoted by $\mathfrak{p}_x$.

Let us see some examples.

Example 2.2.2.

(a) $\operatorname{Spec}(k)$ for a field k consists of one point (0).

[Mum67, Ex. A, p. 135]

(b) (The affine line) $\mathbf{A}^1_k := \operatorname{Spec}(k[x])$ is the *affine line over* k. This space consists of two types of points: The *generic point* $\xi := (0)$ and closed points (f), where $f \in k[x]$ is an irreducible polynomial. If k is algebraically closed, for example when $k = \mathbf{C}$, these points are $(x - \alpha)$ where $\alpha \in k$ by the Nullstellensatz (Theorem 1.1.19). However, if k is not algebraically closed, there are more points. For example, if $k = \mathbf{R}$, then the set of closed points in $\operatorname{Spec}(k[x])$ corresponds to the closed upper half plane in $\mathbf{C}$ because roots of irreducible quadratic polynomials come in conjugate pairs.

[Har77, Ex. II.2.3.1]
[Mum67, Ex. B, p. 136]
[Har77, Ex. II.2.3.3, Exer. II.2.10]

(c) $\operatorname{Spec}(\mathbf{Z})$. This ring is a PID, and we can visualize this space as the "number line" together with the generic point (0).

[Mum67, Ex. C, p. 137]
[Har77, Exer. II.2.5]

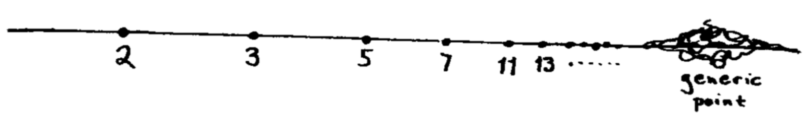


Figure 2.1. $\operatorname{Spec}(\mathbf{Z})$. From [Mum67, p. 137].

(d) (A *trait*) If R is a DVR (i.e., a discrete rank 1 valuation ring), then $T = \operatorname{Spec}(R)$ consists of two points $t_0 := (\varpi)$ and $t_1 := (0)$, where

[Mum67, Ex. D, pp. 137–138]
[Har77, Ex. II.2.3.2]
The letter T is used to stand for the French word *trait* [EGAI_{new}, (5.5.1)]. Some example translations are line,

ϖ is a uniformizer for R. The topology is such that t_0 is closed and $\overline{\{t_1\}} = \operatorname{Spec}(R)$. This relationship can be summarized as

$$t_1 \rightsquigarrow t_0,$$

which is read as "t_0 is a *specialization* of t_1" or "t_1 is a *generization* of t_0."

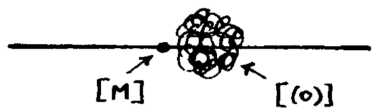


FIGURE 2.2. A trait, i.e., the spectrum of a DVR. From [Mum67, p. 138].

[Mum67, Ex. E, pp. 138–139, Ex. H, pp. 141–143]
[Har77, Ex. II.2.3.4]
[AK21, (2.26)]
[Rei95, (1.5)]

(*e*) (Polynomial rings over PIDs) For R a PID, we have the following description of $\operatorname{Spec}(R[x])$:

THEOREM 2.2.3. *Let R be a PID and consider the polynomial ring $R[x]$ in one variable over R. Let $P \subseteq R[x]$ be a prime ideal.*
(*i*) *$P = (0)$, or $P = (f)$ with f prime, or P is maximal.*
(*ii*) *If P is maximal, then either $P = (f)$ with f prime, or $P = (p, g)$ for $p \in R$ prime and $g \in R[x]$ such that its image in $(R/(p))[x]$ is prime.*

Proof. Suppose $P \neq (0)$ and P is not principal. Then, there exist two polynomials $f_1, f_2 \in P$ with no common factor. After possibly replacing f_1 and f_2 by prime factors (which lie in P by the assumption that P is prime), we may assume that f_1 and f_2 are prime. Set K to be the fraction field of R, i.e., the field obtained from R by adjoining an inverse for every nonzero element in R. Gauss's lemma implies that f_1 and f_2 are relatively prime in $K[x]$. Since $K[x]$ is a PID, there exist $h_1, h_2 \in K[x]$ such that $h_1 f_1 + h_2 f_2 = 1$. Clearing denominators gives $P \cap R \neq 0$. Since R is a PID, we have $P \cap R = (p)$ for a prime element $p \in R$.

Now set $k \coloneqq R/(p)$, which is a field. Set

$$Q = P \cdot \bigl(R[x]/(p)\bigr) \subseteq R[x]/(p) \cong k[x].$$

We then have

$$k[x]/Q \cong R[x]/P.$$

Now since P is prime, these rings are domains, and hence we have $Q = (g')$, where $g' \in k[x]$ is prime. Moreover, $k[x]/Q$ is a field since Q is in fact a maximal ideal by the fact that $k[x]$ is a PID. Now choosing $g \in R[x]$ mapping to g' under the quotient map $R[x] \to k[x]$, we are done. $\square$

For $R = k[y]$ and $R = \mathbf{Z}$, Mumford draws particularly nice pictures. See Figure 2.3. Note that the case when $P = (f)$ is maximal actually occurs: If (V, ϖ) is a DVR, then

$$(\varpi x - 1) \subseteq V[x]$$

is a maximal ideal.

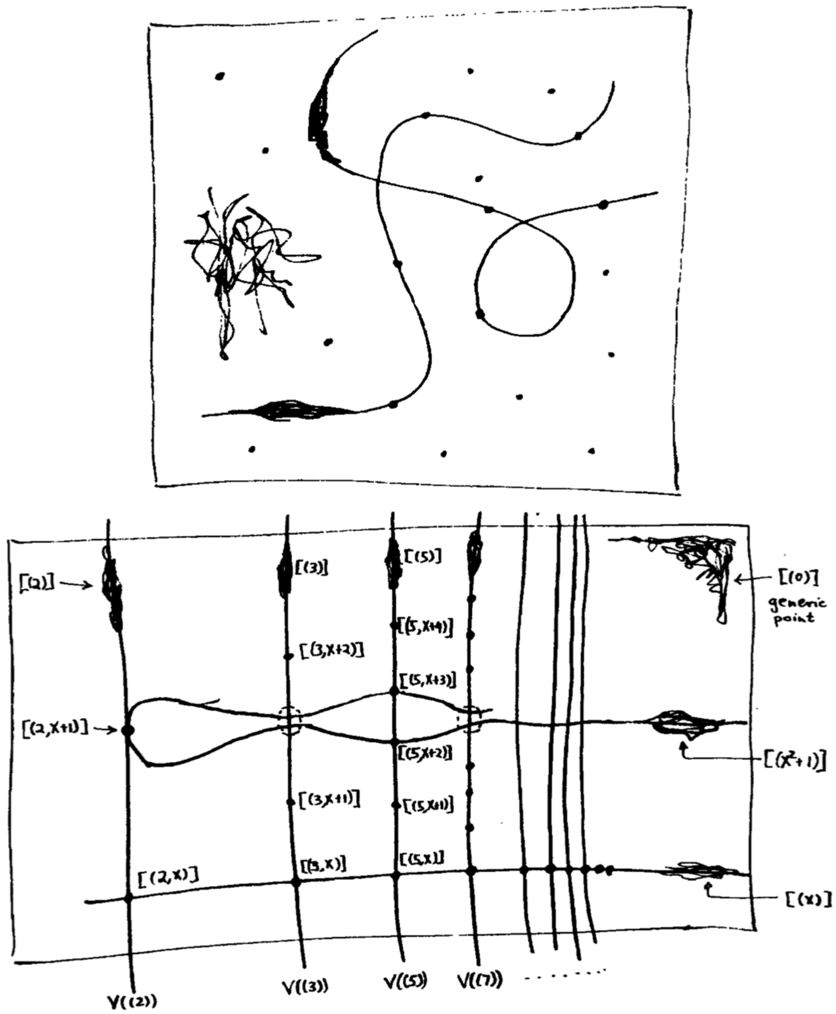


FIGURE 2.3. Spec($k[x,y]$) and Spec($\mathbf{Z}[x]$). From [Mum67, p. 139, p. 141].

2.2.2. The sheaf associated to an A-module. To define the structure sheaf on Spec(A), we will construct sheaves on Spec(A) from A-modules. Recall that the *principal open sets*

$$D(f) \coloneqq \mathrm{Spec}(A) - V(f)$$

form a basis for the Zariski topology on Spec(A) since

$$V(I) = \bigcap_{f \in I} V(f) \Longrightarrow X - V(I) = \bigcup_{f \notin I} D(f).$$

The basic idea of this definition is that we know from commutative algebra that

$$\mathrm{Spec}(A_f) \overset{\text{1-1}}{\longleftrightarrow} D(f) \subseteq \mathrm{Spec}(A)$$

since prime ideals not containing f are in bijection with prime ideals in A_f. There is therefore a nice candidate for what the value of $\mathcal{O}(D(f))$ should be: It should be

A_f! Using [EGAI, (**0**, 3.2); EGAI$_{\text{new}}$, (**0**, 3.2)] (which you studied in Homework 10, Problem 3), we can make the following definition.

In [Har77, p. 70, p. 110], Hartshorne gives a different (but equivalent) definition in terms of the espace étalé.

DEFINITION 2.2.4 [EGAI, Définition 1.3.4; EGAI$_{\text{new}}$, Définition 1.3.4]. Let A be a ring and let M be an A-module. The *sheaf* $\tilde{M}$ *associated to* M is the sheafification of the presheaf defined by the basis of principal open sets by sending

$$D(f) \longmapsto M_f$$

for every principal open set $D(f)$. The *structure sheaf* for $X := \operatorname{Spec}(A)$ is the sheaf $\mathcal{O}_X := \tilde{A}$. We therefore consider $\operatorname{Spec}(A)$ as a ringed space $(\operatorname{Spec}(A), \tilde{A})$.

By definition in [EGAI, (**0**, 3.2); EGAI$_{\text{new}}$, (**0**, 3.2)], we have

$$\tilde{M} := \Biggl(V \longmapsto \varprojlim_{D(f)\subseteq V} M_f \Biggr)^{\#}.$$

Note that by definition, the presheaf in the parentheses on the right maps to $\tilde{M}$, and hence we have group maps

$$\theta_f \colon\ A_f \longrightarrow \Gamma\bigl(D(f), \tilde{A}\bigr)$$
$$\theta_f \colon M_f \longrightarrow \Gamma\bigl(D(f), \tilde{M}\bigr)$$

which are maps of A_f-algebras and A_f-modules, respectively.

We prove some properties of the sheaf $\tilde{M}$.

[Har77, Props. II.2.2(a), II.5.1(b)]

PROPOSITION 2.2.5 [EGAI, p. 85, (**0**, 3.2.4); EGAI$_{\text{new}}$, p. 198, (**0**, 3.2.4)]. *Let A be a ring with spectrum $X := \operatorname{Spec}(A)$ and let $x \in X$ be a point. For every A-module M, we have*

$$\tilde{A}_x \cong A_{\mathfrak{p}_x} \qquad \text{and} \qquad \tilde{M}_x \cong M_{\mathfrak{p}_x}$$

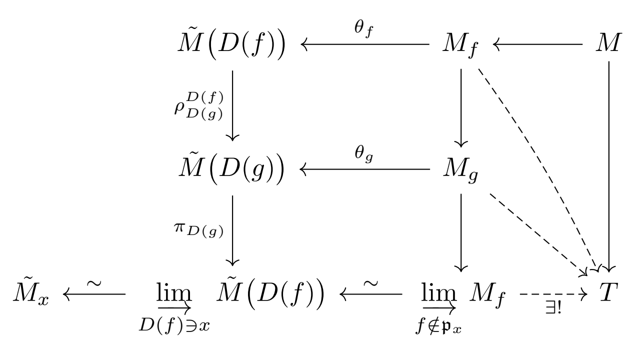
as $A_{\mathfrak{p}_x}$-modules.

Proof. Before we proceed with the proof, we note that

$$D(g) \subseteq D(f) \iff V(g) \subseteq V(f) \iff \sqrt{(g)} \subseteq \sqrt{(f)} \iff \bar{S}_g \supseteq \bar{S}_f$$

where the bars denote saturation [AK21, (3.25)]. Thus, in this situation, we have an induced map $M_f \to M_g$.

We first show the statement for M. For open sets $V \subseteq U$ and principal open sets $D(f) \subseteq U$, $D(g) \subseteq V$ such that $D(g) \subseteq D(f)$, we have the commutative diagram



where the bottom left map is an isomorphism since the directed system of sections on open sets containing x and the drected system of sections on *principal* open sets containing x are cofinal, and the bottom middle map is an isomorphism since sheafification does not affect stalks.

We claim that the direct limit in the right column satisfies the universal property of the localization $M_{\mathfrak{p}_x}$. Consider an A-module T on which all $f \notin \mathfrak{p}_x$ act as units. Consider a map $M \to T$. Then, since $f, g \notin \mathfrak{p}_x$ act as units on T, the universal property of localization of modules shows the diagonal dashed maps exist and are unique. By the universal property of direct limits, the bottom right map exists and is unique.

Finally, the case $M = A$ follows since letting T be an A-algebra on which all $f \notin \mathfrak{p}_x$ act as units, the maps in the diagram above are all ring maps. ☐

Proposition 2.2.6 [EGAI, Proposition 1.3.5; EGAI_{new}, Proposition 1.3.5]. *Let A be a ring. The functor* [Har77, Prop. II.5.2]

$$\begin{aligned} \mathsf{Mod}(A) &\longrightarrow \mathsf{Mod}(\tilde{A}) \\ M &\longmapsto \tilde{M} \end{aligned}$$

is exact.

Proof. This pretty much follows from Proposition 2.2.5 by the exactness of localization. However, since we were not careful about how $M \mapsto \tilde{M}$ acts on *maps*, we prove this more carefully.

Suppose we have an exact sequence $M' \to M \to M''$. Consider the commutative diagram

$$\begin{array}{ccccc} M'_f & \longrightarrow & M_f & \longrightarrow & M''_f \\ \downarrow & & \downarrow & & \downarrow \\ M'_g & \longrightarrow & M_g & \longrightarrow & M''_g \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{M'}_x & \longrightarrow & \tilde{M}_x & \longrightarrow & \widetilde{M''}_x \end{array}$$

for each $f, g \notin \mathfrak{p}_x$ such that $D(f) \supseteq D(g)$. By the exactness of localization, the top two rows are exact. By the proof of Proposition 2.2.5, the direct limit of the top two rows is the bottom row, and is therefore exact by the exactness of filtered direct limits of modules [AK21, (7.9)] (the M_f are filtered since M_f and M_g both map to M_{fg}). ☐

Proposition 2.2.7 [EGAI, Proposition 1.3.6; EGAI_{new}, Proposition 1.3.6]. *Let A be a ring and let M be an A-module. Consider an element $f \in A$. Then,* [Har77, Exer. II.2.1]

$$\widetilde{M_f} \cong \tilde{M}|_{D(f)}.$$

Proof. These two sheaves come from the same presheaf $D(f) \mapsto M_f$ defined on principal open sets. The result now follows since restricting to open subsets does not require sheafification. (Even if it did, we get a map from left to right that induces an isomorphism on stalks.) ☐

Our first goal for today is to show that $\tilde{A}$ satisfies what we wanted last time: On each principal open set, $\Gamma(D(f), \tilde{A})$ is just A_f. To prove this, we need some preliminaries on $\operatorname{Spec}(A)$. 11/20

Lemma 2.2.8 [EGAI, Proposition 1.1.2]. *Let A be a ring.* [Har77, Lem. II.2.1] [BouCA, II.4.3]

(*a*) *If I and J are two ideals of A, then $V(IJ) = V(I) \cup V(J)$.*
(*b*) *If I_i is a set of ideals of A, then $V(\sum I_i) = \bigcap_i V(I_i)$.*
(*c*) *If I and J are two ideals, then $V(I) \subseteq V(J)$ if and only if $\sqrt{I} \supseteq \sqrt{J}$.*

Proof. (*a*). $\supseteq$. If $P \supseteq I$ or $P \supseteq J$, then $P \supseteq IJ$. $\subseteq$. If $P \supseteq IJ$ and $P \not\supseteq J$, then there exists $j \in J$ such that $j \notin P$. Now for any $i \in I$, we have $ij \in P$, so by primeness we must have $i \in P$. We therefore see that $P \supseteq I$.

(*b*). We have $P \supseteq \sum_i I_i$ if and only if $P \supseteq I_i$ since by definition, $\sum_i I_i$ is the smallest ideal containing all the I_i.

(*c*). By the Scheinnullstellensatz [AK21, (3.27)], the radical $\sqrt{I}$ of I is the intersection of all prime ideals containing I. So, we have $\sqrt{I} \supseteq \sqrt{J}$ if and only if $V(I) \subseteq V(J)$. □

[Har77, p. 72]
[BouCA, II.4.3, Prop. 12]
[Hoc17, p. 30]

Proposition 2.2.9 [EGAI, Proposition 1.1.10(*ii*)]. *Let A be a ring. Then, $\operatorname{Spec}(A)$ is quasi-compact.*

Proof. Let $\operatorname{Spec}(A) = \bigcup_{i\in I} U_i$ be an open cover. Since the principal open sets form a basis, we can write

$$U_i = \bigcup_{j\in J_i} D(f_{ij})$$

and hence

$$\operatorname{Spec}(A) = \bigcup_{\substack{i\in I\\ j\in J_i}} D(f_{ij}).$$

Taking complements, we have $V(1) = V((f_{ij})_{i,j})$, and hence there exist powers n_{ij} of finitely many f_{ij} such that we have a "partition of unity"

$$\sum_{i,j} g_{ij} f_{ij}^{n_{ij}} = 1.$$

We therefore have

$$\operatorname{Spec}(A) = \bigcup_{i,j} D(f_{ij}) \subseteq \bigcup_{\{i\in I \mid n_{ij}>0\ \exists j\in J_i\}} U_i. \qquad \square$$

[Har77, Props. II.2.2(*b*),(*c*), II.5.1(*c*),(*d*)]

Theorem 2.2.10 [EGAI, Théorème 1.3.7; EGAI_{new}, Théorème 1.3.7]. *Let A be a ring and let M be an A-module. For every $f \in A$, the map*

$$\theta_f \colon M_f \longrightarrow \Gamma\bigl(D(f), \tilde{M}\bigr)$$

is bijective, and hence the presheaf $D(f) \mapsto M_f$ defined on principal open sets is in fact a sheaf. In particular,

$$\theta_1 \colon M \xrightarrow{\sim} \Gamma\bigl(\operatorname{Spec}(A), \tilde{M}\bigr).$$

We note that if $M = A$, then θ_f is a ring map. Thus, if we identify A_f with $\Gamma(D(f), \tilde{A})$ using θ_f for A, the maps θ_f will be isomorphisms of A_f-modules.

Proof. We first show that θ_f is injective. For each $x \in D(f)$, consider the commutative diagram

$$\begin{array}{ccc} M_f & \xrightarrow{\theta_f} & \Gamma\big(D(f), \tilde{M}\big) \\ \downarrow & & \downarrow \rho^{D(f)}_{D(g)} \\ M_g & \xrightarrow{\theta_g} & \Gamma\big(D(g), \tilde{M}\big) \\ \downarrow & & \downarrow \\ M_{\mathfrak{p}_x} & \xrightarrow{\sim} & \tilde{M}_x \end{array}$$

where the bottom row is the direct limit of the first two rows. If $\theta_f(\xi) = 0$, then ξ maps to 0 in $\tilde{M}_x$. By the commutativity of the diagram, this means that ξ maps to 0 in $M_{\mathfrak{p}_x}$. Since the $\mathfrak{p}_x$ range over all prime ideals in $D(f)$, we see that $\xi|_{D(U_x)} = 0$ for open neighborhoods $U_x \ni x$ inside $D(f)$. Thus, $\xi = 0$ by the sheaf condition (3).

We now show that θ_f is surjective. By Proposition 2.2.7, we may replace A by A_f to reduce to the case when $f = 1$. Let $s \in \Gamma(D(f), \tilde{M})$. Since the stalks of $\tilde{M}$ are $M_{\mathfrak{p}_x}$, for each $x \in \operatorname{Spec}(A)$, there exists a principal open set $D(f) \ni x$ over which $s|_{D(f)}$ is the image of $\xi \in M_f$. Since $\operatorname{Spec}(A)$ is quasi-compact (Proposition 2.2.9), there exists a finite open covering

$$\operatorname{Spec}(A) = \bigcup_{i=1}^{s} D(f_i)$$

such that $s_i \coloneqq s|_{D(f_i)}$ is of the form $\theta_{f_i}(\xi_i)$ for some $\xi_i \in M_{f_i}$. For each $i, j \in I$, we know that

$$\xi_i = \xi_j \tag{2.2.11}$$

in $M_{f_i f_j}$ since they map to the same section $(s_i)|_{D(f_i f_j)} = (s_j)|_{D(f_i f_j)}$ under $\theta_{f_i f_j}$. By definition, for each $i \in I$, we can write $\xi_i = z_i / f_i^{n_i}$ for $z_i \in M$. Since I is finite, after multiplying each z_i by a power of f_i, we may assume that the n_i are all equal to some n. The equations (2.2.11) show that there exist $m_{ij} \geq 0$ such that

$$(f_i f_j)^{m_{ij}} (f_j^n z_i - f_i^n z_j) = 0.$$

Replacing the m_{ij} by their maximum, we may assume that these relations hold for all i, j for a uniform m. Replacing the z_i by $f_i^m z_i$, we now can write $\xi_i = z_i / f_i^n$ for $z_i \in M$ such that

$$f_j^n z_i = f_i^n z_j \tag{2.2.12}$$

for all $i, j \in I$. Now since the $D(f_i^n) = D(f_i)$ cover $\operatorname{Spec}(A)$, the f_i^n generate the unit ideal in A. Thus, there are elements $g_i \in A$ such that

$$\sum_{i \in I} g_i f_i^n = 1.$$

We may therefore consider the element

$$z = \sum_{i \in I} g_i z_i \in M.$$

By (2.2.12), we have

$$f_i^n z = f_i^n \sum_{j \in I} g_j z_j = \sum_{j \in I} g_j (f_i^n z_j) = \sum_{j \in I} g_j (f_j^n z_i) = z_i \sum_{j \in I} g_j f_j^n = z_i,$$

and hence $\xi_i = z/1$ in M_{f_i}. We therefore see that $s_i = \theta_1(z)|_{D(f_i)}$. □

2.2.3. Locally ringed spaces and affine schemes. Now that we have defined a ringed space structure on $(\mathrm{Spec}(A), \mathcal{O})$, we want to glue them together to form other schemes.

[Har77, p. 70] DEFINITION 2.2.13 (Ringed spaces [EGAI, (**0**, 4.1.1); $\mathrm{EGAI}_{\mathrm{new}}$, (**0**, 4.1.1)]). Recall that a *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X called the *structure sheaf*. A *morphism* of ringed spaces

$$(f, f^\#)\colon (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

consists of a continuous map $f\colon X \to Y$ and a map

$$f^\#\colon \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$$

of sheaves of rings on Y. This forms a category RS.

[Har77, pp. 70–71] DEFINITION 2.2.14 (Locally ringed spaces [EGAI, (**0**, 5.5.1); $\mathrm{EGAI}_{\mathrm{new}}$, (**0**, 4.1.9, 4.1.12)]). A ringed space $(X, \mathcal{O}_X)$ is a *locally ringed space* if $\mathcal{O}_{X,P}$ is a local ring for every point $P \in X$. In this situation, we denote by $\mathfrak{m}_{X,P}$ or $\mathfrak{m}_P$ the maximal ideal of $\mathcal{O}_{X,P}$.

To define a morphism of locally ringed spaces, let $P \in X$ be a point and let $V \subseteq Y$ be a neighborhood of $f(P) \in Y$. As V ranges over all open neighborhoods of $f(P)$, we see that $f^{-1}(V)$ ranges over a subset of the open neighborhoods of P. We then obtain the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_Y(U) & \xrightarrow{f^\#(U)} & \mathcal{O}_X\big(f^{-1}(U)\big) \\ \big\downarrow & & \big\downarrow \\ \mathcal{O}_Y(V) & \xrightarrow{f^\#(V)} & \mathcal{O}_X\big(f^{-1}(V)\big) \end{array}$$

and taking direct limits, the left column becomes the stalk $\mathcal{O}_{Y,f(P)}$, while the right column is

$$\varinjlim_{f^{-1}(V)\ni P} \mathcal{O}_X\big(f^{-1}(V)\big) \longrightarrow \mathcal{O}_{X,P}.$$

A *morphism* of locally ringed spaces is a morphism $(f, f^\#)$ of ringed spaces such that taking direct limits in the commutative diagram above and composing with the map to $\mathcal{O}_{X,P}$, the induced map

$$f^\#_P\colon \mathcal{O}_{Y,f(P)} \longrightarrow \mathcal{O}_{X,P}$$

is a *local map*, that is, $f^\#_P(\mathfrak{m}_{Y,f(P)}) \subseteq \mathfrak{m}_{X,P}$. This forms a category LRS.

An *isomorphism* of locally ringed spaces is a morphism with a two-sided inverse. Thus, a morphism $(f, f^\#)$ is an isomorphism if and only if f is a homeomorphism of the underlying topological spaces and $f^\#$ is an isomorphism of sheaves.

Note that $(\mathrm{Spec}(A), \tilde{A})$ is a locally ringed space by Proposition 2.2.5. We can now define the analogue of an affine variety and the affine coordinate ring.

[Har77, p. 74] DEFINITION 2.2.15 (Affine scheme [EGAI, Définition 1.7.1; $\mathrm{EGAI}_{\mathrm{new}}$, Définition 1.6.1]). A locally ringed space $(X, \mathcal{O}_X)$ is an *affine scheme* if it is isomorphic to a locally ringed space of the form $(\mathrm{Spec}(A), \tilde{A})$ where A is a ring. In this case, we say that the ring

$$A(X) \coloneqq \Gamma(X, \mathcal{O}_X),$$

which can be identified with A by Theorem 2.2.10, is the *ring* of the affine scheme.

Our next goal is show that $\mathrm{Spec}(-)$ defines a contravariant functor $\mathsf{Ring}^{\mathrm{op}} \to \mathsf{LRS}$. See Definition A.2.1 for the definition of a functor. We start with describing the map on topological spaces associated to a ring map. 11/22

Proposition 2.2.16 [EGAI, Corollaire 1.2.3; EGAI_{new}, Corollaire 1.2.3]. *If $\varphi\colon A \to B$ is a ring map, then φ induces a continuous map* [Har77, Prop. II.2.3(*b*)]

$$ {}^a\varphi\colon \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A). $$

This association defines a contravariant functor

$$ \begin{aligned} \mathrm{Spec}\colon \mathsf{Ring}^{\mathrm{op}} &\longrightarrow \mathsf{Top} \\ A &\longmapsto \mathrm{Spec}(A) \\ \varphi &\longmapsto {}^a\varphi. \end{aligned} $$

Proof. We define the map by

$$ {}^a\varphi(P) = \varphi^{-1}(P) $$

for a prime ideal $P \in \mathrm{Spec}(B)$, where we recall that the contraction of a prime ideal under a ring map is prime. If $f \in A$ is an element, then

$$ {}^a\varphi^{-1}\bigl(D(f)\bigr) = D\bigl(\varphi(f)\bigr) $$

because $f \in \varphi^{-1}(P)$ if and only if $\varphi(f) \in P$. Since the $D(f)$ form a basis for the topology on $\mathrm{Spec}(A)$, we see that ${}^a\varphi$ is continuous. □

We now note that if we have a morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, taking global sections on the map $f^\#$ yields a ring map $\Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$. This gives a map

$$ \rho\colon \mathrm{Hom}_{\mathsf{RS}}(X, Y) \longrightarrow \mathrm{Hom}\bigl(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X)\bigr). $$

Restricting to LRS, we obtain the analogue of Proposition 1.3.22 for the category of locally ringed spaces.

Proposition 2.2.17 (Greenberg, Tate [Gre61, Proposition 1; EGAII, Errata et addenda, Proposition 1.8.1; EGAI_{new}, Proposition 1.6.3]). *Let $(S, \mathcal{O}_S)$ be an affine scheme and let $(X, \mathcal{O}_X)$ be a locally ringed space. Then, the restriction of ρ to morphisms in LRS:* [Har77, Exer. II.2.4]

$$ \rho_{\mathsf{LRS}}\colon \mathrm{Hom}_{\mathsf{LRS}}(X, S) \longrightarrow \mathrm{Hom}\bigl(A(S), \Gamma(X, \mathcal{O}_X)\bigr) $$

is a bijection natural in X and S.

Proof. The naturality follows from the definition of ρ: it associates to a map of locally ringed spaces the corresponding map on global sections.

We first define a continuous function ${}^a\varphi\colon X \to S$. Set $A \coloneqq A(S) = \Gamma(S, \mathcal{O}_S)$ and consider the ring map

$$ \varphi\colon A \longrightarrow \Gamma(X, \mathcal{O}_X). $$

For every $x \in X$, we can consider the composition

$$ \varphi_x\colon A \longrightarrow \Gamma(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{X,x}. $$

Then, for each $x \in X$, we set

$$ {}^a\varphi(x) \coloneqq \varphi_x^{-1}(\mathfrak{m}_x) = \bigl\{ f \in A \bigm| \varphi_x(f) \in \mathfrak{m}_x \bigr\} \subseteq A \tag{2.2.18} $$

is a prime ideal since it is the contraction of $\mathfrak{m}_x$. To show this assignment is continuous, let $D(f) \subseteq \operatorname{Spec}(A)$ be an open subset. Then,

$$ {}^a\varphi^{-1}\big(D(f)\big) = X_{\varphi(f)} \coloneqq \big\{x \in X \bigm| \varphi(f)_x \notin \mathfrak{m}_x\big\}. $$

This set is open since if $\varphi(f)_x$ has an inverse $g \in \mathcal{O}_{X,x}$ at x, then g is a section of $\mathcal{O}_X$ on an open neighborhood of x such that $\varphi(f), g$ are sections of $\mathcal{O}_X$ on that open neighborhood satisfying $\varphi(f)g = 1$.

We now define a morphism

$$ \tilde{\varphi}\colon \mathcal{O}_S \longrightarrow {}^a\varphi_*\mathcal{O}_X $$

making $({}^a\varphi, \tilde{\varphi})\colon (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ a morphism of locally ringed spaces. Consider the ring maps

$$ \begin{array}{ccccc} A & \xrightarrow{\sim} & \Gamma(S, \mathcal{O}_S) & \xrightarrow{\varphi} & \Gamma(X, \mathcal{O}_X) \\ \downarrow & & \downarrow & & \downarrow \\ A_f & \xrightarrow{\sim} & \Gamma(S_f, \mathcal{O}_S) & \overset{\varphi_f}{\dashrightarrow} & \Gamma(X_{\varphi(f)}, \mathcal{O}_X). \end{array} $$

Since f maps to an invertible element of $\Gamma(X_{\varphi(f)}, \mathcal{O}_X)$ by what we showed in the previous paragraph, we obtain a unique dashed map making the diagram commute by the universal property of localization. Since this map is the unique map making the diagram commute, we have the commutative diagrams

$$ \begin{array}{ccccc} A_f & \xrightarrow{\sim} & \Gamma(S_f, \mathcal{O}_S) & \xrightarrow{\varphi_f} & \Gamma(X_f, \mathcal{O}_X) \\ \downarrow & & \downarrow & & \downarrow \\ A_{fg} & \xrightarrow{\sim} & \Gamma(S_{fg}, \mathcal{O}_S) & \overset{\varphi_{fg}}{\dashrightarrow} & \Gamma(X_{\varphi(fg)}, \mathcal{O}_X) \\ \uparrow & & \uparrow & & \uparrow \\ A_g & \xrightarrow{\sim} & \Gamma(S_g, \mathcal{O}_S) & \xrightarrow{\varphi_g} & \Gamma(X_g, \mathcal{O}_X) \end{array} $$

for all $f, g \in A$. Thus, since the $D(f)$ form a basis, we have a well-defined sheaf morphism $\tilde{\varphi}\colon \tilde{A} \to {}^a\varphi_*\mathcal{O}_X$, since we can specify maps of sheaves on a basis by using the formula with inverse limits in the homework problem on specifying sheaves on a basis. Moreover, this map yields a morphism of locally ringed spaces since $f \in \mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ if and only if $\varphi_x(f) \in \mathfrak{m}_{f(x)} \subseteq \mathcal{O}_{S,f(x)}$ by definition in (2.2.18). We therefore obtain a map

$$ \sigma\colon \operatorname{Hom}\big(A(S), \Gamma(X, \mathcal{O}_X)\big) \longrightarrow \operatorname{Hom}_{\mathsf{LRS}}(X, S). $$

It remains to show that ρ_{LRS} and σ are mutually inverse. We know that $\rho_{\mathsf{LRS}} \circ \sigma = \mathrm{id}$ since $\Gamma(S, \tilde{\varphi}) = \varphi$, and hence ρ_{LRS} is surjective. To show that $\sigma \circ \rho_{\mathsf{LRS}} = \mathrm{id}$, we consider the commutative diagram

$$ \begin{array}{ccc} A & \longrightarrow & \mathcal{O}_{S,f(x)} \\ \downarrow & & \downarrow{\scriptstyle f_x^\#} \\ \Gamma(X, \mathcal{O}_X) & \longrightarrow & \mathcal{O}_{X,x}. \end{array} $$

The right vertical map is local and hence the inverse of $\mathfrak{m}_x$ must be $\mathfrak{m}_{f(x)}$. Thus, the underlying continuous map for $(\sigma \circ \rho_{\mathsf{LRS}})(f)$ is f. The map on sheaves for $(\sigma \circ \rho_{\mathsf{LRS}})(f)$ is $f^\sharp$ because of the universal property of localization. □

As a result, we obtain the following:

COROLLARY 2.2.19 [EGAI, Théorème 1.7.3; EGAI_{new}, (1.6.5)]. *There is a anti-equivalence of categories*

$$\begin{aligned}\operatorname{Spec}\colon \mathsf{Ring}^{\mathrm{op}} &\longrightarrow \mathsf{AffSch}\\ A &\longmapsto \big(\operatorname{Spec}(A), \tilde{A}\big)\\ \varphi &\longmapsto ({}^{a}\varphi, \tilde{\varphi}).\end{aligned}$$

Proof. The functor $\Gamma(-)$ of taking global sections is an inverse for Spec by Proposition 2.2.17. □

COROLLARY 2.2.20 [EGAI, p. 103; EGAI_{new}, p. 226]. $\operatorname{Spec}(\mathbf{Z})$ *is the final object in the category of locally ringed spaces.*

2.2.4. Schemes. We now define schemes.

DEFINITION 2.2.21 (Scheme [EGAI, Définition 2.1.2; EGAI_{new}, Définition 2.1.2]). A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ in which every point has a neighborhood U such that $(U, \mathcal{O}_X|_U)$ is an affine scheme, called an *open affine neighborhood*. We call X the *underlying topological space* of the scheme. If we want to forget the scheme structure on a scheme (or ringed space) we write $\operatorname{sp}(X)$, which we read as "the space of X". A *morphism* of schemes is a morphism of locally ringed spaces. In other words, the category Sch of schemes is a full subcategory of the category LRS of locally ringed spaces.

[Har77, p. 74] Before [EGAI_{new}], schemes were called *preschemes*.

To construct interesting examples of schemes, we prove the following.

LEMMA 2.2.22 (Gluing Lemma [EGAI, (**0**, 4.1.7), (2.3.1); EGAI_{new}, (**0**, 4.1.7), (2.4.1)]). *Let $\{X_i\}$ be a possibly infinite family of schemes. Suppose for each i, j, we are given an open subset $U_{ij} \subseteq X_i$, which we think of as a scheme with structure sheaf $\mathcal{O}_X|_{U_{ij}}$. Suppose also for each i, j, we are given an isomorphism of schemes*

[Har77, Exer. II.2.12]

$$\varphi_{ij}\colon U_{ij} \xrightarrow{\sim} U_{ji}$$

such that

(1) *For each i, j, we have $\varphi_{ji} = \varphi_{ij}^{-1}$.*
(2) *For each i, j, k, we have*

$$\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk},$$
$$\varphi_{ik}|_{U_{ij}\cap U_{ik}} = (\varphi_{jk} \circ \varphi_{ij})|_{U_{ij}\cap U_{ik}}.$$

There exists a scheme X together with morphisms $\psi_i\colon X_i \to X$ for each i such that

(1) *$\psi_i(X_i)$ is an open subset of X and ψ_i induces an isomorphism*

$$(X_i, \mathcal{O}_{X_i}) \xrightarrow{\sim} \big(\psi_i(X_i), \mathcal{O}_X|_{\psi_i(X_i)}\big).$$

(2) *The $\psi_i(X_i)$ cover X.*
(3) *$\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$.*
(4) *$\psi_i|_{U_{ij}} = (\psi_j \circ \varphi_{ij})|_{U_{ij}}$.*

Before proving Lemma 2.2.22, we define some terminology and write down some examples.

DEFINITION 2.2.23 [EGAI, (2.3.1); EGAI_{new}, (2.4.1)]. In the situation of Lemma 2.2.22, we say that X is obtained by *gluing* the schemes X_i along the isomorphisms φ_{ij}.

[Har77, Exer. II.2.12]

Example 2.2.24. When $U_{ij} = \emptyset$ and $\varphi_{ij} \colon \emptyset \to \emptyset$ for every i, j, the scheme X is called the *disjoint union* of the X_i, and is denoted $\coprod_i X_i$. [Har77, Exer. II.2.12]

[Har77, Ex. II.2.3.6] Example 2.2.25 (The affine line with two origins). Let k be a field, let $X_1 = X_2 = \mathbf{A}^1_k$, let $U_1 = U_2 = \mathbf{A}^1_k - \{P\}$ where P is the point corresponding to $(x) \subseteq k[x]$, and let $\varphi \colon U_1 \to U_2$ be the identity map. Let X be the gluing of X_1 and X_2 along U_1 and U_2 via φ. Then, X is the *affine line with two origins*.

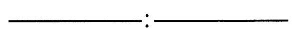


Figure 2.4. Affine line with two origins. From [Har77, p. 76].

This is a scheme that is not an affine scheme! To see this, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(X, \mathcal{O}_X) & \xrightarrow{(1\,1)} & k[x] \oplus k[x] & \xrightarrow{\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)} & k[x, x^{-1}] \\ & & \wr\Big\downarrow \rho^X_{X_1} & & \Big\downarrow \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) & & \\ 0 & \longrightarrow & \Gamma(X_1, \mathcal{O}_X) & \xrightarrow{\sim} & k[x] & \longrightarrow & 0 \end{array}$$

Calculating the kernel in the first row shows that $\rho^X_{X_1}$ is an isomorphism.

Now suppose that X were affine. Then, Corollary 2.2.19 shows that $\psi_1 \colon X_1 \to X$ is an isomorphism of affine schemes since it induces an isomorphism on global sections. This is a contradiction because ψ_1 is not surjective.

We also spell out this last part of the argument more explicitly. Suppose X were affine. Then, ψ_1 maps to the isomorphism $\Gamma(\psi_1) = \rho^X_{X_1}$. Letting $\varphi_1 \coloneqq \Gamma(\psi_1)^{-1}$ be the inverse map in Rings, we have

$$\varphi_1 \circ \Gamma(\psi_1) = \mathrm{id}_{\Gamma(X, \mathcal{O}_X)}$$

and hence

$$\psi_1 \circ {}^a\varphi_1 = \mathrm{id}_X.$$

This is a contradiction because ψ_1 is not surjective.

Proof of Lemma 2.2.22. The underlying topological space of X is obtained by taking the set

$$X \coloneqq \coprod_i X_i \Big/ \varphi_{ij}(x) \sim x \text{ for all } i \le j$$

and giving it the quotient topology for the map $\coprod_i X_i \twoheadrightarrow X$. With this quotient topology, the maps $\psi \colon X_i \to X$ are open embeddings. We can then glue the sheaves $\mathcal{O}_{X_i}$ from each X_i together to obtain the structure sheaf $\mathcal{O}_X$. $\square$

APPENDIX A

Category theory

A.1. Categories

We define some terminology from category theory. These will give us a convenient language to talk about the objects and morphisms we will study.

DEFINITION A.1.1. A *category* $\mathscr{C}$ consists of the following data: [AK21, (6.1)] [Hoc17, p. 8]

(1) A class of *objects*.
(2) For every pair of objects A and B in $\mathscr{C}$, a set $\operatorname{Hom}_{\mathscr{C}}(A,B)$ of *maps* or *morphisms*, such that $\operatorname{Hom}_{\mathscr{C}}(A,B)$ and $\operatorname{Hom}_{\mathscr{C}}(A',B')$ are disjoint unless $A = A'$ and $B = B'$. We write $f\colon A \to B$ to mean that $f \in \operatorname{Hom}_{\mathscr{C}}(A,B)$.
(3) For every triple of objects A, B, and C in $\mathscr{C}$, a *composition law*

$$\begin{aligned} \operatorname{Hom}_{\mathscr{C}}(A,B) \times \operatorname{Hom}_{\mathscr{C}}(B,C) &\longrightarrow \operatorname{Hom}_{\mathscr{C}}(A,C) \\ (f,g) &\longmapsto g \circ f \end{aligned}$$

satisfying the following axioms:
  (a) For every object B, there is a distinguished *identity* morphism $\mathrm{id}_B\colon B \to B$ such that for every morphism $f\colon A \to B$, we have $\mathrm{id}_A \circ f = f$, and for every morphism $g\colon B \to C$, we have $g \circ \mathrm{id}_B = g$.
  (b) Composition is associative: if $f\colon A \to B$, $g\colon B \to C$, and $g\colon C \to D$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

We say that $f\colon A \to B$ is a *isomorphism* with inverse $g\colon B \to A$ if $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. If such an inverse exists, it is unique, and is also an isomorphism $g\colon B \to A$. If there is an isomorphism between a pair of objects A and B, we say that A and B are *isomorphic*.

Given a category $\mathscr{C}$, we can construct the *opposite category* $\mathcal{C}^{\mathrm{op}}$. It has the same objects as $\mathscr{C}$, and the morphisms are given by $\operatorname{Hom}_{\mathscr{C}^{\mathrm{op}}}(A,B) = \operatorname{Hom}_{\mathscr{C}}(B,A)$. If $f \in \operatorname{Hom}_{\mathscr{C}^{\mathrm{op}}}(A,B)$ and $g \in \operatorname{Hom}_{\mathscr{C}^{\mathrm{op}}}(B,C)$, then composition is given by $g \circ_{\mathscr{C}^{\mathrm{op}}} f = f \circ_{\mathscr{C}} g$. [Hoc17, p. 11]

EXAMPLE A.1.2. We give some examples of categories we have already seen in this course. [Hoc17, pp. 9–10]

(1) The category Sets of sets, with functions as morphisms.

So far, objects have underlying sets, morphisms are given by certain functions on those sets, and composition coincides with composition of functions. The following examples are not of this form:

(2) Let $(P, \leq)$ be a partially ordered set. We can consider the category of all elements $x \in P$ where $\operatorname{Hom}(x,y)$ is a set consisting of one element if $x \leq y$, and is empty otherwise. In this category, isomorphic objects are equal.

(3) A category with one object in which every morphism is an isomorphism. This is essentially the same data as a group, where the morphisms of the object correspond to the elements of the group.

A.2. Functors

The utility of categories is really in relationships between them, given by functors.

DEFINITION A.2.1. Given two categories $\mathscr{C}$ and $\mathscr{D}$, a *(covariant) functor* $F\colon \mathscr{C} \to \mathscr{D}$ is a rule that assigns to each object A of $\mathscr{C}$ an object $F(A)$ of $\mathscr{D}$, and assigns to each morphism $f\colon A \to B$ in $\mathscr{C}$ a morphism $F(f)\colon F(A) \to F(B)$ in $\mathscr{D}$, such that

[AK21, (6.2)]
(1) For all objects A in $\mathscr{C}$, we have $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

[Hoc17, p. 11]
(2) For all morphisms $f\colon A \to B$ and $g\colon B \to C$ in $\mathscr{C}$, we have $F(g \circ f) = F(g) \circ F(f)$.

Note that a functor preserves isomorphisms.

A *contravariant functor* from $\mathscr{C}$ to $\mathscr{D}$ is a covariant functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}$.

[Hoc17, pp. 11–12]
EXAMPLE A.2.2. Here are some examples of functors.

(1) Given any category $\mathscr{C}$, there is an identity functor $\mathrm{id}_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{C}$ that sends objects A to A itself and morphisms f to f itself.
(2) There is an *Abelianization* functor $\mathsf{Grp} \to \mathsf{Ab}$ sending a group G to $G/[G,G]$, where
$$[G,G] := \{ghg^{-1}h^{-1} \mid g,h \in G\}$$
is the commutator subgroup.
(3) The composition of two functors is a functor. If both are covariant or both are contravariant, then the composition is covariant. If one is covariant and the other is contravariant, then the composition is contravariant.
(4) Given a category $\mathscr{C}$ whose objects have underlying sets and where composition coincides with composition of functions, there is a *forgetful functor* $\mathsf{Forget}\colon \mathscr{C} \to \mathsf{Sets}$ sending objects to their underlying sets, and morphisms to their underlying functions.
(5) A category $\mathscr{C}$ is a *full subcategory* of another category $\mathscr{D}$ if the objects of $\mathscr{C}$ form a subclass of objects in $\mathscr{D}$, and if $\mathrm{Hom}_{\mathscr{C}}(A,B) = \mathrm{Hom}_{\mathscr{D}}(A,B)$ for every pair of objects A and B in $\mathscr{C}$. For example, finite sets form a full subcategory of Sets, Ab is a full subcategory of Grp.
(6) The spectrum $\mathrm{Spec}(R)$ defines a contravariant functor
$$\mathrm{Spec}\colon \mathsf{Rings}^{\mathrm{op}} \longrightarrow \mathsf{Top}.$$
This is because if $\varphi\colon R \to S$, then we have a map
$$\mathrm{Spec}(\varphi)\colon \mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R)$$
induced by contracting prime ideals. This map is continuous since if $V(I) \subseteq \mathrm{Spec}(R)$ is a closed subset, then $(\mathrm{Spec}(\varphi))^{-1}(V(I)) = V(\varphi(I)S)$.
(7) Here is a non-example: Mimicking the definition of $\mathrm{Spec}(\varphi)$ for the maximal spectrum MaxSpec does not define a functor $\mathsf{Rings}^{\mathrm{op}} \to \mathsf{Top}$ (where $\mathrm{MaxSpec}(R)$ is given the subspace topology), or even to Sets, since the inverse image of a maximal ideal is not always maximal. One example of this is $\mathbf{Z} \to \mathbf{Q}$, where the maximal ideal $(0) \subseteq \mathbf{Q}$ has inverse image $(0) \subseteq \mathbf{Z}$, which is not maximal. The same thing occurs for $k[x] \to k(x)$.

A.3. Natural transformations and equivalences of categories

We also define natural transformations and isomorphisms of functors.

DEFINITION A.3.1. Let $F, G\colon \mathscr{C} \to \mathscr{D}$ be two functors. A *natural transformation* $T\colon F \Rightarrow G$ assigns to every object X in $\mathscr{C}$ a morphism $T_X\colon F(X) \to G(X)$ such that for all morphisms $f\colon X \to Y$ in $\mathscr{C}$, there is a commutative diagram [Hoc17, pp. 13–15]

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle T_X}\downarrow & & \downarrow{\scriptstyle T_Y} \\ G(X) & \xrightarrow{G(f)} & G(Y). \end{array}$$

Natural transformations $S\colon F \Rightarrow G$ and $T\colon G \Rightarrow H$ can be composed to form the natural transformation $T \star S\colon F \Rightarrow H$ given by the rule

$$(T \star S)_X \coloneqq T_X \circ S_X.$$

There is an identity natural transformation id_F from $F\colon \mathscr{C} \to \mathscr{D}$ to itself. Two functors $F, G\colon \mathscr{C} \to \mathscr{D}$ are *isomorphic* if there are natural transformations

$$T\colon F \Longrightarrow G \qquad \text{and} \qquad T'\colon G \Longrightarrow F$$

such that $T' \star T = \mathrm{id}_F$ and $T \star T' = \mathrm{id}_G$. In fact, T is an isomorphism if and only if the morphisms T_X are isomorphisms for all objects X, in which case

$$(T_X^{-1}) = (T_X)^{-1}.$$

Here is an example.

EXAMPLE A.3.2. Let V be a vector space over a field k and write [Hoc17, p. 14]

$$V^* \coloneqq \mathrm{Hom}_k(V, k).$$

Then, $(-)^*$ is a contravariant functor from k-vector spaces Vect_k to Vect_k. Composing $(-)^*$ with itself, we get the covariant functor

$$(-)^{**}\colon \mathsf{Vect}_k \longrightarrow \mathsf{Vect}_k.$$

There is a natural transformation $T\colon \mathrm{id}_{\mathsf{Vect}_k} \Rightarrow (-)^{**}$ defined by

$$\begin{aligned} T_V\colon V &\longrightarrow V^{**} \\ v &\longmapsto \big(g \longmapsto g(v)\big). \end{aligned}$$

To check this defines a natural transformation, we need to check that for every k-linear map $f\colon V \to W$, the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle T_V}\downarrow & & \downarrow{\scriptstyle T_W} \\ V^{**} & \xrightarrow{f^{**}} & W^{**} \end{array}$$

commutes (this follows from the definition). The map $V \to V^{**}$ is always injective, but may not be an isomorphism. It is an isomorphism for example when V is finite-dimensional.

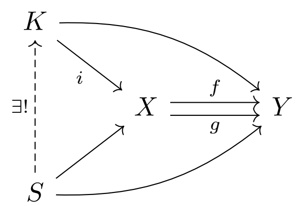
Using isomorphisms of functors, we can define equivalences of categories.

DEFINITION A.3.3. Two categories $\mathscr{C}$ and $\mathscr{D}$ are *equivalent* if there exist functors $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ such that $G \circ F$ is isomorphic to the identity functor on $\mathscr{C}$ and $F \circ G$ is isomorphic to the identity functor on $\mathscr{D}$. Two categories $\mathscr{C}$ and $\mathscr{D}$ are *antiequivalent* if $\mathscr{C}^{\mathrm{op}}$ is equivalent to $\mathscr{D}$. [Hoc17, p. 15]

A.4. Equalizers

We define equalizers. Note that while we mentioned that the definition of a sheaf can be stated in terms of equalizers, it is not necessary for this course.

[Bor94, Def. 2.4.1] DEFINITION A.4.1 (Equalizer). Suppose $f\colon X \to Y$ and $g\colon X \to Y$ are two morphisms in $\mathscr{C}$. An *equalizer* of f, g is an object K in $\mathscr{C}$ together with a morphism $i\colon K \to X$ such that $f \circ i = g \circ i$ and such that for every commutative diagram



such that for every object S in $\mathscr{C}$ such that the two compositions $S \to X \to Y$ are equal, there is a unique morphism $S \to K$ making the triangle on the left commute.

A.5. Direct limits and inverse limits

[Hoc17, p. 153] DEFINITION A.5.1 (Filtered partially ordered set). A partially ordered set $(I, \leq)$ is *filtered* or *directed* if, for any two elements $i, j \in \Lambda$, there exists $k \in \Lambda$ with $i \leq k$ and $j \leq k$.

In other words, any two elements of I have an upper bound.

[Hoc17, p. 153] EXAMPLES A.5.2. Here are examples of filtered partially ordered sets.

(1) Totally ordered sets, for example, the natural numbers $\mathbf{N}$ and the positive integers.
(2) Finite subsets of a given set under $\subseteq$.
(3) Finitely generated R-submodules of an R-module under $\subseteq$.
(4) Finitely generated R-subalgebras of an R-algebra under $\subseteq$.
(5) Open neighborhoods of a point $x \in X$, where X is a topological space, under $\supseteq$.

We now define direct systems and direct limits.

[Hoc17, p. 153] DEFINITION A.5.3 (Direct system). Let $(I, \leq)$ be a partially ordered set. A *direct system* or *direct limit system* indexed by I in a category $\mathscr{C}$ is a functor $I \to \mathscr{C}$ where I is considered as a category as in Example A.1.2(2). Explicitly, for every element $i \in I$, we have an object X_i in $\mathscr{C}$, and for all pairs $i, j \in I$, we have a morphism

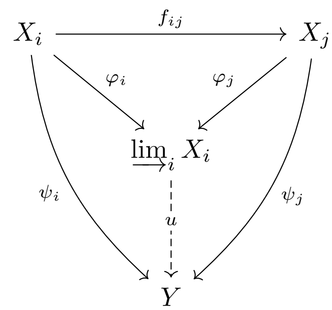
$$f_{ij}\colon X_i \longrightarrow X_j$$

such that

(1) $f_{ii} = \mathrm{id}_{X_i}$ for every $i \in I$.
(2) Whenever $i \leq j \leq k$, we have $f_{ik} = f_{jk} \circ f_{ij}$.

If I is filtered, we say that the direct system is a *filtered direct system*.

Definition A.5.4 (Direct limit). Let $\{X_i, f_{ij}\}$ be a direct system of objects and morphisms in a category $\mathscr{C}$. The *direct limit* of the system $\{X_i, f_{ij}\}$ is an object of $\mathscr{C}$ denoted by $\varinjlim X_i$ with morphisms $\{\varphi_i \colon X_i \to \varinjlim X_i\}$ such that $\varphi_j \circ f_{ij} = \varphi_i$ for all $i \leq j$. Moreover, if $Y \in \mathscr{C}$ is another object with morphisms $\{\psi_i \colon X_i \to Y\}$ such that $\psi_j \circ f_{ij} = \psi_i$ for all $i \leq j$, then there is a unique morphism $u \colon \varinjlim X_i \to Y$ making the following diagram commute: [Hoc17, p. 153] [Lan02, Thm. III.10.1]



If the direct system is filtered, we call the direct limit a *filtered direct limit*.

Examples A.5.5. Let $\mathscr{C}$ be the category of sets, groups, Abelian groups, rings, R-modules, or R-algebras. (The examples below will work for any category where the objects have an underlying set and a morphism is a function possibly satisfying additional conditions.)

(1) Let Z be a fixed object in $\mathscr{C}$ and let $\{X_i\}$ be a filtered set of subobjects of Z, partially ordered by inclusion. Then, the direct limit of the X_i is the union of these subobjects, and is called the *filtered union*.

(2) Let $\{X_i, f_{ij}\}$ be a filtered direct system of objects and morphisms in $\mathscr{C}$. [AK21, (7.4)] Then, the filtered direct limit exists and can be constructed as

$$\varinjlim_i X_i \coloneqq \coprod_i X_i \Big/ f_{ij}(x) \sim x \text{ for all } i \leq j.$$

If $\mathscr{C}$ is the category of Abelian groups or R-modules, then the direct limit exists and can be constructed as

$$\varinjlim_i X_i \coloneqq \bigoplus_i X_i \Big/ \big(f_{ij}(x) - x\big)_{i \leq j}$$

where $(f_{ij}(x) - x)_{i \leq j}$ is the subobject generated by the elements $f_{ij}(x) - x$.

Next, we define inverse systems and inverse limits.

Definition A.5.6 (Inverse system). Let $(I, \leq)$ be a partially ordered set. An *inverse system* or *inverse limit system* indexed by I in a category $\mathscr{C}$ is a functor $I^{\mathrm{op}} \to \mathscr{C}$ where I is considered as a category as in Example A.1.2(2). Explicitly, for every element $i \in I$, we have an object X_i in $\mathscr{C}$, and for all pairs $i, j \in I$, we have a morphism [Hoc17, p. 155]

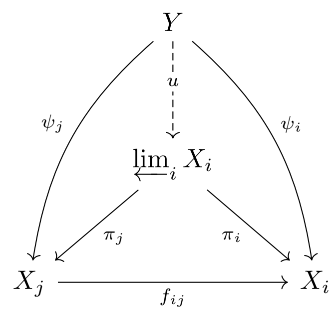
$$f_{ij} \colon X_j \longrightarrow X_i$$

such that

(1) $f_{ii} = \mathrm{id}_{X_i}$ for every $i \in I$.
(2) Whenever $i \leq j \leq k$, we have $f_{ik} = f_{ij} \circ f_{jk}$.

If I is filtered, we say the inverse system is a *cofiltered inverse system*.

[Hoc17, p. 155] Definition A.5.7 (Inverse limit). Let $\{X_i, f_{ij}\}$ be a inverse system of objects and morphisms in a category $\mathscr{C}$. The *inverse limit* of the system $\{X_i, f_{ij}\}$ is an object of $\mathscr{C}$ denoted by $\varprojlim X_i$ with morphisms $\{\pi_i \colon \varprojlim X_i \to X_i\}$ such that $f_{ij} \circ \pi_j = \pi_i$ for all $i \leq j$. Moreover, if Y is another object with morphisms $\{\psi_i \colon Y \to X_i\}$ such that $f_{ij} \circ \psi_j = \psi_i$ for all $i \leq j$, then there is a unique morphism $u \colon Y \to \varprojlim X_i$ making the following diagram commute:



If I is filtered, we say the inverse limit is a *cofiltered inverse limit*.

[Hoc17, pp. 155–156] Example A.5.8. Let $\mathscr{C}$ be the category of sets, Abelian groups, rings, R-modules, or R-algebras. The inverse limit exists and can be constructed as

$$\varprojlim_i X_i \coloneqq \left\{ (x_1, x_2, \ldots) \in \prod_i X_i \;\middle|\; f_{ij}(x_j) = x_i \text{ for all } i \geq j \right\}.$$

A.6. Filtered direct limits are exact

An important result when working with sheaves is the following.

[AK21, (7.9)] Theorem A.6.1 (Filtered direct limits are exact). *Let R be a ring and let I be a filtered partially ordered set. Let $\mathscr{C}$ be the category of 3-term exact sequences of R-modules: its objects are 3-term exact sequences, and its morphisms are commutative diagrams of the form*

$$\begin{array}{ccccc} L & \longrightarrow & M & \longrightarrow & N \\ \downarrow & & \downarrow & & \downarrow \\ L' & \longrightarrow & M' & \longrightarrow & N'. \end{array}$$

Then, for any filtered direct system

$$\left\{ L_i \xrightarrow{\beta_i} M_i \xrightarrow{\gamma_i} N_i \right\}_{i \in I}$$

the induced sequence

$$\varinjlim_i L_i \xrightarrow{\beta} \varinjlim_i M_i \xrightarrow{\gamma} \varinjlim_i N_i$$

is exact.

Proof. Abusing notation, we denote by f_{ij} the transition maps connecting the L_i, M_i, or N_i, and by φ_i the canonical morphisms mapping the L_i, M_i, or N_i to their respective direct limits.

We first show that $\operatorname{im}(\beta) \subseteq \ker(\gamma)$. Suppose $\ell \in \operatorname{im}(\beta)$. By definition of direct limits (Example A.5.5(2)), there exist $i \in I$ and $\ell_i \in L_i$ such that $\varphi_i(\ell_i) = \ell$. We then consider the commutative diagram

$$\begin{array}{ccccc} L_i & \xrightarrow{\beta_i} & M_i & \xrightarrow{\gamma_i} & N_i \\ \downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \varphi_i} \\ \varinjlim_i L_i & \xrightarrow{\beta} & \varinjlim_i M_i & \xrightarrow{\gamma} & \varinjlim_i N_i. \end{array} \tag{A.6.2}$$

Since $\operatorname{im}(\beta_i) \subseteq \ker(\gamma_i)$ by assumption, we see that

$$(\gamma \circ \beta)(\ell) = (\gamma \circ \beta)\big(\varphi_i(\ell_i)\big) = \varphi_i\big((\gamma_i \circ \beta_i)(\ell_i)\big) = 0.$$

We now show that $\operatorname{im}(\beta) \supseteq \ker(\gamma)$. Suppose $m \in \ker(\gamma)$. By construction of filtered direct limits (Example A.5.5(2)), there exist $i \in I$ and $m_i \in M_i$ such that $\varphi_i(m_i) = m$ and hence $\varphi_i(\gamma_i(m_i)) = 0$ by the commutativity of (A.6.2). By construction of filtered direct limits (Example A.5.5(2)), we know there exists $j \geq i$ such that $f_{ij}(m_i) = 0$. We now consider the commutative diagram

$$\begin{array}{ccccc} L_i & \xrightarrow{\beta_i} & M_i & \xrightarrow{\gamma_i} & N_i \\ \downarrow{\scriptstyle f_{ij}} & & \downarrow{\scriptstyle f_{ij}} & & \downarrow{\scriptstyle f_{ij}} \\ L_j & \xrightarrow{\beta_j} & M_j & \xrightarrow{\gamma_j} & N_j \\ \downarrow{\scriptstyle \varphi_j} & & \downarrow{\scriptstyle \varphi_j} & & \downarrow{\scriptstyle \varphi_j} \\ \varinjlim_i L_i & \xrightarrow{\beta} & \varinjlim_i M_i & \xrightarrow{\gamma} & \varinjlim_i N_i. \end{array}$$

By exactness in the j-th row, we know there exists $\ell_j \in L_j$ such that $\beta_j(\ell_j) = f_{ij}(m_i)$. Let $\ell = \varphi_j(\ell_j)$. By the commutativity of the diagram in the bottom two rows, we see that

$$\beta(\ell) = \beta\big(\varphi_j(\ell_j)\big) = \varphi_j\big(\beta_j(\ell_j)\big) = \varphi_j(m_j) = m. \qquad \square$$

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Page numbers in bold refer to definitions. An “f”, “n”, or “t” following a page number refers to a figure, footnote, or table on that page, respectively.

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