MA665: Algebraic Geometry II

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Contents

List of Symbols

Symbol	Description
0	zero element, zero ring, or zero object, 9
1	identity element in a ring R, xiii
$\sim$	homotopy equivalence of morphisms of chain complexes, 33
$\sim$	linear equivalence of Cartier or Weil divisors, 159
$\twoheadrightarrow$	epimorphism, 23
$\rightsquigarrow$	specialization, 57
$\hookrightarrow$	monomorphism, 23
$\dashrightarrow$	rational map, 141
$\langle U, f\rangle$	representative of a germ of a regular or rational function f, or a section f of a sheaf, 5
$\lvert\mathscr{L}\rvert$	complete linear system associated to an invertible sheaf $\mathscr{L}$, 174
$\lvert V\rvert$	linear system associated to a subspace $V \subseteq H^0(X, \mathscr{L})$, 174
$\coprod_i X_i$	coproduct of objects X_i, 9
$\bigoplus_i X_i$	direct sum of objects X_i in an Abelian category, 20
$\prod_i X_i$	product of objects X_i, 9
$\Gamma(U, \mathscr{F})$	sections of a sheaf $\mathscr{F}$ on an open set U, 3
$\Gamma_*(\mathscr{F})$	graded module associated to a sheaf $\mathscr{F}$, 91
$\Gamma_Z(U, \mathscr{F})$	sections of a sheaf $\mathscr{F}$ on an open set U with support in Z, 34
$\Omega^1_{B/A}$	module of relative differentials of B over A, 183
$\Omega^1_{X/Y}$	sheaf of relative differentials of X over Y, 191
ω°_X	dualizing sheaf for X, 198
$\mathbf{C}$	complex numbers
$\mathbf{N}$	natural numbers $\{0, 1, 2, \ldots\}$
$\mathbf{P}^n_A$	projective n-space over A, 84
$\mathbf{Q}$	rational numbers
$\mathbf{R}$	real numbers
Ab	category of Abelian groups, 2, 10
$\mathsf{Ab}(X)$	category of Abelian sheaves on a topological space X, 5
$\mathsf{Ch}(\mathscr{A})$	category of (cochain) complexes in $\mathscr{A}$, 30
$\mathsf{Coh}(\mathcal{O}_X)$	category of coherent sheaves of $\mathcal{O}_X$-modules, 52
$\mathsf{ind}_I(\mathscr{C})$	category of direct systems indexed by I in $\mathscr{C}$, 41
LRS	category of locally ringed spaces, 69
$\mathsf{Mod}(\mathcal{O}_X)$	category of sheaves of $\mathcal{O}_X$-modules, 8
$\mathsf{Mod}(R)$	category of R-modules, 2, 10
${}^*\mathsf{Mod}(S)$	category of graded S-modules, 85
$\mathsf{PAb}(X)$	category of Abelian presheaves on a topological space X, 5

Symbol	**Description**
$\mathsf{PSh}(X)$	category of presheaves of sets on a topological space X, 5
$\mathsf{QCoh}(\mathcal{O}_X)$	category of quasi-coherent sheaves of $\mathcal{O}_X$-modules, 48
RS	category of ringed spaces, 68
Sch	category of schemes, 69
Sets	category of sets, 1
$\mathsf{Sh}(X)$	category of sheaves of sets on a topological space X, 5
$\mathsf{Top}(X)$	category of open subsets of a topological space X, 3
Var_k	category of varieties over k, 105
$\mathfrak{b}(\mathscr{L})$	base ideal of an invertible sheaf $\mathscr{L}$, 175
$\mathfrak{b}(\lvert V\rvert)$	base ideal of a linear system $\lvert V\rvert$, 175
$\mathfrak{m}_x$	maximal ideal of the local ring of a locally ringed space at a point x, 68
$\mathfrak{N}$	nilradical of a ring, 75
$\mathfrak{p}_x$	prime ideal corresponding to a point $x \in \operatorname{Spec}(A)$, 56
$\check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F})$	Čech complex for $\mathscr{F}$ with respect to $\mathfrak{U}$, 121
$\mathscr{C}^{\mathrm{op}}$	opposite category, 1
$\mathscr{C}_{Z/X}$	conormal sheaf of an immersion $Z \hookrightarrow X$, 191
$\mathscr{F}(n)$	n-th twist of an $\mathcal{O}_X$-module $\mathscr{F}$, 86
$\mathscr{F}^\#$	sheafification of a presheaf $\mathscr{F}$, 7
$\mathscr{F}_P$	stalk of a sheaf $\mathscr{F}$ at P, 5
$\mathscr{F}_U$	$j_!j^*\mathscr{F}$ where $j\colon U \hookrightarrow X$ is an open inclusion, 43
${}_U\mathscr{F}$	$j_*j^*\mathscr{F}$ where $j\colon U \hookrightarrow X$ is an open inclusion, 66
$\mathscr{F}_Y$	$i_*i^*\mathscr{F}$ where $i\colon Y \hookrightarrow X$ is a closed inclusion, 43
$\mathscr{K}_X$	sheaf of rational functions, 155
$\mathscr{N}_X$	nilradical of a scheme, 75
$\mathcal{O}_U$	structure sheaf of an open subspace $U \subseteq X$, 47
$\mathcal{O}_X$	structure sheaf of a ringed space, e.g., sheaf of regular functions on a variety X, 4, 8
$\mathcal{O}_X(D)$	sheaf associated to a divisor, 168
$\mathcal{O}_X(n)$	n-th twisting sheaf of Serre, 86
$\mathcal{O}_{X,x}$	stalk of the structure sheaf of a ringed space X at a point x, e.g., local ring of a variety X at a point x, 68
$\mathcal{P}(X)$	power set, 25
$A(X)$	ring of an affine scheme, or affine coordinate ring of an algebraic set $X \subseteq \mathbf{A}^n_k$, 69
$A - B$	set difference, xiii
$A^\bullet$	complex in an Abelian category, 29
$\underline{A}_X$	constant sheaf determined by A, 4
$B^i(A^\bullet)$	i-th coboundaries of a complex $A^\bullet$, 30
$\mathrm{Bl}_{\mathscr{I}}(X)$	blowup of a scheme X along an ideal $\mathscr{I}$, 175
$\mathrm{Bl}_Y(X)$	blowup of a scheme X along a closed subscheme Y, 175
$\mathrm{Bs}(\mathscr{L})$	base scheme of an invertible sheaf $\mathscr{L}$, 175
$\mathrm{Bs}\lvert V\rvert$	base scheme of a linear system $\lvert V\rvert$, 175
$C(X)$	affine cone over $X = \operatorname{Proj}(S)$, 94
$\mathrm{CaCl}(R)$	Cartier divisor class group of a ringed space X, 159
$\mathrm{cd}(X)$	cohomological dimension of X, 128

Symbol	**Description**
$\mathrm{CDiv}(X)$	group of Cartier divisors on X, 156
$\mathrm{CDiv}_{\geq 0}(X)$	monoid of effective Cartier divisors on X, 156
$\mathrm{CDiv}_{\mathrm{princ}}(X)$	group of principal Cartier divisors on X, 158
C_K	abstract nonsingular curve, 163
$\mathrm{Cl}(X)$	divisor class group of a normal Noetherian scheme X, 159
$\mathrm{codim}_X(Z)$	codimension of an arbitrary subset Z of a scheme X, 150
$\mathrm{coim}(f)$	coimage of a morphism, 10
$\mathrm{coker}(f)$	cokernel of a morphism, 10
$D(f)$	distinguished open set in $\mathrm{Spec}(A)$, 59
$\mathrm{Der}_A(B, M)$	module of A-linear derivations $d\colon B \to M$, 183
$D_+(f)$	distinguished open set in $\mathrm{Proj}(S)$, 82
$f^\#$	map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ on sheaves associated to a morphism of ringed spaces, 68
$\mathbf{h}^i(A^\bullet)$	i-th cohomology object of a complex, 30
$H^i(X, \mathscr{F})$	i-th sheaf cohomology module, 34
$H^i_Z(X, \mathscr{F})$	i-th sheaf cohomology module with support in Z, 34
$\check{H}^i(\mathfrak{U}, \mathscr{F})$	i-th Čech cohomology module, 122
$\mathrm{im}(f)$	image of a morphism, 10
$k(x)$	residue field at x, 108
$\mathrm{ker}(f)$	kernel of a morphism, 10
$M(n)$	n-th twist of a graded module M, 86
$\tilde{M}$	sheaf on $\mathrm{Spec}(A)$ (resp. $\mathrm{Proj}(S)$) associated to a module (resp. a graded module), 59, 83
$\mathrm{ord}_A(f)$	order of an element $f \in \mathrm{Frac}(A)$, 157
$\mathrm{Pic}(X)$	Picard group of a ringed space X, 168
$\mathrm{Proj}(S)$	homogeneous spectrum of a graded ring S, 81
$\mathbf{Proj}_Y(\mathscr{S})$	homogeneous spectrum of a graded sheaf of $\mathcal{O}_Y$-algebras $\mathscr{S}$, 140
$\mathbf{P}_Y(\mathscr{E})$	projective bundle, 140
$P_Y(E)$	$\mathbf{P}_Y(\mathscr{E}^\vee)$, 140
s_P	germ of a section of a sheaf at P, 5
$\mathrm{sp}(X)$	underlying topological space of a ringed space or scheme, 69
$\mathrm{Spé}(\mathscr{F})$	the espace étalé of a presheaf $\mathscr{F}$, 7
$\mathrm{Spec}(A)$	spectrum of a ring A, 56
$\mathrm{Sym}^\bullet_{\mathcal{O}_Y}(\mathscr{E})$	symmetric algebra associated to an $\mathcal{O}_Y$-module, 140
t	trace morphism from Serre duality, 198
$T \star S$	composition of two natural transformations T and S, 3
$T_{X/S}$	geometric tangent bundle $\mathbf{V}_X(\Omega^1_{X/S})$, 194
$V(I)$	closed set in $\mathrm{Spec}(A)$ defined by an ideal $I \subseteq A$, 56
$V_+(E)$	vanishing set in $\mathrm{Proj}(S)$, 81
$\mathrm{WDiv}(X)$	group of Weil divisors on X, 156
$\mathrm{WDiv}_{\geq 0}(X)$	monoid of effective Weil divisors on X, 156
$\mathrm{WDiv}_{\mathrm{princ}}(X)$	group of principal Weil divisors on X, 158
X_{red}	reduced scheme associated to X, 76
$X \times_S Y$	fiber product of X and Y over S, 105
X_y	fiber of X over $y \in Y$, 108
$\mathsf{Y}(D)$	closed subscheme associated to an effective Cartier divisor, 156

Symbol	**Description**
$Z^+(f)$	divisor of zeros of f, 159
$Z^-(f)$	divisor of zeros of f, 159
$Z^i(A^\bullet)$	i-th cocycles of a complex $A^\bullet$, 30

Conventions

(1) Let A and B be subsets of a set X. The *set difference* is denoted

$$A - B \coloneqq \left\{ x \in X \;\middle|\; x \in A \text{ and } x \notin B \right\}.$$

(2) All rings R will be assumed to be commutative with an identity element 1, unless stated otherwise. We may sometimes denote 1 by 1_R for clarity.

(3) All ring maps $\varphi \colon R \to S$ will be assumed to respect the identity element, i.e., $\varphi(1_R) = 1_S$.

Preface

These are notes for the second semester of an introductory graduate sequence on algebraic geometry (MA665) taught at Purdue University in Spring 2025. The official course text is [Har77]. We also suggest [EGAI; EGAI_{new}; EGAII; EGAIII_1; EGAIII_2; EGAIV_1; EGAIV_2; EGAIV_3; EGAIV_4] as additional references. The notes in the margins point to where in these texts (and sometimes others) one can find the material written down in these notes.

These notes will be continually updated throughout the semester.

I would like to thank Farrah Yhee for innumerable helpful conversations.

CHAPTER 1

Sheaves and sheaf cohomology

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For the first part of the course, we will develop the theory of sheaf cohomology that will be used throughout the rest of the course. Sheaf cohomology was introduced to algebraic geometry by Serre in [FAC]. The theory works surprisingly well given the coarseness of the Zariski topology! However, since the theory works for basically any topological space, we will start the course by developing the theory of sheaf cohomology for arbitrary topological spaces.

1.1. Categories and functors

We begin with some background material on categories and homological algebra. See also [Mur24ca, §2.1] or [Mur24ag, Appendix A].

1.1.1. Categories.

Definition 1.1.1. A *category* $\mathscr{C}$ consists of the following data:

[Wei94, Def. A.1.1]
[Bor94a, Def. 1.2.1]
[AK21, (6.1)]
[Hoc17, p. 8]

1. A class of *objects*.
2. For every pair of objects A and B in $\mathscr{C}$, a set $\operatorname{Hom}_{\mathscr{C}}(A,B)$ of *maps* or *morphisms*, such that $\operatorname{Hom}_{\mathscr{C}}(A,B)$ and $\operatorname{Hom}_{\mathscr{C}}(A',B')$ are disjoint unless $A = A'$ and $B = B'$. We write $f\colon A \to B$ to mean that $f \in \operatorname{Hom}_{\mathscr{C}}(A,B)$.
3. For every triple of objects A, B, and C in $\mathscr{C}$, a *composition law*

$$\begin{aligned} \operatorname{Hom}_{\mathscr{C}}(A,B) \times \operatorname{Hom}_{\mathscr{C}}(B,C) &\longrightarrow \operatorname{Hom}_{\mathscr{C}}(A,C) \\ (f,g) &\longmapsto g \circ f \end{aligned}$$

satisfying the following axioms:

   - (a) For every object B, there is a distinguished *identity* morphism $\mathrm{id}_B\colon B \to B$ such that for every morphism $f\colon A \to B$, we have $\mathrm{id}_A \circ f = f$, and for every morphism $g\colon B \to C$, we have $g \circ \mathrm{id}_B = g$.
   - (b) Composition is associative: if $f\colon A \to B$, $g\colon B \to C$, and $g\colon C \to D$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

We say that $f\colon A \to B$ is a *isomorphism* with *inverse* $g\colon B \to A$ if $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. If such an inverse exists, it is unique, and is also an isomorphism $g\colon B \to A$. If there is an isomorphism between a pair of objects A and B, we say that A and B are *isomorphic*.

Given a category $\mathscr{C}$, we can construct the *opposite category* $\mathscr{C}^{\mathrm{op}}$. It has the same objects as $\mathscr{C}$, and the morphisms are given by $\operatorname{Hom}_{\mathscr{C}^{\mathrm{op}}}(A,B) = \operatorname{Hom}_{\mathscr{C}}(B,A)$. If $f \in \operatorname{Hom}_{\mathscr{C}^{\mathrm{op}}}(A,B)$ and $g \in \operatorname{Hom}_{\mathscr{C}^{\mathrm{op}}}(B,C)$, then composition is given by $g \circ_{\mathscr{C}^{\mathrm{op}}} f = f \circ_{\mathscr{C}} g$.

[Wei94, A.1.7]
[Hoc17, p. 11]

Example 1.1.2. The category Sets of sets, with functions as morphisms. [Hoc17, pp. 9–10]

[Wei94, Ex. A.1.3]
[Bor94b, Ex. 1.4.6.a]

Example 1.1.3. Let R be a ring. The class of R-modules together with R-module homomorphisms forms a category $\mathsf{Mod}(R)$. When $R = \mathbf{Z}$, the category $\mathsf{Mod}(\mathbf{Z})$ coincides with the category Ab of Abelian groups with group homomorphisms.

1.1.2. Functors. The utility of categories is really in relationships between them, given by functors.

[Wei94, §A.2]
[Bor94a, Def. 1.2.2]
[AK21, (6.2)]
[Hoc17, p. 11]

Definition 1.1.4. Given two categories $\mathscr{C}$ and $\mathscr{D}$, a *(covariant) functor* $F\colon \mathscr{C} \to \mathscr{D}$ is a rule that assigns to each object A of $\mathscr{C}$ an object $F(A)$ of $\mathscr{D}$, and assigns to each morphism $f\colon A \to B$ in $\mathscr{C}$ a morphism $F(f)\colon F(A) \to F(B)$ in $\mathscr{D}$, such that

1. For all objects A in $\mathscr{C}$, we have $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.
2. For all morphisms $f\colon A \to B$ and $g\colon B \to C$ in $\mathscr{C}$, we have $F(g \circ f) = F(g) \circ F(f)$.

Note that a functor preserves isomorphisms.

[Wei94, A.2.5]
[Bor94a, Def. 1.4.1]

A *contravariant functor* from $\mathscr{C}$ to $\mathscr{D}$ is a covariant functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}$. We will say "let $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}$ be a contravariant functor."

[Hoc17, pp. 11–12]

Example 1.1.5. Here are some examples of functors.

1. Given any category $\mathscr{C}$, there is an identity functor $\mathrm{id}_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{C}$ that sends objects A to A itself and morphisms f to f itself.
2. [Wei94, A.2.2] [Bor94a, Ex. 1.2.8.a] Given a category $\mathscr{C}$ whose objects have underlying sets and where composition coincides with composition of functions, there is a *forgetful functor* $\mathsf{Forget}\colon \mathscr{C} \to \mathsf{Sets}$ sending objects to their underlying sets, and morphisms to their underlying functions.
3. [Wei94, A.2.3] [Bor94a, Def. 1.5.4] A category $\mathscr{C}$ is a *full subcategory* of another category $\mathscr{D}$ if the objects of $\mathscr{C}$ form a subclass of objects in $\mathscr{D}$, and if $\mathrm{Hom}_{\mathscr{C}}(A,B) = \mathrm{Hom}_{\mathscr{D}}(A,B)$ for every pair of objects A and B in $\mathscr{C}$. For example, finite sets form a full subcategory of Sets, Ab is a full subcategory of Grp.
4. [Mur24ag, Cor. 2.2.19] The spectrum $\mathrm{Spec}(R)$ defines a contravariant functor
$$\mathrm{Spec}\colon \mathsf{Rings}^{\mathrm{op}} \longrightarrow \mathsf{LRS}$$
to the category of locally ringed spaces.
5. Here is a non-example: Mimicking the definition of $\mathrm{Spec}(\varphi)$ for the maximal spectrum MaxSpec does not define a functor $\mathsf{Rings}^{\mathrm{op}} \to \mathsf{LRS}$ (where $\mathrm{MaxSpec}(R)$ is given the subspace topology and the structure sheaf coming from restriction), or even to Sets, since the inverse image of a maximal ideal is not always maximal. One example of this is $\mathbf{Z} \to \mathbf{Q}$, where the maximal ideal $(0) \subseteq \mathbf{Q}$ has inverse image $(0) \subseteq \mathbf{Z}$, which is not maximal. The same thing occurs for $k[x] \to k(x)$.

1.1.3. Natural transformations and equivalences of categories. We also define natural transformations and isomorphisms of functors.

[Wei94, §A.3]
[Bor94a, Def. 1.3.1]
[Hoc17, pp. 13–15]

Definition 1.1.6. Let $F, G\colon \mathscr{C} \to \mathscr{D}$ be two functors. A *natural transformation* $T\colon F \Rightarrow G$ assigns to every object X in $\mathscr{C}$ a morphism $T_X\colon F(X) \to G(X)$ such that for all morphisms $f\colon X \to Y$ in $\mathscr{C}$, there is a commutative diagram

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "T_X"'] & F(Y) \arrow[d, "T_Y"] \\
G(X) \arrow[r, "G(f)"] & G(Y).
\end{tikzcd}$$

Natural transformations $S\colon F \Rightarrow G$ and $T\colon G \Rightarrow H$ can be composed to form the natural transformation $T \star S\colon F \Rightarrow H$ given by the rule

$$(T \star S)_X \coloneqq T_X \circ S_X.$$

There is an identity natural transformation id_F from $F\colon \mathscr{C} \to \mathscr{D}$ to itself. Two functors $F, G\colon \mathscr{C} \to \mathscr{D}$ are *isomorphic* if there are natural transformations

$$T\colon F \Longrightarrow G \qquad \text{and} \qquad T'\colon G \Longrightarrow F$$

such that $T' \star T = \mathrm{id}_F$ and $T \star T' = \mathrm{id}_G$. In fact, T is an isomorphism if and only if the morphisms T_X are isomorphisms for all objects X, in which case

$$(T_X^{-1}) = (T_X)^{-1}.$$

Using isomorphisms of functors, we can define equivalences of categories.

Definition 1.1.7. Two categories $\mathscr{C}$ and $\mathscr{D}$ are *equivalent* if there exist functors $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ such that $G \circ F$ is isomorphic to the identity functor on $\mathscr{C}$ and $F \circ G$ is isomorphic to the identity functor on $\mathscr{D}$. Two categories $\mathscr{C}$ and $\mathscr{D}$ are *antiequivalent* if $\mathscr{C}^{\mathrm{op}}$ is equivalent to $\mathscr{D}$.

[Wei94, A.3.2]
[Bor94a, Def. 3.4.4]
[Hoc17, p. 15]

1.2. Sheaves

We review the notion of a sheaf. These provide a systematic way to keep track of local algebraic data on a topological space. For example, for a quasi-projective variety X, the structure sheaf $\mathcal{O}_X$ keeps track of the regular functions that are defined on an open subset U.

1.2.1. Presheaves. We start with presheaves.

Definition 1.2.1. Let X be a topological space. We can consider the category $\mathsf{Top}(X)$ whose objects are the open subsets of X, and whose morphisms are the inclusion maps. Let $\mathscr{C}$ be a category. A *presheaf $\mathscr{F}$ on X with values in $\mathscr{C}$* is a contravariant functor

$$\mathscr{F}\colon \mathsf{Top}(X)^{\mathrm{op}} \longrightarrow \mathscr{C}.$$

If $\mathscr{C}$ is the category of Abelian groups, rings, or sets, then we say that $\mathscr{F}$ is a presheaf of Abelian groups, rings, or sets. For Abelian groups, we sometimes say *Abelian presheaf*.

[Mur24ag, Def. 2.1.1]
[Har77, p. 61]
[God73, I.1.9]

[TohokuI, p. 154]

We spell out what a presheaf $\mathscr{F}$ of sets (resp. Abelian groups) is. A presheaf $\mathscr{F}$ of sets (resp. Abelian groups) consists of the following data:

(*a*) For every open subset $U \subseteq X$, a set (resp. an Abelian group) $\mathscr{F}(U)$.
(*b*) For every inclusion $V \subseteq U$ of open subsets of X, a function (resp. a group homomorphism) $\rho^U_V\colon \mathscr{F}(U) \to \mathscr{F}(V)$. We call these *restriction maps*.

These data are subject to the following conditions:

(1) ρ^U_U is the identity map $\mathscr{F}(U) \to \mathscr{F}(U)$.
(2) If $W \subseteq V \subseteq U$ are three open subsets, then $\rho^U_W = \rho^V_W \circ \rho^U_V$.

[Har77, p. 61] has the extra condition that $\mathscr{F}(\emptyset) = 0$. This is not standard.

Definition 1.2.2. If $\mathscr{F}$ is a presheaf on X, we refer to $\mathscr{F}(U)$ as the *sections* over the open set U. We sometimes use the notation

$$\Gamma(U, \mathscr{F}) \coloneqq \mathscr{F}(U).$$

[Mur24ag, Def. 2.1.3]
[Har77, p. 61]
[God73, I.1.9]

If $V \subseteq U$ is an inclusion of open sets, we sometimes write $s|_V$ instead of $\rho^U_V(s)$ for $s \in \mathscr{F}(U)$.

1.2.2. Sheaves. We can now define sheaves as a kind of presheaf that is determined by local data.

[Mur24ag, Def. 2.1.4]
[Har77, p. 61]
[God73, II.1.1]

Definition 1.2.3. Let X be a topological space and let $\mathscr{F}$ be a presheaf on X with values in sets or a category $\mathscr{C}$ like Abelian groups or rings. We say that $\mathscr{F}$ is a *sheaf with values in* $\mathscr{C}$ if it satisfies the following conditions:

(3) Let $\{U_i\}_{i\in I}$ be a family of open subsets of X with union U. Let $s', s'' \in \mathscr{F}(U)$. If $s'|_{U_i} = s''|_{U_i}$ for every i, then $s' = s''$.

(4) Let $\{V_i\}_{i\in I}$ be a family of open subsets of X with union V. Let $s_i \in \mathscr{F}(V_i)$ be elements such that for all $i, j \in I$, we have

$$s_i|_{V_i\cap V_j} = s_j|_{V_i\cap V_j}.$$

Then, there exists $s \in \mathscr{F}(V)$ such that $s|_{V_i} = s_i$ for all $i \in I$. (Note (3) implies that s is unique.)

[Mur24ag, Rem. 2.1.5]
[EGAI$_{\text{new}}$, (**0**, 3.1.1)]

Remark 1.2.4 (The sheaf conditions in terms of an equalizer). A convenient way (that you do not need to know!) to package the definition of a sheaf is that a presheaf $\mathscr{F}$ of sets is a sheaf if the diagram

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r] & \displaystyle\prod_i \mathscr{F}(U_i) \arrow[r, shift left] \arrow[r, shift right] & \displaystyle\prod_{i,j} \mathscr{F}(U_i \cap U_j) \end{tikzcd}$$

is an equalizer diagram for every family of open sets $\{U_i\}_{i\in I}$. See [Mur24ag, Definition A.4.1] for the definition of an equalizer. This means that the elements of $\prod_i \mathscr{F}(U_i)$ that map to the same element in $\prod_{i,j} \mathscr{F}(U_i \cap U_j)$ are *exactly* the elements in the image of $\mathscr{F}(U)$. An Abelian presheaf is a sheaf if the sequence

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\prod_i \rho^U_{U_i}} \prod_i \mathscr{F}(U_i) \xrightarrow{\prod_{i,j}(\rho^{U_i}_{U_i\cap U_j} - \rho^{U_j}_{U_i\cap U_j})} \prod_{i,j} \mathscr{F}(U_i \cap U_j)$$

is exact for every family of open sets $\{U_i\}_{i\in I}$.

[Mur24ag, Ex. 2.1.7]
[Har77, Ex. II.1.0.1]

Example 1.2.5. Let X be a quasi-projective variety over an algebraically closed field k. For each open subset $U \subseteq X$, let $\mathcal{O}_X(U)$ be the ring of regular functions from U to k, and for each $V \subseteq U$, let

$$\rho^U_V \colon \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V)$$

be the restriction map (in the usual sense). Then, $\mathcal{O}_X$ is a presheaf of rings on X by definition of a regular function, and is a sheaf on X since a function that is locally 0 is 0, and a function which is regular locally is regular. We call $\mathcal{O}_X$ the *sheaf of regular functions* on X.

[Mur24ag, Ex. 2.1.8]
[Har77, Ex. II.1.0.2]

Example 1.2.6. Similarly, one can define the sheaf of continuous real-valued functions on any topological space, the sheaf of differentiable functions on a differentiable manifold, the sheaf of holomorphic functions on a complex manifold, etc.

[Mur24ag, Ex. 2.1.9]
[Har77, Ex. II.1.0.3]
[God73, II.1.4]
We use this notation instead of $\mathscr{A}$ since $\underline{\mathbf{Z}}$ is a useful example.

Example 1.2.7. Let X be a topological space and let A be an object in $\mathscr{C}$. The *constant sheaf* $\underline{A}_X$ on X determined by A is defined as follows. Consider A with the discrete topology. For any open set $U \subseteq X$, we set

$$\underline{A}_X(U) \coloneqq \left\{ \begin{matrix} \text{continuous maps} \\ U \to A \end{matrix} \right\}.$$

Together with the usual restriction maps, we obtain a sheaf $\underline{A}_X$. We sometimes drop the subscript X.

If U is an open set whose connected components are open (which is always true if X is locally connected), then $\underline{A}_X(U)$ is a direct product of copies of A, one for each connected component of U.

1.2.3. Stalks. The following definition gives the analogue of a germ of a function and the local rings of a quasi-projective variety for arbitrary sheaves.

[Mur24ag, Def. 2.1.10]
[Har77, p. 62]
[God73, II.1.2]

Definition 1.2.8. Let X be a topological space and let $\mathscr{F}$ be a presheaf on X with values in a cocomplete category (i.e., a category with arbitrary small colimits, in particular direct limits). Let $P \in X$ be a point. The *stalk* of $\mathscr{F}$ at P is

$$\mathscr{F}_P := \varinjlim_{U \ni P} \mathscr{F}(U)$$

where the transition maps in the direct system are the restriction maps. The elements of $\mathscr{F}_P$ are called *germs* of sections of $\mathscr{F}$ at the point P. If $s \in \mathscr{F}(U)$ is a section, then the germ of s at $P \in U$ is s_P.

Spelling out the definition of a direct limit, the elements of $\mathscr{F}_P$ are represented by pairs $\langle U, s\rangle$ where U is an open neighborhood of P and $s \in \mathscr{F}(U)$, subject to the equivalence relation that

$$\langle U, s\rangle \sim \langle V, t\rangle \iff \begin{array}{c}\text{there exists an open neighborhood}\\ W \subseteq U \cap V \text{ of } P \text{ such that } s|_W = t|_W.\end{array}$$

1.2.4. Morphisms. We now define morphisms of (pre)sheaves.

[Mur24ag, Def. 2.1.12]
[Har77, p. 62]
[God73, I.1.9]

Definition 1.2.9. Let X be a topological space. Let $\mathscr{F}, \mathscr{G}$ be two presheaves on X with values in $\mathscr{C}$. A *morphism* $\varphi\colon \mathscr{F} \to \mathscr{G}$ is a natural transformation $\mathscr{F} \Rightarrow \mathscr{G}$. More explicitly, φ consists of the data of a morphism $\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$ in $\mathscr{C}$ such that whenever $V \subseteq U$ is an inclusion, the diagram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho^U_V"'] & \mathscr{G}(U) \arrow[d, "\rho'^U_V"] \\
\mathscr{F}(V) \arrow[r, "\varphi(V)"] & \mathscr{G}(V)
\end{tikzcd}$$

commutes, where the vertical maps are the restriction maps for $\mathscr{F}$ and $\mathscr{G}$, respectively. We use the same definition for sheaves. An *isomorphism* is a morphism which has a two-sided inverse.

We therefore obtain the categories

$$\mathsf{Sh}(X) \hookrightarrow \mathsf{PSh}(X)$$

of sheaves and presheaves of sets on X, respectively, where the arrow denotes that $\mathsf{Sh}(X)$ forms a full subcategory of $\mathsf{PSh}(X)$. The analogues for Abelian (pre)sheaves are

$$\mathsf{Ab}(X) \hookrightarrow \mathsf{PAb}(X)$$

which are connected to the categories for sheaves via forgetful functors. Since these forgetful functors reflect isomorphisms, we can apply statements about sheaves of sets to Abelian sheaves.

To prove Proposition 1.2.11 below, we define the following:

[Stacks, Tag 006X] [God73, II.3.1] [Har77, Exer. II.1.16(e)]

Example 1.2.10. Let X be a topological space and let $\{A_P\}_{P\in X}$ be a family of sets indexed by X. Then, the assignment

$$U \longmapsto \prod_{P\in U} A_P$$

defines a sheaf Π.

By the definition of direct limits, a morphism $\varphi\colon \mathscr{F} \to \mathscr{G}$ of presheaves on X induces a morphism

$$\varphi_P \colon \mathscr{F}_P \longrightarrow \mathscr{G}_P$$

on stalks for any point $P \in X$. The following result illustrates the local nature of sheaves.

[Har77, Prop. II.1.1]

Proposition 1.2.11. *Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves of sets on a topological space X. If φ is an isomorphism, then the induced maps $\varphi_P \colon \mathscr{F}_P \to \mathscr{G}_P$ on stalks are isomorphisms for every $P \in X$. Conversely:*

- (a) *If φ_P is injective for every $P \in X$ and if $\mathscr{F}$ satisfies the sheaf property (3), then $\varphi(U)$ is injective for every open set $U \subseteq X$.*
- (b) *If φ_P is an isomorphism for every $P \in X$, the presheaf $\mathscr{F}$ is a sheaf, and $\mathscr{G}$ satisfies the sheaf property (3), then $\varphi(U)$ is an isomorphism for every open set $U \subseteq X$.*

As a consequence, if $\mathscr{F}, \mathscr{G}$ are sheaves and the maps φ_P are isomorphisms for every P, then φ is an isomorphism.

Proof. The direction $\Rightarrow$ follows from the construction of direct limits.

For the converse, denote by $\Pi(\mathscr{F})$ the sheaf in Example 1.2.10 obtained from the family $\{\mathscr{F}_P\}_{P\in X}$. We first show:

Claim 1.2.12. *If $\mathscr{F}$ satisfies the sheaf property (3), then the map*

$$\mathscr{F}(U) \longrightarrow \pi(\mathscr{F})(U)$$

is injective.

If $s, t \in \mathscr{F}(U)$ map to the same elements, then there is an open covering W_P of U for which $s|_{W_P} = t|_{W_P}$ for every P by the definition of germs. The sheaf property (3) then shows that $s = t$.

We now show (a). If φ_P is injective for every $\varphi(U)$, then the commutativity of the diagram

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, hook] & \mathscr{G}(U) \arrow[d] \\ \displaystyle\prod_{P\in U} \mathscr{F}_P \arrow[r, "\prod_P \varphi_P"] & \displaystyle\prod_{P\in U} \mathscr{G}_P. \end{tikzcd} \tag{1.2.13}$$

shows that the top horizontal map $\varphi(U)$ is injective. Here we use Claim 1.2.12 to show that the left vertical map is injective.

We now show (b). Suppose we have a section $t \in \mathscr{G}(U)$. Then, for every $P \in U$, there are an open neighborhood $V_P \ni P$ and a section $s(P) \in \mathscr{F}(V_P)$ such that

$$\varphi_P\bigl(s(P)_P\bigr) = t_P$$

for every $P \in U$. After possibly shrinking the V_P, we may assume that

$$\varphi\big(s(P)\big)|_{V_P} = t|_{V_P}$$

for every P. Now if P, Q are two points, then $s(P)|_{V_P \cap V_Q}$ and $s(Q)|_{V_P \cap V_Q}$ are two sections of $\mathscr{F}(V_P \cap V_Q)$, which are both sent by φ to $t|_{V_P \cap V_Q}$. By the injectivity shown in (*a*), this implies that these two sections $s(P)|_{V_P \cap V_Q}$ and $s(Q)|_{V_P \cap V_Q}$ are equal. By the sheaf property (4), we then see there is a section $s \in \mathscr{F}(U)$ such that $s|_{V_P} = s(P)$ for every P. Finally, we have $\varphi(U)(s) = t$ by the commutativity of (1.2.13) and Claim 1.2.12 applied to $\mathscr{G}$.

We can therefore define an inverse morphism $\varphi^{-1} \colon \mathscr{G} \to \mathscr{F}$, where compatibility with restriction maps holds by the fact that $\varphi^{-1}(U)$ is the inverse for $\varphi(U)$ on every open subset $U \subseteq X$. □

1.2.5. Espace étalé and sheafification. We now recall the *sheafification* construction, which turns a presheaf into a sheaf.

DEFINITION 1.2.14 (Espace étalé and sheafification). Let $\mathscr{F}$ be a presheaf of sets on X. We define a topological space $\operatorname{Spé}(\mathscr{F})$ called the *espace étalé* of $\mathscr{F}$ as follows.

[Har77, Exer. II.1.13, Prop.-Def. II.1.2] [God73, II.1.2]

(1) As a set,

$$\operatorname{Spé}(\mathscr{F}) \coloneqq \coprod_{P \in X} \mathscr{F}_P.$$

We define a projection map

$$\begin{aligned} \pi \colon \operatorname{Spé}(\mathscr{F}) &\longrightarrow X \\ \mathscr{F}_P \ni s &\longmapsto P. \end{aligned}$$

For each open set $U \subseteq X$ and each section $s \in \mathscr{F}(U)$, we obtain a map

$$\begin{aligned} \bar{s} \colon U &\longrightarrow \operatorname{Spé}(\mathscr{F}) \\ P &\longmapsto s_P \end{aligned}$$

which is a section of π over U, i.e., $\pi \circ \bar{s} = \mathrm{id}_U$. Note that for every inclusion $V \subseteq U$, we have

$$\bar{s}|_V = \overline{s|_V}.$$

(2) The topology on $\operatorname{Spé}(\mathscr{F})$ is the topology generated by all subsets of $\operatorname{Spé}(\mathscr{F})$ of the form $\bar{s}(U)$ for all $U \subseteq X$ open and all $s \in \mathscr{F}(U)$. Note that π is a continuous map.

[TohokuI, p. 154]

The espace étalé defines a *sheaf* $\mathscr{F}^{\#}$ where

$$\mathscr{F}^{\#}(U) \coloneqq \big\{\text{continuous sections } U \to \operatorname{Spé}(\mathscr{F})\big\}.$$

We obtain a morphism

$$\theta \colon \mathscr{F} \longrightarrow \mathscr{F}^{\#}$$

which we call the *sheafification* of $\mathscr{F}$.

The espace étalé description of a sheaf is from [SHC50/51, Exp. XIV] and predates the one given above. The notation for the sheafification in [Har77] is $\mathscr{F}^{+}$. We use the notation $\mathscr{F}^{\#}$ to match other sources, for example [Art62, §2.1].

We now have the following:

PROPOSITION 1.2.15. *Let X be a topological space and let $\mathscr{F}$ be a presheaf of sets on X.*

[Har77, Exer. II.1.13, Prop.-Def. II.1.2] [God73, II.1.2]

(*i*) *θ_P is bijective for every $P \in X$.*

(*ii*) *If $\mathscr{F}$ satisfies the sheaf property (3), then $\theta(U)$ is injective for every open set $U \subseteq X$.*

(*iii*) *If $\mathscr{F}$ is a sheaf, then θ is an isomorphism.*

[Stacks, Tag 0080]

(*iv*) *The sheafification is the left adjoint of the forgetful functor $\mathsf{Sh}(X) \hookrightarrow \mathsf{PSh}(X)$ or $\mathsf{Ab}(X) \hookrightarrow \mathsf{PAb}(X)$, i.e., for every sheaf $\mathscr{G}$ on X, there is a bijection*

$$\theta^*\colon \operatorname{Hom}(\mathscr{F}^\#, \mathscr{G}) \xrightarrow{\sim} \operatorname{Hom}(\mathscr{F}, \mathscr{G})$$

induced by precomposition by θ that is functorial in both $\mathscr{F}$ and $\mathscr{G}$.

Proof. (*i*) holds by definition of the stalk $\mathscr{F}_P$. (*ii*) and (*iii*) follow from (*i*) and Proposition 1.2.11.

For (*iv*), naturality in $\mathscr{G}$ holds by definition as precomposition by θ, and naturaliy in $\mathscr{F}$ holds by the fact that if $\mathscr{F}_1 \to \mathscr{F}_2$ is a morphism of presheaves, we have a corresponding morphism $\mathscr{F}_1^\# \to \mathscr{F}_2^\#$ of sheaves.

We want to show that θ^* is bijective. This follows from the commutative diagram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\theta_{\mathscr{F}}(U)"] \arrow[d] & \mathscr{F}^\#(U) \arrow[r, hook] \arrow[d] & \prod_{P\in U} \mathscr{F}_P \arrow[d] \\
\mathscr{G}(U) \arrow[r, "\theta_{\mathscr{G}}(U)", "\sim"'] & \mathscr{G}^\#(U) \arrow[r, hook] & \prod_{P\in U} \mathscr{G}_P
\end{tikzcd}$$

where $\theta_{\mathscr{G}}(U)$ is an isomorphism by (*iii*): There exists an induced map $\mathscr{F}^\#(U) \to \mathscr{G}(U)$ using that $\theta_{\mathscr{G}}(U)$ is an isomorphism, and this map is unique since the horizontal maps on the right are injective by Proposition 1.2.11. ☐

1.3. The category of sheaves of modules has enough injectives

We now discuss sheaves of modules. These include sheaves of Abelian groups as a special case, and will be the category of sheaves that we will work with for much of the first part of this course.

1.3.1. The category of sheaves of modules.

[Mur24ag, Def. 2.1.29]
[God73, II.2.1, II.2.2]
[Har77, pp. 64–65, p. 109]

Definition 1.3.1. Let X be a topological space. Let $\mathcal{O}$ be a sheaf of rings on X. A *sheaf $\mathscr{F}$ of $\mathcal{O}$-modules*, or an *$\mathcal{O}$-module* for short, is a sheaf of Abelian groups such that every $\mathscr{F}(U)$ is a $\mathcal{O}(U)$-module and the restriction maps $\rho^U_V\colon \mathscr{F}(U) \to \mathscr{F}(V)$ are maps of modules compatible with the ring maps $\rho^U_V\colon \mathcal{O}(U) \to \mathcal{O}(V)$. A *morphism* $\varphi\colon \mathscr{F} \to \mathscr{G}$ of $\mathcal{O}$-modules is a morphism of sheaves of sets such that each $\mathscr{F}(U) \to \mathscr{G}(U)$ is a morphism of $\mathcal{O}(U)$-modules. This forms the category $\mathsf{Mod}(\mathcal{O})$.

A pair $(X, \mathcal{O})$ where $\mathcal{O}$ is a sheaf of rings on X is called a *ringed space*. In this situation, we often denote $\mathcal{O}$ by $\mathcal{O}_X$ and call it the *structure sheaf*.

[Mur24ag, Ex. 2.1.30]

Example 1.3.2. The 0 sheaf is a sheaf of $\mathcal{O}$-modules. By properties of categories of modules, we know that there are always unique morphisms $0 \to \mathscr{F}$ and $\mathscr{F} \to 0$, i.e., the 0 sheaf is a *0 object* in the category $\mathsf{Mod}(\mathcal{O})$.

[Mur24ag, Ex. 2.1.31]

Example 1.3.3. If $\mathcal{O} = \underline{\mathbf{Z}}_X$, then $\mathsf{Mod}(\underline{\mathbf{Z}}_X) = \mathsf{Ab}(X)$.

A very important fact about the category of sheaves of modules is that it "behaves similarly" to the category of modules over a ring. The precise version of this statement is that $\mathsf{Mod}(\mathcal{O}_X)$ is an *Abelian category*. For the next few subsections, we define the necessary terminology from category theory to make sense of this notion.

1.3.2. (Pre)additive categories. To define (pre)additive categories, we need to define (co)products.

Definition 1.3.4. Let $\mathscr{C}$ be a category. Let $\{X_i\}$ be a set of objects in $\mathscr{C}$. [Wei94, A.1.9]

(1) The *product* of the $\{X_i\}$ is an object $\prod_i X_i$ together with projection maps $\pi_i \colon \prod_i X_i \to X_i$ such that for every object S with maps $f_i \colon S \to X_i$, there is a unique dashed morphism making the diagram [Bor94a, Def. 2.1.1]

$$\begin{tikzcd}
 & & X_i \\
S \arrow[rru, bend left, "f_i"] \arrow[r, dashed, "\exists!"] \arrow[rrd, bend right, "f_j"'] & \prod\limits_i X_i \arrow[ru, "\pi_i"'] \arrow[rd, "\pi_j"] & \\
 & & X_j
\end{tikzcd}$$

commute for all i, j. If $\{X_i\}$ is indexed over the natural numbers, we also write $X_1 \times X_2 \times \cdots$ for $\prod_i X_i$, especially for finite products.

(2) The *coproduct* of the $\{X_i\}$ is an object $\coprod_i X_i$ together with inclusion maps $\iota_i \colon X_i \to \coprod_i X_i$ such that for every object T with maps $g_i \colon X_i \to T$, there is a unique dashed morphism making the diagram [Bor94a, Def. 2.2.1]

$$\begin{tikzcd}
X_i \arrow[rrd, bend left, "g_i"] \arrow[rd, "\iota_i"'] & & \\
 & \coprod\limits_i X_i \arrow[r, dashed, "\exists!"] & T \\
X_j \arrow[ru, "\iota_j"] \arrow[rru, bend right, "g_j"'] & &
\end{tikzcd}$$

commute for all i, j. If $\{X_i\}$ is indexed over the natural numbers, we also write $X_1 \oplus X_2 \oplus \cdots$ for $\coprod_i X_i$, especially for finite coproducts.

Definition 1.3.5. Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ is an *preadditive category* if the following condition holds. [Bor94b, Defs. 1.2.1, Def. 1.2.6]

(1) For all objects X, Y, Z in $\mathscr{C}$, the hom sets $\operatorname{Hom}_{\mathscr{C}}(X, Y)$ are equipped with the structure of an Abelian group such that the composition maps

$$\operatorname{Hom}_{\mathscr{C}}(X, Y) \times \operatorname{Hom}_{\mathscr{C}}(Y, Z) \longrightarrow \operatorname{Hom}_{\mathscr{C}}(X, Z)$$

are bilinear.

We say that $\mathscr{C}$ is an *additive category* if it is preadditive and the following condition holds. [TohokuI, §1.3] [Wei94, A.4.1]

(2) Finite products (including the empty product!) of objects in $\mathscr{C}$ exist.

The empty product is called the *zero object* in $\mathscr{C}$ and is denoted 0. On Homework 11 last semester, you showed that this zero object is also the empty coproduct, and is simultaneously the initial object and the final object of the category.

1.3.3. Abelian categories. To define Abelian categories, we need to define kernels, cokernels, coimages, and images.

[TohokuI, §1.3]
[Wei94, A.1.6]
[Bor94b, Def. 1.1.5]

Definition 1.3.6. Let $\mathscr{C}$ be a preadditive category. Let $f\colon X \to Y$ be a morphism.

(1) A *kernel* for f is an object $\ker(f)$ together with a morphism $i\colon \ker(f) \to X$ such that $f \circ i = 0$ and such that for every commutative diagram

$$\begin{tikzcd}
\ker(f) \arrow[dr, "i"'] \arrow[drr, bend left, "0"] & & \\
& X \arrow[r, "f"] & Y \\
S \arrow[uu, dashed, "\exists!"] \arrow[ur] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

there is a unique dashed morphism making the diagram commute.

(2) A *cokernel* for f is an object $\operatorname{coker}(f)$ together with a morphism $p\colon Y \to \operatorname{coker}(f)$ such that $p \circ f = 0$ and such that for every commutative diagram

$$\begin{tikzcd}
& & \operatorname{coker}(f) \arrow[dd, dashed, "\exists!"] \\
X \arrow[r, "f"] \arrow[urr, bend left, "0"] \arrow[drr, bend right, "0"'] & Y \arrow[ur, "p"'] \arrow[dr] & \\
& & T
\end{tikzcd}$$

there is a unique dashed morphism making the diagram commute.

(3) If a kernel and cokernel for f exist, then a *coimage* for f is a cokernel for $\ker(f) \to X$, and is denoted $X \to \operatorname{coim}(f)$.

(4) If a kernel and cokernel for f exist, then an *image* for f is a kernel for $Y \to \operatorname{coker}(f)$, and is denoted $\operatorname{im}(f) \to Y$.

We can finally define Abelian categories. The definition below is due to Grothendieck [TohokuI] but an equivalent definition due to Buchsbaum [Buc55] appeared earlier.

[Buc55, Pt. I, §1]
[Har77, p. 202]
[Wei94, Def. A.4.2]
[Bor94b, Def. 1.4.1]

Definition 1.3.7 [TohokuI, §1.4]. Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ is an *Abelian category* if it is additive and the following conditions hold.

(AB1) All kernels and cokernels of morphisms in $\mathscr{C}$ exist.

(AB2) For every morphism f in $\mathscr{C}$, the morphism

$$\operatorname{coim}(f) \longrightarrow \operatorname{im}(f)$$

induced by the universal property is an isomorphism.

[TohokuI, p. 128]
[Wei94, p. 426]
[Bor94b, Def. 1.8.1]
[God73, II.2.5]

We say that a sequence

$$\cdots \longrightarrow X^{i-1} \xrightarrow{f^{i-1}} X^i \xrightarrow{f^i} X^{i+1} \longrightarrow \cdots$$

in an Abelian category is *exact* if $\ker(f^i) = \operatorname{im}(f^{i-1})$ for every i.

[Wei94, Ex. A.4.4]

Example 1.3.8. Here are some examples of Abelian categories.

[Bor94b, Ex. 1.4.6.a]

(1) The category Ab of Abelian groups.

[Har77, Ex. III.1.0.1]

(2) The category $\mathsf{Mod}(R)$ of modules over a ring R.

[Har77, Ex. III.1.0.2]

1.3.4. Sheaves of modules form an Abelian category. For this class, the following example of an Abelian category is (one of) the most important.

THEOREM 1.3.9 [TohokuI, Prop. 3.1.1]. *Let* $(X, \mathcal{O}_X)$ *be a ringed space. The category* $\mathsf{Mod}(\mathcal{O}_X)$ *of* $\mathcal{O}_X$*-modules is Abelian.* [Har77, Ex. III.1.0.3]

We break up this theorem into parts. The following is a worked out solution to a problem appearing on Homework 11 from last semester. We denote by $\varphi\colon \mathscr{F} \to \mathscr{G}$ a morphism in $\mathsf{Mod}(\mathcal{O}_X)$.

CLAIM 1.3.10. *The kernel presheaf* $\ker(\varphi) := \{U \mapsto \ker(\varphi(U))\}$ *is a sheaf and satisfies* $\ker(\varphi_P) \cong \ker(\varphi)_P$ *for every* $P \in X$*. Moreover,* $\ker(\varphi)$ *satisfies the universal property of the kernel.* [Har77, Exer. II.1.2(*a*)]

Proof. We first show that the kernel presheaf is a sheaf. Suppose that $U = \bigcup_i V_i$ is an open covering of an open set U. For sheaf condition (3), let $s, t \in \ker(\varphi(U))$ such that $s|_{V_i} = t|_{V_i}$ for all i. Considering s, t as elements of $\mathscr{F}(U)$, we see that $s = t$ in $\mathscr{F}(U)$ by the sheaf condition (3) for $\mathscr{F}$, and hence $s = t$ in $\ker(\varphi(U))$. For sheaf condition (4), let $s_i \in \ker(\varphi(V_i))$ be such that

$$s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$$

for all i, j. Considering the s_i as elements of $\mathscr{F}(V_i)$, we see that there exists an element $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$ in $\mathscr{F}(V_i)$ for all i by the sheaf condition (4) for $\mathscr{F}$. It remains to show that $s \in \ker(\varphi(U))$. Consider the commutative diagram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d] & \mathscr{G}(U) \arrow[d] \\
\mathscr{F}(V_i) \arrow[r, "\varphi(V_i)"] & \mathscr{G}(V_i)
\end{tikzcd}$$

where the vertical maps are the restriction maps. For each i, we see that

$$\varphi(U)(s)|_{V_i} = \varphi(V_i)(s_i) = 0$$

by the commutativity of the diagram. By the sheaf condition (3) for $\mathscr{G}$, we see that $\varphi(U)(s) = 0$.

We now show that $\ker(\varphi_P) \cong \ker(\varphi)_P$ for every $P \in X$. For every open set U, we have the exact sequence

$$0 \longrightarrow \ker\bigl(\varphi(U)\bigr) \longrightarrow \mathscr{F}(U) \xrightarrow{\varphi(U)} \mathscr{G}(U).$$

These exact sequences fit into the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker\bigl(\varphi(U)\bigr) \arrow[r] \arrow[d] & \mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d] & \mathscr{G}(U) \arrow[d] \\
0 \arrow[r] & \ker\bigl(\varphi(V)\bigr) \arrow[r] & \mathscr{F}(V) \arrow[r, "\varphi(V)"] & \mathscr{G}(V)
\end{tikzcd}$$

for all inclusions of open sets $V \subseteq U$, where the vertical maps are the restriction maps. Taking direct limits over all open sets containing P, we have the exact sequence

$$0 \longrightarrow \ker(\varphi)_P \longrightarrow \mathscr{F}_P \xrightarrow{\varphi_P} \mathscr{G}_P$$

since filtered direct limits are exact [Mur24ag, Theorem A.6.1]. By the universal property of kernels in the category of $\mathcal{O}_{X,P}$-modules, we therefore have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker(\varphi)_P \arrow[r] \arrow[d, dashed, "\exists!"'] & \mathscr{F}_P \arrow[r, "\varphi_P"] \arrow[d, equal] & \mathscr{G}_P \arrow[d, equal] \\
0 \arrow[r] & \ker(\varphi_P) \arrow[r] & \mathscr{F}_P \arrow[r, "\varphi_P"] & \mathscr{G}_P
\end{tikzcd}$$

with exact rows. Since kernels of isomorphic maps of modules are isomorphic, we see that $\ker(\varphi)_P \to \ker(\varphi_P)$ is an isomorphism.

We now show that $\ker(\varphi)$ satisfies the universal property of the kernel. Let $i\colon \ker(\varphi) \to \mathscr{F}$ be the morphism defined by the inclusions $\ker(\varphi(U)) \subseteq \mathscr{F}(U)$ on every open subset U. We then see that $\varphi \circ i = 0$ since $(\varphi \circ i)(U) = 0$ for every open subset U. Now consider the commutative diagram

$$\begin{tikzcd}
\ker(\varphi) \arrow[dr, "i"'] \arrow[drr, bend left, "0"] & & \\
& \mathscr{F} \arrow[r, "\varphi"] & \mathscr{G}. \\
\mathscr{S} \arrow[ur] \arrow[urr, bend right, "0"'] \arrow[uu, dashed, "\psi", "\exists!"'] & &
\end{tikzcd}$$

We need to show that for every $\mathscr{S}$ fitting into this commutative diagram, there is a unique map $\mathscr{S} \to \ker(\varphi)$ fitting into this diagram making the diagram commute. On every open subset U, we have the commutative diagram

$$\begin{tikzcd}
\ker\bigl(\varphi(U)\bigr) \arrow[dr, "i(U)"'] \arrow[drr, bend left, "0"] & & \\
& \mathscr{F}(U) \arrow[r, "\varphi(U)"] & \mathscr{G}(U). \\
\mathscr{S}(U) \arrow[ur] \arrow[urr, bend right, "0"'] \arrow[uu, dashed, "\psi(U)", "\exists!"'] & &
\end{tikzcd}$$

By the universal property of kernels in the category of $\mathcal{O}_X(U)$-modules, there is a unique map $\psi(U)\colon \mathscr{S}(U) \to \ker(\varphi(U))$ making the diagram commute. We then consider the diagram

$$\begin{tikzcd}
 & \ker\big(\varphi(U)\big) \arrow[dr, "i(U)"] \arrow[drr, bend left, "0"] \arrow[ddd] & & \\
 & & \mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[ddd] & \mathscr{G}(U) \arrow[ddd] \\
\mathscr{S}(U) \arrow[uur, "\psi(U)"] \arrow[urr] \arrow[urrr, bend right, "0"'] \arrow[ddd] \arrow[dr, dashed, red] & & & \\
 & \ker\big(\varphi(V)\big) \arrow[dr, "i(V)"] \arrow[drr, bend left, "0"] & & \\
 & & \mathscr{F}(V) \arrow[r, "\varphi(V)"] & \mathscr{G}(V) \\
\mathscr{S}(V) \arrow[uur, "\psi(V)"] \arrow[urr] \arrow[urrr, bend right, "0"'] & & &
\end{tikzcd}$$

where the vertical maps are the restriction maps and where every triangle or square commutes except possibly for the leftmost square face involving $\ker(\varphi)$ and $\mathscr{S}$. We need to show that the compositions along this leftmost square face from $\mathscr{S}(U)$ to $\ker(\varphi(V))$ are equal. It suffices to note that the composition $\mathscr{S}(U) \to \mathscr{G}(V)$ is the 0 map, and hence the universal property of $\ker(\varphi(V))$ implies that there is a unique map $\mathscr{S}(U) \to \ker(\varphi(V))$ shown as a dashed red arrow in the diagram above compatible with the map $\mathscr{S}(U) \to \mathscr{F}(V)$. We therefore see that the leftmost square face involving $\ker(\varphi)$ and $\mathscr{S}$ commutes. ☐

Claim 1.3.11. *Let $\mathscr{F}'$ be a sub-$\mathcal{O}_X$-module of $\mathscr{F}$. Then, the natural morphism $\mathscr{F} \to \mathscr{F}/\mathscr{F}'$ is surjective with kernel $\mathscr{F}'$, and hence there is a short exact sequence*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}/\mathscr{F}' \longrightarrow 0.$$

[Har77, Exer. II.1.6(a)]

Proof. Denote by $\mathscr{G}$ the quotient presheaf $U \mapsto \mathscr{F}(U)/\mathscr{F}'(U)$. For every inclusion $V \subseteq U$ of open sets, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}'(U) \arrow[r] \arrow[d] & \mathscr{F}(U) \arrow[r] \arrow[d] & \mathscr{G}(U) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}'(V) \arrow[r] & \mathscr{F}(V) \arrow[r] & \mathscr{G}(V) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Taking direct limits over all open sets containing P, we obtain the short exact sequence

$$0 \longrightarrow \mathscr{F}'_P \longrightarrow \mathscr{F}_P \longrightarrow \mathscr{G}_P \longrightarrow 0$$

since filtered direct limits are exact [Mur24ag, Theorem A.6.1].

We now consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}' \arrow[r] \arrow[d, equal] & \mathscr{F} \arrow[r] \arrow[d, equal] & \mathscr{G} \arrow[r] \arrow[d, "\theta"] & 0 \\
0 \arrow[r] & \mathscr{F}' \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{F}/\mathscr{F}' \arrow[r] & 0.
\end{tikzcd}$$

Taking stalks, we obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}'_P \arrow[r] \arrow[d, equal] & \mathscr{F}_P \arrow[r] \arrow[d, equal] & \mathscr{G}_P \arrow[r] \arrow[d, "\theta", "\wr"'] & 0 \\
0 \arrow[r] & \mathscr{F}'_P \arrow[r] & \mathscr{F}_P \arrow[r] & (\mathscr{F}/\mathscr{F}')_P \arrow[r] & 0
\end{tikzcd}$$

where the top row is exact and the right vertical map is an isomorphism since sheafification induces an isomorphism on stalks. We therefore see that the bottom row is exact, and hence the sequence

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}/\mathscr{F}' \longrightarrow 0.$$

is exact. Exactness at $\mathscr{F}/\mathscr{F}'$ says that $\pi\colon \mathscr{F} \to \mathscr{F}/\mathscr{F}'$ is surjective. To show that $\mathscr{F}'$ is the kernel of π, consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}' \arrow[r] \arrow[d, dashed, "\exists!"] & \mathscr{F} \arrow[r] \arrow[d, equal] & \mathscr{F}/\mathscr{F}' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker(\pi) \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{F}/\mathscr{F}' \arrow[r] & 0
\end{tikzcd}$$

where the top row is exact and the left dashed map is induced by the universal property of kernels from Claim 1.3.10. The bottom row is also exact by the fact that $\ker(\pi)_P \to \ker(\pi_P)$ is an isomorphism by Claim 1.3.10. Finally, since on every U, kernels of isomorphic maps of $\mathcal{O}_X(U)$-modules are isomorphic, we see that the map $\mathscr{F}' \to \ker(\pi)$ is an isomorphism. □

[Har77, Exer. II.1.7(a)]

Claim 1.3.12 (The first isomorphism theorem in $\mathsf{Mod}(\mathcal{O}_X)$). *We have*

$$\operatorname{im}(\varphi) \cong \mathscr{F}/\ker(\varphi).$$

Soluion. Recall that $\operatorname{im}(\varphi)$ is defined to be the sheafification of the presheaf $U \mapsto \operatorname{im}(\varphi(U))$. For every inclusion $V \subseteq U$ of open sets, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker\bigl(\varphi(U)\bigr) \arrow[r] \arrow[d] & \mathscr{F}(U) \arrow[r] \arrow[d] & \operatorname{im}\bigl(\varphi(U)\bigr) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker\bigl(\varphi(V)\bigr) \arrow[r] & \mathscr{F}(V) \arrow[r] & \operatorname{im}\bigl(\varphi(V)\bigr) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Taking direct limits over all open sets containing a point $P \in X$, we obtain the top exact sequence in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker(\varphi)_P \arrow[r] \arrow[d, equal] & \mathscr{F}_P \arrow[r] \arrow[d, equal] & \varinjlim_{U \ni P} \operatorname{im}\bigl(\varphi(U)\bigr) \arrow[r] \arrow[d, "\theta_P", "\wr"'] & 0 \\
0 \arrow[r] & \ker(\varphi)_P \arrow[r] & \mathscr{F}_P \arrow[r] & \operatorname{im}(\varphi)_P \arrow[r] & 0
\end{tikzcd}$$

since filtered direct limits are exact. The right vertical map is an isomorphism since sheafification induces an isomorphism on stalks. Thus, the bottom row is exact for every $P \in X$, and hence

$$0 \longrightarrow \ker(\varphi) \longrightarrow \mathscr{F} \longrightarrow \operatorname{im}(\varphi) \longrightarrow 0$$

is exact. We therefore see that $\operatorname{im}(\varphi) \cong \mathscr{F}/\ker(\varphi)$ by Claim 1.3.11. □

[Har77, Exer. II.1.7(b)]

Claim 1.3.13. *We have* $\operatorname{coker}(\varphi) \cong \mathscr{G}/\operatorname{im}(\varphi)$*. Moreover, the cokernel sheaf* $\operatorname{coker}(\varphi)$ *satisfies the universal property of the cokernel.*

Proof. Recall that $\operatorname{coker}(\varphi)$ is the sheafification of the presheaf $\operatorname{coker}_{\mathcal{P}}(\varphi)$ defined by $U \mapsto \operatorname{coker}(\varphi(U))$. For every inclusion $V \subseteq U$ of open sets, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im}\big(\varphi(U)\big) \arrow[r] \arrow[d] & \mathscr{G}(U) \arrow[r] \arrow[d] & \operatorname{coker}\big(\varphi(U)\big) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{im}\big(\varphi(V)\big) \arrow[r] & \mathscr{G}(V) \arrow[r] & \operatorname{coker}\big(\varphi(V)\big) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Taking direct limits over all open sets containing a point $P \in X$, we obtain the top exact sequence in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim_{U \ni P} \operatorname{im}\big(\varphi(U)\big) \arrow[r] \arrow[d, "\theta", "\wr"'] & \mathscr{G}_P \arrow[r] \arrow[d, equal] & \varinjlim_{U \ni P} \operatorname{coker}\big(\varphi(U)\big) \arrow[r] \arrow[d, "\theta", "\wr"'] & 0 \\
0 \arrow[r] & \operatorname{im}(\varphi)_P \arrow[r] & \mathscr{G}_P \arrow[r] & \operatorname{coker}(\varphi)_P \arrow[r] & 0
\end{tikzcd}$$

since filtered direct limits are exact. The outer vertical maps are isomorphisms since sheafification induces isomorphisms on stalks. We therefore see that the bottom row is exact for every $P \in X$, and hence

$$0 \longrightarrow \operatorname{im}(\varphi) \longrightarrow \mathscr{G} \longrightarrow \operatorname{coker}(\varphi) \longrightarrow 0$$

is exact. We therefore see that $\operatorname{coker}(\varphi) \cong \mathscr{G}/\operatorname{im}(\varphi)$ by Claim 1.3.11.

We now show that $\operatorname{coker}(\varphi)$ satisfies the universal property of the cokernel. Let $p\colon \mathscr{G} \to \operatorname{coker}(\varphi)$ be the morphism defined by taking the sheafification of the map defined by $\mathscr{G}(U) \to \operatorname{coker}(\varphi(U))$ on presheaves. We then see that $p \circ \varphi = 0$ since $(p \circ \varphi)(U) = 0$ for every open subset U. Now consider the commutative diagram

$$\begin{tikzcd}
& & \operatorname{coker}(\varphi) \arrow[d, dashed, "\exists!"', "\psi"] \\
\mathscr{F} \arrow[r, "\varphi"] \arrow[urr, bend left, "0"] \arrow[drr, bend right, "0"'] & \mathscr{G} \arrow[ur, "p"'] \arrow[dr] & \\
& & \mathscr{T}.
\end{tikzcd} \tag{1.3.14}$$

We need to show that for every $\mathscr{T}$ fitting into this commutative diagram, there is a unique map $\operatorname{coker}(\varphi) \to \mathscr{T}$ fitting into this diagram making the diagram commute. On every open subset U, we have the commutative diagram

$$\begin{tikzcd}
& & \operatorname{coker}\big(\varphi(U)\big) \arrow[d, dashed, "\exists!"', "\psi(U)"] \\
\mathscr{F}(U) \arrow[r, "\varphi"] \arrow[urr, bend left, "0"] \arrow[drr, bend right, "0"'] & \mathscr{G}(U) \arrow[ur, "p"'] \arrow[dr] & \\
& & \mathscr{T}(U).
\end{tikzcd}$$

By the universal property of cokernels in the category of $\mathcal{O}_X(U)$-modules, there is a unique map $\psi(U)\colon \operatorname{coker}(\varphi(U)) \to \mathscr{T}(U)$ making the diagram commute. We then consider the diagram

$$\begin{tikzcd}
& & & \operatorname{coker}\big(\varphi(U)\big) \arrow[ddd] \arrow[ddl, "\psi(U)"] \arrow[dddddl, dashed, red] \\
\mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[urrr, bend left, "0"] \arrow[drr, bend right, "0"'] \arrow[ddd] & \mathscr{G}(U) \arrow[urr, "p(U)"] \arrow[dr] \arrow[ddd] & & \\
& & \mathscr{T}(U) \arrow[ddd] & \\
& & & \operatorname{coker}\big(\varphi(V)\big) \arrow[ddl, "\psi(V)"] \\
\mathscr{F}(V) \arrow[r, "\varphi(V)"] \arrow[urrr, bend left, "0"] \arrow[drr, bend right, "0"'] & \mathscr{G}(V) \arrow[urr, "p(V)"] \arrow[dr] & & \\
& & \mathscr{T}(V) &
\end{tikzcd}$$

where the vertical maps are the restriction maps and where every triangle or square commutes except possibly for the rightmost square face involving $\operatorname{coker}(\varphi)$ and $\mathscr{T}$. We need to show that the compositions along this rightmost square face from $\operatorname{coker}(\varphi(U))$ to $\mathscr{T}(V)$ are equal. It suffices to note that the composition $\mathscr{F}(U) \to \mathscr{T}(V)$ is the 0 map, and hence the universal property of $\operatorname{coker}(\varphi(U))$ implies that there is a unique map $\operatorname{coker}(\varphi(U)) \to \mathscr{T}(V)$ shown as a dashed red arrow in the diagram above compatible with the map $\mathscr{G}(U) \to \mathscr{T}(V)$. We therefore see that the rightmost square face involving $\operatorname{coker}(\varphi)$ and $\mathscr{T}$ commutes. Finally, taking sheafifications, we see that there is a unique morphism $\psi^{\#}\colon \operatorname{coker}(\varphi) \to \mathscr{T}$ making the diagram (1.3.14) commute since sheafification induces a bijection

$$\operatorname{Hom}_{\mathsf{PMod}(\mathcal{O}_X)}\big(\operatorname{coker}_{\mathcal{P}}(\varphi), \mathscr{T}\big) \xrightarrow{\sim} \operatorname{Hom}_{\mathsf{Mod}(\mathcal{O}_X)}\big(\operatorname{coker}(\varphi), \mathscr{T}\big)$$

by Proposition 1.2.15 (note that the proof can be performed in the category of (pre-)sheaves of $\mathcal{O}_X$-modules since the commutative diagram in the proof consists of maps of $\mathcal{O}_X(U)$-modules). ☐

[Har77, Exer. II.1.2(*a*)]

CLAIM 1.3.15. *We have* $\operatorname{im}(\varphi_P) \cong \operatorname{im}(\varphi)_P$ *for every* $P \in X$. *Moreover, the image sheaf* $\operatorname{im}(\varphi)$ *satisfies the universal property of the image.*

Proof. By the universal property of kernels in the category of $\mathcal{O}_X$-modules, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im}(\varphi) \arrow[r] \arrow[d, dashed, "\exists!"'] & \mathscr{G} \arrow[r, "p"] \arrow[d, equal] & \operatorname{coker}(\varphi) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker(p) \arrow[r] & \mathscr{G} \arrow[r, "p"] & \operatorname{coker}(\varphi) \arrow[r] & 0
\end{tikzcd}$$

where the top row is the exact sequence obtained by combining Claim 1.3.11 and Claim 1.3.13, and the bottom row is the exact sequence obtained by combining

Claim 1.3.11 and Claim 1.3.12. Taking stalks, we obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im}(\varphi)_P \arrow[r] \arrow[d, dashed, "\exists!"'] & \mathscr{G}_P \arrow[r, "p_P"] \arrow[d, equal] & \operatorname{coker}(\varphi)_P \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker(p)_P \arrow[r] & \mathscr{G}_P \arrow[r, "p_P"] & \operatorname{coker}(\varphi)_P \arrow[r] & 0
\end{tikzcd}$$

with exact rows. We therefore see that the left vertical map is an isomorphism since kernels of isomorphic maps of $\mathcal{O}_{X,P}$-modules are isomorphic, and hence $\operatorname{im}(\varphi) \to \ker(p)$ is an isomorphism by [Mur24ag, Proposition 2.1.14]. Since $\ker(p)_P \to \ker(p_P)$ is an isomorphism and both satisfy the universal property of $\operatorname{im}(\varphi_P)$, we therefore see that $\operatorname{im}(\varphi)_P \to \operatorname{im}(\varphi_P)$ is an isomorphism. □

We can finally show that Theorem 1.3.9.

Claim 1.3.16. $\mathsf{Mod}(\mathcal{O}_X)$ *is Abelian.*

$\boxed{1/16}$
[TohokuI, Prop. 3.1.1]
[Har77, Ex. III.1.0.3]

Proof. By Homework 11, Problem 2 from last semester, the category $\mathsf{Mod}(\mathcal{O}_X)$ is additive. By Claim 1.3.10 and Claim 1.3.13, $\mathsf{Mod}(\mathcal{O}_X)$ has kernels and cokernels. It remains to show that for every morphism $\varphi\colon \mathscr{F} \to \mathscr{G}$, the morphism $\operatorname{coim}(\varphi) \to \operatorname{im}(\varphi)$ induced by universal properties is an isomorphism.

We first construct this map. We have the commutative diagram

$$\begin{tikzcd}
 & & \operatorname{coim}(\varphi) \arrow[dr] \arrow[dd, dashed, "\exists!"', "\psi"] \arrow[drr, bend left, "0"] & & \\
\ker(\varphi) \arrow[r] \arrow[urr, bend left, "0"] \arrow[drr, bend right, "0"'] & \mathscr{F} \arrow[ur] \arrow[rr, "\varphi"] \arrow[dr] & & \mathscr{G} \arrow[r] & \operatorname{coker}(\varphi). \\
 & & \operatorname{im}(\varphi) \arrow[ur] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

Since $\operatorname{im}(\varphi) \to \operatorname{coker}(\varphi)$ (resp. $\ker(\varphi) \to \operatorname{coim}(\varphi)$) is the 0 map, there exists a unique dashed map $\operatorname{coim}(\varphi) \to \operatorname{im}(\varphi)$ that makes the diagram commute by the universal property of the coimage (resp. image). Taking stalks, we obtain the commutative diagram

$$\begin{tikzcd}
 & & \operatorname{coim}(\varphi_P) \arrow[dr] \arrow[dd, "\psi_P"] \arrow[drr, bend left, "0"] & & \\
\ker(\varphi_P) \arrow[r] \arrow[urr, bend left, "0"] \arrow[drr, bend right, "0"'] & \mathscr{F}_P \arrow[ur] \arrow[rr, "\varphi_P"] \arrow[dr] & & \mathscr{G}_P \arrow[r] & \operatorname{coker}(\varphi_P) \\
 & & \operatorname{im}(\varphi_P) \arrow[ur] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

where $\ker(\varphi)_P \xrightarrow{\sim} \ker(\varphi_P)$ by Claim 1.3.10, $\operatorname{coim}(\varphi)_P \xrightarrow{\sim} \operatorname{coim}(\varphi_P)$ by Claim 1.3.11 and Claim 1.3.12, $\operatorname{coker}(\varphi)_P \xrightarrow{\sim} \operatorname{coker}(\varphi_P)$ by Claim 1.3.11 and 1.3.13, and $\operatorname{im}(\varphi)_P \xrightarrow{\sim} \operatorname{im}(\varphi_P)$ by Claim 1.3.15. The map ψ_P is the same as that induced by the universal property of coimages (or images) of $\mathcal{O}_{X,P}$-modules by the uniqueness part of the universal property. Since the category of $\mathcal{O}_{X,P}$-modules is Abelian, the map ψ_P is an isomorphism. □

1.3.5. Direct limits and inverse limits (not all covered in class).

[Hoc17, p. 153]

Definition 1.3.17 (Filtered partially ordered set). A partially ordered set $(I, \leq)$ is *filtered* or *directed* if, for any two elements $i, j \in \Lambda$, there exists $k \in \Lambda$ with $i \leq k$ and $j \leq k$.

In other words, any two elements of I have an upper bound.

[Hoc17, p. 153]

Examples 1.3.18. Here are examples of filtered partially ordered sets.

(1) Totally ordered sets, for example, the natural numbers $\mathbf{N}$ and the positive integers.
(2) Finite subsets of a given set under $\subseteq$.
(3) Finitely generated R-submodules of an R-module under $\subseteq$.
(4) Finitely generated R-subalgebras of an R-algebra under $\subseteq$.
(5) Open neighborhoods of a point $x \in X$, where X is a topological space, under $\supseteq$.

We now define direct systems and direct limits.

[Hoc17, p. 153]

Definition 1.3.19 (Direct system). Let $(I, \leq)$ be a partially ordered set. A *direct system* or *direct limit system* indexed by I in a category $\mathscr{C}$ is a functor $I \to \mathscr{C}$ where I is considered as a category as in [Mur24ag, Example A.1.2(2)]. Explicitly, for every element $i \in I$, we have an object X_i in $\mathscr{C}$, and for all pairs $i, j \in I$, we have a morphism

$$f_{ij} \colon X_i \longrightarrow X_j$$

such that

(1) $f_{ii} = \mathrm{id}_{X_i}$ for every $i \in I$.
(2) Whenever $i \leq j \leq k$, we have $f_{ik} = f_{jk} \circ f_{ij}$.

If I is filtered, we say that the direct system is a *filtered direct system*.

[Hoc17, p. 153]
[Lan02, Thm. III.10.1]
[TohokuI, §1.8]

Definition 1.3.20 (Direct limit). Let $\{X_i, f_{ij}\}$ be a direct system of objects and morphisms in a category $\mathscr{C}$. The *direct limit* of the system $\{X_i, f_{ij}\}$ is an object of $\mathscr{C}$ denoted by $\varinjlim X_i$ with morphisms $\{\varphi_i \colon X_i \to \varinjlim X_i\}$ such that $\varphi_j \circ f_{ij} = \varphi_i$ for all $i \leq j$. Moreover, if $Y \in \mathscr{C}$ is another object with morphisms $\{\psi_i \colon X_i \to Y\}$ such that $\psi_j \circ f_{ij} = \psi_i$ for all $i \leq j$, then there is a unique morphism $u \colon \varinjlim X_i \to Y$ making the following diagram commute:

$$\begin{tikzcd}
X_i \arrow[rr, "f_{ij}"] \arrow[dr, "\varphi_i"] \arrow[ddr, bend right, "\psi_i"'] & & X_j \arrow[dl, "\varphi_j"'] \arrow[ddl, bend left, "\psi_j"] \\
& \varinjlim_i X_i \arrow[d, dashed, "u"] & \\
& Y &
\end{tikzcd}$$

If the direct system is filtered, we call the direct limit a *filtered direct limit*.

Examples 1.3.21. Let $\mathscr{C}$ be the category of sets, groups, Abelian groups, rings, R-modules, or R-algebras. (The examples below will work for any category where the objects have an underlying set and a morphism is a function possibly satisfying additional conditions.)

(1) Let Z be a fixed object in $\mathscr{C}$ and let $\{X_i\}$ be a filtered set of subobjects of Z, partially ordered by inclusion. Then, the direct limit of the X_i is the union of these subobjects, and is called the *filtered union*.

(2) Let $\{X_i, f_{ij}\}$ be a filtered direct system of objects and morphisms in $\mathscr{C}$. [AK21, (7.4)]
Then, the filtered direct limit exists and can be constructed as

$$\varinjlim_i X_i := \bigsqcup_i X_i \Big/ f_{ij}(x) \sim x \text{ for all } i \leq j.$$

If $\mathscr{C}$ is the category of Abelian groups or R-modules, then the direct limit exists and can be constructed as

$$\varinjlim_i X_i := \bigoplus_i X_i \Big/ \big(f_{ij}(x) - x\big)_{i \leq j}$$

where $(f_{ij}(x) - x)_{i \leq j}$ is the subobject generated by the elements $f_{ij}(x) - x$.

Next, we define inverse systems and inverse limits.

Definition 1.3.22 (Inverse system). Let $(I, \leq)$ be a partially ordered set. An [Hoc17, p. 155]
inverse system or *inverse limit system* indexed by I in a category $\mathscr{C}$ is a functor $I^{\mathrm{op}} \to \mathscr{C}$ where I is considered as a category as in [Mur24ag, Example A.1.2(2)]. Explicitly, for every element $i \in I$, we have an object X_i in $\mathscr{C}$, and for all pairs $i, j \in I$, we have a morphism

$$f_{ij} \colon X_j \longrightarrow X_i$$

such that

(1) $f_{ii} = \mathrm{id}_{X_i}$ for every $i \in I$.
(2) Whenever $i \leq j \leq k$, we have $f_{ik} = f_{ij} \circ f_{jk}$.

If I is filtered, we say the inverse system is a *cofiltered inverse system*.

Definition 1.3.23 (Inverse limit). Let $\{X_i, f_{ij}\}$ be a inverse system of objects [Hoc17, p. 155]
and morphisms in a category $\mathscr{C}$. The *inverse limit* of the system $\{X_i, f_{ij}\}$ is an object of $\mathscr{C}$ denoted by $\varprojlim X_i$ with morphisms $\{\pi_i \colon \varprojlim X_i \to X_i\}$ such that $f_{ij} \circ \pi_j = \pi_i$ for all $i \leq j$. Moreover, if Y is another object with morphisms $\{\psi_i \colon Y \to X_i\}$ such that $f_{ij} \circ \psi_j = \psi_i$ for all $i \leq j$, then there is a unique morphism $u \colon Y \to \varprojlim X_i$ making the following diagram commute:

$$\begin{tikzcd}
 & Y \arrow[d, dashed, "u"] \arrow[ddl, bend right, "\psi_j"'] \arrow[ddr, bend left, "\psi_i"] & \\
 & \varprojlim_i X_i \arrow[dl, "\pi_j"] \arrow[dr, "\pi_i"'] & \\
X_j \arrow[rr, "f_{ij}"'] & & X_i
\end{tikzcd}$$

If I is filtered, we say the inverse limit is a *cofiltered inverse limit*.

Example 1.3.24. Let $\mathscr{C}$ be the category of sets, Abelian groups, rings, R- [Hoc17, pp. 155–156]
modules, or R-algebras. The inverse limit exists and can be constructed as

$$\varprojlim_i X_i := \left\{ (x_1, x_2, \ldots) \in \prod_i X_i \;\middle|\; f_{ij}(x_j) = x_i \text{ for all } i \geq j \right\}.$$

1.3.6. Grothendieck's axioms. As we saw in the proof of Theorem 1.3.9, the specific Abelian categories that we like to work with satisfy additional properties. Grothendieck identified these properties as useful and defined the following additional axioms (among others) for Abelian categories, and used them to give a general criterion for the existence of "enough" injectives.

[Wei94, A.4.3] Definition 1.3.25 [TohokuI, §1.5]. Let $\mathscr{A}$ be an Abelian category. We consider the following axiom:

(AB3) For every family of objects $\{A_i\}$ in $\mathscr{A}$, the coproduct $\coprod_i A_i$ exists.

Traditionally, in an Abelian category coproducts are called *direct sums* and are denoted $\bigoplus_i X_i$.

We also formulate the dual axiom:

(AB3*) For every family of objects $\{A_i\}$ in $\mathscr{A}$, the product $\prod_i A_i$ exists.

If $\mathscr{A}$ is an Abelian category satisfying (AB3), then it has all direct limits.

[Wei94, Exer. A.5.2] Proposition 1.3.26 [TohokuI, Proposition 1.8]. *Let $\mathscr{A}$ be an Abelian category satisfying* (AB3). *Then, $\mathscr{A}$ has arbitrary direct limits (not necessarily filtered).*

Proof. Consider the objects

$$S = \bigoplus_{i \in I} A_i \qquad\qquad T = \bigoplus_{j \in I} \bigoplus_{i \le j} A_i$$

with insertion maps $v_i \colon A_i \to S$ and $w_{ij} \colon A_i \to T$. We then have two maps $d, e \colon T \to S$:

$$\begin{tikzcd} A_i \arrow[d, "w_{ij}"'] \arrow[dr, "v_i"] & \\ T \arrow[r, dashed, "d", "\exists!"'] & S \end{tikzcd} \qquad \begin{tikzcd} A_i \arrow[r, "f_{ij}"] \arrow[d, "w_{ij}"'] & A_j \arrow[d, "v_j"] \\ T \arrow[r, dashed, "e", "\exists!"'] & S. \end{tikzcd}$$

The cokernel for $d - e$ is the direct limit we wanted. ☐

We can ask whether filtered direct limits are exact, which was the case for $\mathsf{Mod}(R)$.

[Wei94, A.4.6] Definition 1.3.27 [TohokuI, §1.5]. Let $\mathscr{A}$ be an Abelian category. We consider the following axiom:

(AB5) The axiom (AB3) holds and filtered direct limits in $\mathscr{A}$ are exact.

We also formulate the dual axiom:

(AB5*) The axiom (AB3*) holds and cofiltered inverse limits in $\mathscr{A}$ are exact.

Remark 1.3.28. (AB5) is not Grothendieck's original formulation. Our version is equivalent to Grothendieck's by [TohokuI, Proposition 1.8].

There are also axioms (AB4) and (AB6) which we will not define or use.

1.3.7. Additive functors and exact functors. Recall our goal is to describe functors between Abelian categories like sheaf cohomology and how to work with them. One way to think about this is that now that we know $\mathsf{Mod}(\mathcal{O}_X)$ forms an Abelian category, we want a good way to extract invariants from it with values in Abelian groups, vector spaces, modules, or even in another category (for our discussion, another Abelian category). For those of you taking the algebraic topology class, this is the idea underlying homology and cohomology.

We first define classes of functors that preserve additivity and exactness properties that exist in an Abelian category.

[Har77, pp. 203–204]
[Wei94, Def. 1.6.6]
[Bor94b, Def. 1.11.1]
[TohokuI, p. 126]

Definition 1.3.29. Let $F\colon \mathscr{A} \to \mathscr{B}$ be a covariant functor between Abelian categories. We say that F is

(*i*) *additive* if, for every pair of objects A, A' in $\mathscr{A}$, the induced map

$$\operatorname{Hom}(A, A') \longrightarrow \operatorname{Hom}(FA, FA')$$

is a map of Abelian groups. (This makes sense even if $\mathscr{A}$ and $\mathscr{B}$ are preadditive.)

[TohokuI, p. 128]

(*ii*) *left exact* (resp. *right exact*, *middle exact*) if it is additive and if it sends short exact sequences

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

to exact sequence of the form

$$\begin{aligned} 0 \longrightarrow FA' \longrightarrow FA \longrightarrow FA''& \\ FA' \longrightarrow FA \longrightarrow FA'' &\longrightarrow 0 \\ FA' \longrightarrow FA \longrightarrow FA''& \end{aligned}$$

respectively.

(*iii*) *exact* if it is both left and right exact.

We use the same terminology for contravariant functors $F\colon \mathscr{A}^{\mathrm{op}} \to \mathscr{B}$. (The way to remember which one is left or right is to think about what the final result looks like—this is also why I prefer to denote contravariant functors by $\mathscr{A}^{\mathrm{op}} \to \mathscr{B}$ instead of $\mathscr{A} \to \mathscr{B}^{\mathrm{op}}$.)

Let us see some examples. We showed the following last semester (you will prove the analogous statement for $\Gamma_Z(X, -)$ on Homework 1):

[Har77, Exer. II.1.8]
[God73, p. 133]

Proposition 1.3.30. *Let $(X, \mathcal{O}_X)$ be a rined space and let $U \subseteq X$ be an open subset. The functor*

$$\begin{aligned} \mathsf{Mod}(\mathcal{O}_X) &\longrightarrow \mathsf{Mod}\bigl(\Gamma(U, \mathcal{O}_X)\bigr) \\ \mathscr{F} &\longmapsto \Gamma(U, \mathscr{F}) \end{aligned}$$

is left exact.

Proof. Consider an exact sequence

$$0 \longrightarrow \mathscr{F}' \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}''.$$

We then have the sequence

$$0 \longrightarrow \Gamma(U, \mathscr{F}') \xrightarrow{\varphi(U)} \Gamma(U, \mathscr{F}) \xrightarrow{\psi(U)} \Gamma(U, \mathscr{F}'')$$

that we need to show is exact. The sequence is automatically exact at $\Gamma(U, \mathscr{F}')$ since a morphism of sheaves is injective if and only if it is injective on sections by [Mur24ag, (2.1.34)] (the morphism $\ker(\varphi) \to 0$ is an isomorphism if and only if it is an isomorphism on stalks).

By functoriality, we know that the composition $\psi(U) \circ \varphi(U)$ is the 0 map (you can check the map is 0 on stalks since you can check that the image is 0 on stalks). We therefore have the inclusion

$$\operatorname{im}(\varphi)(U) \subseteq \ker(\psi)(U)$$

as sub-$\mathcal{O}_X(U)$-modules of $\Gamma(U, \mathscr{F})$. It remains to show the opposite inclusion $\operatorname{im}(\varphi)(U) = \ker(\psi)(U)$. We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(U, \mathscr{F}') \arrow[r, "\varphi(U)"] \arrow[d] & \Gamma(U, \mathscr{F}) \arrow[r, "\psi(U)"] \arrow[d] & \Gamma(U, \mathscr{F}'') \arrow[d] \\
0 \arrow[r] & \mathscr{F}'_P \arrow[r, "\varphi_P"] & \mathscr{F}_P \arrow[r, "\psi_P"] & \mathscr{F}''_P
\end{tikzcd}$$

Suppose $t \in \ker(\psi)(U)$. Then, $\psi(U)(t) = 0$, and hence

$$\psi(U)(t)_P = \psi_P(t_P) = 0$$

for all $P \in U$. Since the bottom row is exact, there exists $s_P \in \mathscr{F}'_P$ such that $\varphi_P(s_P) = t_P$. We claim we can lift s_P to some $s \in \mathscr{F}'(U)$. For each P, pick an open set $V_P \ni P$ and $r_P \in \mathscr{F}'(V_P)$ such that $s_P = \langle V_P, r_P \rangle$. For each $W \coloneqq V_P \cap V_Q$, we have

$$\varphi(W)(r_P|_W) = \varphi(W)(r_Q|_W) = t|_W.$$

By injectivity of φ we have that

$$r_P|_W = r_Q|_W.$$

By the sheaf property (4), the r_P therefore glue to form a section $s \in \mathscr{F}(U)$. Thus, since $\varphi(U)(s) = t$ for all P, we have $\ker(\psi)(U) \subseteq \operatorname{im}(\varphi)(U)$. □

Another important example for us is:

[Har77, Ex. III.1.0.8]
[TohokuI, p. 128]
[CE56, Prop. II.4.4]

Proposition 1.3.31. *Let $\mathscr{A}$ be an Abelian category and let M be an object in $\mathscr{A}$. Then, $\operatorname{Hom}_{\mathscr{A}}(M, \cdot\,)$ is a covariant left exact functor $\mathscr{A} \to \mathsf{Ab}$ and $\operatorname{Hom}_{\mathscr{A}}(\,\cdot\,, M)$ is a contravariant left exact functor $\mathscr{A}^{\mathrm{op}} \to \mathsf{Ab}$.*

Proof. We can apply the first statement to $\mathscr{A}^{\mathrm{op}}$ to prove the second statement. (Here, we are using the fact that the opposite category of an Abelian category is Abelian. This is because additivity and the axioms (AB1) and (AB2) are self-dual!) It therefore suffices to show the first statement.

Consider a short exact sequence

$$0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{p} A'' \longrightarrow 0.$$

We want to show that

$$0 \longrightarrow \operatorname{Hom}_{\mathscr{A}}(M, A') \longrightarrow \operatorname{Hom}_{\mathscr{A}}(M, A) \longrightarrow \operatorname{Hom}_{\mathscr{A}}(M, A'')$$

is exact. Exactness on the left holds since if $i \circ g = 0$ for $g \in \operatorname{Hom}_{\mathscr{A}}(M, A')$, then the exactness of the original sequence and the universal property of kernels says that g factors through 0, and hence must be the 0 map.

It remains to show that

$$\operatorname{im}\bigl(\operatorname{Hom}_{\mathscr{A}}(M, A') \to \operatorname{Hom}_{\mathscr{A}}(M, A)\bigr) = \ker\bigl(\operatorname{Hom}_{\mathscr{A}}(M, A) \to \operatorname{Hom}_{\mathscr{A}}(M, A'')\bigr).$$

The inclusion $\subseteq$ holds since $p \circ i = 0$. Conversely, suppose $f \in \operatorname{Hom}_{\mathscr{A}}(M, A)$ satisfies $p \circ f = 0$. My the universal property of kernels, we obtain the commutative diagram

$$\begin{tikzcd}
 & & M \arrow[dl, dashed, "g"', "\exists!"] \arrow[d, "f"] \arrow[dr, "0"] & & \\
0 \arrow[r] & A' \arrow[r, "i"'] & A \arrow[r, "p"'] & A'' \arrow[r] & 0.
\end{tikzcd}$$

By the commutativity of the diagram, we have $g = f \circ i$. □

Proposition 1.3.31 motivates the following:

Definition 1.3.32 [Bae40, p. 804; CE56, p. 6 and p. 8]. Let $\mathscr{A}$ be an Abelian category. We say that an object M of $\mathscr{A}$ is *injective* if the functor $\mathrm{Hom}_{\mathscr{A}}(\,\cdot\,, M)$ is exact. In other words, given the solid commutative diagram below where $A' \hookrightarrow A$ is a monomorphism, there is a dashed map (not necessarily unique) making the diagram commute:

[Wei94, p. 33, p. 37]
[TohokuI, §1.10]
[Har77, p. 204]

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[dl, dashed] \\
& M. &
\end{tikzcd}$$

We say that an object M of $\mathscr{A}$ is *projective* if the functor $\mathrm{Hom}_{\mathscr{A}}(M, \,\cdot\,)$ is exact. In other words, given the solid commutative diagram below where $A \twoheadrightarrow A''$ is an epimorphism, there is a dashed map (not necessarily unique) making the diagram commute:

$$\begin{tikzcd}
P \arrow[d, dashed] \arrow[dr] & & \\
A \arrow[r] & A'' \arrow[r] & 0.
\end{tikzcd}$$

The theme for a while will be: How can we fix the non-exactness of these two functors? For Hom, what we want to do is to approximate a given object A by an injective (resp. projective) one. For Γ, it turns out that injective objects will still give us the correct answer. (We will show this later.)

In MA 557, we worked with many examples of projective modules. Free modules and taking direct summands thereof gave many examples. A particularly interesting example due to Kaplansky [Mur24ca, Example 7.9.1] used the hedgehog theorem from differential topology to prove that a specific module over

$$\frac{\mathbf{R}[X, Y, Z]}{(X^2 + Y^2 + Z^2 - 1)}$$

is projective but not free.

In an arbitrary Abelian category, you cannot always surject onto objects with a projective object. But there *is* a general recipe to find injective objects due to Grothendieck.

1.3.8. Monomorphisms, epimorphisms, and generators. To make sense of Grothendieck's result, we need a few more pieces of terminology.

Definition 1.3.33. Let $\mathscr{C}$ be a category and consider a morphism $u\colon A \to B$ in $\mathscr{C}$. We say that u is a *monomorphism*, write $A \hookrightarrow B$, and say A is a *subobject* of B (resp. *epimorphism*, write $A \twoheadrightarrow B$, and say B is a *quotient* of A) if

[TohokuI, p. 122]

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{C}}(C, A) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(C, B) \\
v &\longmapsto u \circ v
\end{aligned}$$

(resp. if

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{C}}(B, C) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(A, C) \\
w &\longmapsto w \circ u
\end{aligned}$$

) is injective for every object C in $\mathscr{C}$. Monomorphisms are the categorical versions of injections one way to visualize monomorphisms is that in the diagram

$$\begin{tikzcd} C \arrow[r, shift left, "v"] \arrow[r, shift right, "v'"'] & A \arrow[r, "u"] & B \end{tikzcd}$$

if the two compositions are equal, then $v = v'$. Similarly, epimorphisms are the categorical versions of surjections: in the diagram

$$\begin{tikzcd} A \arrow[r, "u"] & B \arrow[r, shift left, "w"] \arrow[r, shift right, "w'"'] & C \end{tikzcd}$$

if the two compositions are equal, then $w = w'$.

We now define generators.

Definition 1.3.34 [TohokuI, p. 134]. Let $\mathscr{C}$ be a category. We say that U is a *generator of* $\mathscr{C}$ if, for every object A in $\mathscr{C}$ and every monomorphism $B \hookrightarrow A$ that is not the identity, there exists a morphism $u\colon U \to A$ that does not come from a morphism $U \to B$.

1.3.9. Grothendieck Abelian categories have enough injectives. We can now formulate Grothendieck's result. The case for Abelian groups is due to Baer [Bae40, Theorem 3].

$\boxed{1/21}$

[Stacks, Tag 079H]

Theorem 1.3.35 (Grothendieck [TohokuI, Théorème 1.10.1]). *Let $\mathscr{A}$ be an Abelian category satisfying* (AB5) *and admitting a generator. Then, for every object A in $\mathscr{A}$, there exists a monomorphism $A \hookrightarrow M$ where M is an injective object.*

In fact, we will construct a *functorial* embedding of objects into injective objects: a functor $A \mapsto M(A)$ from $\mathscr{A}$ to $\mathscr{A}$ and a natural transformation $\mathrm{id}_{\mathscr{A}} \Rightarrow M$ such that for every object A in $\mathscr{A}$, $M(A)$ is injective and $f(A)$ is a monomorphism $A \hookrightarrow M(A)$. This theorem motivates the following:

[Har77, p. 204]

Definition 1.3.36. An Abelian category satisfying (AB5) and admitting a generator is called a *Grothendieck Abelian category*. An Abelian category *has enough injective objects* if every objects admits a monomorphism into an injective object.

To prove Theorem 1.3.35, we need to make sure we are working with sets. The existence of a generator will ensure that nothing we are working with is "too large." The following is a spelled-out version of the parenthetical remark in [TohokuI, p. 136]. The *cofinality* of an ordinal α is the smallest ordinal that is the order type of a cofinal subset of α [Jec03, p. 31].

[Stacks, Tag 0E8N]

I did not do (*ii*) in class.

Lemma 1.3.37. *Let $\mathscr{A}$ be an Abelian category with a generator U. Let A be an object of $\mathscr{A}$. Let κ be the cardinality of* $\mathrm{Hom}_{\mathscr{A}}(U, A)$*.*

- (*i*) *There does not exist a strictly increasing (or strictly decreasing) chain of subobjects of A indexed by a cardinal larger than κ.*
- (*ii*) *If α is an ordinal of cofinality $> \kappa$ then any increasing (or decreasing) sequence of subobjects of A indexed by α is eventually constant.*
- (*iii*) *The cardinality of the set of subobjects of A is $\leq 2^{\kappa}$.*

Proof. (*i*). Suppose $\kappa' > \kappa$ is a cardinal and assume $(A_i)_{i\in\kappa'}$ is strictly increasing. By definition of a generator, for each i, there exists $\phi_i \in \mathrm{Hom}_{\mathscr{A}}(U, A)$ such that ϕ_i

factors through A_{i+1} but not A_i. The morphisms ϕ_i are distinct, contradicting the definition of κ.

(ii) follows from the definition of cofinality and (i).

(iii). For any subobject $B \hookrightarrow A$, let $S_B \in \mathcal{P}(\mathrm{Hom}_{\mathscr{A}}(U, A))$, the power set of $\mathrm{Hom}_{\mathscr{A}}(U, A)$, be the subset

$$S_B = \bigl\{\phi \in \mathrm{Hom}_{\mathscr{A}}(U, A) \bigm| \phi \text{ factors through } B\bigr\}.$$

Then, $B = B'$ if and only if $S_B = S_{B'}$. Thus, the cardinality of the set of subobjects is at most the cardinality of this power set. □

We now prove the following result, which is a version of Baer's criterion [Bae40, Theorem 1] but for arbitrary Grothendieck Abelian categories. Baer's criterion is the special case when $\mathscr{A} = \mathsf{Mod}(R)$ and $U = R$.

Lemma 1.3.38 [TohokuI, Lemme 1]. *Let $\mathscr{A}$ be an Abelian category satisfying* (AB5) *and admitting a generator U. Consider an object M of $\mathscr{A}$. Then, M is injective if and only if for every subobject $V \hookrightarrow U$, any morphism $V \to M$ can be extended to a morphism $U \to M$.*

Proof. $\Rightarrow$ follows by definition of injectivity.

$\Leftarrow$. We want to construct an extension

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, "u"'] & A \arrow[dl, dashed] \\ & M & \end{tikzcd}$$

of a morphism $u\colon B \to M$. Consider the set P of extensions of u to subobjects of A containing B, partially ordered by saying that $(v_1\colon B_1 \to M) \le (v_2\colon B_2 \to M)$ if and only if $B_1 \hookrightarrow B_2$ and $v_2|_{B_1} = v_1$. Note that P is a set since there is a set of subobjects of A by Lemma 1.3.37.

We want to apply Zorn's lemma [Kur1922; Zor35] to P. Let T be a totally ordered subset of P. By (AB5), we can then take the direct limit over elements in T to obtain an upper bound (in P) for the totally ordered set T. By Zorn's lemma, there exists a maximal element in P. Replacing u by this maximal element, we may assume that $u\colon B \to M$ is maximal.

We want to show that $B = A$. Suppose not. Since U is a generator, there exists a morphism $j\colon U \to A$ that does not come from a morphism $U \to B$. Consider the Cartesian diagram

$$\begin{tikzcd} V \arrow[r, "j'"] \arrow[d, hook] & B \arrow[d, hook] \\ U \arrow[r, "j"] & A. \end{tikzcd}$$

Comparing universal properties, V can be constructed as the kernel making the sequence

$$\begin{tikzcd} V \arrow[r, "\begin{pmatrix} \mathrm{id} \\ -j' \end{pmatrix}", "\varphi'"'] & U \oplus B \arrow[r, "(j\ \mathrm{id})"] & A \end{tikzcd}$$

exact. Let $B' = \operatorname{im}(U \oplus B \to A)$, which is a subobject of A strictly containing B. We have the commutative diagram

$$\begin{tikzcd}
V \arrow[r, "\begin{pmatrix}\mathrm{id}\\ -j'\end{pmatrix}", "\varphi'"'] & U \oplus B \arrow[r, "(j\ \mathrm{id})", "\varphi"'] \arrow[d, dashed, "w"'] & B' \arrow[r] \arrow[dl, dashed, "v"] & 0 \\
 & M. & &
\end{tikzcd}$$

By the universal property of cokernels, to define a morphism $v\colon B' \to M$, it suffices to define a morphism $w\colon U \oplus B \to M$ such that $w \circ \varphi' = 0$. Let k be an extension to U of $u \circ j'\colon V \to M$. Let w be the morphism $(k, u)\colon U \oplus B \to M$. Then, $w \circ \varphi' = 0$ and $v\colon B' \to M$ extends u, contradicting the maximality of u. ☐

We can now prove Grothendieck's Theorem 1.3.35. A key technique in the proof is *transfinite induction* [Sup72, pp. 195–197; Jec03, Theorem 2.14].

Proof of Theorem 1.3.35. Let A be an object of $\mathscr{A}$. Consider the co-Cartesian diagram

$$\begin{tikzcd}
\bigoplus\limits_{V \hookrightarrow U} \bigoplus\limits_{u \in \mathrm{Hom}(V,A)} V \arrow[r, "\sum u"] \arrow[d, hook, "\bigoplus \mathrm{id}"'] & A \arrow[d, "f_1(A)"] \\
\bigoplus\limits_{V \hookrightarrow U} \bigoplus\limits_{u \in \mathrm{Hom}(V,A)} U \arrow[r] & M_1(A).
\end{tikzcd} \tag{1.3.39}$$

In other words, we consider the morphism

$$\bigoplus_{V \hookrightarrow U} \bigoplus_{u \in \mathrm{Hom}(V,A)} V \longrightarrow A \oplus \bigoplus_{V \hookrightarrow U} \bigoplus_{u \in \mathrm{Hom}(V,A)} U$$

where on each summand V the components are $-u$ and the canonical insertion map from V to the u-th direct summand on the right, and set $M_1(A)$ to be the cokernel of this morphism. Note that the outer direct sums are indexed by a set by Lemma 1.3.37 and that $\bigoplus \mathrm{id}$ is a monomorphism by (AB5). Note that every morphism $u\colon V \to A$ can be extended to a morphism $U \to M_1(A)$ by restricting to the u-th direct summand.

We want to show that $f_1(A)$ is a monomorphism. First, since

$$\bigoplus_{V \hookrightarrow U} \bigoplus_{u \in \mathrm{Hom}(V,A)} V \hookrightarrow \bigoplus_{V \hookrightarrow U} \bigoplus_{u \in \mathrm{Hom}(V,A)} U$$

is injective, we see that the sequence

$$0 \longrightarrow \bigoplus_{V \hookrightarrow U} \bigoplus_{u \in \mathrm{Hom}(V,A)} V \longrightarrow A \oplus \bigoplus_{V \hookrightarrow U} \bigoplus_{u \in \mathrm{Hom}(V,A)} U \longrightarrow M_1(A) \longrightarrow 0$$

is a short exact sequence. Comparing universal properties again, this means (1.3.39) is also a Cartesian diagram. If $K = \ker(A \to M_1(A))$, then we obtain the commutative diagram

$$\begin{tikzcd}
K \arrow[dr, dashed] \arrow[ddr, bend right, "0"'] \arrow[drr, hook, bend left] & & \\
& \bigoplus\limits_{V\hookrightarrow U}\ \bigoplus\limits_{u\in\mathrm{Hom}(V,A)} V \arrow[r, "\sum u"'] \arrow[d, hook, "\bigoplus \mathrm{id}"'] & A \arrow[d, "f_1(A)"] \\
& \bigoplus\limits_{V\hookrightarrow U}\ \bigoplus\limits_{u\in\mathrm{Hom}(V,A)} U \arrow[r] & M_1(A).
\end{tikzcd}$$

Since the left vertical map is a monomorphism, we see that the dashed map is 0. This shows that $K \to A$ is the 0 map, i.e., $K = 0$.

We now repeat this construction transfinitely many times. Let α be an ordinal.

- (Base case) If $\alpha = 0$, we set $M_0(A) \coloneqq A$.
- (Successor ordinals) Given $M_\alpha(A)$, we set $M_{\alpha+1}(A) \coloneqq M_1(M_\alpha(A))$.
- (Limit ordinals) Let β be a limit ordinal. We set

$$M_\beta(A) \coloneqq \varinjlim_{\alpha<\beta} M_\alpha(A).$$

By transfinite induction [Sup72, §7.1, Theorem Schema 4; Jec03, Theorem 2.14], we have now constructed functorial morphisms $A \to M_\alpha(A)$ for every ordinal α. These morphisms are monomorphisms by (AB5).

To finish, let κ be the smallest ordinal whose cardinality is strictly larger than that of the set of subobjects of U, which exists by Lemma 1.3.37. We claim that

$$f_\kappa(A)\colon A \hookrightarrow M_\kappa(A)$$

works. We need to show that $M_\kappa(A)$ is injective. By Lemma 1.3.38, we need to show that in the diagram

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] \arrow[d, "v"'] & U \arrow[dl, dashed, "u"] \\
& M_\kappa(A) &
\end{tikzcd}$$

there exists a morphism $u\colon U \to M_\kappa(A)$ making the diagram commute. Since

$$M_\kappa(A) = \varinjlim_{\alpha\le\kappa} M_\alpha(A),$$

we know that

$$V = \varinjlim_{\alpha\le\kappa} v^{-1}\bigl(M_\alpha(A)\bigr)$$

by (AB5) (which we apply to say that filtered direct limits preserve kernels).

We claim that V factors through some $M_\alpha(A)$. The inverse images $v^{-1}(M_\alpha(A))$ form an increasing chain of subobjects of V indexed by a cardinal strictly larger than the cardinality of the set of subobjects of U by Lemma 1.3.37(*iii*). By Lemma 1.3.37(*i*), this means that the sequence of $v^{-1}(M_\alpha(A))$ is eventually stationary, and hence must equal V for some α. Finally, the morphism $v\colon V \to M_\alpha(A)$ is extended by $u\colon U \to M_{\alpha+1}(A)$ using the v-th direct summand in (1.3.39). □

1.3.10. The category of sheaves of modules has enough injectives. We now apply Grothendieck's Theorem 1.3.35. Our convention for rings is that rings are always commutative and have an identity element 1.

Proposition 1.3.40. *Let R be a ring. Then, $\mathsf{Mod}(R)$ has enough injectives.*

Proof using Theorem 1.3.35. By Theorem 1.3.35, we need to check that $\mathsf{Mod}(R)$ is a Grothendieck Abelian category: (AB5) holds by [Mur24ag, Theorem A.6.1], and R is a generator. $\square$

We also give another proof, which has the advantage of being more concrete. Note, however, that there are many categories that one comes across where there is no obvious way to make a concrete construction like this.

Alternative proof. We first construct *one* injective module for $R = \mathbf{Z}$. We claim that $\mathbf{Q}/\mathbf{Z}$ is injective. By Baer's criterion (Lemma 1.3.38), since R is a generator, it suffices to show that in the diagram

$$\begin{tikzcd}
0 \arrow[r] & n\mathbf{Z} \arrow[r] \arrow[d] & \mathbf{Z} \arrow[dl, dashed, "\exists ?"] \\
& \mathbf{Q}/\mathbf{Z} &
\end{tikzcd}$$

the dashed map exists and makes the diagram commute. If $\bar{a} \in \mathbf{Q}/\mathbf{Z}$ is the image of n, then mapping $1 \mapsto \overline{a/n} \in \mathbf{Q}/\mathbf{Z}$ works.

[Wei94, Exer. 2.3.5] [God73, Thm. I.1.2.2]

We now consider the general case. First, $\operatorname{Hom}_{\mathbf{Z}}(R, \mathbf{Q}/\mathbf{Z})$ is an injective R-module since

$$\operatorname{Hom}_R\bigl(\,\cdot\,, \operatorname{Hom}_{\mathbf{Z}}(R, \mathbf{Q}/\mathbf{Z})\bigr) \cong \operatorname{Hom}_{\mathbf{Z}}(\,\cdot\,, \mathbf{Q}/\mathbf{Z})$$

is an exact functor, where the isomorphism holds by tensor–Hom adjunction (the version in [BouAlgI, Chapter II, §4, no. 1, Proposition 1], for example). Now consider

$$I(M) := \prod_{\varphi \in \operatorname{Hom}_R(M, \operatorname{Hom}_{\mathbf{Z}}(R, \mathbf{Q}/\mathbf{Z}))} \operatorname{Hom}_{\mathbf{Z}}(R, \mathbf{Q}/\mathbf{Z}).$$

This is injective since it is a direct product of injective modules. Moreover,

$$\begin{aligned} M &\longrightarrow I(M) \\ m &\longmapsto \bigl(\varphi(m)\bigr)_{\varphi} \end{aligned}$$

is injective. $\square$

[Har77, Prop. II.2.2]

Proposition 1.3.41 [TohokuI, Proposition 3.1.1 and Corollaire]. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then, $\mathsf{Mod}(\mathcal{O}_X)$ has enough injectives.*

Proof using Theorem 1.3.35. By Theorem 1.3.35, we need to check that $\mathsf{Mod}(\mathcal{O}_X)$ is a Grothendieck Abelian category: (AB5) holds by [Mur24ag, Theorem A.6.1] since exactness can be checked on stalks. To construct a generator, let

$$j_U \colon U \hookrightarrow X$$

be all possible open inclusions. We claim the object

$$\bigoplus_{U \subseteq X} (j_U)_! \mathcal{O}_U$$

is a generator. Consider a monomorphism $\mathscr{F} \hookrightarrow \mathscr{G}$ that is not an isomorphism. After replacing $\mathscr{F}$ by its image, we may assume that $\mathscr{F} \subseteq \mathscr{G}$. Let $U \subseteq X$ be an open subset such that $\mathscr{F}(U) \subsetneq \mathscr{G}(U)$. Then, we can consider a morphism $\mathcal{O}_U(U) \to \mathscr{G}(U)$ mapping $1 \in \mathcal{O}_U(U)$ to an element not in the image. We obtain a morphism $\mathcal{O}_U \to \mathscr{G}|_U$ and applying $(j_U)_!$, the composition

$$(j_U)_!\mathcal{O}_U \longrightarrow (j_U)_!\bigl(\mathscr{G}|_U\bigr) \lhook\joinrel\longrightarrow \mathscr{G}$$

is a morphism not factoring through $\mathscr{F}$. Here, the injection on the right holds by [Har77, Exercise II.1.19(c)] (Homework 1, Problem 2(c)). Combining this map with 0 maps for $U' \neq U$, we see that $\bigoplus_{U \subseteq X}(j_U)_!\mathcal{O}_U$ is a generator. □

We can give a direct proof due to Godement [God73, Chapitre II, Théorème 7.1.1] (see also [TohokuI, p. 156]).

Alternative proof. Let $\mathscr{F}$ be an $\mathcal{O}_X$-module. For every $x \in X$, let $\mathscr{F}_x \hookrightarrow I_x$ be an injection into an injective $\mathcal{O}_{X,x}$-module. Denote

$$i_x \colon \{x\} \lhook\joinrel\longrightarrow X$$

and consider the sheaf

$$I(\mathscr{F}) \coloneqq \prod_{x \in X}(i_x)_*I_x.$$

We have a morphism $\mathscr{F} \to I(\mathscr{F})$ since

$$\begin{aligned}\operatorname{Hom}_{\mathcal{O}_X}\bigl(\mathscr{F}, I(\mathscr{F})\bigr) &\cong \prod_{x \in X}\operatorname{Hom}_{\mathcal{O}_X}\bigl(\mathscr{F}, (i_x)_*I_x\bigr)\\ &\cong \prod_{x \in X}\operatorname{Hom}_{\mathcal{O}_{X,x}}(i_x^*\mathscr{F}, I_x)\\ &= \prod_{x \in X}\operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathscr{F}_x, I_x)\end{aligned}$$

where the isomorphism hold by definition of products and by [Har77, p. 110] (Homework 1, Problem 5), respectively. The morphisms $\mathscr{F}_x \hookrightarrow I_x$ therefore glue together to give a morphism $\mathscr{F} \to I(\mathscr{F})$, which is injective since it is injective on stalks. Finally, to show that $I(\mathscr{F})$ is injective, it suffices to show that each $(i_x)_*I_x$ is injective since direct products of injective objects are injective (direct products are exact for the category Ab of Abelian groups). But this holds since

$$\operatorname{Hom}_{\mathcal{O}_X}\bigl(\,\cdot\,, (i_x)_*I_x\bigr) \cong \operatorname{Hom}_{\mathcal{O}_{X,x}}\bigl((i_x)^*\,\cdot\,, I_x\bigr)$$

is the composition of the stalks functor, which is exact, followed by the exact functor obtained by Hom-ing into I_x. Note that here, we are considering the space $\{x\}$ with the structure sheaf $i_x^{-1}\mathcal{O}_X$, which is just the constant sheaf associated to $\mathcal{O}_{X,x}$ on $\{x\}$. □

1.4. Derived functors and sheaf cohomology

1.4.1. Complexes. To define derived functors and cohomology, we need to discuss complexes.

Definition 1.4.1. Let $\mathscr{A}$ be an Abelian category. A *(cochain) complex* $A^\bullet$ in $\mathscr{A}$ is a collection of objects A^i together with a collection of *coboundary maps* $d^i \colon A^i \to A^{i+1}$ in $\mathscr{A}$, both indexed by $i \in \mathbf{Z}$, such that $d^{i+1} \circ d^i = 0$. We visualize a complex as a sequence of objects and maps:

$$\cdots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \longrightarrow \cdots.$$

[Wei94, pp. 2–3, p. 7]
[TohokuI, §1.7, Ex. c]
[Har77, p. 203]

If the A^i are only specified in a certain range (e.g., $i \geq 0$), then we set $A^i = 0$ for the other i. A *morphism of complexes* $f\colon A^\bullet \to B^\bullet$ is a collection of morphisms $f^i\colon A^i \to B^i$ for each i that commute with the coboundary maps d^i, i.e., such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "d_A^{i-2}"] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, "f^{i-1}"] & A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"] & A^{i+1} \arrow[r, "d_A^{i+1}"] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r, "d_B^{i-2}"] & B^{i-1} \arrow[r, "d_B^{i-1}"] & B^i \arrow[r, "d_B^i"] & B^{i+1} \arrow[r, "d_B^{i+1}"] & \cdots.
\end{tikzcd} \tag{1.4.2}$$

Complexes form a category $\mathsf{Ch}(\mathscr{A})$.

Remark 1.4.3. By a *complex* in this course, we will mean cochain complexes. This is the "algebraic" or "cohomological" convention—indices on complexes go *up* as you compose maps in a complex. In topology, the opposite, "homological" convention with *chain complexes* is often used—indices go *down* as you compose maps in a complex. Traditionally, chain complexes are denoted with subscripts $A_\bullet$.

[TohokuI, Prop. 1.6.1]

Proposition 1.4.4. *Let $\mathscr{A}$ be an additive category. Then, $\mathsf{Ch}(\mathscr{A})$ is an additive category. If $\mathscr{A}$ satisfies (AB1), (AB2), (AB3), or (AB5), then $\mathsf{Ch}(\mathscr{A})$ satisfies the same axiom.*

Proof idea. Compute kernels, cokernels, and coproducts termwise. Check they satisfy the desired universal properties. For (AB5), the fact that kernels and cokernels are computed termwise means that exactness can be checked termwise. ☐

As a result, we can make sense of exact sequences of complexes.

The following is an intermediate definition. Ultimately, we will define sheaf cohomology as the i-th cohomology module of a certain complex obtained from a sheaf.

[Wei94, p. 3, p. 7]
[Har77, p. 203]

I prefer to use a boldface $\mathbf{h}^i$ instead of Hartshorne's notation h^i because the latter is often used to denote the k-vector space dimension of the sheaf cohomology of a projective variety over k.

Definition 1.4.5. Let $\mathscr{A}$ be an Abelian category. The *i-th cocycles* of a complex $A^\bullet$ are

$$Z^i(A^\bullet) \coloneqq \ker(d_A^i)$$

and the *i-th coboundaries* of $A^\bullet$ are

$$B^i(A^\bullet) \coloneqq \operatorname{im}(d_A^{i-1}).$$

The *i-th cohomology object* of $A^\bullet$ is

$$\mathbf{h}^i(A^\bullet) \coloneqq \frac{Z^i(A^\bullet)}{B^i(A^\bullet)}.$$

(The way to remember the indices is that $\mathbf{h}^i(A^\bullet)$ is a quotient of a subobject of A^i.)

If $f\colon A^\bullet \to B^\bullet$ is a morphism of complexes, then f induces a map

$$\mathbf{h}^i(f)\colon \mathbf{h}^i(A^\bullet) \longrightarrow \mathbf{h}^i(B^\bullet)$$

by the commutativity of (1.4.2). We therefore obtain a functor

$$\mathbf{h}^i\colon \mathsf{Ch}(\mathscr{A}) \longrightarrow \mathscr{A}.$$

[Wei94, Exer. 1.1.2]

Explicitly, the map is obtained as follows. First, we look at the following square in

(1.4.2):

$$\begin{tikzcd}
A^i \arrow[r, "d^i_A"] \arrow[d, "f^i"'] & A^{i+1} \arrow[d, "f^{i+1}"] \\
B^i \arrow[r, "d^i_B"] & B^{i+1}.
\end{tikzcd}$$

We now consider the kernels of the d^i to obtain the commutative diagram of solid maps below:

$$\begin{tikzcd}
Z^i(A^\bullet) \arrow[r] \arrow[d, dashed, "\exists!"'] & A^i \arrow[r, "d^i_A"] \arrow[d, "f^i"'] & A^{i+1} \arrow[d, "f^{i+1}"] \\
Z^i(B^\bullet) \arrow[r] & B^i \arrow[r, "d^i_B"] & B^{i+1}.
\end{tikzcd}$$

By definition of $Z^i(A^\bullet) \coloneqq \ker(d^i_A)$, the composition $Z^i(A^\bullet) \to A^i \to A^{i+1}$ is 0. Thus, the composition $Z^i(A^\bullet) \to B^{i+1}$ is 0. By the universal property of $Z^i(B^\bullet) \coloneqq \ker(d^i_B)$, there is a unique dashed map $Z^i(A^\bullet) \to Z^i(B^\bullet)$ making the diagram commute. Now the fact that $A^\bullet$ and $B^\bullet$ are complexes means that the maps $B^i(A^\bullet) \to A^i$ and $B^i(B^\bullet) \to B^i$ factor through $A^i(A^\bullet)$ and $Z^i(B^\bullet)$, i.e., we can enlarge the commutative diagram to get the commutative diagram of solid maps below:

$$\begin{tikzcd}
A^{i-1} \arrow[r] \arrow[d, "f^{i-1}"'] & B^i(A^\bullet) \arrow[r] \arrow[d, dashed, "\exists!"'] & Z^i(A^\bullet) \arrow[r] \arrow[d] & A^i \arrow[r, "d^i_A"] \arrow[d, "f^i"'] & A^{i+1} \arrow[d, "f^{i+1}"] \\
B^{i-1} \arrow[r] & B^i(B^\bullet) \arrow[r] & Z^i(B^\bullet) \arrow[r] & B^i \arrow[r, "d^i_B"] & B^{i+1}.
\end{tikzcd}$$

By the universal property of $B^i(A^\bullet) \coloneqq \operatorname{im}(d^{i-1}_A)$, there is a unique dashed map $B^i(A^\bullet) \to B^i(B^\bullet)$ making the diagram commute. Finally, by the universal property of

$$\mathbf{h}^i(A^\bullet) \coloneqq \operatorname{coker}\bigl(B^i(A^\bullet) \to Z^i(A^\bullet)\bigr),$$

we obtain the map $\mathbf{h}^i(f)$. The fact that this defines a functor can be seen by imagining running through this argument with *three* rows in each commutative diagram for a composition of morphisms

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet.$$

Using universal properties, the maps induced by $g \circ f$ on cocycles, coboundaries, and cohomology objects coincide with the maps obtained by composing those obtained for g and f. Thus,

$$\mathbf{h}^i(g \circ f) = \mathbf{h}^i(g) \circ \mathbf{h}^i(f).$$

[Har77, p. 203]
[Wei94, Thm. 1.3.1]

Lemma 1.4.6. *Let $\mathscr{A}$ be an Abelian category and consider a short exact sequence*

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

in $\mathsf{Ch}(\mathscr{A})$. Then, there are maps $\delta^i \colon \mathbf{h}^i(C^\bullet) \to \mathbf{h}^{i+1}(A^\bullet)$ which yield a long exact sequence

$$\cdots \longrightarrow \mathbf{h}^{i-1}(C^\bullet) \xrightarrow{\delta^{i-1}} \mathbf{h}^i(A^\bullet) \longrightarrow \mathbf{h}^i(B^\bullet) \longrightarrow \mathbf{h}^i(C^\bullet) \xrightarrow{\delta^i} \mathbf{h}^{i+1}(A^\bullet) \longrightarrow \cdots$$

natural in $A^\bullet, B^\bullet, C^\bullet$.

We use the snake lemma below. See [KS06, Lemma 12.1.1] for a proof of the snake lemma that works in an arbitrary Abelian category.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Z^i(A^\bullet) \arrow[r] \arrow[d] & Z^i(B^\bullet) \arrow[r] \arrow[d] & Z^i(C^\bullet) \arrow[d] & \\
0 \arrow[r] & A^i \arrow[r] \arrow[d, "d"'] & B^i \arrow[r] \arrow[d, "d"'] & C^i \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & A^{i+1} \arrow[r] \arrow[d] & B^{i+1} \arrow[r] \arrow[d] & C^{i+1} \arrow[r] \arrow[d] & 0 \\
 & \dfrac{A^{i+1}}{dA^i} \arrow[r] \arrow[d] & \dfrac{B^{i+1}}{dB^i} \arrow[r] \arrow[d] & \dfrac{C^{i+1}}{dC^i} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

The snake lemma implies that the top and bottom rows are exact. The rows in the commutative diagram

$$\begin{tikzcd}
 & \dfrac{A^i}{dA^{i-1}} \arrow[r] \arrow[d, "d"'] & \dfrac{B^i}{dB^{i-1}} \arrow[r] \arrow[d, "d"'] & \dfrac{C^i}{dC^{i-1}} \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & Z^{i+1}(A^\bullet) \arrow[r, "f"] & Z^{i+1}(B^\bullet) \arrow[r, "g"] & Z^{i+1}(C^\bullet) &
\end{tikzcd}$$

are therefore exact. Applying the snake lemma again, we obtain the commutative diagram

$$\begin{tikzcd}
 & \mathbf{h}^i(A^\bullet) \arrow[r] \arrow[d] & \mathbf{h}^i(B^\bullet) \arrow[r] \arrow[d] & \mathbf{h}^i(C^\bullet) \arrow[d] \arrow[dddll, out=0, in=180, looseness=2] & \\
 & \dfrac{A^i}{dA^{i-1}} \arrow[r] \arrow[d, "d"'] & \dfrac{B^i}{dB^{i-1}} \arrow[r] \arrow[d, "d"'] & \dfrac{C^i}{dC^{i-1}} \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & Z^{i+1}(A^\bullet) \arrow[r, "f"] \arrow[d] & Z^{i+1}(B^\bullet) \arrow[r, "g"] \arrow[d] & Z^{i+1}(C^\bullet) \arrow[d] & \\
 & \mathbf{h}^{i+1}(A^\bullet) \arrow[r] & \mathbf{h}^{i+1}(B^\bullet) \arrow[r] & \mathbf{h}^{i+1}(C^\bullet). &
\end{tikzcd}$$

The snake map in the diagram is the morphism δ^i. The long exact sequence is natural in $A^\bullet, B^\bullet, C^\bullet$ because we can perform the same proof with two copies of each commutative diagram connected to each other via the maps induced by $A^\bullet \to A'^\bullet$, $B^\bullet \to B'^\bullet$, $C^\bullet \to C'^\bullet$, respectively. $\square$

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For later use, we also define the following:

[Har77, p. 203]
[Wei94, Def. 1.4.4]

Definition 1.4.7. Let $\mathscr{A}$ be an Abelian category. Two morphisms of complexes

$f, g\colon A^\bullet \to B^\bullet$ are *homotopic* (written $f \sim g$) if there is a collection of morphisms $k^i\colon A^i \to B^{i-1}$ for each i (which need not commute with the d^i) such that

$$f - g = dk + kd.$$

The collection of morphisms $k = (k^i)_{i\in\mathbf{Z}}$ is called a *homotopy operator*. If $f \sim g$, then f and g induce the *same* morphism $\mathbf{h}^i(A^\bullet) \to \mathbf{h}^i(B^\bullet)$ on cohomology objects, for every i [Wei94, Lemma 1.4.5].

1.4.2. Right derived functors and the definition of sheaf cohomology.

We are now ready to define right derived functors.

Definition 1.4.8. Let $\mathscr{A}$ be an Abelian category. A *right resolution* of an object A of $\mathscr{A}$ consists of a complex [TohokuI, p. 143n3]

$$C^\bullet = \bigl\{C^0 \longrightarrow C^1 \longrightarrow \cdots\bigr\}$$

indexed by the non-negative integers together with a *augmentation morphism* $\varepsilon\colon A \to C^\bullet$ such that

$$0 \longrightarrow A \xrightarrow{\varepsilon} C^0 \longrightarrow C^1 \longrightarrow \cdots$$

is exact. An *injective resolution* of A is a right resolution such that each C^i is injective. If $\mathscr{A}$ has enough injectives, then injective resolutions always exist by applying Theorem 1.3.35 repeatedly to $\mathrm{coker}(\varepsilon)$ and $\mathrm{coker}(d^i)$:

$$\begin{tikzcd}
& & & \mathrm{coker}(\varepsilon) \arrow[dr, hook] & & & & \\
0 \arrow[r] & A \arrow[r, "\varepsilon"] & I^0 \arrow[rr, "d^0"] \arrow[ur, two heads] & & I^1 \arrow[rr, "d^1"] \arrow[dr, two heads] & & I^2 \arrow[r] & \cdots \\
& & & & & \mathrm{coker}(d^0). \arrow[ur, hook] & &
\end{tikzcd}$$

Moreover, if $A \to B$ is a morphism, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & I^0 \arrow[r] \arrow[d] & I^1 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & B \arrow[r] & I'^0 \arrow[r] & I'^1 \arrow[r] & \cdots
\end{tikzcd} \tag{1.4.9}$$

by repeatedly using the definition of injective objects. Any two injective resolutions are homotopy equivalent [Wei94, Comparison Theorem 2.3.7] (see [Wei94, Comparison Theorem 2.2.6]) by using injectivity to construct the homotopy inductively.

Definition 1.4.10 [TohokuI, §2.3]. Let $F\colon \mathscr{A} \to \mathscr{A}'$ be an additive (covariant) functor between two Abelian categories. Suppose that $\mathscr{A}$ has enough injectives. For every object A in $\mathscr{A}$, the *right derived functors* of F are defined by setting

$$R^iF(A) \coloneqq \mathbf{h}^i\bigl(F(I^\bullet)\bigr)$$

for an injective resolution $A \to I^\bullet$. The action of R^iF on morphisms is determined by applying F to the diagram (1.4.9) after deleting A and B.

As a definition, it is not clear $R^iF(A)$ is well-defined: Why is the definition independent of the injective resolution chosen? There are of course more things we want to show about R^iF. For now, let us state the two main examples we will study in this course.

Definition 1.4.11. Let $(X, \mathcal{O}_X)$ be a topological space. Let $\mathscr{F}$ be an Abelian sheaf.

[TohokuI, p. 157]
[Har77, p. 207, Exer. III.2.3]

(i) Consider the global sections functor with support in a closed subset $Z \subseteq X$

$$\Gamma_Z(X, \cdot\,)\colon \mathsf{Ab}(X) \longrightarrow \mathsf{Ab}$$

defined by $\Gamma_Z(X, \cdot\,) \coloneqq \Gamma(X, \mathscr{H}^0_Z(\,\cdot\,))$. The *$i$-th sheaf cohomology group of $\mathscr{F}$ with support in Z* is

$$H^i_Z(X, \mathscr{F}) \coloneqq (R^i\Gamma_Z)(\mathscr{F}).$$

If $Z = X$, we call this module the *i-th sheaf cohomology group of $\mathscr{F}$* and write $H^i(X, \mathscr{F})$. These modules are also called the *i-th local cohomology groups of $\mathscr{F}$ with support in Z.*

[TohokuII, §4.2]
[Har77, p. 233]

(ii) The functors $\mathrm{Ext}^i(\mathscr{F}, \cdot\,)$ are the right derived functors of $\mathrm{Hom}(\mathscr{F}, \cdot\,)$ and the functors $\mathscr{E}xt^i(\mathscr{F}, \cdot\,)$ are the right derived functors of $\mathscr{H}om(\mathscr{F}, \cdot\,)$. We will discuss these more later.

Here is the main existence result.

[Har77, Thm. III.1.1A]
[Wei94, Thm. 2.4.6]
[Wei94, Lem. 2.4.1]

Theorem 1.4.12. *Let $F\colon \mathscr{A} \to \mathscr{A}'$ be a left exact functor between Abelian categories such that $\mathscr{A}$ has enough injectives.*

(a) *For every $i \geq 0$, R^iF as defined above is an additive functor $\mathscr{A} \to \mathscr{A}'$. Furthermore, it is independent (up to isomorphism of functors) of the choices of injective resolutions made.*

(b) *There is an isomorphism of functors $F \cong R^0F$.*

(c) *For every short exact sequence $0 \to A' \to A \to A'' \to 0$ and for every $i \geq 0$, there is a morphism $\delta^i\colon R^iF(A'') \to R^{i+1}F(A')$ natural in the short exact sequence fitting in the long exact sequence*

$$\cdots \longrightarrow R^iF(A') \longrightarrow R^iF(A) \longrightarrow R^iF(A'') \xrightarrow{\delta^i} R^{i+1}F(A') \longrightarrow R^{i+1}F(A) \longrightarrow \cdots.$$

(d) *For every commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0
\end{tikzcd}$$

with exact rows, the δ's fit into the commutative diagrams

$$\begin{tikzcd}
R^iF(A'') \arrow[r, "\delta^i"] \arrow[d] & R^{i+1}F(A') \arrow[d] \\
R^iF(B'') \arrow[r, "\delta^i"] & R^{i+1}F(B').
\end{tikzcd}$$

(e) *For every injective object I of $\mathscr{A}$ and for every $i > 0$, we have $R^iF(I) = 0$.*

Proof Sketch. For (a), we compare two injective resolutions $I^\bullet, J^\bullet$ using the diagram (1.4.9) for $A = B$ to obtain a chain map $f\colon I^\bullet \to J^\bullet$. Any other chain map f' comparing the two is homotopic to f, so the action on $\mathbf{h}^i$ is well-defined. Similarly, we can find a chain map $g\colon J^\bullet \to I^\bullet$. The compositions $f \circ g$ and $g \circ f$ are both homotopic to the identity. (e) now follows by letting I be its own injective resolution. (b) holds by left-exactness.

(*c*) and (*d*) are harder. The idea is as follows. Given a short exact sequence as written, we find resolutions fitting into the short exact sequence

$$0 \longrightarrow I'^{\bullet} \longrightarrow I^{\bullet} \longrightarrow I''^{\bullet} \longrightarrow 0$$

using [Wei94, Horseshoe Lemma 2.2.8]: we consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & I'^0 \\
0 \arrow[r] & A \arrow[d] \arrow[ur, dashed, "\exists"'] & \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & I''^0 \\
 & 0 &
\end{tikzcd}$$

and using the two maps $A \to I'^0$ and $A \to I''^0$, we can fill in the diagram with $I^0 := I'^0 \oplus I''^0$. Iterating with the correspoding short exact sequence of cokernels, we obtain an injective resolution for A. Now since $I'^{\bullet}$ is a complex of injectives, we can find a splitting of the inclusion $I'^{\bullet} \hookrightarrow I^{\bullet}$. Thus, the sequence

$$0 \longrightarrow F(I'^{\bullet}) \longrightarrow F(I^{\bullet}) \longrightarrow F(I''^{\bullet}) \longrightarrow 0$$

remains exact. We then apply Lemma 1.4.6 to prove (*c*). Finally, to prove (*d*), we need to construct injective resolutions for all the objects involved such that

$$\begin{tikzcd}
0 \arrow[r] & I'^{\bullet} \arrow[r] \arrow[d] & I^{\bullet} \arrow[r] \arrow[d] & I''^{\bullet} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J'^{\bullet} \arrow[r] & J^{\bullet} \arrow[r] & J''^{\bullet} \arrow[r] & 0
\end{tikzcd}$$

commutes, and then apply the naturality in Lemma 1.4.6. See [Wei94, Theorem 2.4.6] for details. ☐

1.4.3. δ-functors. We want to give a universal property of derived functors. To do this, we introduce a more general notion.

Definition 1.4.13 [TohokuI, §2.1] (cf. [CE56, Chapter III]). Let $\mathscr{A}$ be an Abelian category and let $\mathscr{A}'$ be an additive category. Let $a, b \in \mathbf{Z} \cup \{\pm\infty\}$ such that $a + 1 < b$. A *(covariant) δ-functor $T\colon \mathscr{A} \to \mathscr{A}'$ defined in degrees $a < i < b$* is a sequence of covariant additive functors

[Har77, p. 205]

$$T = \bigl(T^i\colon \mathscr{A} \longrightarrow \mathscr{A}'\bigr)_{i \in (a,b)}$$

and *connecting morphisms*

$$\delta^i\colon T^i(A'') \longrightarrow T^{i+1}(A')$$

for each short exact sequence $0 \to A' \to A \to A'' \to 0$ and $a < i < b - 1$ satisfying the following axioms:

(*i*) For every commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0 \end{tikzcd}$$

with exact rows, the corresponding diagram

$$\begin{tikzcd} T^i(A'') \arrow[r, "\delta^i"] \arrow[d] & T^{i+1}(A') \arrow[d] \\ T^i(B'') \arrow[r, "\delta^i"] & T^{i+1}(B') \end{tikzcd}$$

commutes.

(*ii*) For every short exact sequence $0 \to A' \to A \to A'' \to 0$, the sequence

$$\cdots \longrightarrow T^i(A') \longrightarrow T^i(A) \longrightarrow T^i(A'') \xrightarrow{\delta^i} T^{i+1}(A') \longrightarrow \cdots \tag{1.4.14}$$

is a complex.

Suppose that $\mathscr{A}'$ is also Abelian. We say that T is *exact* if the sequence (1.4.14) is exact for every short exact sequence $0 \to A' \to A \to A'' \to 0$. A *cohomological functor* is an exact δ-functor defined for all degrees.

If T and T' are two δ-functors defined for the same degrees, a *natural transformation* $f\colon T \Rightarrow T'$ is a sequence of natural transformations

$$f^i\colon T^i \Longrightarrow T'^i$$

such that for every short exact sequence $0 \to A' \to A \to A'' \to 0$, the diagrams

$$\begin{tikzcd} T^i(A'') \arrow[r, "\delta^i"] \arrow[d] & T^{i+1}(A') \arrow[d] \\ T'^i(A'') \arrow[r, "\delta^i"] & T'^{i+1}(A') \end{tikzcd}$$

commute for all i.

[Har77, pp. 205–206] See [McL10, p. 368] for some discussion of the set-theoretic subtleties in this definition.

Definition 1.4.15 [TohokuI, §2.2]. Let $\mathscr{A}, \mathscr{A}'$ be Abelian categories. Let $T = (T^i)_{0 \le i \le a}$ be a covariant δ-functor from $\mathscr{A}$ to $\mathscr{A}'$ where $a > 0$. We say that T is a *universal δ-functor* if for every δ-functor $T' = (T'^i)_i$ defined for the same degrees and every natural transformation $f^0\colon T^0 \Rightarrow T'^0$, there exists a unique natural transformation $f\colon T \Rightarrow T'$ reducing to f^0 in degree 0.

[TohokuI, p. 140] [Har77, Rem. III.1.2.1]

Since universal δ-functors are defined using a universal property, a universal δ-functor (if it exists!) is unique.

To prove universal δ-functors exist, we use the following notion.

[Har77, p. 206]

Definition 1.4.16 [TohokuI, p. 141]. Let $F\colon \mathscr{A} \to \mathscr{A}'$ be an additive functor between Abelian categories. We say that F is *effaceable* if, for every object A in $\mathscr{A}$, there is a monomorphism $u\colon A \hookrightarrow M$ such that $F(u) = 0$. If $\mathscr{A}$ has enough injectives, this is equivalent to saying that $F(M) = 0$ for every injective object M.

[Har77, Thm. III.1.3A]

Proposition 1.4.17 [TohokuI, Proposition 2.2.1]. *Let $\mathscr{A}, \mathscr{A}'$ be two Abelian categories and consider an exact δ-functor*

$$T = (T^i)_{i \in [0,a)}\colon \mathscr{A} \longrightarrow \mathscr{A}'$$

where $a > 1$. If T^i is effaceable for every $i > 0$, then T is a universal δ-functor.

The converse holds if every object A in $\mathscr{A}$ admits an injective effacement [TohokuI, Théorème 2.2.2], in particular if enough injectives exist.

Proof. By induction and relabeling, it suffices to show that if (T'^0, T'^1) is a δ-functor defined in degrees $0, 1$ and f^0 is a natural transformation $T^0 \Rightarrow T'^0$, then there exists a unique natural transformation $(f^0, f^1)\colon (T^0, T^1) \Rightarrow (T'^0, T'^1)$ reducing to f^0 in degree 0. Let A be an object of $\mathscr{A}$ and consider a short exact sequence

$$0 \longrightarrow A \xrightarrow{u} M \longrightarrow A' \longrightarrow 0$$

such that $T^1(u) = 0$. Consider the commutative diagram

$$\begin{tikzcd}
T^0(M) \arrow[r] \arrow[d] & T^0(A') \arrow[r] \arrow[d, "f^0(A')"] & T^1(A) \arrow[r, "0"] \arrow[d, dashed, "f^1(A)"] & T^1(M) \\
T'^0(M) \arrow[r] & T'^0(A') \arrow[r] & T'^1(A)
\end{tikzcd} \tag{1.4.18}$$

where the first row is exact. There exists a dashed morphism making the diagram commute by the universal property of cokernels since the composition $T^0(M) \to T'^1(A)$ is 0 by the fact that the bottom row is a complex. Since the first row is exact, $T^0(A') \to T^1(A)$ is an epimorphism. Thus, the morphism $f^1(A)$ is unique.

Next, we show that $f^1(A)$ does not depend on the choice of effacement $u\colon A \hookrightarrow M$. If we have another effacement $v\colon A \hookrightarrow N$, then consider

$$v\colon A \longrightarrow M \oplus N$$

which is still an effacement by additivity of the functor T^1. We can then consider the commutative diagram

$$\begin{tikzcd}
T^0(M \oplus N) \arrow[r] \arrow[d] & T^0(A'_\oplus) \arrow[r] \arrow[d] & T^1(A) \arrow[r, "0"] \arrow[d, equal] & T^1(M \oplus N) \\
T^0(M) \arrow[r] \arrow[d] & T^0(A') \arrow[r] \arrow[d, "f^0(A')"] & T^1(A) \arrow[r, "0"] \arrow[d, "f^1(A)"] & T^1(M) \\
T'^0(M) \arrow[r] & T'^0(A') \arrow[r] & T'^1(A) \\
T'^0(M \oplus N) \arrow[r] \arrow[u] & T'^0(A'_\oplus) \arrow[r] \arrow[u] & T'^1(A). \arrow[u, equal]
\end{tikzcd}$$

The composition $T^1(A) \to T'^1(A)$ in the third column is unique by using the universal property of cokernels as before.

Finally, it remains to show that f^1 defined as above commutes with connecting morphisms. Consider a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Let $B \hookrightarrow M$ be an effacement. Then, $A \to B \to M$ is also an effacement. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, hook] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & M \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Constructing the f^1 maps as before, we have the diagram

$$\begin{tikzcd}
T^0(C) \arrow[rr] \arrow[dr] \arrow[dd, "f^0(C)"'] & & T^1(A) \arrow[dd, "f^1(A)"] \\
& T^0(C') \arrow[ur] \arrow[dd, "f^0(C')" near start] & \\
T'^0(C) \arrow[rr, crossing over] \arrow[dr] & & T'^1(A) \\
& T'^0(C') \arrow[ur] &
\end{tikzcd}$$

where the front left face commutes by the functoriality of f^0, the front right face commutes by the construction of $f^1(A)$ in (1.4.18), and the top and bottom faces commute by the definition of δ-functors. This shows the back face commutes. $\square$

The following is a special case of [TohokuI, Théorème 2.2.2].

[Har77, Cor. III.1.4]

Theorem 1.4.19 [TohokuI, Théorème 2.2.2]. *Let $F\colon \mathscr{A} \to \mathscr{A}'$ be a left exact functor between Abelian categories such that $\mathscr{A}$ has enough injectives. Then, the derived functors R^iF exist and form a universal δ-functor with $F \cong R^0F$. Conversely, if $T = (T^i)_{i\geq 0}$ is a universal δ-functor, then T^0 is left exact, and the T^i are isomorphic to R^iT^0 for all $i \geq 0$.*

Proof. The derived functors R^iF exist by Theorem 1.4.12. Since $R^iF(I) = 0$ for all $i > 0$ and $\mathscr{A}$ has enough injectives, we see that R^iF is effaceable for every $i > 0$, and hence the R^iF form a universal δ-functor.

For the converse, T^0 is left exact by the definition of δ-functors. Thus, the derived functors R^iT^0 exist by Theorem 1.4.12, and also form a universal δ-functor. Since $R^0T^0 = T^0$, we see that $R^iT^0 \cong T^i$ for all i by uniqueness. $\square$

1.4.4. Computing derived functors using acyclic objects. One difficult aspect of our theory so far is: How can we actually compute derived functors in practice? Injective resolutions are hard to compute, so it is not easy to use them in practice.

We will explore multiple ways to compute sheaf cohomology. When we get back to talking about schemes, we will explore different conditions on schemes that make sheaf cohomology computable and satisfy finiteness properties. For now, we want to find a way to use flasque sheaves to compute sheaf cohomology. This is the definition taken in [God73, Chapitre II, §4].

We start with a general definition.

[Har77, p. 205]
[Wei94, 2.4.3]

Definition 1.4.20 [TohokuI, Théorème 2.4.1, Remarque 3]. Consider a left exact functor $F\colon \mathscr{A} \to \mathscr{A}'$ between Abelian categories with right derived functors R^iF. An object I of $\mathscr{A}$ is *F-acyclic* if $R^iF(I) = 0$ for all $i > 0$.

$\boxed{1/28}$
[Har77, Prop. III.1.2A]
[Wei94, Exer. 2.4.3]

Proposition 1.4.21 [TohokuI, Théorème 2.4.1, Remarque 3]. *Consider a left exact functor $F\colon \mathscr{A} \to \mathscr{A}'$ between Abelian categories with right derived functors R^iF. Suppose there is a right resolution*

$$0 \longrightarrow A \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots \tag{1.4.22}$$

where each J^i is F-acyclic for all $i \geq 0$. For every $i \geq 0$, there is an isomorphism

$$R^iF(A) \cong \mathbf{h}^i\bigl(F(J^\bullet)\bigr).$$

A resolution of the form (1.4.22) is called an *F-acyclic resolution*. The proof method below is called "dimension shifting" in [Wei94, Exercise 2.4.3].

Proof. We proceed by induction on i (where A is allowed to vary!). For $i = 0$, we consider the exact sequence

$$0 \longrightarrow A \longrightarrow J^0 \longrightarrow J^1.$$

Applying F, we obtain

$$0 \longrightarrow F(A) \longrightarrow F(J^0) \longrightarrow F(J^1).$$

We therefore see that $F(A)$ is isomorphic to $\mathbf{h}^0(F(J^\bullet))$.

For $i > 0$, we consider the exact sequence

$$0 \longrightarrow A \longrightarrow J^0 \longrightarrow A' \longrightarrow 0.$$

Note that $0 \to A' \to J^1 \to \cdots$ is an F-acyclic resolution. Looking at the long exact sequence for R^iF, we obtain the isomorphisms

$$\mathbf{h}^{i-1}\big(F(J^1 \to J^2 \to \cdots)\big) \cong R^{i-1}F(A') \xrightarrow{\sim} R^iF(A)$$

for every $i \geq 1$, where the isomorphism on the left holds by inductive hypothesis. We are now done since

$$\mathbf{h}^{i-1}\big(F(J^1 \to J^2 \to \cdots)\big) = \mathbf{h}^i\big(F(J^0 \to J^1 \to J^2 \to \cdots)\big). \qquad \square$$

1.4.5. Flasque sheaves are acyclic. Our goal now is to show that flasque sheaves are acyclic for the global sections functor $\Gamma(X, \cdot)$.

LEMMA 1.4.23 [God73, Chapitre II, Lemme 7.3.2]. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then, every injective $\mathcal{O}_X$-module $\mathscr{I}$ is flasque.* [Har77, Lem. III.2.4]

Proof. Let $\mathscr{I}$ be an injective $\mathcal{O}_X$-module and let $V \subseteq U$ be open sets. We then have the inclusion

$$0 \longrightarrow (j_V)_!\mathcal{O}_V \longrightarrow (j_U)_!\mathcal{O}_U$$

of sheaves of $\mathcal{O}_X$-modules, where $j_V\colon V \hookrightarrow X$ and $j_U\colon U \hookrightarrow X$ are the respective inclusion maps. Applying $\operatorname{Hom}(\cdot, \mathscr{I})$, we obtain the surjection

$$\operatorname{Hom}\big((j_U)_!\mathcal{O}_U, \mathscr{I}\big) \longrightarrow \operatorname{Hom}\big((j_V)_!\mathcal{O}_V, \mathscr{I}\big) \longrightarrow 0.$$

This surjection is isomorphic to $\mathscr{I}(U) \to \mathscr{I}(V)$, and hence $\mathscr{I}$ is flasque. $\square$

We use dimension shifting again to show the following:

PROPOSITION 1.4.24 [TohokuI, Proposition 3.3.2, Corollaire]. *Let X be a topological space. Suppose $\mathscr{F}$ is a flasque Abelian sheaf on X. Then, $H^i(X, \mathscr{F}) = 0$ for all $i > 0$.* [Har77, Prop. III.2.5]

Proof. We induce on i (where $\mathscr{F}$ is allowed to vary). Since $\mathsf{Ab}(X)$ has enough injectives, we can find an injective $\mathcal{O}_X$-module $\mathscr{I}$ and a short exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I} \longrightarrow \mathscr{G} \longrightarrow 0.$$

Then, $\mathscr{F}$ is flasque by hypothesis, $\mathscr{I}$ is flasque by Lemma 1.4.23, and $\mathscr{G}$ is flasque by [Har77, Exercise II.1.16(*c*)] (Homework 1, Problem 1(*c*)).

For the case $i = 1$, since $\mathscr{F}$ is flasque, we have the short exact sequence

$$0 \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{I}) \longrightarrow \Gamma(X, \mathscr{G}) \longrightarrow 0$$

by [Har77, Exercise II.1.16(b)] (Homework 1, Problem 1(b)). Since $\mathscr{I}$ is injective, we have $H^i(X,\mathscr{I})=0$ for all $i>0$. Looking at the associated long exact sequence, the sequence

$$\Gamma(X,\mathscr{I}) \longrightarrow \Gamma(X,\mathscr{G}) \xrightarrow{0} H^1(X,\mathscr{F}) \longrightarrow H^1(X,\mathscr{I}) = 0$$

is exact, and hence $H^1(X,\mathscr{F})=0$.

For the case $i \geq 2$, the long exact sequence yields the isomorphisms

$$H^{i-1}(X,\mathscr{G}) \xrightarrow{\sim} H^i(X,\mathscr{F})$$

for every $i \geq 2$. By the inductive hypothesis, the left-hand side is 0. ☐

In particular, we have the following result, which says that to compute sheaf cohomology of an $\mathcal{O}_X$-module, it does not matter whether we think of it as an Abelian sheaf or an $\mathcal{O}_X$-module.

[Har77, Prop. III.2.6]

Proposition 1.4.25 [TohokuI, Théorème 2.4.1, Remarque 3]. *Let $(X,\mathcal{O}_X)$ be a ringed space. We have the following commutative diagram of categories and functors*

$$\begin{tikzcd}
\mathsf{Mod}(\mathcal{O}_X) \arrow[r, "R^i\Gamma(X,\,\cdot\,)"] \arrow[d, "\mathsf{Forget}"'] & \mathsf{Mod}\big(\Gamma(X,\mathcal{O}_X)\big) \arrow[d, "\mathsf{Forget}"] \\
\mathsf{Ab}(X) \arrow[r, "H^i(X,\,\cdot\,)"] & \mathsf{Ab}
\end{tikzcd}$$

where in the first row, $R^i\Gamma(X,\,\cdot\,)$ is computed using injective resolutions in $\mathsf{Mod}(\mathcal{O}_X)$.

As a result, there is no ambiguity when we use the notation $H^i(X,\mathscr{F})$.

Proof. We calculate the derived functors as $\mathcal{O}_X$-modules using injective resolutions in $\mathsf{Mod}(\mathcal{O}_X)$. But injective sheaves are flasque (Lemma 1.4.23), and flasque sheaves are acyclic (Proposition 1.4.24). We are done by Proposition 1.4.21. ☐

We can also compute sheaf cohomology of sheaves of the form $i_*\mathscr{F}$ where i is the inclusion of a closed set.

[Har77, Lem. III.2.10]

Lemma 1.4.26 [God73, Chapitre II, Théorème 4.9.1, Corollaire; TohokuI, Théorème 3.5.1]. *Let Y be a closed subset of a topological space X with inclusion map $i\colon Y \hookrightarrow X$. Let $\mathscr{F}$ be an Abelian sheaf on Y. Then, we have isomorphisms*

$$H^i(X,i_*\mathscr{F}) \cong H^i(Y,\mathscr{F}).$$

Proof. Let $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ be an injective resolution on Y. The pushforward $0 \to i_*\mathscr{F} \to i_*\mathscr{I}^\bullet$ is a flasque resolution on X: Exactness holds by [Har77, Exercise II.1.19(a)] and flasqueness holds by [Har77, Exercise II.1.16(d)]. Both these exercises were solved on Homework 1. ☐

1.4.6. Grothendieck's vanishing theorem. We now come to the main application of the theory so far: Sheaf cohomology vanishes beyond the Krull dimension of a Noetherian topological space.

[Har77, Thm. III.2.7]

Theorem 1.4.27 (Grothendieck vanishing [TohokuI, Théorème 3.6.5]). *Let X be a Noetherian topological space of dimension $\leq n$. Then, $H^i(X,\mathscr{F})=0$ for all $i>n$ and every Abelian sheaf $\mathscr{F}$ on X.*

When X is an algebraic variety and $\mathscr{F}$ is *coherent* (we will define this soon!), Theorem 1.4.27 is due to Serre. See [FAC, n° 53, Proposition 4] for the case of curves, see [FAC, n° 52, Proposition 3] for the case of quasi-projective varieties, and see [Ser57, Théorème 2] for the case of algebraic varieties (in the sense of [FAC, n° 34, Définition]) in general.

Proving Theorem 1.4.27 requires some work. We start by proving how derived functors behave under direct limits.

[Har77, Prop. III.2.9]

PROPOSITION 1.4.28 [TohokuI, Proposition 3.6.3(1)]. *Let $\mathscr{A}$ and $\mathscr{A}'$ be two Abelian categories satisfying* (AB5). *Suppose that $\mathscr{A}$ admits a generator. Let $T\colon \mathscr{A} \to \mathscr{A}'$ be a left exact functor. For the right derived functors R^iT to commute with filtered direct limits, it suffices for the following to hold:*

(a) *T commutes with filtered direct limits.*
(b) *If $M = \varinjlim M_i$ is a filtered direct limit of injective objects in $\mathscr{A}$, then M is T-acyclic.*

Proof. Let $\{A_i\}_{i\in I}$ be a filtered direct system in $\mathscr{A}$ and let A be its direct limit. We want to show that

$$\varinjlim R^pT(A_i) \longrightarrow R^pT(A)$$

is an isomorphism.

We claim there exists a direct system $\{C_i^\bullet\}_{i\in I}$ of complexes (with values in $\mathscr{A}$) and an augmentation map $\{A_i\} \to \{C_i^\bullet\}$ such that for every $i \in I$, $A_i \to C_i^\bullet$ is an injective resolution of A_i. To show this, consider the category $\mathsf{ind}_I(\mathscr{A})$ consisting of all direct systems of objects of $\mathsf{Ab}(X)$ indexed by I. If U is a generator for $\mathscr{A}$, then

$$\bigoplus_{i\in I} \varepsilon_i(U)$$

is a generator for $\mathsf{ind}_I(\mathscr{A})$, where $\varepsilon_i(U)$ is the inductive system that is U at i and 0 elsewhere (see [TohokuI, Proposition 1.9.2]). Now by Theorem 1.3.35, we see that $\mathsf{ind}_I(\mathscr{A})$ has enough injectives. Since exactness is computed entrywise, we see that if $\{M_i\}$ is an injective object, then each M_i is injective. Now we can construct the desired direct system $\{C_i^\bullet\}_{i\in I}$ by taking an injective resolution in $\mathsf{ind}_I(\mathscr{A})$.

Now by hypothesis, each $C^j := \varinjlim_i C_i^j$ is acyclic. (AB5) ensures that $A \to C^\bullet$ is an acyclic resolution. Since T commutes with filtered direct limits by the hypothesis, the exactness of $\varinjlim$ on $\mathsf{ind}_I(\mathscr{A}')$ implies

$$\mathbf{h}^p\bigl(T(C^\bullet)\bigr) = \varinjlim_{i\in I} \mathbf{h}^p\bigl(T(C_i^\bullet)\bigr) = \varinjlim_{i\in I} R^pT(A_i). \qquad \square$$

In order to apply Proposition 1.4.28, we need to verify the two hypotheses. We start with $\Gamma(X, \cdot\,)$ permuting with direct limits.

[Har77, Exer. II.1.11]

LEMMA 1.4.29 [God73, Chapitre II, Théorème 3.10.1]. *Let X be a topological space and consider a filtered direct system $\{\mathscr{F}_i\}_{i\in I}$ of sheaves of sets on X. Consider the direct limit presheaf*

$$\varinjlim_{i\in I}{}^{\mathcal{P}} \mathscr{F}_i := \Bigl(U \longmapsto \varinjlim_{i\in I} \mathscr{F}_i(U)\Bigr)$$

and its sheafification

$$\varinjlim_{i\in I} \mathscr{F}_i := \left(U \longmapsto \varinjlim_{i\in I} \mathscr{F}_i(U)\right)^{\#}.$$

Suppose that X is Noetherian. Then, the sheafification map

$$\varinjlim_{i\in I}{}^{\mathcal{P}} \mathscr{F}_i \longrightarrow \varinjlim_{i\in I} \mathscr{F}_i$$

is an isomorphism. In particular,

$$\varinjlim_{i\in I} \Gamma(X, \mathscr{F}_i) \longrightarrow \Gamma\left(X, \varinjlim_{i\in I} \mathscr{F}_i\right)$$

is an isomorphism.

Remark 1.4.30. The forgetful functors

$$\mathsf{Mod}(\mathcal{O}_X) \longrightarrow \mathsf{Ab}(X) \longrightarrow \mathsf{Sh}(X)$$

reflect isomorphisms, that is, a morphism in one of these categories is an isomorphism if it is an isomorphism after applying a forgetful functor. This is because in the category of modules or of Abelian groups, bijective morphisms are isomorphisms. Thus, Lemma 1.4.29 applies to $\mathcal{O}_X$-modules or Abelian sheaves as well.

Proof of Lemma 1.4.29. Set $\mathscr{F} := \varinjlim \mathscr{F}_i$. For injectivity, suppose that

$$s', s'' \in \varinjlim_{i\in I} \mathscr{F}_i(U)$$

map to the same section in $\mathscr{F}(U)$. Since a presheaf and its sheafification have the same stalks, there exists an open covering $U = \bigcup_j U_j$ such that

$$s'|_{U_j} = s''|_{U_j}$$

for all j. Since X is Noetherian, U is quasi-compact, and hence we can take a finite subcover of the $\{U_j\}$ to assume that there are only finitely many U_j. Now let i_0 be an index for which s', s'' are represented by elements

$$s'_0, s''_0 \in \mathscr{F}_{i_0}(U).$$

For each j, we can find an index $i_j \geq i_0$ such that $s'_0|_{U_j}, s''_0|_{U_j}$ are equal after mapping to $\mathscr{F}_{i_j}(U_j)$. We then see that s', s'' are represented by equal elements in $\mathscr{F}_{\max\{i_j\}}(U)$.

For surjectivity, let $s \in \mathscr{F}(U)$. Since sheafification induces bijections on stalks (Proposition 1.2.15(*i*)), for each $P \in U$, there exists a section

$$s^P \in \mathscr{F}_{i_P}(U_P)$$

representing the germ $s_P \in \mathscr{F}_P$ of s at P. Replacing U_P by smaller subsets and using the definition of stalks as a direct limit, we may assume that the image of s^P in $\mathscr{F}(U_P)$ and $s|_{U_P}$ coincide. Since X is Noetherian, U is quasi-compact, and hence we can take a finite subcover of the $\{U_P\}$ to assume that there are only finitely many U_P, which we call $U_1, U_2, \ldots, U_n$ with corresponding sections

$$s_1, s_2, \ldots, s_n \in \mathscr{F}_j(U_j).$$

Replacing the s_j by their images in $\mathscr{F}_{\max\{j\}}(U_j)$, we can then glue the s_j to a section in $\mathscr{F}_{\max\{j\}}(U)$. This section maps to $s \in \mathscr{F}(U)$ by definition of the direct limit. $\square$

As a consequence, we obtain:

Lemma 1.4.31 [TohokuI, Lemme 3.6.4; God73, p. 163]. *Let X be a Noetherian topological space. Then, filtered direct limits of flasque sheaves of sets on X are flasque.*

[Har77, Lem. III.2.8]

Proof. Let $V \subseteq U$ be an inclusion of open subsets of X. This follows from the commutative diagram

$$\begin{tikzcd}
\varinjlim_{i\in I} \mathscr{F}_i(U) \arrow[r, "\sim"] \arrow[d, two heads] & \left(\varinjlim_{i\in I} \mathscr{F}_i\right)(U) \arrow[d] \\
\varinjlim_{i\in I} \mathscr{F}_i(V) \arrow[r, "\sim"] & \left(\varinjlim_{i\in I} \mathscr{F}_i\right)(V)
\end{tikzcd}$$

where the left vertical map is surjective by the fact that each $\mathscr{F}_i$ is flasque and the horizontal maps are bijective by Lemma 1.4.29. ☐

Corollary 1.4.32 [TohokuI, Proposition 3.6.3(1)]. *Let X be a Noetherian topological space. Then, $H^i(X, \cdot\,)$ commutes with filtered direct limits of Abelian sheaves.*

[Har77, Prop. III.2.9]

Proof. The two hypotheses in Proposition 1.4.28 hold by Lemma 1.4.29 and by Lemma 1.4.31 since injective sheaves are flasque by Lemma 1.4.23. ☐

We are now ready to prove Grothendieck's vanishing Theorem 1.4.27. The proof is a good example of how lots of proofs in algebraic geometry go: One uses induction with respect to closed subsets to reduce to the irreducible case, and then limit arguments to reduce to the finitely generated case. Eventually, we will reduce to the case of considering sheaves of the form $\mathbf{Z}_U$.

Proof of Grothendieck's vanishing Theorem 1.4.27. We fix some notation. If $Y \subseteq X$ is a closed subset with complement U, denote

$$U \xhookrightarrow{j} X \xhookleftarrow{i} Y.$$

For any sheaf $\mathscr{F}$ on X, we set $\mathscr{F}_Y := i_*(\mathscr{F}|_Y)$ and $\mathscr{F}_U := j_!(\mathscr{F}|_U)$, in which case we have the short exact sequence

$$0 \longrightarrow \mathscr{F}_U \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}_Y \longrightarrow 0$$

by [Har77, Exercise II.1.19(c)].

We proceed by induction on n.

Step 1. The case $n = 0$.

If $n = 0$, then every $\mathscr{F}$ is flasque, and hence we are done by Proposition 1.4.24.

Step 2. The case $n \geq 1$. Reduction to the irreducible case.

Let X_k be the irreducible components of X [Mur24ag, Proposition 1.1.32]. Setting $\mathscr{F}_k := \mathscr{F}_{X_k}$, we have the short exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \bigoplus_k \mathscr{F}_k \longrightarrow \mathscr{R} \longrightarrow 0,$$

where $\mathscr{R}$ has support contained in $Y = \bigcup_{k \neq l} X_k \cap X_l$, which is of dimension $\leq n-1$. Here, the injection on the left holds by considering how functions glue on a closed covering of an open subset $U \subseteq X$. We then have the exact sequence

$$H^{i-1}(X, \mathscr{R}) \longrightarrow H^i(X, \mathscr{F}) \longrightarrow \bigoplus_k H^i(X, \mathscr{F}_k).$$

If $i > n$, then $i - 1 > n - 1$, and hence

$$H^{i-1}(X, \mathscr{R}) \cong H^{i-1}(Y, \mathscr{R}|_Y) = 0$$

by the inductive hypothesis, since $\mathscr{R} \cong i_*(\mathscr{R}|_Y)$ by looking at the short exact sequence

$$0 \longrightarrow \mathscr{R}_{X-Y} \longrightarrow \mathscr{R} \longrightarrow \mathscr{R}_Y \longrightarrow 0$$

and noting that $\mathscr{R}_{X-Y} = 0$. It therefore suffices to show that $H^i(X, \mathscr{F}_k) = 0$ for all $i > n$. Since

$$H^i(X, \mathscr{F}_k) \cong H^i\bigl(X_k, \mathscr{F}|_{X_k}\bigr)$$

and each X_k is of dimension $\leq n$, we may work one irreducible component at a time to assume that X is irreducible.

[TohokuI, Prop. 3.6.1]

Step 3. The case $n \geq 1$. The irreducible case.

To simplify notation, we introduce the following:

Definition 1.4.33 [TohokuI, p. 167]. Let $\mathscr{F}$ be an Abelian sheaf on a topological space X. Let $\{f_i \in \mathscr{F}(U_i)\}_{i \in I}$ be a family of sections on open subsets $U_i \subseteq X$. Each f_i defines a morphism $\mathbf{Z}_{U_i} \to \mathscr{F}$, and hence the family $\{f_i\}$ induces a morphism

$$\bigoplus_{i \in I} \mathbf{Z}_{U_i} \longrightarrow \mathscr{F}. \tag{1.4.34}$$

We say that the family $\{f_i\}$ is a *system of generators of* $\mathscr{F}$ if the morphism (1.4.34) is a surjection.

Taking $\{f_i\} = \bigcup_{U \subseteq X} \mathscr{F}(U)$, we see that every Abelian sheaf $\mathscr{F}$ is generated by some family of sections, and that every Abelian sheaf is the filtered direct limit of subsheaves generated by a *finite* system of generators. By Lemma 1.4.29, we may therefore assume that $\mathscr{F}$ is generated by a finite system of generators $\{f_i\}_{1 \leq i \leq m}$.

We now induce on the number of generators m.

Substep 3.1. The inductive step.

Let $\mathscr{F}_j$ be the subsheaf of $\mathscr{F}$ generated by $f_1, f_2, \ldots, f_j$ for each $j \leq m$. The long exact sequence associated to the short exact sequence

$$0 \longrightarrow \mathscr{F}_{m-1} \longrightarrow \mathscr{F} \longrightarrow \frac{\mathscr{F}}{\mathscr{F}_{m-1}} \longrightarrow 0$$

together with the inductive hypothesis and the $m = 1$ case proves the inductive case.

Substep 3.2. The case $m = 1$.

We have the short exact sequence

$$0 \longrightarrow \mathscr{R} \longrightarrow \mathbf{Z}_U \longrightarrow \mathscr{F} \longrightarrow 0.$$

By the long exact sequence on cohomology, it suffices to show that $H^i(X, \mathbf{Z}_U) = 0$ and that $H^{i+1}(X, \mathscr{R}) = 0$ for all $i > n$. The vanishing $H^i(X, \mathbf{Z}_U) = 0$ holds by considering the long exact sequence associated to

$$0 \longrightarrow \mathbf{Z}_U \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Z}_Y \longrightarrow 0$$

and induction on $n = \dim(X)$ since $\mathbf{Z}$ is flasque [Har77, Exercise II.1.16(a)] and $\dim(Y) < n$ by the irreducibility of X. The vanishing $H^{i+1}(X, \mathscr{R}) = 0$ holds if $\mathscr{R} = 0$, and hence it remains to consider the case when $\mathscr{R} \neq 0$. Let d be the least positive integer that occurs in any of the groups $\mathscr{R}_x$ as $x \in X$ varies. Then, there is a nonempty open subset $V \subseteq U$ such that

$$\mathscr{R}|_V \cong d \cdot \mathbf{Z}|_V \subseteq \mathbf{Z}|_V.$$

We therefore see that $\mathscr{R}_V \cong \mathbf{Z}_V$ and we have a short exact sequence

$$0 \longrightarrow \mathbf{Z}_V \xrightarrow{d} \mathscr{R} \longrightarrow \frac{\mathscr{R}}{\mathbf{Z}_V} \longrightarrow 0.$$

The sheaf $\mathscr{R}/\mathbf{Z}_V$ is supported on $\overline{(U - V)} \subseteq X$, which is a closed subset of dimension $< n$ since X is irreducible. Thus, by the induction on $n = \dim(X)$ and the vanishing for $\mathbf{Z}_V$ shown above, we are done. $\square$

CHAPTER 2

Schemes

Now that we have set up the theory of sheaf cohomology, our goal in this chapter to apply the theory to schemes. We have seen so far (especially on Homework 2) that it is pretty difficult to compute sheaf cohomology using just what we have done so far. $\boxed{1/30}$

On schemes and many other ringed spaces, we will largely consider the subcategory of *coherent* or *quasi-coherent* sheaves in $\mathsf{Mod}(\mathcal{O}_X)$. For schemes, these subcategories consists of $\mathcal{O}_X$-modules that locally look like sheaves associated to modules (in the coherent case, *coherent* modules) and contain many modules of interest, like sheaves associated to vector bundles. We will see that on sufficiently nice schemes, the associated sheaf cohomology sheaves can be computed in a simpler way (using Čech cohomology) and satisfy nice finiteness properties on projective varieties or more generally, relative to proper morphisms.

2.1. Quasi-coherent and coherent sheaves

To simplify notation, we use the following notation.

Definition 2.1.1 [EGAI$_{\text{new}}$, Chapitre 0, (4.1.2)]. Let $(X,\mathcal{O}_X)$ be a ringed space. If $j\colon U\hookrightarrow X$ is an open subset, we set $\mathcal{O}_U := \mathcal{O}_X|_U$ and call $(U,\mathcal{O}_U)$ an *open subspace* of $(X,\mathcal{O}_X)$ with the *canonical inclusion morphism* [Har77, Exer. II.2.2]

$$j\colon (U,\mathcal{O}_U)\longhookrightarrow (X,\mathcal{O}_X)$$

where $j^{\#}\colon \mathcal{O}_X\to j_*\mathcal{O}_U$ is the restriction map.

Remark 2.1.2. Note that this notation clashes with the notation $\mathscr{F}_U$ from the proof of Grothendieck's vanishing Theorem 1.4.27! Whenever structure sheaves are involved, you should use this new definition and not the old one.

2.1.1. Quasi-coherent sheaves. We start by defining quasi-coherent sheaves. We will eventually show that on schemes, this is the class of $\mathcal{O}_X$-modules that "come from algebra," that is, they are sheaves associated to modules on affine open subsets.

Definition 2.1.3 [EGAI$_{\text{new}}$, Chapitre 0, (5.1.1)]. Let $(X,\mathcal{O}_X)$ be a ringed space and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. To give a morphism $u\colon \mathcal{O}_X\to\mathscr{F}$ of $\mathcal{O}_X$-modules is equivalent to giving a section $s=u(1)\in\Gamma(X,\mathscr{F})$. This is because if s is given, then for every section $t\in\Gamma(U,\mathcal{O}_X)$, we must have $u(t)=t\cdot(s|_U)$. We say that u is the morphism *defined by the global section* s. If I is an indexing set, consider the direct sum $\mathcal{O}_X^{(I)}$ with injection maps $h_i\colon \mathcal{O}_X\to\mathcal{O}_X^{(I)}$. We see that there is a bijection [Har77, p. 121]

$$\begin{tikzcd} \operatorname{Hom}_{\mathcal{O}_X}\bigl(\mathcal{O}_X^{(I)},\mathscr{F}\bigr) \arrow[r, leftrightarrow, "\text{1-1}"] & \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X,\mathscr{F})^I \\ u \arrow[r, mapsto] & (u\circ h_i)_{i\in I} \end{tikzcd}$$

where the superscript on the right denotes the direct product. We therefore see that morphisms $u\colon \mathcal{O}_X^{(I)} \to \mathscr{F}$ are in one-to-one correspondence with families $(s_i)_{i\in I}$ of global sections of $\mathscr{F}$.

We say that $\mathscr{F}$ is *generated by the family* $(s_i)_{i\in I}$ if the morphism

$$\mathcal{O}_X^{(I)} \longrightarrow \mathscr{F} \tag{2.1.4}$$

defined by the family is surjective. We say that $\mathscr{F}$ is *globally generated* if it is generated by $\Gamma(X,\mathscr{F})$, which by the previous paragraph is equivalent to saying that there exists some surjective morphism of the form (2.1.4).

[Har77, p. 111] This definition is equivalent to the definition in [Har77]. See [Har77, Exer. II.5.4].

Definition 2.1.5 [EGAI$_{\text{new}}$, Chapitre 0, (5.1.3)]. Let $(X,\mathcal{O}_X)$ be a ringed space and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. We say that $\mathscr{F}$ is *quasi-coherent* if, for every $x \in X$, there exists an open neighborhood U of x and a morphism $u\colon \mathcal{O}_U^{(I)} \to \mathcal{O}_U^{(J)}$ for some indexing sets I and J such that

$$\mathscr{F}|_U \cong \operatorname{coker}\bigl(\mathcal{O}_U^{(I)} \xrightarrow{u} \mathcal{O}_U^{(J)}\bigr). \tag{2.1.6}$$

Equivalently, $\mathscr{F}$ is quasi-coherent if, for every $x \in X$, there exists an open neighborhood U of x and an exact sequence of the form

$$\mathcal{O}_U^{(I)} \xrightarrow{u} \mathcal{O}_U^{(J)} \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

[Har77, Prop. II.5.8(a)]

for some indexing sets I and J. Quasi-coherence is preserved under f^* by taking the pullback of the map (2.1.6) [EGAI$_{\text{new}}$, Chapitre 0, (5.1.4)].

For simplicity, we say *quasi-coherent sheaf on* X instead of *quasi-coherent* $\mathcal{O}_X$-*module*. Quasi-coherent $\mathcal{O}_X$-modules form a full subcategory of $\mathsf{Mod}(\mathcal{O}_X)$, which we denote by $\mathsf{QCoh}(\mathcal{O}_X)$.

[Har77, Ex. II.5.2.1]

Example 2.1.7 [EGAI$_{\text{new}}$, Chapitre 0, (5.1.3)]. Let $(X,\mathcal{O}_X)$ be a ringed space. Then, $\mathcal{O}_X$ and finite direct sums thereof are quasi-coherent.

Let us see one example of a non-quasi-coherent sheaf. When setting up the theory for sheaf cohomology, we used the construction $j_!$ many times. This example shows why the extra flexibility of working in $\mathsf{Mod}(\mathcal{O}_X)$ is so useful despite the fact that not all $\mathcal{O}_X$-modules "come from algebra."

[Har77, Ex. II.5.2.3]

Example 2.1.8 (An $\mathcal{O}_X$-module that is not quasi-coherent). Let X be a quasi-projective variety in the sense of [Har77, Chapter I, §2] and let $j\colon U \hookrightarrow X$ be a nonempty open subset.

We claim that $\mathscr{F} = j_!(\mathcal{O}_U)$ is not quasi-coherent. Let $x \in X - U$. We will show that there cannot be an open neighborhood V of x such that $\mathscr{F}|_V$ is the cokernel of a morphism

$$\mathcal{O}_V^{(I)} \longrightarrow \mathcal{O}_V^{(J)}.$$

By the discussion in Definition 2.1.3, it suffices to show that $\mathscr{F}|_V$ has no global sections for all $V \ni x$.

For our first argument, we consider the injection $\mathscr{F} \hookrightarrow \mathcal{O}_X$. We then have the commutative diagram:

$$\begin{tikzcd}
\Gamma\bigl(V,\mathscr{F}|_V\bigr) \arrow[r, hook] \arrow[d] & \Gamma(V,\mathcal{O}_V) \arrow[d, hook] \\
\bigl(\mathscr{F}|_V\bigr)_x \arrow[r, hook] & \mathcal{O}_{X,x}
\end{tikzcd}$$

where the top horizontal map is injective by the injectivity of $\mathscr{F} \hookrightarrow \mathcal{O}_X$ and the right vertical arrow is injective by the fact that V is a variety [Mur24ag, Lemma 1.3.14]. Note that $(\mathscr{F}|_V)_x \cong \mathscr{F}_x = 0$ by [Har77, Exercise II.1.19(b)] (Homework 1, Problem 2(b)). Since the composition $\Gamma(V, \mathscr{F}|_V) \to \mathcal{O}_{X,x}$ is injective, this shows that $\Gamma(V, \mathscr{F}|_V) = 0$.

We give another proof following [MM16]. Let $i\colon Z \hookrightarrow X$ be the closed complement of U. Consider the short exact sequence

$$0 \longrightarrow j_!\mathcal{O}_U \longrightarrow \mathcal{O}_X \longrightarrow i_*\bigl(\mathcal{O}_X|_Z\bigr) \longrightarrow 0$$

from [Har77, Exercise II.1.19(c)] (Homework 1, Problem 2(c)). Restricting to V and taking sections, we obtain the left exact sequence

$$0 \longrightarrow \Gamma\bigl(V, \mathscr{F}|_V\bigr) \longrightarrow \Gamma(V, \mathcal{O}_V) \longrightarrow \Gamma\bigl(V \cap Z, \mathcal{O}_V|_{V\cap Z}\bigr).$$

We claim the map on the right is injective. Let $i_x\colon \{x\} \hookrightarrow X$ be the inclusion of x in X. We have the sequence of maps

$$\mathcal{O}_V \longrightarrow i_*i^{-1}\mathcal{O}_V \longrightarrow i_*i_{x*}i_x^{-1}i^{-1}\mathcal{O}_V \longrightarrow i_*i_{x*}i_x^{-1}i^{-1}\mathscr{K}_V$$

where we use i to also denote the inclusion $V \cap Z \hookrightarrow V$ and where $\mathscr{K}_V$ is the constant sheaf of rational functions on V. Restricting then taking sections over V, we obtain the map

$$\Gamma(V, \mathcal{O}_V) \longrightarrow \Gamma\bigl(V \cap Z, \mathcal{O}_V|_{V\cap Z}\bigr) \longrightarrow \mathcal{O}_{V,x} \longrightarrow K(V),$$

which is injective since regular functions on V form a subring of the ring of rational functions on V. Thus, $\Gamma(V, \mathcal{O}_V) \to \Gamma(V \cap Z, \mathcal{O}_X|_{V\cap Z})$ is injective. This shows that $\Gamma(V, \mathscr{F}|_V) = 0$.

Remark 2.1.9. The notion of quasi-coherence defined in Definition 2.1.5 works well for schemes but not for arbitrary ringed spaces. See, for example, [Con06, §2.1], where Conrad discusses the case of rigid-analytic spaces.[1]

2.1.2. Sheaves of finite type and of finite presentation.

In MA557 (Commutative Algebra), we saw that over Noetherian rings, various finiteness properties of modules coincide, like finite generation (also called finite type) and finite presentation [Mur24ca, Theorem 7.5.2]. For $\mathcal{O}_X$-modules, we have to deal with the same ambiguity. The possible lack of quasi-coherence also causes issues.

Definition 2.1.10 [FAC, n° 12, Définition 1; TohokuII, p. 185]. Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. We say that $\mathscr{F}$ is *of finite type* if, for every $x \in X$, there exists an open neighborhood U of x such that $\mathscr{F}|_U$ is generated by a *finite* family of sections on U, i.e., $\mathscr{F}|_U$ is isomorphic to a quotient of a sheaf of the form $\mathcal{O}_U^p$ for a non-negative integer p. Equivalently, $\mathscr{F}$ is of finite type if, for every $x \in X$, there exists an open neighborhood U of x such that $\mathscr{F}|_U$ fits into an exact sequence of the form [EGAI_{new}, ($\mathbf{0}$, 5.2.1)]

$$\mathcal{O}_U^p \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

for a non-negative integer p. "Finite type" is preserved under f^* by taking the pullback of the surjection $\mathcal{O}_U^p \twoheadrightarrow \mathscr{F}|_U$ [EGAI_{new}, Chapitre 0, (5.2.4)].

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[1] Technically, rigid-analytic spaces are not ringed spaces, but Conrad's discussion applies to other types of non-Archimedean analytic spaces as well.

Remark 2.1.11. An $\mathcal{O}_X$-module of finite type is not necessarily quasi-coherent. For example, consider the quotient $\mathcal{O}_X/\mathscr{F}$, where $\mathscr{F}$ is as constructed in Example 2.1.8. This is because we have a short exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathscr{F} \longrightarrow 0$$

and if $\mathcal{O}_X/\mathscr{F}$ were quasi-coherent, then $\mathscr{F}$ would be by [EGAI$_{\text{new}}$, Corollaire 2.2.2(i)] (to be shown later).

We prove some properties of sheaves of finite type for all ringed spaces. While usually stated for coherent sheaves in algebraic geometry, this shows how the tools of commutative algebra are useful even if you are not working on a scheme.

Proposition 2.1.12 [FAC, n$^{\text{o}}$ 12, Proposition 1; EGAI$_{\text{new}}$, Chapitre 0, Proposition 5.2.2, Corollaire 5.2.2.1, and Corollaire 5.2.2.2]. *Let $(X, \mathcal{O}_X)$ be a ringed space, let $\mathscr{F}, \mathscr{G}$ be $\mathcal{O}_X$-modules, and let $x \in X$ be a point. Suppose that $\mathscr{F}$ is of finite type.*

- (i) *Suppose $(s_i)_{1\le i\le n}$ are sections of $\mathscr{F}$ on an open subset $U \subseteq X$. Suppose the $s_{i,x}$ generate $\mathscr{F}_x$. Then, there exists an open subset $V \subseteq U$ containing x such that the $s_{i,y}$ generate $\mathscr{F}_y$ for all $y \in V$.*
- (ii) *If $u\colon \mathscr{F} \to \mathscr{G}$ is a morphism such that $u_x = 0$, then there exists an open neighborhood U of x such that $u|_U = 0$.*
- (iii) *If $v\colon \mathscr{G} \to \mathscr{F}$ is a morphism such that v_x is surjective, then there exists an open neighborhood V of x such that $v|_V$ is surjective.*
- (iv) Supp$(\mathscr{F})$ *is closed.* [Har77, Exer. II.5.6(c)]

For the following statements, suppose that $(X, \mathcal{O}_X)$ is a locally *ringed space.*

- (v) Supp$(\mathscr{F}) = \{x \in X \mid \mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x \neq 0\}$.
- (vi) *If $\mathscr{G}$ is also of finite type, then* $\operatorname{Supp}(\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}) = \operatorname{Supp}(\mathscr{F}) \cap \operatorname{Supp}(\mathscr{G})$.

Proof. (i). Let $(t_j)_{1\le j\le m}$ be sections of $\mathscr{F}$ on an open set $U' \subseteq U$ containing x that generate $\mathscr{F}|_{U'}'$. Since the $s_{i,x}$ generate $\mathscr{F}_x$, there exist sections a_{ij} on an open subset $U'' \subseteq U'$ containing x such that

$$t_{j,x} = \sum_{i=1}^{n} a_{ij,x}\, s_{i,x}$$

for all $1 \le j \le m$. We conclude that there exists an open subset $V \subseteq U''$ containing x such that for all $y \in V$, we have

$$t_{j,y} = \sum_{i=1}^{n} a_{ij,y}\, s_{i,y}$$

for all $1 \le j \le m$, in which case the $s_{i,y}$ generate $\mathscr{F}_y$ for all $y \in V$.

(iv) holds by applying (i) to the case $n = 1$ and $s_{1,x} = 0$. (iv) implies (iii) and (ii) since coker(v) and im(u) are of finite type.

(v) holds by NAK [Mur24ca, Lemma 2.3.8], and implies (vi) since the tensor product of nonzero $\kappa(x)$-vector spaces is nonzero. □

Lemma 2.1.13 [EGAI$_{\text{new}}$, Chapitre 0, (5.2.9)]. *Let $(X, \mathcal{O}_X)$ be a ringed space. Consider a short exact sequence* [Har77, Prop. II.5.7]

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

of $\mathcal{O}_X$-modules.

- (i) *If $\mathscr{G}$ is of finite type, then $\mathscr{H}$ is of finite type.*

(ii) If $\mathscr{F}$ and $\mathscr{H}$ are of finite type, then $\mathscr{G}$ is of finite type.

Proof. (*i*) holds by definition.

(*ii*). Since the question is local, after replacing X by an open subset, we may assume that $\mathscr{F}$ and $\mathscr{H}$ are generated by finitely many sections $(s_i)_{1\le i\le n}$ and $(t_j)_{1\le j\le m}$ on X, respectively, and that the sections t_j lift to sections t'_j of $\mathscr{G}$. Then, $\mathscr{G}$ is generated by the t'_j and the images of the s_i. □

Next, we consider sheaves of finite presentation. These are called *pseudo-coherent* in [TohokuII, p. 185], but the term pseudo-coherent means something different today.

[EGAI$_{\text{new}}$, (**0**, 5.2.5)]

Definition 2.1.14 [TohokuII, p. 185]. Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. We say that $\mathscr{F}$ is *of finite presentation* if, for every $x \in X$, there exists an open neighborhood U of x and a morphism $u\colon \mathcal{O}_U^p \to \mathcal{O}_U^q$ for p, q non-negative integers such that

$$\mathscr{F}|_U \cong \operatorname{coker}\bigl(\mathcal{O}_U^p \xrightarrow{u} \mathcal{O}_U^q\bigr). \tag{2.1.15}$$

Equivalently, $\mathscr{F}$ is of finite presentation if, for every $x \in X$, there exists an open neighborhood U of x such that $\mathscr{F}|_U$ fits into an exact sequence of the form

$$\mathcal{O}_U^p \xrightarrow{u} \mathcal{O}_U^q \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

for some p, q non-negative integers. Such an $\mathcal{O}_X$-module is both of finite type and is quasi-coherent. Finite presentation is preserved under f^* by taking the pullback of the map in (2.1.15).

A useful fact about sheaves of finite type and finite presentation is the following result, which for affine schemes is a special case of “Hom commutes with flat base change” [Mur24ca, Proposition 7.7.4].

[Har77, Prop. III.6.8]

Proposition 2.1.16 [TohokuII, Proposition 4.1.1]. *Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathscr{F}, \mathscr{G}$ be $\mathcal{O}_X$-modules. For every $x \in X$, there is a natural morphism*

$$\mathscr{H}\!om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})_x \longrightarrow \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x).$$

This morphism is injective (resp. bijective) if $\mathscr{F}$ is of finite type (resp. finitely presented).

Proof. For every open neighborhood U of x, there is a natural morphism

$$\operatorname{Hom}_{\mathcal{O}_U}\bigl(\mathscr{F}|_U, \mathscr{G}|_U\bigr) \longrightarrow \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x).$$

Taking direct limits over all $U \ni x$ on the left-hand side, we obtain the morphism.

Since the question is local, after replacing X by an open neighborhood of x, we may assume that $\mathscr{F}$ has a presentation

$$\mathcal{O}_X^{(J)} \longrightarrow \mathscr{F} \longrightarrow 0$$

where J is finite in the finite type case and

$$\mathcal{O}_X^{(I)} \longrightarrow \mathcal{O}_X^{(J)} \longrightarrow \mathscr{F} \longrightarrow 0$$

where both I and J are finite in the finite presentation case. Applying $\mathscr{H}\!om_{\mathcal{O}_X}(\,\cdot\,, \mathscr{G})$, which is left exact (since it is on sections over each U), we obtain the left exact sequence in the top row of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\mathscr{G})_x \arrow[r] \arrow[d] & \mathscr{H}om_{\mathcal{O}_X}\big(\mathcal{O}_X^{(J)},\mathscr{G}\big)_x \arrow[r] \arrow[d, "\wr"] & \mathscr{H}om_{\mathcal{O}_X}\big(\mathcal{O}_X^{(I)},\mathscr{G}\big)_x \arrow[d] \\
0 \arrow[r] & \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathscr{F}_x,\mathscr{G}_x) \arrow[r] & \operatorname{Hom}_{\mathcal{O}_{X,x}}\big(\mathcal{O}_{X,x}^{(J)},\mathscr{G}_x\big) \arrow[r] & \operatorname{Hom}_{\mathcal{O}_{X,x}}\big(\mathcal{O}_{X,x}^{(I)},\mathscr{G}_x\big)
\end{tikzcd}$$

where in the finite type case, we only work with the left square. The middle vertical map is an isomorphism in either case since J is finite and both modules represent the same data as specifying $|J|$ elements of $\mathscr{G}_x$. When $\mathscr{F}$ is of finite presentation, the right vertical map is an isomorphism since I is finite. □

Corollary 2.1.17 [EGAI$_{\text{new}}$, Chapitre 0, (5.2.7)]. *Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathscr{F},\mathscr{G}$ be $\mathcal{O}_X$-modules of finite presentation. Let $x \in X$ be a point such that $\mathscr{F}_x \cong \mathscr{G}_x$. Then, there exists an open neighborhood U of x such that $\mathscr{F}|_U \cong \mathscr{G}|_U$.*

Proof. If $\varphi\colon \mathscr{F}_x \to \mathscr{G}_x$ and $\psi\colon \mathscr{G}_x \to \mathscr{F}_x$ are mutually inverse, then Proposition 2.1.16 implies there exists an open neighborhood $V \ni x$ and sections u and v of $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\mathscr{G})$ and $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{G},\mathscr{F})$ over V such that $u_x = \varphi$ and $v_x = \psi$. Since $(u \circ v)_x = \mathrm{id}_{\mathscr{G}_x}$ and $(v \circ u)_x = \mathrm{id}_{\mathscr{F}_x}$, by Proposition 2.1.16, there exists an open subset $U \subseteq V$ containing x such that $(u \circ v)|_U = \mathrm{id}_{\mathscr{F}|_U}$ and $(v \circ u)|_V = \mathrm{id}_{\mathscr{G}|_U}$. □

2.1.3. Coherent sheaves. We can now define coherent sheaves. The first version of this definition is from complex analytic geometry [SHC51/52, Exposé 15, Définition 3 and Exposé 18, Définition 1]. The fact that the sheaf of holomorphic functions on a complex manifold is coherent is a deep result known as Oka's coherence theorem [Oka50].

Serre introduced the definition for algebraic varieties in [FAC, n° 12, Définition 2]. It is important to remember that in algebraic geometry, we are pretty spoiled: The fact that our spaces will usually come from gluing spectra of Noetherian rings means that the algebraic analogue of Oka's coherence theorem basically follows from definition of Noetherianity. As an aside, this shows how far-reaching and powerful Noether's original insights are!

Definition 2.1.18 [FAC, n° 12, Définition 2; EGAI$_{\text{new}}$, Chapitre 0, (5.3.1)]. Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. We say that $\mathscr{F}$ is *coherent* if it satisfies the following two conditions:

(*a*) $\mathscr{F}$ is of finite type.

(*b*) For every open subset $U \subseteq X$, every integer $n \geq 0$, and every morphism

$$u\colon \mathcal{O}_U^n \longrightarrow \mathscr{F}|_U,$$

the kernel of u is of finite type.

This is not equivalent to the definition in [Har77, p. 111] in general. They *are* equivalent when X is a locally Noetherian scheme. See [Har77, Exer. II.5.4].

By definition, every sub-$\mathcal{O}_X$-module *of finite type* of a coherent $\mathcal{O}_X$-module is coherent. By definition, every coherent $\mathcal{O}_X$-module is of finite presentation.

For simplicity, we say *coherent sheaf on X* instead of *coherent $\mathcal{O}_X$-module*. Coherent $\mathcal{O}_X$-modules form a full subcategory of $\mathsf{QCoh}(\mathcal{O}_X)$, which we denote by $\mathsf{Coh}(\mathcal{O}_X)$.

The four classes of $\mathscr{O}_X$-modules are related in the following manner:

$$\begin{tikzcd}
\text{coherent} \arrow[r, Rightarrow] & \text{finite presentation} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{quasi-coherent} \\
 & \text{finite type.} &
\end{tikzcd}$$

Note that the three classes on the right are preserved under f^*. This is not the case for coherence without extra hypotheses.

So far, we have seen that finite type sheaves satisfy some nice formal properties with respect to exact sequences (Lemma 2.1.13). The following result shows why coherent sheaves are even nicer to work with.

Proposition 2.1.19 ("2 out of 3" property for coherent sheaves [FAC, n° 13, Théorème 1; EGAI$_{\text{new}}$, Chapitre 0, Proposition 5.3.2]). *Let* $(X, \mathscr{O}_X)$ *be a ringed space and consider a short exact sequence* [Har77, Prop. II.5.7]

$$0 \longrightarrow \mathscr{F} \xrightarrow{u} \mathscr{G} \xrightarrow{v} \mathscr{H} \longrightarrow 0$$

be a short exact sequence of $\mathscr{O}_X$*-modules. If two of the three sheaves are coherent, then the third sheaf is also coherent.*

Below, we use the snake lemma [KS06, Lemma 12.1.1] a few times. If you are worried about the fact that we are working with $\mathscr{O}_X$-modules and cannot chase elements, we are using the snake lemma to check surjectivity of morphisms, and hence one could also pass to stalks and apply the usual snake lemma for modules.

Proof. We proceed in steps.

Step 1. If $\mathscr{G}$ and $\mathscr{H}$ are coherent, then $\mathscr{F}$ is coherent.

Since the question is local, after possibly replacing X by an open subset, we may assume there exists a surjective morphism $w\colon \mathscr{O}_X^p \to \mathscr{G}$. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \ker(v \circ w) \arrow[r] \arrow[d, dashed, two heads] & \mathscr{O}_X^p \arrow[r, "v \circ w"] \arrow[d, two heads, "w"'] & \mathscr{H} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{F} \arrow[r, "u"] & \mathscr{G} \arrow[r, "v"] & \mathscr{H} \arrow[r] & 0
\end{tikzcd}$$

Since $\mathscr{H}$ is coherent, $\ker(v \circ w)$ is of finite type. Now by the snake lemma [KS06, Lemma 12.1.1], since w is surjective, the map $\mathscr{I} \to \mathscr{F}$ induced by the universal property of the kernel is also surjective. Thus, $\mathscr{F}$ is of finite type, and hence coherent.

Step 2. If $\mathscr{F}$ and $\mathscr{G}$ are coherent, then $\mathscr{H}$ is coherent.

Since $\mathscr{G}$ is of finite type, we know that $\mathscr{H}$ is also of finite type by Lemma 2.1.13(i). It remains to show that $\mathscr{H}$ satisfies the condition (b). Consider a morphism $f\colon \mathscr{O}_U^n \to \mathscr{H}|_U$ and let $(s_i)_{1 \le i \le n}$ be the corresponding sections of $\mathscr{H}$ over U. Since the question is local, after replacing X by an open subset of U, we may assume that there exist n sections s_i' of $\mathscr{G}$ over U such that $s_i = v(s_i')$, and that $\mathscr{F}|_U$ is generated by sections $(t_j)_{1 \le j \le m}$ over U. Now consider the morphism

$$w\colon \mathscr{O}_U^{n+m} \longrightarrow \mathscr{G}|_U$$

defined by the n sections s'_i and the m sections $u(t_j)$. These morphisms fit into the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \ker(w) \arrow[r] \arrow[d] & \ker(f) \arrow[r, dashed] \arrow[d] & 0 \\
 & & \mathscr{O}_U^{n+m} \arrow[r, "p"] \arrow[d, "w"'] & \mathscr{O}_U^n \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \mathscr{F}|_U \arrow[r, "u"] & \mathscr{G}|_U \arrow[r, "v"] & \mathscr{H}|_U \arrow[r] & 0.
\end{tikzcd}$$

By the universal property of $\ker(f)$, there is a map $\ker(w) \to \ker(f)$ that we claim is surjective. It suffices to show surjectivity after passing to stalks at every $x \in U$. Suppose $q_x \in \ker(f)_x$. Let $q'_x \in \mathscr{O}_{U,x}^{n+m}$ be a lift of q_x. Then, $(v \circ w)q'_x = 0$, and hence by exactness, there exist $h_{j,x} \in \mathscr{O}_{X,x}$ such that

$$w(q'_x) + \sum_{j=1}^{m} h_{j,x}\, u(t_j)_x = 0.$$

Thus, the germ $r_x = q'_x + (h_{j,x})_j \in \mathscr{O}_{U,x}^{n+m}$ is in $\ker(w)_x$ and $p(r_x) = q_x$. We therefore see that $\ker(f)$ is of finite type as desired.

Step 3. If $\mathscr{F}$ and $\mathscr{H}$ are coherent, then $\mathscr{G}$ is coherent.

By Lemma 2.1.13(*ii*), we know that $\mathscr{G}$ is of finite type. It remains to show that $\mathscr{G}$ satisfies the condition (*b*). Consider a morphism $f\colon \mathscr{O}_U^n \to \mathscr{G}|_U$ and consider the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \ker(f) \arrow[d] & & \\
0 \arrow[r] & \ker(v \circ f) \arrow[r] \arrow[d] & \mathscr{O}_U^n \arrow[r, "v \circ f"] \arrow[d, "f"] & \mathscr{H}|_U \arrow[d, equal] & \\
0 \arrow[r] & \mathscr{F}|_U \arrow[r, "u"] & \mathscr{G}|_U \arrow[r, "v"] & \mathscr{H}|_U \arrow[r] & 0
\end{tikzcd}$$

where the morphism $\ker(v \circ f) \to \mathscr{F}|_U$ exists by the universal property of kernels. Since $\mathscr{H}$ is coherent, $\ker(v \circ f)$ is of finite type. Since the question is local, after possibly replacing U by a smaller subset, we may assume that $\ker(v \circ f)$ is generated by finitely many sections over U. Let $\mathscr{O}_U^m \twoheadrightarrow \ker(v \circ f)$ be the corresponding surjection. Note that the image of $\mathscr{O}_U^m$ in $\mathscr{O}_U^n$ is $\ker(v \circ f)$. We therefore have the commutative diagram

$$\begin{tikzcd}
 & \ker(w) \arrow[r] \arrow[d] & \ker(f) \arrow[r] \arrow[d] & 0 \arrow[d] & \\
 & \mathcal{O}_U^m \arrow[r] \arrow[d, "w"'] & \mathcal{O}_U^n \arrow[r, "v\circ f"] \arrow[d, "f"] & \operatorname{im}(v\circ f) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}|_U \arrow[r, "u"] & \mathscr{G}|_U \arrow[r, "v"] & \mathscr{H}|_U \arrow[r] & 0
\end{tikzcd}$$

where the middle row is exact. By the snake lemma [KS06, Lemma 12.1.1], the top row is exact. Since $\ker(w)$ is of finite type by the coherence of $\mathscr{F}$, we see that $\ker(f)$ is of finite type as desired. □

Corollary 2.1.20 [FAC, n° 13, Théorème 1, Corollaire and Théorème 2, n° 14, Propositions 4 and 6; EGAI$_{\text{new}}$, Chapitre 0, Corollaire 5.3.5 and Corollaire 5.3.8]. *Let $(X,\mathcal{O}_X)$ be a ringed space.*

$\boxed{2/4}$

(*i*) *Finite direct sums of coherent $\mathcal{O}_X$-modules are coherent.* [Har77, Prop. II.5.7]

(*ii*) *Suppose $u\colon \mathscr{F}\to\mathscr{G}$ is a morphism of coherent $\mathcal{O}_X$-modules. Then,* $\operatorname{im}(u)$, $\ker(u)$, *and* $\operatorname{coker}(u)$ *are coherent, and hence* $\mathsf{Coh}(\mathcal{O}_X)$ *is an Abelian category.*

(*iii*) *Suppose $\mathscr{F}$ and $\mathscr{G}$ are coherent $\mathcal{O}_X$-modules. Then, $\mathscr{F}\otimes_{\mathcal{O}_X}\mathscr{G}$ and $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\mathscr{G})$ are coherent.* [Har77, Exer. III.6.3(*a*)]

Proof. (*i*) follows immediately from Proposition 2.1.19.

(*ii*). By Lemma 2.1.13(*i*), $\operatorname{im}(u)$ is of finite type. Since it is a subsheaf of a coherent sheaf, we see that $\operatorname{im}(u)$ is coherent. For $\ker(u)$ and $\operatorname{coker}(u)$, we apply Proposition 2.1.19 to the short exact sequences

$$0 \longrightarrow \ker(u) \longrightarrow \mathscr{F} \longrightarrow \operatorname{im}(u) \longrightarrow 0,$$
$$0 \longrightarrow \operatorname{im}(u) \longrightarrow \mathscr{G} \longrightarrow \operatorname{coker}(u) \longrightarrow 0.$$

(*iii*). Since the questons are local, after possibly replacing X by an open subset, we have the right exact sequence

$$\mathcal{O}_X^p \longrightarrow \mathcal{O}_X^q \longrightarrow \mathscr{F} \longrightarrow 0.$$

Since tensor products are right exact, we obtain the right exact sequence

$$\mathcal{O}_X^p\otimes_{\mathcal{O}_X}\mathscr{G} \longrightarrow \mathcal{O}_X^q\otimes_{\mathcal{O}_X}\mathscr{G} \longrightarrow \mathscr{F}\otimes_{\mathcal{O}_X}\mathscr{G} \longrightarrow 0.$$

The left and middle sheaves are isomorphic to $\mathscr{G}^p$ and $\mathscr{G}^q$, respectively. By (*i*) and Proposition 2.1.19, we are done. For $\mathscr{H}om$, we use the left exactness of Hom and Proposition 2.1.16 to obtain the left exact sequence

$$0 \longrightarrow \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\mathscr{G}) \longrightarrow \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X^q,\mathscr{G}) \longrightarrow \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X^p,\mathscr{G}).$$

The middle and right sheaves are isomorphic to $\mathscr{G}^q$ and $\mathscr{G}^p$, respectively. By (*i*) and Proposition 2.1.19, we are done. □

To guarantee nice behavior under pullbacks, we have the following:

Definition 2.1.21 [FAC, n° 15, Définition 3; EGAI$_{\text{new}}$, Chapitre 0, (5.3.9)]. Let $(X,\mathcal{O}_X)$ be a ringed space. We say that $\mathcal{O}_X$ is a *coherent sheaf of rings* if $\mathcal{O}_X$ is coherent as a $\mathcal{O}_X$-module.

If $\mathcal{O}_X$ is a coherent sheaf of rings, every $\mathcal{O}_X$-module $\mathscr{F}$ of finite presentation is coherent: Locally, $\mathscr{F}$ is the cokernel of a map between direct sums of finitely many copies of $\mathcal{O}_X$, which are coherent by Corollary 2.1.20(*i*). Cokernels of such maps are coherent by Corollary 2.1.20(*ii*).

We add this last comment to the chart of implications from before:

$$\begin{tikzcd} \text{coherent} \arrow[r, Rightarrow] & \text{finite presentation} \arrow[r, Rightarrow] \arrow[l, dashed, bend right, "\mathcal{O}_X\text{ coh.}"'] \arrow[d, Rightarrow] & \text{quasi-coherent} \\ & \text{finite type.} & \end{tikzcd}$$

Example 2.1.22. Let X be a complex manifold. Then, the sheaf $\mathcal{O}_X$ of germs of holomorphic functions on X is coherent by Oka's coherence theorem [Oka50]. This is a deep theorem from several complex variables. For a textbook account, see [Hör90, Theorem 6.4.1].

We can also say something about pullbacks.

[Har77, Prop. II.5.8(b)]

Lemma 2.1.23 [EGAI$_{\text{new}}$, Chapitre 0, (5.3.14)]. *Let $f\colon X \to Y$ be a morphism of ringed spaces and suppose that $\mathcal{O}_X$ is coherent. Then, for every finitely presented $\mathcal{O}_Y$-module $\mathscr{G}$, the pullback $f^*\mathscr{G}$ is coherent. In particular, if $\mathscr{G}$ is coherent, then $f^*\mathscr{G}$ is coherent.*

Proof. Pullbacks of finitely presented sheaves are finitely presented and the coherence of $\mathcal{O}_X$ implies finitely presented $\mathcal{O}_X$-modules are coherent. □

2.2. Affine schemes

Our next goal is to understand quasi-coherent and coherent sheaves on schemes. Before doing so, we review the definition of schemes.

2.2.1. Review of spectra of rings. We start with spectra of rings.

[Har77, p. 70]

Definition 2.2.1 [EGAI$_{\text{new}}$, (1.1)]. Let A be a ring. As a set, the *spectrum* $\operatorname{Spec}(A)$ of A is the set of prime ideals in A. As a topological space, the closed sets in $\operatorname{Spec}(A)$ are sets of the form

$$V(I) := \bigl\{\mathfrak{p} \in \operatorname{Spec}(A) \bigm| I \subseteq \mathfrak{p}\bigr\}$$

for every ideal $I \subseteq A$. This topology is called the *Zariski topology*.

When we think of $\operatorname{Spec}(A)$ geometrically, we sometimes denote its points by x, in which case the corresponding prime ideals are denoted by $\mathfrak{p}_x$.

Let us see some examples.

Example 2.2.2.

[Mum67, Ex. A, p. 135]
[Har77, Ex. II.2.3.1]
[Mum67, Ex. B, p. 136]
[Har77, Ex. II.2.3.3, Exer. II.2.10]

(a) $\operatorname{Spec}(k)$ for a field k consists of one point (0).
(b) (The affine line) $\mathbf{A}^1_k := \operatorname{Spec}(k[x])$ is the *affine line over* k. This space consists of two types of points: The *generic point* $\xi := (0)$ and closed points (f), where $f \in k[x]$ is an irreducible polynomial. If k is algebraically closed, for example when $k = \mathbf{C}$, these points are $(x-\alpha)$ where $\alpha \in k$ by the Nullstellensatz [Mur24ca, Theorem 5.3.3]. However, if k is not algebraically closed, there are more points. For example, if $k = \mathbf{R}$, then

the set of closed points in Spec($k[x]$) corresponds to the closed upper half plane in $\mathbf{C}$ because roots of irreducible quadratic polynomials come in conjugate pairs.

(c) Spec($\mathbf{Z}$). This ring is a PID, and we can visualize this space as the "number line" together with the generic point (0).

[Mum67, Ex. C, p.!137]
[Har77, Exer. II.2.5]

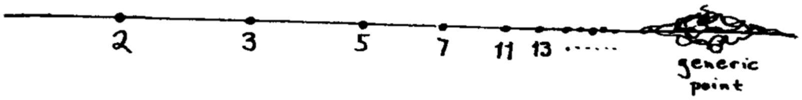

FIGURE 2.1. Spec($\mathbf{Z}$). From [Mum67, p. 137].

(d) (A *trait*) If R is a DVR (i.e., a discrete rank 1 valuation ring), then $T = \operatorname{Spec}(R)$ consists of two points $t_0 := (\varpi)$ and $t_1 := (0)$, where ϖ is a uniformizer for R. The topology is such that t_0 is closed and $\overline{\{t_1\}} = \operatorname{Spec}(R)$. This relationship can be summarized as

$$t_1 \rightsquigarrow t_0,$$

which is read as "t_0 is a *specialization* of t_1" or "t_1 is a *generization* of t_0."

[Mum67, Ex. D, pp. 137–138]
[Har77, Ex. II.2.3.2]
The letter T is used to stand for the French word *trait* [EGAI$_{\text{new}}$, (5.5.1)]. Some example translations are line, stroke, or dash, but the word also refers to connections as in "avoir trait à."

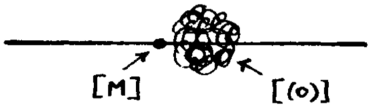

FIGURE 2.2. A trait, i.e., the spectrum of a DVR. From [Mum67, p. 138].

(e) (Polynomial rings over PIDs) For R a PID, we have the following description of Spec($R[x]$):

[Mum67, Ex. E, pp. 138–139, Ex. H, pp. 141–143]
[Har77, Ex. II.2.3.4]
[AK21, (2.26)]
[Rei95, (1.5)]

THEOREM 2.2.3. *Let R be a PID and consider the polynomial ring $R[x]$ in one variable over R. Let $P \subseteq R[x]$ be a prime ideal.*

(i) *$P = (0)$, or $P = (f)$ with f prime, or P is maximal.*

(ii) *If P is maximal, then either $P = (f)$ with f prime, or $P = (p, g)$ for $p \in R$ prime and $g \in R[x]$ such that its image in $(R/(p))[x]$ is prime.*

Proof. Suppose $P \neq (0)$ and P is not principal. Then, there exist two polynomials $f_1, f_2 \in P$ with no common factor. After possibly replacing f_1 and f_2 by prime factors (which lie in P by the assumption that P is prime), we may assume that f_1 and f_2 are prime. Set K to be the fraction field of R, i.e., the field obtained from R by adjoining an inverse for every nonzero element in R. Gauss's lemma implies that f_1 and f_2 are relatively prime in $K[x]$. Since $K[x]$ is a PID, there exist $h_1, h_2 \in K[x]$ such that $h_1 f_1 + h_2 f_2 = 1$. Clearing denominators gives $P \cap R \neq 0$. Since R is a PID, we have $P \cap R = (p)$ for a prime element $p \in R$.

Now set $k := R/(p)$, which is a field. Set

$$Q = P \cdot \bigl(R[x]/(p)\bigr) \subseteq R[x]/(p) \cong k[x].$$

We then have

$$k[x]/Q \cong R[x]/P.$$

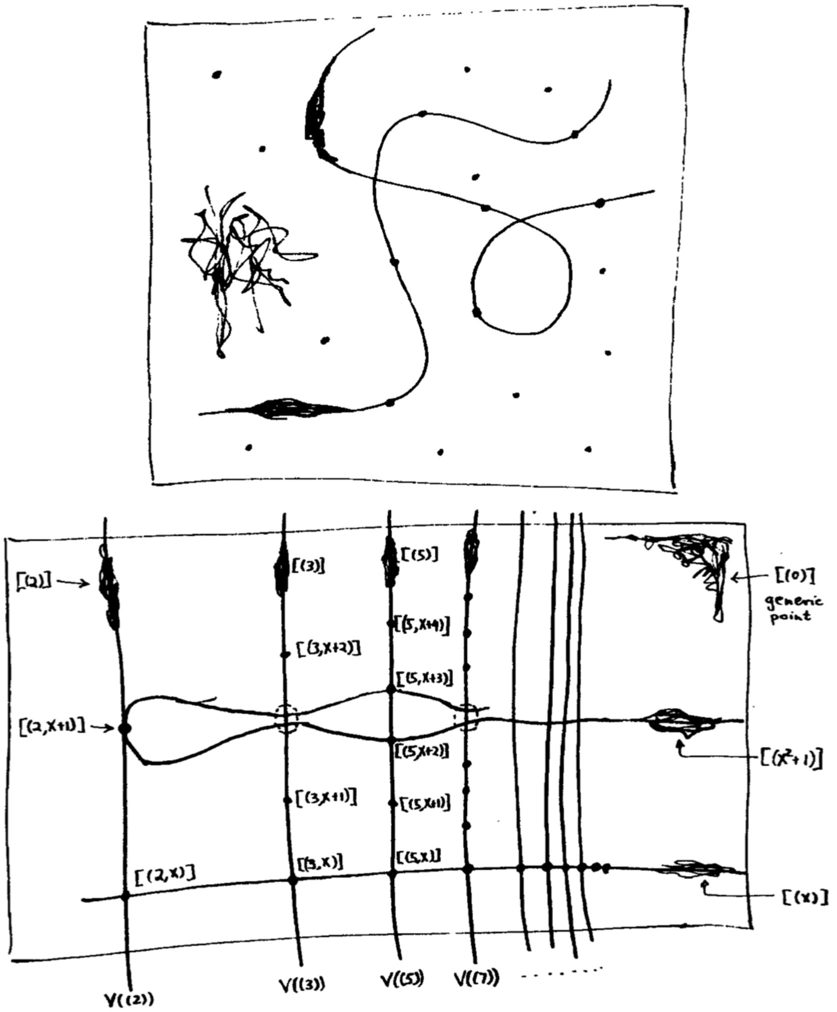

Figure 2.3. Spec($k[x,y]$) and Spec($\mathbf{Z}[x]$). From [Mum67, p. 139, p. 141].

Now since P is prime, these rings are domains, and hence we have $Q = (g')$, where $g' \in k[x]$ is prime. Moreover, $k[x]/Q$ is a field since Q is in fact a maximal ideal by the fact that $k[x]$ is a PID. Now choosing $g \in R[x]$ mapping to g' under the quotient map $R[x] \to k[x]$, we are done. $\square$

For $R = k[y]$ and $R = \mathbf{Z}$, Mumford draws particularly nice pictures. See Figure 2.3. Note that the case when $P = (f)$ is maximal actually occurs: If (V, ϖ) is a DVR, then

$$(\varpi x - 1) \subseteq V[x]$$

is a maximal ideal.

2.2.2. Review of sheaves associated to modules. To define the structure sheaf on Spec(A), we will construct sheaves on Spec(A) from A-modules. The *distinguished open sets*

$$D(f) \coloneqq \operatorname{Spec}(A) - V(f)$$

form a basis for the Zariski topology on $\operatorname{Spec}(A)$ since

$$V(I) = \bigcap_{f \in I} V(f) \Longrightarrow X - V(I) = \bigcup_{f \notin I} D(f).$$

The basic idea of this definition is that we know from commutative algebra that

$$\operatorname{Spec}(A_f) \overset{\text{1-1}}{\longleftrightarrow} D(f) \subseteq \operatorname{Spec}(A)$$

since prime ideals not containing f are in bijection with prime ideals in A_f. There is therefore a nice candidate for what the value of $\mathcal{O}_{\operatorname{Spec}(A)}(D(f))$ should be: It should be A_f! Using [EGAI$_{\text{new}}$, Chapitre 0, §3.2] (which you studied in Homework 10, Problem 3 last semester), we can make the following definition.

Definition 2.2.4 [EGAI$_{\text{new}}$, Définition 1.3.4]. Let A be a ring and let M be an A-module. The *sheaf $\tilde{M}$ associated to M* is the sheafification of the presheaf defined by the basis of principal open sets by sending

$$D(f) \longmapsto M_f$$

for every principal open set $D(f)$. The *structure sheaf* for $X \coloneqq \operatorname{Spec}(A)$ is the sheaf $\mathcal{O}_X \coloneqq \tilde{A}$. We therefore consider $\operatorname{Spec}(A)$ as a ringed space $(\operatorname{Spec}(A), \tilde{A})$.

In [Har77, p. 70, p. 110], Hartshorne gives a different (but equivalent) definition in terms of the espace étalé.

By definition in [EGAI$_{\text{new}}$, Chapitre 0, §3.2], we have

$$\tilde{M} \coloneqq \Bigl(V \longmapsto \varprojlim_{D(f) \subseteq V} M_f\Bigr)^{\#}.$$

Note that by definition, the presheaf in the parentheses on the right maps to $\tilde{M}$, and hence we have maps

$$\theta_f \colon A_f \longrightarrow \Gamma\bigl(D(f), \tilde{A}\bigr)$$
$$\theta_f \colon M_f \longrightarrow \Gamma\bigl(D(f), \tilde{M}\bigr)$$

of A_f-algebras and A_f-modules, respectively.

We recall some properties of the sheaf $\tilde{M}$.

Proposition 2.2.5 [EGAI$_{\text{new}}$, Chapitre 0, (3.2.4), p. 198]. *Let A be a ring with spectrum $X \coloneqq \operatorname{Spec}(A)$ and let $x \in X$ be a point. For every A-module M, we have*

$$\tilde{A}_x \cong A_{\mathfrak{p}_x} \qquad \text{and} \qquad \tilde{M}_x \cong M_{\mathfrak{p}_x}$$

as $A_{\mathfrak{p}_x}$-modules.

[Har77, Props. II.2.2(a), II.5.1(b)]

Proof. See [Mur24ag, Proposition 2.2.5]. □

Proposition 2.2.6 [EGAI$_{\text{new}}$, Proposition 1.3.5]. *Let A be a ring. The functor*

$$\begin{aligned} \mathsf{Mod}(A) &\longrightarrow \mathsf{Mod}(\tilde{A}) \\ M &\longmapsto \tilde{M} \end{aligned}$$

is exact.

[Har77, Prop. II.5.2]

Proof. See [Mur24ag, Proposition 2.2.6]. □

[Har77, Exer. II.2.1]

Proposition 2.2.7 [EGAI$_{\text{new}}$, Proposition 1.3.6]. *Let A be a ring and let M be an A-module. Consider an element $f \in A$. Then,*

$$\widetilde{M_f} \cong \tilde{M}|_{D(f)}.$$

Proof. See [Mur24ag, Proposition 2.2.7]. □

[Har77, Props. II.2.2(b),(c), II.5.1(c),(d)]

Theorem 2.2.8 [EGAI$_{\text{new}}$, Théorème 1.3.7]. *Let A be a ring and let M be an A-module. For every $f \in A$, the map*

$$\theta_f \colon M_f \longrightarrow \Gamma\bigl(D(f), \tilde{M}\bigr)$$

is bijective, and hence the presheaf $D(f) \mapsto M_f$ defined on principal open sets is in fact a sheaf. In particular,

$$\theta_1 \colon M \xrightarrow{\sim} \Gamma\bigl(\operatorname{Spec}(A), \tilde{M}\bigr).$$

Proof. See [Mur24ag, Theorem 2.2.10]. □

2.2.3. Serre's equivalence for affine schemes. Our next goal is to show that $M \mapsto \tilde{M}$ is an equivalence of categories $\mathsf{Mod}(A) \to \mathsf{QCoh}(\mathcal{O}_{\operatorname{Spec}(A)})$. We first show that the functor $M \mapsto \tilde{M}$ is fully faithful.

[Har77, Exer. III.6.7]

Corollary 2.2.9 [EGAI$_{\text{new}}$, Corollaire 1.3.8]. *Let A be a ring and let M, N be an A-module. Then, the homomorphism*

$$\begin{tikzcd} \operatorname{Hom}_A(M,N) \arrow[r] & \operatorname{Hom}_{\tilde{A}}(\tilde{M},\tilde{N}) \\ u \arrow[r, mapsto] & \tilde{u} \end{tikzcd}$$

is bijective. In particular, $M = 0$ if and only if $\tilde{M} = 0$.

Proof. Consider the canonical map

$$\begin{tikzcd} \operatorname{Hom}_{\tilde{A}}(\tilde{M},\tilde{N}) \arrow[r] & \operatorname{Hom}_{\Gamma(\tilde{A})}\bigl(\Gamma(\tilde{M}),\Gamma(\tilde{N})\bigr) \\ v \arrow[r, mapsto] & \Gamma(v). \end{tikzcd}$$

By Theorem 2.2.8, the A-module on the right is naturally isomorphic to $\operatorname{Hom}_A(M,N)$. We want to show that $u \mapsto \tilde{u}$ and $v \mapsto \Gamma(v)$ are mutually inverse. We know that $\Gamma(\tilde{u}) = u$ by definition of $\tilde{u}$. Conversely, given $v \in \operatorname{Hom}_{\tilde{A}}(\tilde{M},\tilde{N})$, the induced map

$$w \colon \Gamma\bigl(D(f), \tilde{M}\bigr) \longrightarrow \Gamma\bigl(D(f), \tilde{N}\bigr)$$

fits into the commutative diagram

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d] & N \arrow[d] \\ M_f \arrow[r, "w"] & N_f \end{tikzcd}$$

where the vertical maps are the localization maps by Theorem 2.2.8. By the universal property of localization [Mur24ca, Theorem 10.2.6], we know that $w = u_f$ for every $f \in A$. By the definition of the action of $M \mapsto \tilde{M}$ on maps, we see that $(\Gamma(v))^{\sim} = v$. □

We now prove the following theorem, which explains why quasi-coherent sheaves are sheaves that "come from algebra." Condition (a) is the definition of quasi-coherence in [Har77, p. 111].

[Har77, p. 111, Exer. II.5.4]

Theorem 2.2.10 [EGAI$_{\text{new}}$, Théorème 1.4.1]. *Let A be a ring, let $X = \operatorname{Spec}(A)$, and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. The following three conditions are equivalent.*

- (a) *There exists an A-module M such that $\mathscr{F} \cong \tilde{M}$.*
- (b) *The sheaf $\mathscr{F}$ is quasi-coherent.* [Har77, Exer. II.5.4]
- (c) *The following two properties hold:* [Har77, Lem. II.5.3]
  - $(c1)$ *For every $f \in A$ and every section $s \in \Gamma(D(f), \mathscr{F})$, there exists an integer $n \geq 0$ such that $f^n s$ extends to a section of $\mathscr{F}$ on X.*
  - $(c2)$ *For every $f \in A$ and every section $t \in \Gamma(X, \mathscr{F})$ such that $t|_{D(f)} = 0$, there exists an integer $n \geq 0$ such that $f^n t = 0$.*

Proof. $(a) \Rightarrow (b)$. Let M be an A-module such that $\mathscr{F} \cong \tilde{M}$. Then, M has a presentation

$$A^{(I)} \longrightarrow A^{(J)} \longrightarrow M \longrightarrow 0.$$

Applying the functor $\tilde{(\cdot)}$, which is exact by Proposition 2.2.6, we obtain the presentation

$$\mathcal{O}_X^{(I)} \longrightarrow \mathcal{O}_X^{(J)} \longrightarrow \mathscr{F} \longrightarrow 0$$

which shows that $\mathscr{F}$ is quasi-coherent.

$(b) \Rightarrow (c)$. We proceed in steps.

Step 1. The case when $\mathscr{F} = \tilde{M}$ for an A-module M.

By Proposition 2.2.7 and Theorem 2.2.8, we have the isomorphism

$$M_f \xrightarrow{\sim} \Gamma\bigl(D(f), \tilde{M}\bigr).$$

We first show $(c1)$. A section $s \in \Gamma(D(f), \tilde{M})$ corresponds to an element $z/f^n \in M_f$ where $z \in M$. We then see that $f^n s$ corresponds to the element $z/1 \in M$ under this isomorphism. The element $z \in M$ gives a section of $\mathscr{F} = \tilde{M}$ on X extending $f^n s$, proving $(c1)$.

We now show $(c2)$. A section $t \in \Gamma(X, \tilde{M})$ corresponds to an element $z' \in M$, and the restriction of z' to $D(f)$ corresponds to the image $z'/1$ of z' in M_f. By the definition of localization, $z'/1 = 0$ in M_f implies $f^n z' = 0$ in M for some $n \geq 0$. We see that for the same n, we have $f^n t = 0$.

Step 2. The general case.

Since X is quasi-compact [Mur24ag, Proposition 2.2.9], the definition of quasi-coherence implies there exist finitely many elements $g_i \in A$ such that $X = \bigcup_i D(g_i)$ and we have presentations

$$\tilde{A}_{g_i}^{(I_i)} \longrightarrow \tilde{A}_{g_i}^{(J_i)} \longrightarrow \mathscr{F}|_{D(g_i)} \longrightarrow 0.$$

In this case, the exactness (Proposition 2.2.6) and the fully faithfulness (Corollary 2.2.9) of the functor $\tilde{(\cdot)}$ shows that $\mathscr{F}|_{D(g_i)} \cong \tilde{N}_i$ for an A_{g_i}-module N_i. By Step 1, $\mathscr{F}|_{D(g_i)}$ satisfies $(c1)$ and $(c2)$. Step 1 also shows that

$$\mathscr{F}|_{D(g_i) \cap D(g_j)} = \mathscr{F}|_{D(g_i g_j)}$$

satisfies $(c1)$ and $(c2)$.

We first show $(c2)$. Since $D(f) \cap D(g_i) = D(fg_i)$, for every i, there exists an integer n_i such that $(fg_i)^{n_i} t|_{D(g_i)} = 0$. Since the image of g_i in A_{g_i} is invertible, we have $f^{n_i} t|_{D(g_i)} = 0$. Setting $n = \max\{n_i\}$, we see that $f^n t = 0$ on X.

To prove ($c1$) for $\mathscr{F}$, we apply ($c1$) to $\mathscr{F}|_{D(g_i)}$. For every i, there exists an integer $n_i \geq 0$ and a section $s_i' \in \Gamma(D(g_i), \mathscr{F})$ extending $(fg_i)^{n_i} s|_{D(fg_i)}$. Since the image of g_i in A_{g_i} is invertible, there exists a section $s_i \in \Gamma(D(g_i), \mathscr{F})$ such that $s_i' = g_i^{n_i} s_i$ and such that s_i extends $f^{n_i} s|_{D(fg_i)}$. After possibly replacing the n_i by $n = \max\{n_i\}$, we may assume the n_i are equal to a fixed number n. By construction,

$$(s_i - s_j)|_{D(f) \cap D(g_i) \cap D(g_j)} = (s_i - s_j)|_{D(fg_ig_j)} = 0.$$

By ($c2$) applied to $\mathscr{F}|_{D(g_ig_j)}$, there exists an integer $m_{ij} \geq 0$ such that

$$(fg_ig_j)^{m_{ij}}(s_i - s_j)|_{D(g_ig_j)} = 0.$$

Since the image of g_ig_j in $A_{g_ig_j}$ is invertible, we see that

$$f^{m_{ij}}(s_i - s_j)|_{D(g_ig_j)} = 0.$$

After possibly replacing the m_{ij} by $m = \max\{m_{ij}\}$, we see there exists a section $s' \in \Gamma(X, \mathscr{F})$ extending the $f^m s_i$. This section extends $f^{m+n}s$, proving ($c1$).

[Har77, Exer. II.5.3]
[EGAI$_{\text{new}}$, (1.7.1)]

(c) ⇒ (a). We first describe an auxiliary construction. Consider the morphism

$$i \colon \bigl(\operatorname{Spec}(A), \tilde{A}\bigr) \longrightarrow \bigl(\{*\}, \underline{A}\bigr)$$

of ringed spaces. We then have the isomorphism

$$\operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathscr{F}) \xrightarrow{\sim} \operatorname{Hom}_A\bigl(M, \Gamma(X, \mathscr{F})\bigr) \tag{2.2.11}$$

by [Har77, p. 110] (Homework 1, Problem 5) since $\tilde{M} \cong i^*M$. Plugging in $M = \Gamma(X, \mathscr{F})$, we obtain a morphism $u \colon \tilde{M} \to \mathscr{F}$ of $\tilde{A}$-modules. Explicitly, since the $D(f)$ form a basis for the topology on X, it suffices to define a morphism

$$u_f \colon M_f \longrightarrow \Gamma\bigl(D(f), \mathscr{F}\bigr)$$

satisfying the usual compatibility conditions. Since the image of f in A_f is invertible, the universal property of localization [Mur24ca, Theorem 10.2.6] yields the commutative diagram

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & \Gamma(X, \mathscr{F}) \arrow[d, "\rho^X_{D(f)}"] \\
M_f \arrow[r, "u_f"] & \Gamma\bigl(D(f), \mathscr{F}\bigr).
\end{tikzcd}$$

The compatibility of the morphisms u_f follows by the uniqueness part of the universal property.

We now use property ($c1$) (resp. ($c2$)) to show that the u_f are surjective (resp. injective), which will show (a). If $s \in \Gamma(D(f), \mathscr{F})$, then by ($c1$), there exists an integer $n \geq 0$ such that $f^n s$ extends to a section $z \in M$. Since $u_f(z/f^n) = s$, this shows that u_f is surjective. If $z \in M$ is such that $u_f(z/1) = 0$, then $z|_{D(f)} = 0$. By ($c2$), there exists an integer $n \geq 0$ such that $f^n z = 0$, and hence $z/1 = 0$ in M_f. This shows that u_f is injective. □

We can restate what we have shown as an equivalence of categories. I am used to calling it Serre's equivalence for affine schemes, although this terminology does not seem to be widespread. Serre proved this statement for coherent sheaves on affine varieties [FAC, n$^{\text{o}}$ 49, Théorème 1].

[Har77, Cor. II.5.5]

Corollary 2.2.12 [EGAI$_{\text{new}}$, Corollaire 1.4.2 and Corollaire 1.4.3]. *Let A be a ring and let $X = \operatorname{Spec}(A)$. We have an exact equivalence of categories*

$$\begin{tikzcd}
\mathsf{Mod}(A) \arrow[r, leftrightarrow, "\sim"] & \mathsf{QCoh}(\mathcal{O}_X) \\
M \arrow[r, mapsto] & \tilde{M} \\
\Gamma(X, \mathscr{F}) & \mathscr{F}. \arrow[l, mapsto]
\end{tikzcd}$$

In particular, the functor $\Gamma(X, \cdot\,)$ is exact on $\mathsf{QCoh}(X)$. Moreover, $\tilde{M}$ is of finite type (resp. of finite presentation) if and only if M is of finite type (resp. of finite presentation).

Proof. The equivalence of categories follows by combining Corollary 2.2.9 and Theorem 2.2.10.

It remains to show the "Moreover" statement. The direction $\Leftarrow$ follows by choosing a presentation for M and using the exactness of the functor $M \mapsto \tilde{M}$ (Proposition 2.2.6).

For the direction $\Rightarrow$, since X is quasi-compact, we know there exist finitely many elements $f_1, f_2, \ldots, f_r \in A$ such that $D(f_i)$ cover X and such that for every i, the localization M_{f_i} is an A_{f_i}-module of finite type (resp. of finite presentation). In the finite type case, let $z_{i1}, z_{i2}, \ldots, z_{is_i}$ be a set of generators for M_{f_i}. Take n large enough such that $f_i^n z_{ij} \in M$ for all i, j. The collection $\{f_i^n z_{ij}\}_{i,j}$ generates M since [BouCA, II.5.2, Prop. 3, Cor.]

$$A^{\sum_i s_i} \xrightarrow{[\, f_1^n z_{11} \ \cdots \ f_1^n z_{1s_1} \ f_2^n z_{21} \ \cdots \ f_r^n z_{rs_r} \,]} M$$

is surjective after localizing at every prime ideal $\mathfrak{p} \subseteq A$. For finite presentation, we do the same thing once to obtain a surjection $A^q \twoheadrightarrow M$ and another time to obtain a surjection $A^p \twoheadrightarrow \ker(A^q \to M)$. Here, we use [Mur24ca, Lemma 7.10.2] to say that the kernel of $A^q_{f_i} \to M_{f_i}$ is finitely presented. $\square$

As a consequence, we obtain the following:

Corollary 2.2.13 [EGAI$_{\text{new}}$, Corollaire 1.3.12(i),(ii) and Corollaire 1.7.7]. *Let A be a ring and let M, N be two A-modules.*

(i) *There is a natural isomorphism* [Har77, Prop. II.5.2(b)]

$$(M \otimes_A N)^\sim \xrightarrow{\sim} \tilde{M} \otimes_{\tilde{A}} \tilde{N}.$$

(ii) *If M is of finite presentation, then there is a natural isomorphism* [Har77, Exer. III.6.7]

$$\big(\operatorname{Hom}_A(M, N)\big)^\sim \xrightarrow{\sim} \mathscr{H}\!om_{\tilde{A}}(\tilde{M}, \tilde{N}).$$

(iii) *Let $\varphi\colon A \to B$ be a ring map and let $u\colon X \to S$ be the morphism of corresponding affine schemes. For every A-module M, we have* [Har77, Prop. II.5.2(d),(e)]

$$u^* \tilde{M} \xrightarrow{\sim} (M \otimes_A B)^\sim$$

and for every B-module N, we have

$$\big(N_{[\varphi]}\big)^\sim \xrightarrow{\sim} u_* \tilde{N},$$

where $(\,\cdot\,)_{[\varphi]}$ denotes restriction of scalars.

Proof. Briefly, (i) and (ii) hold because tensor products commute with localization and Hom commutes with localization when the first argument is of finite presentation. We describe what maps are used in each of the arguments below.

(*ii*). The sheaf $\mathscr{G} = (\operatorname{Hom}_A(M,N))^\sim$ is the sheaf associated to the presheaf

$$D(f) \longmapsto \mathscr{G}\big(D(f)\big) = \operatorname{Hom}_{A_f}(M_f, N_f)$$

on principal open sets $U = D(f) \subseteq \operatorname{Spec}(A)$. On the other hand, $\mathscr{G}(D(f))$ is naturally isomorphic to

$$\big(\operatorname{Hom}_A(M,N)\big)_f \xrightarrow{\sim} \Gamma\big(D(f), \mathscr{G}\big)$$

by the fact that Hom commutes with flat base change [Mur24ca, Proposition 7.7.4], where the isomorphism is from Theorem 2.2.8. Since the construction of these isomorphisms is compatible with restriction in Proposition 2.2.7 and Theorem 2.2.8, we are done.

(*i*). Ideally one can just observe that they satisfy the same universal property using bilinear maps and then apply the equivalence in Corollary 2.2.12. However, because the universal property of tensor products talks about bilinear maps, and the equivalence as stated does not know about these, we need to proceed differently.

One proof is via tensor–Hom adjunction, which you show in [Har77, Exercise II.5.1(*c*)] (Homework 3, Problem 1(*c*)). We have

$$\begin{aligned} \operatorname{Hom}_{\mathcal{O}_X}\big((M \otimes_A N)^\sim, \cdot\,\big) &\xleftarrow{\sim} \operatorname{Hom}_A\big(M \otimes_A N, \Gamma(X, \cdot\,)\big) \\ &\xrightarrow{\sim} \operatorname{Hom}_A\big(M, \operatorname{Hom}_A\big(N, \Gamma(X, \cdot\,)\big)\big) \\ &\xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}_X}\big(\tilde{M}, \mathscr{H}om_{\mathcal{O}_X}(\tilde{N}, \cdot\,)\big) \\ &\xleftarrow{\sim} \operatorname{Hom}_{\mathcal{O}_X}(\tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}, \cdot\,) \end{aligned}$$

where the first and third isomorphisms hold by Corollary 2.2.9, the third isomorphism uses (*ii*), and the second and fourth isomorphisms hold by tensor–Hom adjunction for modules [Mur24ca, Theorem 7.6.1] and $\mathcal{O}_X$-modules [Har77, Exercise II.5.1(*c*)] (Homework 3, Problem 1(*c*)), respectively. By the Yoneda lemma [Mur24ca, Corollary 4.6.3], we are done.

Another proof uses the definition of $(\tilde{\cdot})$. The sheaf $\mathscr{F} = (M \otimes_A N)^\sim$ is the sheaf associated to the presheaf

$$U \longmapsto \mathscr{F}(U) = \Gamma(U, \tilde{M}) \otimes_{\Gamma(U,\tilde{A})} \Gamma(U, \tilde{N})$$

on principal open sets $U = D(f) \subseteq \operatorname{Spec}(A)$. On the other hand, $\mathscr{F}(D(f))$ is isomorphic (via the universal property of localization) to $M_f \otimes_{A_f} N_f$ by Proposition 2.2.7. In turn, this module is naturally isomorphic to

$$(M \otimes_A N)_f \xrightarrow{\sim} \Gamma\big(D(f), \mathscr{F}\big)$$

by [Mur24ca, Lemma 7.12.7], where the isomorphism is from Theorem 2.2.8. Since the construction of these isomorphisms is compatible with restriction in Proposition 2.2.7 and Theorem 2.2.8, we are done.

$\boxed{2/6}$ (*iii*). For direct images, the left-hand side is the sheaf associated to

[EGAI, Prop. 1.6.3]

$$D(f) \longmapsto \big(N_{[\varphi]}\big)_f \cong \big(N_{\varphi(f)}\big)_{[\varphi f]} = \Gamma\Big(D\big(\varphi(f)\big), \tilde{N}\Big)_{[\varphi f]}$$

[EGAI, Prop. 1.6.5] on principal open sets $D(f) \subseteq \operatorname{Spec}(A)$. For inverse images, we show that the two sides represent the same functor. We have

$$\begin{aligned} \operatorname{Hom}_{\mathcal{O}_X}(u^*\tilde{M}, \,\cdot\,) &\xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}_S}(\tilde{M}, u_*\,\cdot\,) \\ &\xrightarrow{\sim} \operatorname{Hom}_A\big(M, \Gamma(S, u_*\,\cdot\,)\big) \\ &\xrightarrow{\sim} \operatorname{Hom}_A\big(M, \Gamma(X, \,\cdot\,)_{[\varphi]}\big) \\ &\xleftarrow{\sim} \operatorname{Hom}_B\big(M \otimes_A B, \Gamma(X, \,\cdot\,)\big) \\ &\xleftarrow{\sim} \operatorname{Hom}_{\mathcal{O}_X}\big((M \otimes_A B)^{\sim}, \,\cdot\,\big) \end{aligned}$$

where the first isomorphism holds by [Har77, p. 110] (Homework 1, Problem 5), the second and fifth isomorphisms hold by the adjunction in (2.2.11), the third isomorphism holds by definition of direct images, and the fourth isomorphism holds by tensor–Hom adjunction [Mur24ca, Corollary 7.6.2]. By the Yoneda lemma [Mur24ca, Corollary 4.6.3], we are done. ☐

In the Noetherian case, we obtain Hartshorne's definition for coherent sheaves.

Theorem 2.2.14 [EGAI$_{\text{new}}$, Théorème 1.5.1]. *Let A be a* Noetherian *ring, let $X = \operatorname{Spec}(A)$, and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. The following conditions are equivalent.*

(a) *$\mathscr{F}$ is coherent.*
(b) *$\mathscr{F}$ is quasi-coherent and of finite type.*
(c) *There exists a finite type A-module M such that $\mathscr{F} \cong \tilde{M}$.*

[Har77, Exer. II.5.4]

Proof. This follows from Theorem 2.2.10 and Corollary 2.2.12 using the fact that for A Noetherian, sub-modules of the Noetherian module A^q are finitely generated [Mur24ca, Theorem 7.5.2]. ☐

2.2.4. Cartan's Theorem B for affine schemes. One consequence of Corollary 2.2.12 is that taking global sections is exact for exact sequences of quasi-coherent sheaves on $\operatorname{Spec}(A)$. In fact, we can say something stronger: For a short exact sequence

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0,$$

of $\mathcal{O}_{\operatorname{Spec}(A)}$-modules on $\operatorname{Spec}(A)$ *and assuming that $\mathscr{F}'$ is quasi-coherent*, taking global sections is exact. This exactness result will be a consequence of the cohomological result proved below. See [Har77, Proposition II.5.6] for a direct proof of this exactness result using techniques from this course. See [EGAI$_{\text{new}}$, Proposition 1.4.6] for an alternative proof using Yoneda Ext as defined in [Yon54] instead of sheaf cohomology.

In the complex analytic case, the following result is due to Cartan [SHC51/52, Exposé 18, Théorème B], who showed that complex Stein manifolds have vanishing cohomology for *coherent* sheaves. Because of the name of Cartan's theorem in [SHC51/52, Exposé XVIII], Cartan's result is known as *Cartan's Theorem B*. Serre [FAC, n° 46, Théorème 1, Corollaire 1] proved Cartan's Theorem B for affine varieties. For another analogue of Cartan's Theorem B, see [Kie67, Satz 2.4.2], where Kiehl shows Cartan's Theorem B for coherent sheaves on affinoid spaces (or more generally, quasi-Stein spaces) in rigid analytic geometry.

Theorem 2.2.15 (Cartan's Theorem B for affine schemes [EGAIII$_1$, Théorème 1.3.1]). *Let A be a ring and let $X = \operatorname{Spec}(A)$. Consider a quasi-coherent sheaf $\mathscr{F}$ on X. Then, we have*

$$H^i(X, \mathscr{F}) = 0$$

[Har77, Prop. II.5.6,
Thm. III.3.5,
Rem. III.3.5.1]

for all $i > 0$.

We prove Theorem 2.2.15 following [Kem80, §1]. If $j\colon U \hookrightarrow X$ is an open inclusion, we set

$$_U\mathscr{F} \coloneqq j_*\bigl(\mathscr{F}|_U\bigr).$$

We start with the following preliminary result, which is another "dimension shifting" type argument.

Proposition 2.2.16 [Kem80, Proposition 1]. *Let X be a topological space and let $\mathscr{F}$ be an Abelian sheaf. Suppose X has a basis $\mathscr{U}$ such that for some positive integer $n > 0$, we have*

$$H^i\bigl(U, \mathscr{F}|_U\bigr) = 0$$

for all $0 < i < n$ and for all $U \in \mathscr{U}$. Given any element $\alpha \in H^n(X, \mathscr{F})$, there exists a covering $\mathscr{V}$ of X by members of $\mathscr{U}$ such that the image of α in $H^n(X, {}_V\mathscr{F})$ is 0 for all $V \in \mathscr{V}$.

Proof. Let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be a short exact sequence where $\mathscr{G}$ is flasque. Then, for every open subset $U \subseteq X$, we have $H^i(U, \mathscr{G}|_U) = 0$ for all $i > 0$. By the long exact sequence on sheaf cohomology, we have the exact sequence

$$0 \longrightarrow \Gamma\bigl(U, \mathscr{F}|_U\bigr) \longrightarrow \Gamma\bigl(U, \mathscr{G}|_U\bigr) \longrightarrow \Gamma\bigl(U, \mathscr{H}|_U\bigr) \longrightarrow H^1\bigl(U, \mathscr{F}|_U\bigr) \longrightarrow 0 \tag{2.2.17}$$

and the isomorphisms

$$H^i\bigl(U, \mathscr{H}|_U\bigr) \xrightarrow{\sim} H^{i+1}\bigl(U, \mathscr{F}|_U\bigr) \tag{2.2.18}$$

for all $i > 0$.

For any open subset $V \subseteq X$, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{G} \arrow[r] \arrow[d] & \mathscr{H} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & {}_V\mathscr{F} \arrow[r] & {}_V\mathscr{G} \arrow[r] & {}_V\mathscr{H} &
\end{tikzcd}$$

with exact rows, where ${}_V\mathscr{G}$ is still flasque. The image $\mathscr{K}$ of $\mathscr{H} \to {}_V\mathscr{H}$ and of ${}_V\mathscr{G} \to {}_V\mathscr{H}$ coincide since they are both the sheafification of the presheaf

$$W \longmapsto \operatorname{im}\bigl(\mathscr{G}(W \cap V) \to \mathscr{H}(W \cap V)\bigr) = \operatorname{im}\bigl(\mathscr{G}(V) \to \mathscr{H}(W \cap V)\bigr)$$

where the equality holds by the flasqueness of $\mathscr{G}$. Replacing ${}_V\mathscr{H}$ by $\mathscr{K}$, the same long exact sequence argument as in the previous paragraph yields the exact sequence

$$0 \longrightarrow \Gamma\bigl(X, {}_V\mathscr{F}\bigr) \longrightarrow \Gamma\bigl(X, {}_V\mathscr{G}\bigr) \longrightarrow \Gamma\bigl(X, \mathscr{K}\bigr) \longrightarrow H^1\bigl(X, {}_V\mathscr{F}\bigr) \longrightarrow 0 \tag{2.2.19}$$

and the isomorphisms

$$H^i(X, \mathscr{K}) \xrightarrow{\sim} H^{i+1}\bigl(X, {}_V\mathscr{F}\bigr) \tag{2.2.20}$$

for all $i > 0$.

We proceed by induction on n. Suppose that $n = 1$. We consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X, \mathscr{F}) \arrow[r] \arrow[d] & \Gamma(X, \mathscr{G}) \arrow[r] \arrow[d] & \Gamma(X, \mathscr{H}) \arrow[r] \arrow[d] & H^1(X, \mathscr{F}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma\bigl(X, {}_V\mathscr{F}\bigr) \arrow[r] & \Gamma\bigl(X, {}_V\mathscr{G}\bigr) \arrow[r] & \Gamma\bigl(X, \mathscr{K}\bigr) \arrow[r] & H^1\bigl(X, {}_V\mathscr{F}\bigr) \arrow[r] & 0
\end{tikzcd}$$

with exact rows obtained by combining the exact sequences (2.2.17) for $U = X$ and (2.2.19). By exactness in the top row, $\alpha \in H^1(X, \mathscr{F})$ is $\delta(\beta)$ for some $\beta \in \Gamma(X, \mathscr{H})$. By exactness in the bottom row, the image of α in $H^1(X, {}_V\mathscr{F})$ is 0 if and only if the image of β in

$$\Gamma(X, \mathscr{K}) \subseteq \Gamma\bigl(X, {}_V\mathscr{H}\bigr) = \Gamma(V, \mathscr{H})$$

lifts to an element of $\Gamma(X, {}_V\mathscr{G}) = \Gamma(V, \mathscr{G})$. Since $\mathscr{U}$ contains arbitrarily small open subsets, we can find a suitable covering $\mathscr{V}$ of X because $\mathscr{G} \to \mathscr{H} \to 0$ is exact.

Now suppose that $n > 0$. If V and W are members of $\mathscr{U}$, the sequence (2.2.17) for $U = V \cap W$ shows that

$$\Gamma\bigl(U, {}_V\mathscr{G}\bigr) \longrightarrow \Gamma\bigl(U, {}_V\mathscr{H}\bigr) \longrightarrow 0$$

is surjective. Thus, $\mathscr{K} = {}_V\mathscr{H}$. Moreover, (2.2.18) shows that $\mathscr{H}$ satisfies the assumption in the proposition for the integer $n - 1$. The isomorphisms

$$H^i\bigl(X, {}_V\mathscr{H}\bigr) \xrightarrow{\sim} H^{i+1}\bigl(X, {}_V\mathscr{F}\bigr)$$

from (2.2.20) then show that the covering $\mathscr{V}$ works for $\mathscr{F}$ by inductive hypothesis. □

We now show Cartan's Theorem B for affine schemes (Theorem 2.2.15).

Proof of Theorem 2.2.15. Let $\mathscr{U}$ be the basis of X consisting of principal open subsets $D(f)$. We will show by induction on n that $H^i(X, \mathscr{F}) = 0$ for all $0 < i < n$ and for all quasi-coherent sheaves $\mathscr{F}$ on spectra of rings. The hypothesis of Proposition 2.2.16 holds for $\mathscr{U}$. Thus, given any element $\alpha \in H^n(X, \mathscr{F})$, there exists a covering $X = \bigcup_{j=1}^{p} V_i$ by members of $\mathscr{U}$ such that the image of α in [Kem80, Thm. 2]

$$H^n\left(X, \bigoplus_{j=1}^{p} {}_{V_j}\mathscr{F}\right)$$

is 0. The long exact sequence on sheaf cohomology associated to

$$0 \longrightarrow \mathscr{F} \longrightarrow \bigoplus_{j=1}^{p} {}_{V_j}\mathscr{F} \longrightarrow \mathscr{G} \longrightarrow 0$$

shows that α is in the image of $\delta(H^{n-1}(X, \mathscr{G}))$.

If $n = 1$, then $\delta = 0$ because $\Gamma(X, \cdot)$ is exact for quasi-coherent sheaves by Corollary 2.2.12. If $n > 0$, then $H^{n-1}(X, \mathscr{G}) = 0$ by inductive hypothesis. In either case, we see that $\alpha = 0$. □

As a consequence, we have

Corollary 2.2.21 ("2 out of 3" property for quasi-coherent sheaves on affine schemes [EGAI$_{\text{new}}$, Corollaire 1.4.7]). *Let A be a ring and let $X = \operatorname{Spec}(A)$. Consider a short exact sequence* [Har77, Prop. II.5.7]

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

of $\mathcal{O}_X$-modules. If two of the three sheaves are quasi-coherent, then the third sheaf is also quasi-coherent.

Proof. If $\mathscr{F}$ and $\mathscr{G}$ (resp. $\mathscr{G}$ and $\mathscr{H}$) are quasi-coherent, then this is also the case for $\mathscr{H}$ (resp. $\mathscr{F}$) because kernels and cokernels can be computed in terms of their associated A-modules.

Suppose that $\mathscr{F}$ and $\mathscr{H}$ are quasi-coherent. The long exact sequence on sheaf cohomology yields the exact sequence

$$0 \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{G}) \longrightarrow \Gamma(X, \mathscr{H}) \longrightarrow H^1(X, \mathscr{F}) = 0$$

by Theorem 2.2.15. Taking associated sheaves, we obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X,\mathscr{F})^\sim \arrow[r] \arrow[d] & \Gamma(X,\mathscr{G})^\sim \arrow[r] \arrow[d] & \Gamma(X,\mathscr{H})^\sim \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{G} \arrow[r] & \mathscr{H} \arrow[r] & 0
\end{tikzcd}$$

with exact rows, where the vertical maps are those constructed during the proof of (c) ⇒ (a) in Theorem 2.2.10 (see (2.2.11)). Since the outer maps are isomorphisms, the middle map is an isomorphism by the snake lemma [KS06, Lemma 12.1.1]. $\square$

2.3. Schemes and Serre's criterion for affineness

We now turn to schemes in general.

2.3.1. Review of locally ringed spaces and schemes. Now that we have defined a ringed space structure on $(\operatorname{Spec}(A), \mathcal{O})$, we glue them together to form other schemes.

[Har77, p. 70]

Definition 2.3.1 (Ringed spaces [EGAI_{new}, Chapitre 0, (4.1.1)]). Recall that a *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X called the *structure sheaf*. A *morphism* of ringed spaces

$$(f, f^\#) \colon (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

consists of a continuous map $f \colon X \to Y$ and a map

$$f^\# \colon \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X$$

of sheaves of rings on Y. This forms a category RS.

[Har77, pp. 70–71]

Definition 2.3.2 (Locally ringed spaces [EGAI_{new}, Chapitre 0, (4.1.9) and (4.1.12)]). A ringed space $(X, \mathcal{O}_X)$ is a *locally ringed space* if $\mathcal{O}_{X,x}$ is a local ring for every point $x \in X$. In this situation, we denote by $\mathfrak{m}_{X,x}$ or $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$.

Morphisms of locally ringed spaces are morphisms of ringed spaces with an additional condition. To define a morphism of locally ringed spaces, consider a morphism $(f, f^\#) \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces let $x \in X$ be a point and let $V \subseteq Y$ be a neighborhood of $f(x) \in Y$. As V ranges over all open neighborhoods of $f(x)$, we see that $f^{-1}(V)$ ranges over a subset of the open neighborhoods of x. We then obtain the commutative diagram

$$\begin{tikzcd}
\mathcal{O}_Y(U) \arrow[r, "f^\#(U)"] \arrow[d] & \mathcal{O}_X\bigl(f^{-1}(U)\bigr) \arrow[d] \\
\mathcal{O}_Y(V) \arrow[r, "f^\#(V)"] & \mathcal{O}_X\bigl(f^{-1}(V)\bigr)
\end{tikzcd}$$

and taking direct limits, the left column becomes the stalk $\mathcal{O}_{Y,f(x)}$, while the right column is

$$\varinjlim_{f^{-1}(V) \ni x} \mathcal{O}_X\bigl(f^{-1}(V)\bigr)$$

which has a natural map to $\mathcal{O}_{X,x}$. A *morphism* of locally ringed spaces is a morphism $(f, f^{\#})$ of ringed spaces such that taking direct limits in the commutative diagram above and composing with the map to $\mathcal{O}_{X,x}$, the induced map

$$f_x^{\#}\colon \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$

is a *local map*, that is,

$$f_x^{\#}\bigl(\mathfrak{m}_{Y,f(x)}\bigr) \subseteq \mathfrak{m}_{X,x}.$$

Locally ringed spaces form a category LRS.

An *isomorphism* of locally ringed spaces is a morphism with a two-sided inverse. Thus, a morphism $(f, f^{\#})$ is an isomorphism if and only if f is a homeomorphism of the underlying topological spaces and $f^{\#}$ is an isomorphism of sheaves.

Note that $(\operatorname{Spec}(A), \tilde{A})$ is a locally ringed space by Proposition 2.2.5. Moreover, we showed last semester [Mur24ag, Corollary 2.2.19] that Spec defines a fully faithful functor

$$\operatorname{Spec}\colon \mathsf{Ring}^{\mathrm{op}} \longrightarrow \mathsf{LRS}$$

and that there is an "adjunction" [Mur24ag, Proposition 2.2.17]

$$\operatorname{Hom}_{\mathsf{LRS}}\bigl(X, \operatorname{Spec}(A)\bigr) \xrightarrow{\sim} \operatorname{Hom}_{\mathsf{Ring}}\bigl(A, \Gamma(X, \mathcal{O}_X)\bigr).$$

In particular, $A \to B$ is an isomorphism if and only if $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is an isomorphism.

We define the analogue of an affine variety and the affine coordinate ring.

Definition 2.3.3 (Affine scheme [EGAI_{new}, Définition 1.6.1]). A locally ringed space $(X, \mathcal{O}_X)$ is an *affine scheme* if it is isomorphic to a locally ringed space of the form $(\operatorname{Spec}(A), \tilde{A})$ where A is a ring. In this case, we say that the ring

$$A(X) \coloneqq \Gamma(X, \mathcal{O}_X),$$

which can be identified with A by Theorem 2.2.8, is the *ring* of the affine scheme.

[Har77, p. 74]

We now define schemes.

Definition 2.3.4 (Scheme [EGAI_{new}, Définition 2.1.2]). A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ in which every point has a neighborhood U such that $(U, \mathcal{O}_X|_U)$ is an affine scheme, called an *affine open neighborhood*. We call X the *underlying topological space* of the scheme. If we want to forget the scheme structure on a scheme (or ringed space) we write $\operatorname{sp}(X)$ (or $|X|$), which we read as "the space of X". A *morphism* of schemes is a morphism of locally ringed spaces. In other words, the category Sch of schemes is a full subcategory of the category LRS of locally ringed spaces.

[Har77, p. 74] Before [EGAI_{new}], schemes were called *preschemes*. See [EGAI_{new}, p. 3].

Something we will use all the time is that affine open subsets form a basis for the topology on $\operatorname{sp}(X)$ [EGAI_{new}, Proposition 2.1.3]. This is because principal open subsets form a basis of opens in an affine scheme, and principal open subsets are still affine.

Example 2.3.5 (The affine line with two origins I). Let k be a field, let $X_1 = X_2 = \mathbf{A}_k^1$, let $U_1 = U_2 = \mathbf{A}_k^1 - \{P\}$ where P is the point corresponding to $(t) \subseteq k[t]$, and let $\varphi\colon U_1 \to U_2$ be the identity map. Let X be the gluing of X_1 and X_2 along U_1 and U_2 via φ. Then, X is the *affine line with two origins*.

[Har77, Ex. II.2.3.6]

This is a scheme that is not an affine scheme! See [Mur24ag, Example 2.2.25] for one proof. For another proof, we claim that $H^1(X, \mathcal{O}_X) \neq 0$, which shows that

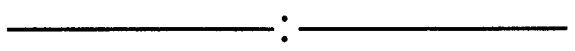

Figure 2.4. Affine line with two origins. From [Har77, p. 76].

X is not affine by Cartan's Theorem B for affine schemes (Theorem 2.2.15). We have the left exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} {}_{X_1}\mathcal{O}_X \oplus {}_{X_2}\mathcal{O}_X \xrightarrow{[1\ -1]} {}_{X_1\cap X_2}\mathcal{O}_X \longrightarrow 0 \tag{2.3.6}$$

which we show is exact on the right by taking stalks. For $x \in X_1 \cap X_2$, we have the short exact sequence

$$0 \longrightarrow \mathcal{O}_{X,x} \xrightarrow{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} \mathcal{O}_{X,x} \oplus \mathcal{O}_{X,x} \xrightarrow{[1\ -1]} \mathcal{O}_{X,x} \longrightarrow 0.$$

For the first origin $x_1 \in X_1 - X_2$, we have the short exact sequence

$$0 \longrightarrow \mathcal{O}_{X,x} \xrightarrow{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} \mathcal{O}_{X,x} \oplus \mathscr{K}_{X,x} \xrightarrow{[1\ -1]} \mathscr{K}_{X,x} \longrightarrow 0$$

and for the second origin $x_2 \in X_2 - X_1$, we have the short exact sequence

$$0 \longrightarrow \mathcal{O}_{X,x} \xrightarrow{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} \mathscr{K}_{X,x} \oplus \mathcal{O}_{X,x} \xrightarrow{[1\ -1]} \mathscr{K}_{X,x} \longrightarrow 0$$

where $\mathscr{K}_X = \underline{k(t)}$ is the constant sheaf of rational functions on X. We now take global sections in (2.3.6):

$$\begin{tikzcd}
\Gamma(X_1, \mathcal{O}_X) \oplus \Gamma(X_2, \mathcal{O}_X) \arrow[r, "{[1\ -1]}"] \arrow[d, equal] & \Gamma(X_1 \cap X_2, \mathcal{O}_X) \arrow[d, equal] \\
k[t] \oplus k[t] \arrow[r, "{[1\ -1]}"] & k[t, t^{-1}].
\end{tikzcd}$$

The bottom map is not surjective. Thus, $H^1(X, \mathcal{O}_X) \neq 0$, and hence X is not affine by Cartan's Theorem B (Theorem 2.2.15).

Remark 2.3.7. A subtle point in this computation is that we do not know (yet) that the middle and right terms in (2.3.6) are quasi-coherent. Thus, we actually need Cartan's Theorem B (Theorem 2.2.15) instead of Corollary 2.2.12.

2.3.2. Quasi-coherent sheaves on schemes.

Using what we know in the affine case, we immediate have the following. This is the definition of quasi-coherence in [Har77, p. 111].

[Har77, Exer. II.5.4]

Proposition 2.3.8 [EGAI$_{\text{new}}$, Proposition 2.2.1]. *Let X be a scheme and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. Then, $\mathscr{F}$ is quasi-coherent if and only if, for every affine open subset $V = \operatorname{Spec}(A) \subseteq X$, we have $\mathscr{F}|_V \cong \tilde{M}$ for some A-module M.*

Proof. The definition of quasi-coherence is local. We can therefore apply Theorem 2.2.10. ☐

Corollary 2.3.9 [EGAI$_{\text{new}}$, Corollaire 2.2.2]. *Let X be a scheme.*

[Har77, Prop. II.5.7]

(*i*) ("2 out of 3" property for quasi-coherent sheaves on schemes) *Consider an exact sequence*

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

of $\mathcal{O}_X$-modules. If two of the three sheaves are quasi-coherent, then the third sheaf is also quasi-coherent.

(*ii*) *Tensor products and $\mathscr{H}om$'s of quasi-coherent $\mathcal{O}_X$-modules are quasi-coherent.*

(*iii*) *Images, kernels, and cokernels of maps of quasi-coherent $\mathcal{O}_X$-modules are quasi-coherent.*

[Har77, Prop. II.5.7]

(*iv*) *Direct limits and (possibly infinite) direct sums of quasi-coherent $\mathcal{O}_X$-modules are quasi-coherent.*

Proof. By Proposition 2.3.8, we reduce to the affine case, which is shown in Corollary 2.2.21 and Corollary 2.2.13. The last two statements hold since $M \mapsto \tilde{M}$ is an equivalence of categories (Corollary 2.2.12). □

2.3.3. Noetherian schemes and coherent sheaves on Noetherian schemes.

To get a nice description of coherent sheaves, we introduce the following class of schemes.

Definition 2.3.10 [EGAI$_{\text{new}}$, Définition 2.7.1]. Let X be a scheme. We say that X is *Noetherian* (resp. *locally Noetherian*) if there exists a finite affine open covering (resp. an affine open covering)

[Har77, p. 83]

$$X = \bigcup_i \operatorname{Spec}(A_i)$$

of X such that A_i is Noetherian for every i.

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Example 2.3.11. Noetherian schemes have Noetherian underlying topological spaces because it is a finite union of Noetherian spaces. The converse does not hold! Here are some examples. Below, k is a field.

[Har77, Caution II.3.1.1, Exer. II.2.13(*d*)]

(1) $\operatorname{Spec}(k[x_1, x_2, \ldots]/(x_1^2, x_2^2, \ldots)) = \{*\}$.

(2) The spectrum of a finite-dimensional non-Noetherian valuation ring. One example of such a ring is from last semester [Mur24ag, Example 1.6.2], the ring

[Har77, Exer. II.4.12(*b*)(3)]

$$k\left[x, y, \frac{x}{y}, \frac{x}{y^2}, \ldots\right]_{\mathfrak{m}} \subseteq k(x,y)$$

where $\mathfrak{m}$ is generated by all the generators on the left-hand side. This is the valuation ring of $k(x,y)$ with value group $\mathbf{Z}^2$ with the lexicographic order, where

$$v(x^i y^j) = (i,j).$$

This ring is non-Noetherian since $\mathfrak{m}$ cannot be generated by finitely many elements: The set of powers of y appearing in the denominators of elements in $\mathfrak{m}$ is bounded below.

We want to characterize Noetherianity in terms of having a property for *every* affine open subset.

Proposition 2.3.12 [EGAI$_{\text{new}}$, Proposition 2.7.2]. *Let X be a scheme. The following are equivalent.*

[Har77, p. 83]

(*i*) *X is Noetherian.*

(*ii*) *X is locally Noetherian and quasi-compact.*

Proof. (*i*) $\Leftrightarrow$ (*ii*) holds since affine schemes are quasi-compact, and finite unions of quasi-compact sets are quasi-compact. □

[Har77, Prop. II.3.2] PROPOSITION 2.3.13 [EGAI$_{\text{new}}$, Proposition 2.7.3]. *Let X be a scheme. The following are equivalent.*

(*i*) *X is locally Noetherian.*
(*ii*) *For every affine open subset $\operatorname{Spec}(B) \subseteq X$, the ring B is Noetherian.*

In particular, an affine scheme $\operatorname{Spec}(A)$ is Noetherian if and only if A is Noetherian.

Proof. (*ii*) ⇒ (*i*) holds since we can choose an arbitrary affine open cover.

(*i*) ⇒ (*ii*). Consider an affine open subset $\operatorname{Spec}(B) \subseteq X$ and a chain of ideals

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$$

in B. Taking associated sheaves yields an ascending chain of quasi-coherent subsheaves of $\mathcal{O}_{\operatorname{Spec}(B)}$. Now let

$$\operatorname{Spec}(B) \cap \operatorname{Spec}(A_i) = \bigcup_j \operatorname{Spec}\bigl(A_{i,f_{ij}}\bigr)$$

be an affine open cover, which exists since principal opens form a basis. Since $\operatorname{Spec}(B)$ is quasi-compact [Mur24ca, Theorem 3.6.1], the resulting affine open cover

$$\operatorname{Spec}(B) = \bigcup_{i,j} \operatorname{Spec}\bigl(A_{i,f_{ij}}\bigr)$$

has a finite subcover. For each i, j, the ascending chains

$$\tilde{\mathfrak{a}}_1\big|_{\operatorname{Spec}(A_{i,f_{ij}})} \subseteq \tilde{\mathfrak{a}}_2\big|_{\operatorname{Spec}(A_{i,f_{ij}})} \subseteq \cdots$$

of quasi-coherent subsheaves of $\mathcal{O}_{\operatorname{Spec}(B)}$ stabilize at some index n_{ij} since the localizations $(A_i)_{f_{ij}}$ are Noetherian and taking associated sheaves is an equivalence of categories (Corollary 2.2.12). Taking $n = \max\{n_{ij}\}$, the original chain stabilizes at n on $\operatorname{Spec}(B)$. □

CAUTION 2.3.14. It is not true that Noetherian schemes X have Noetherian rings of global sections $\Gamma(X, \mathcal{O}_X)$. In [Oja08], Ojanguren constructs a quasi-projective example of this phenomenon. We will discuss this more after we define Proj.

[Har77, Exer. II.5.4] PROPOSITION 2.3.15 [EGAI$_{\text{new}}$, p. 228]. *Let X be a locally Noetherian scheme and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. The following are equivalent.*

(*a*) *$\mathscr{F}$ is coherent.*
(*b*) *$\mathscr{F}$ is quasi-coherent and of finite type.*

Moreover, every quasi-coherent sub-$\mathcal{O}_X$-module or quotient $\mathcal{O}_X$-module of a coherent sheaf is coherent.

Proof. We can apply Theorem 2.2.14 to an arbitrary affine open cover of X using Proposition 2.3.12. □

EXAMPLE 2.3.16. As we mentioned before (Remark 2.1.11), an $\mathcal{O}_X$-module of finite type is not necessarily quasi-coherent, even if X is Noetherian. For example, let $X = \mathbf{A}^1_k = \operatorname{Spec}(k[t])$ for a field k, let $\mathscr{F} = j_!(\mathcal{O}_X|_U)$ where $U = X - \{\mathbf{0}\}$, and consider the quotient $\mathcal{O}_X/\mathscr{F}$. We showed that $\mathscr{F}$ is not quasi-coherent in Example 2.1.8. The short exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathscr{F} \longrightarrow 0$$

shows that $\mathcal{O}_X/\mathscr{F}$ is also not quasi-coherent, for otherwise $\mathscr{F}$ would be by Corollary 2.3.9(*i*).

2.3.4. Ideal sheaves, subschemes, and immersions. Before moving on to Serre's criteiron for affineness, we have some preliminaries to take care of. So far, we have talked about schemes and sheaves on schemes. In commutative algebra, ideals are the first examples of modules we see. We define the analogous notion for ringed spaces and schemes, which already appeared in the proof of Proposition 2.3.12.

Definition 2.3.17 [$\mathrm{EGAI}_{\mathrm{new}}$, (4.1.3)]. Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of ideals* in $\mathcal{O}_X$ or *ideal sheaf* on X is a sub-$\mathcal{O}_X$-module of $\mathcal{O}_X$.

[Har77, p. 109]

Example 2.3.18. Sheaves of ideals are not necessarily quasi-coherent, for example $j_!j^*\mathcal{O}_X \hookrightarrow \mathcal{O}_X$ for an open inclusion $j\colon U \hookrightarrow X$ (see Example 2.1.8).

Using ideal sheaves, we can construct closed subschemes of a given scheme.

Proposition 2.3.19 [$\mathrm{EGAI}_{\mathrm{new}}$, Proposition 4.1.1]. *Let X be a scheme and let $\mathscr{I}$ be a quasi-coherent ideal sheaf on X. The support $Y := \mathrm{Supp}(\mathcal{O}_X/\mathscr{I})$ of the sheaf of rings $\mathcal{O}_X/\mathscr{I}$ is closed. Setting*

$$\mathcal{O}_Y := \bigl(\mathcal{O}_X/\mathscr{I}\bigr)\big|_Y,$$

the locally ringed space $(Y, \mathcal{O}_Y)$ is a scheme.

[Har77, p. 85,
Ex. II.3.2.3,
Exer. II.3.11(b),
Prop. II.5.9,
Cor. II.5.10]

Definition 2.3.20 [$\mathrm{EGAI}_{\mathrm{new}}$, p. 257]. With notation as in Proposition 2.3.19, we say that $(Y, \mathcal{O}_Y)$ is the *(closed) subscheme of $(X, \mathcal{O}_X)$ defined by the quasi-coherent ideal sheaf $\mathscr{I}$.*

Proof of Proposition 2.3.19. Since affine open subsets of X form a basis and one can check whether a locally ringed space is a scheme on an open cover, it suffices to prove the case when $X = \mathrm{Spec}(A)$ is affine. In this situation, $\mathscr{I} = \tilde{\mathfrak{a}}$ for an ideal $\mathfrak{a} \subseteq A$ by the fact that $M \mapsto \tilde{M}$ is an exact equivalence of categories (Corollary 2.2.12). By [Har77, Exercise II.5.6(b)] (Homework 3, Problem 2(b)), we have $Y = \mathrm{Supp}(A/\mathfrak{a})$.

Consider the canonical ring map $\varphi\colon A \to A/\mathfrak{a}$. The induced map on affine schemes satisfies

$$\mathrm{Spec}(\varphi)_*\Bigl(\widetilde{A/\mathfrak{a}}\Bigr) \cong \tilde{A}/\tilde{\mathfrak{a}} = \mathcal{O}_X/\mathscr{I}$$

by Corollary 2.2.13(iii) and Corollary 2.2.12. By the 1-1 correspondence between ideals containing $\mathfrak{a}$ in A and ideals in $A/\mathfrak{a}$ [Mur24ca, Proposition 1.3.12], we know that $\mathrm{Spec}(\varphi)$ factors as

$$\begin{tikzcd}
\mathrm{Spec}(A/\mathfrak{a}) \arrow[rr, "\mathrm{Spec}(\varphi)"] \arrow[dr, "\sim"'] & & X \\
& Y \arrow[ur, hook] &
\end{tikzcd}$$

where $\mathrm{Spec}(A/\mathfrak{a}) \xrightarrow{\sim} Y$ is a homeomorphism. $\square$

The subspace Y in Definition 2.3.20 is a special case of the following notion.

Definition 2.3.21 [$\mathrm{EGAI}_{\mathrm{new}}$, Définition 4.1.2, (4.1.4)]. We say that a ringed space $(Y, \mathcal{O}_Y)$ is a *(locally closed) subscheme* of a scheme $(X, \mathcal{O}_X)$ if the following conditions hold.

[Har77, p. 85]

1. Y is a locally closed subspace of X.
2. If U denotes the largest open subset of X containing Y such that Y is closed in U (i.e., $U = (X - \bar{Y}) \cup Y$), then $(Y, \mathcal{O}_Y)$ is the subscheme of $(U, \mathcal{O}_U)$ defined by a quasi-coherent sheaf of ideals in $\mathcal{O}_U$.

The morphism $(Y,\mathcal{O}_Y)\hookrightarrow(X,\mathcal{O}_X)$ is called the *canonical inclusion morphism*. We say that a subscheme $(Y,\mathcal{O}_Y)$ of $(X,\mathcal{O}_X)$ is a *closed subscheme* of $(X,\mathcal{O}_X)$ if Y is closed in X. We say that a subscheme $(Y,\mathcal{O}_Y)$ of $(X,\mathcal{O}_X)$ is an *open subscheme* of $(X,\mathcal{O}_X)$ if it is an open subspace of $(X,\mathcal{O}_X)$ in the sense of Definition 2.1.1. In this situation, Y is open in X in (1) and the ideal sheaf in (2) is the 0 ideal.

Caution 2.3.22. In the literature, the word "subscheme" can mean "closed subscheme."

[EGAI$_{\text{new}}$, p. 257] [Har77, Prop. II.5.9, Cor. II.5.10]

By Proposition 2.3.19 and Definition 2.3.21, we have the 1-1 correspondence

$$\begin{tikzcd} \left\{\begin{matrix}\text{quasi-coherent ideal sheaves}\\ \mathscr{I}\subseteq\mathcal{O}_X\end{matrix}\right\} \arrow[r, leftrightarrow, "\text{1-1}"] & \left\{\begin{matrix}\text{closed subschemes}\\ i\colon Y\hookrightarrow X\end{matrix}\right\} \\ \mathscr{I} \arrow[r, mapsto] & \Bigl(Y, \bigl(\mathcal{O}_X/\mathscr{I}\bigr)\big|_Y\Bigr) \\ \ker\bigl(\mathcal{O}_X\to i_*\mathcal{O}_Y\bigr) & (Y,\mathcal{O}_Y). \arrow[l, mapsto] \end{tikzcd} \tag{2.3.23}$$

quasi-coherent ideal sheaves in $\mathcal{O}_X$ and closed subschemes of X.

Example 2.3.24. Let $(Y,\mathcal{O}_Y)$ be an open subscheme of X. Then, Y is defined by the ideal $0\subseteq\mathcal{O}_X|_Y$ in (2).

We now define immersions, which are morphisms that "look like" the canonical inclusion map for a subscheme.

[Har77, p. 85, p. 120]

Definition 2.3.25 [EGAI$_{\text{new}}$, Définition 4.2.1]. Let $f\colon Y\to X$ be a morphism of schemes. We say that f is a (locally closed) *immersion* (resp. *closed immersion*, *open immersion*) if it factors as

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "g"', "\sim"] & & X \\ & Z \arrow[ur, hook, "j"'] & \end{tikzcd}$$

where g is an isomorphism and Z is a subscheme (resp. closed subscheme, open subscheme) of X and j is the canonical inclusion morphism.

Caution 2.3.26. In [Har77, p. 120], Hartshorne defines an immersion to be a morphism with a factorization as above where j factors as an open immersion followed by a closed immersion. If we need to distinguish between the two notions, we will call immersion in our sense a *locally closed immersion* and call Hartshorne's definition an *H-immersion* (the latter terminology is not standard). These definitions are not equivalent in general [Stacks, Tag 01QW].

[Har77, Exer. II.3.11(b)]

Remark 2.3.27 [EGAI$_{\text{new}}$, Remarque 4.2.1.1]. Let X be an affine scheme. By Proposition 2.3.19 and [Mur24ag, Proposition 2.2.17], a morphism $Y\to X$ of schemes is a closed immersion if and only if Y is an affine scheme and the homomorphism $\Gamma(X,\mathcal{O}_X)\to\Gamma(Y,\mathcal{O}_Y)$ is surjective.

Remark 2.3.28. Our definition of a closed immersion is equivalent to Hartshorne's [Har77, p. 85], although this takes work to show. The key point is that a consequence of Hartshorne's definition is that the kernel of $\mathcal{O}_X\to i_*\mathcal{O}_Y$ is quasi-coherent. In the Stacks project, part of the definition of a closed immersion is that the ideal sheaf $\ker(\mathcal{O}_X\to i_*\mathcal{O}_Y)$ is locally generated by sections. We define closed immersions following [EGAI$_{\text{new}}$] because it avoids some confusing aspects of the definition of a closed immersion in [Har77]. For example, the fact that a closed immersion is affine is part of our definition since it is the canonical inclusion of the closed subscheme defined by a quasi-coherent ideal up to isomorphism. See [Stacks, Tag 01IM] for more discussion. See [Har77, Exercise II.3.11(b)] for the relevant exercise in [Har77].

2.3.5. Reduced schemes and the reduced subscheme structure. We also want a way to think of closed subsets of a scheme as a scheme itself. By the previous section, what we would like to do is to find an ideal that cuts out a closed subset exactly. On affine open subsets, this is easy to do: just take the radical ideal corresponding to a closed subset. We need to globalize this construction.

Definition 2.3.29 [EGAI$_{\text{new}}$, Chapitre 0, (4.1.4)]. Let X be a scheme. We say that X is *reduced* if $\mathcal{O}_{X,x}$ is reduced for every $x \in X$. [Har77, p. 82, Exer. II.2.3]

Proposition 2.3.30 [EGAI$_{\text{new}}$, Proposition 4.5.1]. *Let $(X, \mathcal{O}_X)$ be a scheme and let $\mathscr{B}$ be a quasi-coherent $\mathcal{O}_X$-algebra. There exists a unique quasi-coherent ideal $\mathscr{N}$ in $\mathscr{B}$ such that the stalk $\mathscr{N}_x$ is the nilradical of $\mathscr{B}_x$ for every $x \in X$. If $X = \operatorname{Spec}(A)$ is affine and $\mathscr{B} = \tilde{B}$ for an A-algebra B, then $\mathscr{N} = \tilde{\mathfrak{N}}_B$, where $\mathfrak{N}_B$ is the nilradical of B.* [Har77, Ex. II.3.0.1]

Proof. The question is local and hence it suffices to prove the affine case. The sheaf $\tilde{\mathfrak{N}}_B$ is a quasi-coherent $\mathcal{O}_X$-module by Corollary 2.2.12. Its stalks are equal to $(\mathfrak{N}_B)_{\mathfrak{p}_x}$ for every $x \in X$ by [Mur24ca, Proposition 3.5.2] and Proposition 2.2.5. ☐

Definition 2.3.31 [EGAI$_{\text{new}}$, p. 268]. Let X be a scheme and let $\mathscr{B}$ be a quasi-coherent $\mathcal{O}_X$-algebra. The quasi-coherent ideal sheaf $\mathscr{N}$ in $\mathscr{B}$ constructed in Proposition 2.3.30 is the *nilradical* of $\mathscr{B}$. We denote by $\mathscr{N}_X$ the nilradical of $\mathcal{O}_X$.

Corollary 2.3.32 [EGAI$_{\text{new}}$, Corollaire 4.5.2]. *Let X be a scheme. The closed subscheme X_{red} defined by $\mathscr{N}_X$ is the unique reduced subscheme of X whose underlying space is X. It is the "smallest" subscheme of X whose underlying subspace in X, i.e., for every subscheme $Z \hookrightarrow X$ such that $\operatorname{sp}(Z) = \operatorname{sp}(X)$, the canonical inclusion morphism factors through X_{red}:* [Har77, Exer. II.2.3(b), Exer. II.3.11(c)]

$$\begin{tikzcd}
Z \arrow[rr, hook] & & X \\
& X_{\text{red}}. \arrow[ul, dashed, "\exists!"] \arrow[ur, hook] &
\end{tikzcd}$$

Proof. For every $x \in X$, we have $\mathscr{N}_x \subseteq \mathfrak{m}_x \subsetneq \mathcal{O}_{X,x}$. We therefore see that $\operatorname{sp}(X_{\text{red}}) = X$.

For the remaining assertions, let Z be a subscheme of X such that $\operatorname{sp}(Z) = \operatorname{sp}(X)$. Then, Z is the closed subscheme defined by a quasi-coherent ideal sheaf $\mathscr{I}$ in $\mathcal{O}_X$ such that $\mathscr{I}_x \subseteq \mathfrak{m}_x \subsetneq \mathcal{O}_{X,x}$ for every $x \in X$. We want to show that $\mathscr{I}_x \subseteq \mathscr{N}_{X,x}$ for every $x \in X$. This would imply that the map $\mathscr{I} \to \mathcal{O}_X/\mathscr{N}_X$ is the 0 map, and hence the inclusion $\mathscr{I} \hookrightarrow \mathcal{O}_X$ factors through $\mathscr{N}_X$ by the universal property of kernels:

$$\begin{tikzcd}
& & \mathscr{I} \arrow[dl, dashed, "\exists!"'] \arrow[d, hook] \arrow[dr, "0"] & & \\
0 \arrow[r] & \mathscr{N}_X \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{O}_X/\mathscr{N}_X \arrow[r] & 0.
\end{tikzcd}$$

Since we can check that $\mathscr{I}_x \subseteq \mathscr{N}_{X,x}$ locally, we can reduce to the affine case. Suppose $X = \operatorname{Spec}(A)$ and $\mathscr{I} = \tilde{I}$ for an ideal $I \subseteq A$. Since $\mathscr{I}_x \subseteq \mathfrak{m}_x$ for every $x \in X$, we see that I is contained in the intersection of all prime ideals in A, and hence $I \subseteq \mathfrak{N}_A$ by a special case of the Scheinnullstellensatz [Mur24ca, Corollary 1.5.6]. This shows that X_{red} is the smallest subscheme of X whose underlying subspace in X. □

[Har77, Exer. II.2.3(*b*)]

Definition 2.3.33 [EGAI$_{\text{new}}$, Définition 4.5.3]. Let X be a scheme. The *reduced scheme associated to* X is the scheme X_{red} constructed in Corollary 2.3.32.

Using the universal property in Corollary 2.3.32, we can put a scheme structure on any closed subset of a scheme.

[Har77, Ex. II.3.2.6]

Proposition 2.3.34 [EGAI$_{\text{new}}$, Proposition 4.6.1]. *Let X be a scheme. For every locally closed subset $Y \subseteq \operatorname{sp}(X)$, there exists a unique reduced subscheme of X whose underlying topological space is Y.*

Proof. Uniqueness holds by Corollary 2.3.32. It therefore suffices to show existence.

First consider the case when $X = \operatorname{Spec}(A)$ for a ring A and Y is closed in X. The largest ideal $I(Y)$ such that $V(I(Y)) = Y$ is a radical ideal by [Mur24ca, Proposition 1.6.3], and hence $A/I(Y)$ is reduced.

We now consider the general case. For every affine open subset $U \subseteq X$ such that $U \cap Y$ is closed in U, we consider the closed subscheme $Y_U \hookrightarrow U$ defined by the quasi-coherent ideal sheaf associated to $I(U \cap Y)$ of $A(U)$. The subscheme Y_U is reduced by the previous paragraph. For every open subset $V \subseteq U$, the open subscheme $Y_U \cap V \hookrightarrow Y_U$ is reduced with underlying topological space $Y \cap V$. Using the universal property in Corollary 2.3.32, we may therefore glue the scheme structures on the Y_U to obtain a scheme structure on Y. □

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2.3.6. Serre's criterion for affineness and the qcqs lemma. Cartan's Theorem B (Theorem 2.2.15) says that affine schemes have vanishing higher cohomology for quasi-coherent sheaves. We were able to use this to prove that certain schemes are not affine (Example 2.3.5). This raises the question: Does vanishing of higher cohomology for quasi-coherent sheaves characterize affine schemes? The answer is yes, by Serre's criterion for affineness below.

As was the case with Cartan's Theorem B, the complex analytic case predates the algebraic case: For complex manifolds, Serre showed that cohomology vanishing for coherent sheaves characterizes Stein manifolds [SHC51/52, Exposé XX, n° 2]. The case for algebraic varieties is also due to Serre [Ser57, Théorème 1].

[Stacks, Tag 01XF]
[Har77, Thm. III.3.7, Exer. II.2.17(*b*)]

Theorem 2.3.35 (Serre's criterion for affineness [EGAII, Théorème 5.2.1; EGAIV$_1$, (1.7.17)]). *Let X be a quasi-compact scheme. Set $A = \Gamma(X, \mathcal{O}_X)$. The following conditions are equivalent.*

- (*a*) *X is affine.*
- (*b*) *There exists a family of elements $f_\alpha \in A$ such that the open sets*

$$X_{f_\alpha} \coloneqq \{x \in X \mid f_{\alpha,x} \notin \mathfrak{m}_x\}$$

  are affine and the f_α generate the unit ideal in A.
- (*c*) *The functor $\Gamma(X, \cdot\,)$ is exact on $\mathsf{QCoh}(X)$.*
- (*c'*) *The functor $\Gamma(X, \cdot\,)$ is exact on short exact sequences*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

in $\mathsf{QCoh}(X)$ *where* $\mathscr{F}$ *is a sub-*$\mathcal{O}_X$*-module of* $\mathcal{O}_X^q$ *for some non-negative integer* q*.*

(*d*) $H^i(X,\mathscr{F})=0$ *for every quasi-coherent* $\mathcal{O}_X$*-module* $\mathscr{F}$ *and for every* $i>0$*.*

(*d*′) $H^1(X,\mathscr{I})=0$ *for every quasi-coherent ideal sheaf* $\mathscr{I}\subseteq\mathcal{O}_X$*.*

Note that the X_{f_α} are open since they are the preimages of [Har77, Exer. II.2.16(*a*)]

$$D(f_\alpha)\subseteq\operatorname{Spec}\bigl(\Gamma(X,\mathcal{O}_X)\bigr)$$

under the morphism $X\to\operatorname{Spec}(\Gamma(X,\mathcal{O}_X))$ from [Mur24ag, Proposition 2.2.17].

Remark 2.3.36. It is tempting to try to use the dimension shifting argument from Proposition 1.4.24 to show that (*c*) ⇔ (*d*). This is not so easy because we have not shown that $\mathsf{QCoh}(\mathcal{O}_X)$ has enough injectives. Hartshorne gets around this issue by assuming that X is Noetherian (see [Har77, Corollary III.3.6]).

It is known that $\mathsf{QCoh}(\mathcal{O}_X)$ has enough injectives because it is Grothendieck Abelian and has a generator. This is a relatively new result due to Gabber [Stacks, Tag 077K], first stated without proof in [Con00, Lemma 2.1.7]. To prove Theorem 2.3.35 in the qcqs case, one can instead use [SGA6, Exposé II, Lemme 3.1], which says that $\mathsf{QCoh}(\mathcal{O}_X)$ is a Grothendieck Abelian category with a generator when X is qcqs.

To prove Serre's criterion (Theorem 2.3.35), we need some preliminaries.

Definition 2.3.37 [EGAI$_{\text{new}}$, Définition 6.1.3 and Proposition 6.1.12]. Let X be a scheme. We say that X is *quasi-separated* if, for every pair U,V of affine open subsets of X, the intersection $U\cap V$ is quasi-compact.

We say that X is *qcqs* if it is both quasi-compact and quasi-separated.

Example 2.3.38. Let X be a scheme. If X is Noetherian, or more generally, if $\mathrm{sp}(X)$ is Noetherian, then X is qcqs.

Remark 2.3.39. Definition 2.3.37 is a preliminary definition for quasi-separatedness. It is equivalent to the general topological definition in [SGA4$_2$, Exposé VI, Définition 1.13] because affine open subsets form a basis for the Zariski topology on a scheme. The definition for schemes is usually stated in terms of the diagonal morphism (to be defined later).

There are at least three alternatives to the terminology "qcqs." In [SGA4$_2$, Exposé VI, Exemple 1.22], Grothendieck and Verdier proposed the terminology *coherent*. In [Kem80, p. 640], Kempf proposed the terminology *quasi-Noetherian*. In [ATJLL97, p. 18], Alonso Tarrío, Jeremías López, and Lipman proposed the terminology *concentrated*.

The following theorem allows us to "clear denominators" even when we are not working on an affine scheme. This is called the "qcqs lemma" in [Vak, Lemma 6.2.9].

Theorem 2.3.40 (Extending sections of quasi-coherent sheaves [EGAI$_{\text{new}}$, Théorème 6.8.1]). *Let* X *be a scheme, let* $\mathscr{L}$ *be an invertible* $\mathcal{O}_X$*-module on* X*, let* $f\in\Gamma(X,\mathscr{L})$*, and let* [Har77, Lem. II.5.14]

$$X_f:=\bigl\{x\in X \bigm\vert f_x\notin\mathfrak{m}_x\mathscr{L}_x\bigr\}.$$

Consider a quasi-coherent $\mathcal{O}_X$*-module* $\mathscr{F}$ *on* X*.*

(*i*) *Suppose that* X *is quasi-compact and let* $s\in\Gamma(X,\mathscr{F})$ *such that* $s|_{X_f}=0$*. Then, for some* $n>0$*, the section*

$$s\otimes f^{\otimes n}\in\Gamma\bigl(X,\mathscr{F}\otimes_{\mathcal{O}_X}\mathscr{L}^{\otimes n}\bigr)$$

satisfies $s \otimes f^{\otimes n} = 0$.

(*ii*) *Suppose that* X *is qcqs. For every section* $t \in \Gamma(X_f, \mathscr{F})$, *there exists* $n > 0$ *such that the section*

$$t \otimes f^{\otimes n} \in \Gamma\bigl(X_f, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n}\bigr)$$

extends to a section $t' \in \Gamma(X, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n})$.

Thus, if X *is qcqs, then the map*

$$\Gamma\bigl(X, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n}\bigr)_f \longrightarrow \Gamma\bigl(X_f, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n}\bigr)$$

induced by the universal property of localization is bijective.

This generalizes one of the implications in Theorem 2.2.10 and also generalizes [Har77, Exercise II.2.16(*b*),(*c*)]. The exact condition we use in the proof of (*ii*) below appears weaker than "qcqs," but the two conditions are equivalent. You can show this using Nike's trick [Vak, Proposition 5.3.1] (Homework 4, Problem 1).

Proof of Theorem 2.3.40. (*i*). Since X is quasi-compact, we may cover X by finitely many affine open subsets U_i such that $\mathscr{L}|_{U_i} \cong \mathcal{O}_{U_i}$. We may therefore reduce to the case when X is affine and $\mathscr{L} = \mathcal{O}_X$, which was shown in Theorem 2.2.10.

(*ii*). Since X is qcqs, there exists a finite covering by affine open subsets U_i such that $U_i \cap U_j$ is quasi-compact for every i, j and such that $\mathscr{F}|_{U_i}$ is free for every i. By Theorem 2.2.10, there exists an integer n such that the sections $(t \otimes f^{\otimes n})|_{U_i \cap X_f}$ extend to sections

$$t_i \in \Gamma\bigl(U_i, \mathscr{F} \otimes \mathscr{L}^{\otimes n}\bigr)$$

for every i. Now set $t_{i|j} \coloneqq t_i|_{U_i \cap U_j}$. By definition, we have

$$\bigl(t_{i|j} - t_{j|i}\bigr)\big|_{X_f \cap U_i \cap U_j} = 0.$$

Since $U_i \cap U_j$ is quasi-compact, by (*i*), there exists an integer m (independent of i, j) such that

$$\bigl(t_{i|j} - t_{j|i}\bigr) \otimes f^{\otimes m} = 0$$

for all i, j. We can therefore glue the sections $t_i \otimes f^{\otimes m}$ together to obtain a section

$$t' \in \Gamma\bigl(X, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes(n+m)}\bigr)$$

extending $t \otimes f^{\otimes(n+m)}$. □

Proof of Theorem 2.3.35. We proceed in steps.

[Stacks, Tag 01QF]

STEP 1. (*a*) ⇔ (*b*). We do not need quasi-compactness for this equivalence.

The direction ⇒ holds by choosing $\{f_\alpha\} = \{1\}$. For ⇐, write

$$1 = \sum_\alpha g_\alpha f_\alpha$$

for some $g_\alpha \in \Gamma(X, \mathcal{O}_X)$. Then, the f_α that appear in this sum generate the unit ideal, and hence we may assume there are finitely many f_α. We have $X = \bigcup_\alpha X_{f_\alpha}$ for otherwise, we would have $1 \in \mathfrak{m}_x$ for some $x \in X$. The X_{f_α} are quasi-compact since they are affine. Since X is a finite union of quasi-compact open sets, X is quasi-compact.

We now consider the canonical morphism

$$\begin{aligned} j\colon X &\longrightarrow \operatorname{Spec}\bigl(\Gamma(X,\mathcal{O}_X)\bigr) \\ x &\longmapsto \bigl\{f \in \Gamma(X,\mathcal{O}_X) \bigm| f_x \in \mathfrak{m}_x\bigr\} \end{aligned} \tag{2.3.41}$$

from [Mur24ag, Proposition 2.2.17]. Note that $j^{-1}(D(f_\alpha)) = X_{f_\alpha}$ by construction of the map in [Mur24ag, Proposition 2.2.17]. We claim that to show that j is an isomorphism, it suffices to show that

$$j|_{X_{f_\alpha}}\colon X_{f_\alpha} \longrightarrow D(f_\alpha)$$

is an isomorphism for every α. First, the $D(f_\alpha)$ cover $\operatorname{Spec}(\Gamma(X,\mathcal{O}_X))$ because the f_α generate the unit ideal. Second, the inverses defined on each $D(f_\alpha)$ glue to a homeomorphism of X with $\operatorname{Spec}(\Gamma(X,\mathcal{O}_X))$. To check that the map $j^\#$ on structure sheaves is an isomorphism, it suffices to show it is an isomorphism on an open cover.

It remains to show that $j|_{X_{f_\alpha}}$ is an isomorphism for every α. Since both X_{f_α} and $D(f_\alpha)$ are affine, it suffices to show that

$$\varphi_\alpha\colon \Gamma(X,\mathcal{O}_X)_{f_\alpha} \longrightarrow \Gamma(X_{f_\alpha},\mathcal{O}_X)$$

is an isomorphism by [Mur24ag, Corollary 2.2.19]. This holds by Theorem 2.3.40 for $\mathscr{F} = \mathscr{L} = \mathcal{O}_X$. Note that for each α,β, the intersection

$$X_{f_\alpha} \cap X_{f_\beta} = D\bigl(f_\beta|_{X_{f_\alpha}}\bigr) \subseteq X_{f_\alpha}$$

of affine open sets is also affine, and in particular, quasi-compact.

STEP 2. (a) $\Rightarrow$ (c) $\Rightarrow$ (c') $\Rightarrow$ (b).

(a) $\Rightarrow$ (c) holds by Corollary 2.2.12. (c) $\Rightarrow$ (c') holds since (c') is a special case of (c). It therefore suffices to show (c') $\Rightarrow$ (b). If $X = \emptyset$, we have $X = \operatorname{Spec}(0)$. We may therefore assume that X is nonempty.

SUBSTEP 2.1. Let X be a nonempty quasi-compact topological space that is T_0. Then, X has a closed point. In particular, every point in a quasi-compact scheme has a closed point in its closure.

[Stacks, Tag 005E]
[Sch05, Prop. 4.1]

Consider the set

$$\mathcal{T} = \Bigl\{\overline{\{x\}} \subseteq X \Bigm| x \in X\Bigr\}$$

partially ordered by inclusion, which is nonempty since X is nonempty. We claim that $\mathcal{T}$ has a minimal element. By Zorn's lemma [Kur1922; Zor35], it suffices to show that every descending chain

$$Z_1 \supseteq Z_2 \supseteq \cdots \tag{2.3.42}$$

in $\mathcal{T}$ has a lower bound in $\mathcal{T}$. Let $Z = \bigcap_i Z_i$. We claim that $Z \neq \emptyset$, and hence choosing a point $z \in Z$, the closure $\overline{\{z\}} \in \mathcal{T}$ is a lower bound for the chain. Suppose that $Z = \emptyset$. Then,

$$X = \bigcup_i (X - Z_i)$$

is an open cover. Since X is quasi-compact, there is a finite subcover. Thus, the descending chain (2.3.42) stabilizes at some $Z_i = \overline{\{x_i\}}$, which is nonempty, a contradiction.

By Zorn's lemma [Kur1922; Zor35], there exists a minimal element $\overline{\{x\}} \in \mathcal{T}$. We claim that $\{x\} = \overline{\{x\}}$, that is, x is closed in X. If not, choose $y \in \overline{\{x\}} - x$. Since

X is T_0, either there exists an open neighborhood U_x of x that does not contain y, or there exists an open neighborhood U_y of y that does not contain x. In the first case, we have $\overline{\{x\}} \supsetneq \overline{\{y\}}$, contadicting minimality. The second case is impossible since it would imply $y \notin \overline{\{x\}}$.

The "in particular" statement holds by applying what we showed above to the closure of a given point.

SUBSTEP 2.2. Assume (c'). Then, for every closed point $x \in X$ and every open neighborhood U of x, there exists $f \in A$ such that $x \in X_f \subseteq U$.

By Proposition 2.3.34, there exist quasi-coherent ideal sheaves $\mathscr{I}, \mathscr{I}'$ defining $X - U$ and $(X-U) \cup \{x\}$ with the reduced scheme structures, respectively. Consider the short exact sequence

$$0 \longrightarrow \mathscr{I}' \longrightarrow \mathscr{I} \longrightarrow \mathscr{I}'' \longrightarrow 0$$

in $\mathsf{QCoh}(\mathcal{O}_X)$, where we note that $\operatorname{Supp}(\mathscr{I}'') = \{x\}$ and $\mathscr{I}''_x = k(x) \coloneqq \mathcal{O}_{X,x}/\mathfrak{m}_x$. By ($c'$), the map

$$\Gamma(X, \mathscr{I}) \longrightarrow \Gamma(X, \mathscr{I}'')$$

is surjective. Thus, there exists $f \in \Gamma(X, \mathscr{I}) \subseteq A$ such that $f(x) = 1$ and $f(y) = 0$ for all $y \in X - U$, where $f(\,\cdot\,)$ denotes the image of f in the residue field at a point.

SUBSTEP 2.3. (c') $\Rightarrow$ (b).

For every closed $x \in X$, which exist by Substep 2.1, choose an affine open neighborhood U of x. By Substep 2.2, we can find an open neighborhood X_{f_x} of x such that $x \in X_{f_x} \subseteq U$. Since U is affine, we see that X_{f_x} is affine. Setting

$$Z = \bigcup_{\substack{x \in X \\ \text{closed}}} X_{f_x},$$

we see that Z contains all closed points of X. Since X is quasi-compact and T_0, this is only possible if $Z = X$ by applying Substep 2.1 to $X - Z$. Next, since X is quasi-compact, there exist finitely many $f_i \in A$ such that $X = \bigcup_i X_{f_i}$. Consider the morphism

$$\mathcal{O}_X^n \longrightarrow \mathcal{O}_X$$

defined by these f_i. Since for every $x \in X$, at least one of the $f_{i,x}$ is a unit, we see that this morphism is surjective. Now consider the short exact sequence

$$0 \longrightarrow \mathscr{R} \longrightarrow \mathcal{O}_X^n \longrightarrow \mathcal{O}_X \longrightarrow 0$$

where $\mathscr{R}$ is a quasi-coherent sub-$\mathcal{O}_X$-module of $\mathcal{O}_X^n$. Then, (c') implies

$$\Gamma(X, \mathcal{O}_X^n) \xrightarrow{[\,f_1\ f_2\ \cdots\ f_n\,]} \Gamma(X, \mathcal{O}_X) = A$$

is surjective, showing (b).

STEP 3. (a) $\Rightarrow$ (d) $\Rightarrow$ (d') $\Rightarrow$ (c').

(a) $\Rightarrow$ (d) is Cartan's Theorem B (Theorem 2.2.15), and (d) $\Rightarrow$ (d') holds since (d') is a special case of (d). It remains to show (d') $\Rightarrow$ (c').

Let $\mathscr{F}'$ be a quasi-coherent sub-$\mathcal{O}_X$-module of $\mathcal{O}_X^n$. Then, the filtration

$$0 \subseteq \mathcal{O}_X \subseteq \mathcal{O}_X^2 \subseteq \cdots \subseteq \mathcal{O}_X^n$$

defines a filtration $\mathscr{F}'_k = \mathscr{F}' \cap \mathcal{O}_X^k$ on $\mathscr{F}'_k$ where $k \in \{0, 1, \ldots, n\}$. Each $\mathscr{F}'_k$ is quasi-coherent since it is the kernel of the morphism

$$\mathscr{F}' \longrightarrow \mathcal{O}_X^n / \mathcal{O}_X^k.$$

Now (c') implies that $H^1(X, \mathscr{F}'_{k+1}/\mathscr{F}'_k) = 0$ for all k. By inducing on k, this shows that $H^1(X, \mathscr{F}'_k) = 0$ for all k. In particular, $H^1(X, \mathscr{F}') = 0$, showing (c'). ☐

Remark 2.3.43. In our discussion of Cartan's Theorem B (Theorem 2.2.15), I mentioned that there are versions of Cartan's Theorem B in other contexts, for example for rigid analytic spaces. For rigid analytic spaces, it is an *open question* whether there exists a cohomological characterization of being affinoid, which is the rigid analytic analogue of affineness. See, for example, [mst16]. Liu [Liu88] constructed an example of an non-affinoid space such that $H^i(X, \mathscr{F}) = 0$ for all coherent $\mathscr{F}$ and all $i > 0$. Gabber [Con06, Example 2.1.6] constructed an example of a quasi-coherent sheaf on an affinoid space such that $H^1(X, \mathscr{F}) \neq 0$, where quasi-coherence is defined as in [Con06, Definition 2.1.1].

2.4. The Proj construction

Now that we have developed some tools that apply to schemes in general, it is time to turn our attention to more special constructions of schemes that will ultimately be our main interest for much of the rest of the course.

2.4.1. Proj as a topological space. Last semester, a very important construction was that of projective space and varieties in projective space. We want an analogous construction for schemes.

Definition 2.4.1 (Proj as a set [EGAII, (2.3.1)]). Let S be an $\mathbf{N}$-graded ring. The *homogeneous spectrum* or the *Proj* of S is the set

[Har77, p. 76]

$$\operatorname{Proj}(S) := \left\{ \mathfrak{p} \in \operatorname{Spec}(S) \;\middle|\; \begin{array}{c} \mathfrak{p} \text{ is homogeneous} \\ S_+ \not\subseteq \mathfrak{p} \end{array} \right\}$$

where we recall that $S_+ = \bigoplus_{d>0} S_d$.

To define the topology on $\operatorname{Proj}(S)$, we define certain subsets of $\operatorname{Proj}(S)$.

Definition 2.4.2 (The topology on $\operatorname{Proj}(S)$ [EGAII, (2.3.2)]). Let S be an $\mathbf{N}$-graded ring and let $E \subseteq S$ be a subset. We set

[Har77, p. 76] Hartshorne omits the subscripts "+." I prefer to keep them in to distinguish between subsets of $\operatorname{Spec}(S)$ and of $\operatorname{Proj}(S)$.

$$V_+(E) := \{\mathfrak{p} \in \operatorname{Proj}(S) \mid \mathfrak{p} \supseteq E\}.$$

Viewing $\operatorname{Proj}(S)$ as a subset of $\operatorname{Spec}(S)$, we see that

$$V_+(E) = V(E) \cap \operatorname{Proj}(S) \subseteq \operatorname{Spec}(S).$$

By [Mur24ag, Lemma 2.2.8], we see that

[Har77, Lem. II.2.4]

$$\begin{aligned} V_+(0) &= \operatorname{Proj}(S) \\ V_+(S) &= V_+(S_+) = \emptyset \\ V_+\Big(\bigcup_\lambda E_\lambda\Big) &= \bigcap_\lambda V_+(E_\lambda) \\ V_+(EE') &= V_+(E) \cup V_+(E'). \end{aligned}$$

We give $\operatorname{Proj}(S)$ the subspace topology induced by $\operatorname{Spec}(S)$. In other words, the closed sets are the sets $V_+(E)$. The *distinguished open sets*

$$D_+(f) \coloneqq D(f) \cap \operatorname{Proj}(S) = \operatorname{Proj}(S) - V_+(f)$$

[EGAII, Prop. 2.3.4]
[Har77, Prop. 2.5(*b*)]

for f homogeneous of degree $d \geq 1$ form a basis for this topology, and we have

$$D_+(fg) = D_+(f) \cap D_+(g).$$

To define sheaves on $\operatorname{Proj}(S)$, we identify certain principal open subsets with affine schemes. Recall that if $f \in S_+$ is a homogeneous element of degree $d > 0$, then

$$S_{(f)} = \left\{ \frac{x}{f^n} \;\middle|\; \deg(x) = dn,\ n \geq 0 \right\} \subseteq S_f.$$

[Har77, Prop. II.2.5(*b*)]

Proposition 2.4.3 [EGAII, (2.3.5) and Proposition 2.3.6]. *Let S be an $\mathbf{N}$-graded ring and let $f \in S_+$ be a homogeneous element of degree $d > 0$. Then, the map*

$$\begin{aligned} \psi_f \colon D_+(f) &\longrightarrow \operatorname{Spec}\bigl(S_{(f)}\bigr) \\ \mathfrak{p} &\longmapsto (\mathfrak{p}S_f) \cap S_{(f)} \end{aligned}$$

is a homeomorphism and fits into the commutative diagram

$$\begin{tikzcd} D_+(f) \arrow[r, "\psi_f"] & \operatorname{Spec}\bigl(S_{(f)}\bigr) \\ D_+(fg) \arrow[u, hook] \arrow[r, "\psi_{fg}"] & \operatorname{Spec}\bigl(S_{(fg)}\bigr) \arrow[u] \end{tikzcd} \tag{2.4.4}$$

for every homogeneous element $g \in S_+$. Here, the left vertical map is the inclusion and the right vertical map is induced by the ring map $S_{(f)} \to S_{(fg)}$.

Proof. We first show that the diagram (2.4.4) commutes. Suppose $\mathfrak{p} \in D_+(fg)$. We need to show that

$$(\mathfrak{p}S_f) \cap S_{(f)} = \bigl((\mathfrak{p}S_{fg}) \cap S_{(fg)}\bigr) \cap S_{(f)}.$$

The inclusion $\subseteq$ holds by definition of extension/contraction of ideals. Conversely, suppose

$$\frac{x}{f^n} \in \bigl((\mathfrak{p}S_{fg}) \cap S_{(fg)}\bigr) \cap S_{(f)}.$$

Since $\mathfrak{p} \in D_+(fg)$, we have $fg \notin \mathfrak{p}$. Thus,

$$\frac{g^n x}{(fg)^n} \in \bigl((\mathfrak{p}S_{fg}) \cap S_{(fg)}\bigr) \cap S_{(f)}.$$

Since $g^n x \in \mathfrak{p}$ but $g \notin \mathfrak{p}$, we have $x \in \mathfrak{p}$.

[Stacks, Tag 00JP(6)]

Next, we show that ψ_f is a homeomorphism. We will show that the ring maps

$$S \longrightarrow S_f \longleftarrow S_{(f)}$$

induce homeomorphisms

$$D_+(f) \longleftarrow \bigl\{\mathfrak{p} \in \operatorname{Spec}(S_f) \bigm| \mathfrak{p} \text{ is } \mathbf{Z}\text{-graded}\bigr\} \longrightarrow \operatorname{Spec}\bigl(S_{(f)}\bigr).$$

The first map is a homeomorphism since it is obtained from the homeomorphism $D(f) \leftarrow \operatorname{Spec}(S_f)$ by restricting to homogeneous prime ideals, that is, prime ideals that are generated by homogeneous elements. For the second map, we claim that

[Stacks, Tag 00JO]

$$\begin{aligned}\operatorname{Spec}\bigl(S_{(f)}\bigr) &\longrightarrow \bigl\{\mathfrak{p} \in \operatorname{Spec}(S_f) \bigm| \mathfrak{p} \text{ is } \mathbf{Z}\text{-graded}\bigr\}\\ \mathfrak{p}_0 &\longmapsto \sqrt{\mathfrak{p}_0 S_f}\end{aligned}$$

is a continuous inverse. First, we show that the ideal $\sqrt{\mathfrak{p}_0 S_f}$ is prime. If $ab \in \mathfrak{p}_0 S_f$ with a, b homogeneous, then

$$\frac{a^d b^d}{f^{\deg(a)+\deg(b)}} \in \mathfrak{p}_0.$$

Since $\mathfrak{p}_0$ is prime, we have either $a^d/f^{\deg(a)} \in \mathfrak{p}_0$ or $b^d/f^{\deg(b)} \in \mathfrak{p}_0$, and hence either $a^d \in \mathfrak{p}_0 S_f$ or $b^d \in \mathfrak{p}_0 S_f$. Thus, $\sqrt{\mathfrak{p}_0 S_f}$ is prime. Next, we note that

$$\sqrt{\mathfrak{p}_0 S_f} \cap S_{(f)} = \sqrt{\mathfrak{p}_0 S_f \cap S_{(f)}} = \mathfrak{p}_0$$

since $\mathfrak{p}_0$ is radical and since $S_{(f)} \hookrightarrow S_f$ has a $S_{(f)}$-linear retraction. Thus, the map $\mathfrak{p}_0 \mapsto \sqrt{\mathfrak{p}_0 S_f}$ is an inverse.

It remains to show that $\mathfrak{p} \mapsto \mathfrak{p} \cap S_{(f)}$ is continuous. We claim the image of

$$\bigl\{\mathfrak{p} \in \operatorname{Spec}(S_f) \bigm| \mathfrak{p} \text{ is } \mathbf{Z}\text{-graded}\bigr\} \cap D(g)$$

is open for any $g \in S_f$. Write $g = \sum_i g_i$ for $\deg(g_i) = i$. The image of this set is

$$\bigcup_i D\biggl(\frac{g_i^d}{f^i}\biggr),$$

which is open. $\square$

2.4.2. Sheaves associated to graded modules. We now discuss the projective analogue of the construction $M \mapsto \tilde{M}$ we used to define the structure sheaf on $\operatorname{Spec}(A)$. $\boxed{2/18}$

Definition 2.4.5 [EGAII, Définition 2.5.3]. Let S be an $\mathbf{N}$-graded ring and consider a $\mathbf{Z}$-graded S-module M. The *sheaf* $\tilde{M}$ *associated to* M is the sheaf associated to the presheaf defined on distinguished open sets $D_+(f)$ as [Har77, p. 116]

$$D_+(f) \longmapsto M_{(f)}.$$

The *structure sheaf* on $\operatorname{Proj}(S)$ is $\tilde{S}$. By construction, we see that $\tilde{M}|_{D_+(f)} \cong \widetilde{M_{(f)}}$, and hence $(\operatorname{Proj}(S), \tilde{S})$ is a scheme. [Har77, p. 76]

Proposition 2.4.6 [EGAII, Proposition 2.4.6]. *Let A be a ring and let S be an $\mathbf{N}$-graded A-algebra, that is, each S_d is an A-module. Then, there is a natural morphism* $\operatorname{Proj}(S) \to \operatorname{Spec}(A)$.

In this situation (and similar ones in the future), we say that $\operatorname{Proj}(S)$ is a scheme *over* $\operatorname{Spec}(A)$ or *over* A for simplicity.

Proof. For homogeneous elements $f, g \in S$, we have the commutative diagram

$$\begin{tikzcd} S_{(f)} \arrow[r] & S_{(fg)} & S_{(g)} \arrow[l] \\ & A. \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

The morphisms $\operatorname{Spec}(S_{(f)}) \to \operatorname{Spec}(A)$ therefore glue together to yield a morphism $\operatorname{Proj}(S) \to \operatorname{Spec}(A)$. $\square$

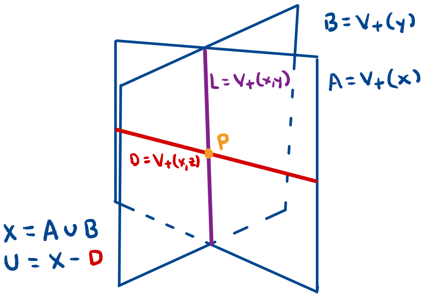

Figure 2.5. Ojanguren's example of a Noetherian scheme with non-Noetherian global sections [Oja08].

[Har77, Ex. II.2.5.1] Example 2.4.7 (Projective n-space). Let A be a ring and let $n \geq 0$. The *projective n-space over A* is

$$\mathbf{P}^n_A \coloneqq \operatorname{Proj}\big(A[x_0, x_1, \ldots, x_n]\big).$$

Using this construction, we can construct an example of a Noetherian scheme with non-Noetherian global sections. Below, the exact choices of A, B, L, D are not too relevant, but I have chosen specific ones for concreteness.

We use the language of regular functions to mean sections of the structure sheaf defined on some open set.

Example 2.4.8 [Oja08]. Let k be an algebraically closed field. Let $A = V_+(x)$ and $B = V_+(y)$ be two planes in $\mathbf{P}^3_k$ intersecting along $L = V_+(x, y)$. Let $X = A \cup B$. Let $D = V_+(x, z) \subseteq A$ be another line, distinct from L, such that $D \cap L = \{P\}$ for a unique closed point P. See Figure 2.5 for an illustration.

We claim that $U = X - D$ is such that $\Gamma(U, \mathcal{O}_X)$ is non-Noetherian. Note that since U is a subscheme of the Noetherian scheme $\mathbf{P}^3_k$, it is Noetherian. Every $f \in \Gamma(U, \mathcal{O}_U)$ is constant on

$$B - D = B - \{P\} \cong \mathbf{P}^2_k - \{P\}$$

since in the diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\Gamma\big(B - \{P\}, \mathcal{O}_X\big) \arrow[r, equal] \arrow[d, "\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]"'] & \Gamma\big(B - \{P\}, \mathcal{O}_X\big) \arrow[d, "\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]"] \\
\Gamma\big(B - V_+(x), \mathcal{O}_X\big) \oplus \Gamma\big(B - V_+(z), \mathcal{O}_X\big) \arrow[r, equal] \arrow[d, "{[1\ -1]}"'] & k\left[\dfrac{y}{x}, \dfrac{z}{x}\right] \oplus k\left[\dfrac{x}{z}, \dfrac{y}{z}\right] \arrow[d, "{[1\ -1]}"] \\
\Gamma\big(B - V_+(xz), \mathcal{O}_X\big) \arrow[r, hookrightarrow] & k\left(\dfrac{y}{x}, \dfrac{x}{z}\right)
\end{tikzcd}$$

the bottom right map $[1\ -1]$ is injective. The restriction of f to

$$A_0 \coloneqq A - D \cong \mathbf{A}_k^2$$

is therefore constant on $L_0 \coloneqq A_0 \cap L$. On the other hand, if f is a regular function on A_0 that is equal to a constant c on L_0, then it extends to a regular function on U. Thus, the ring $\Gamma(U, \mathcal{O}_U)$ is isomorphic to the ring

$$\begin{aligned} R &= \bigl\{ f(x,y) \in k[x,y] \bigm| f(x,0) \equiv c \text{ for some } c \in k \bigr\} \\ &= \bigl\{ c + g(x,y) \bigm| c \in k,\ g(x,y) \in y \cdot k[x,y] \bigr\} \\ &= k + y \cdot k[x,y]. \end{aligned}$$

[use19]

The k-algebra R is generated by the monomials $x^m y^{1+n}$ for $m, n \geq 0$. The ideal $I \subseteq R$ of polynomials vanishing along the line $L = V_+(x,y)$ is generated by these monomials. The ideal I cannot be generated by finitely many elements f_i because the f_i will have a term with largest power N on x, in which case

$$x^{N+1} y \in I - (f_i)_i.$$

Proposition 2.4.9 [EGAII, Propositions 2.5.2, 2.5.4, and 2.5.5]. *Let S be an* $\mathbf{N}$*-graded ring and consider a* $\mathbf{Z}$*-graded S-module M.*

(i) *There exists on $X = \operatorname{Proj}(S)$ exactly one quasi-coherent $\mathcal{O}_X$-module $\tilde{M}$ such that for every homogeneous $f \in S_+$, we have*

$$\Gamma\bigl(D_+(f), \tilde{M}\bigr) = M_{(f)}$$

and for every $f, g \in S_+$, the diagram

$$\begin{tikzcd} \Gamma\bigl(D_+(f), \tilde{M}\bigr) \arrow[r, equal] \arrow[d] & M_{(f)} \arrow[d] \\ \Gamma\bigl(D_+(fg), \tilde{M}\bigr) \arrow[r, equal] & M_{(fg)} \end{tikzcd}$$

commutes.

(ii) *The functor*

$$\begin{aligned} {}^*\mathsf{Mod}(S) &\longrightarrow \mathsf{QCoh}(\mathcal{O}_X) \\ M &\longmapsto \tilde{M} \end{aligned}$$

from the category ${}^*\mathsf{Mod}(S)$ of graded *S-modules with* graded *homomorphisms is exact and commutes with direct limits and with direct sums.*

I chose this notation to match the notation for graded Hom in [BH98, p. 33].

(iii) *For every $\mathfrak{p} \in X = \operatorname{Proj}(S)$, we have $\tilde{M}_{\mathfrak{p}} = M_{(\mathfrak{p})}$.*

[Har77, Prop. II.2.5(a), II.5.11(a)]

Proof. (i) holds by definition and the construction of sheaves defined on a basis. (ii) holds since exactness is computed stalkwise, and since direct limits and direct sums can be computed in terms of the behavior on each distinguished open set $D_+(f)$ because they will have the same associated sheaves. (iii) holds by computing the direct limit along the restriction maps in (i). ☐

2.4.3. Serre twists and sheaves associated to tensor products, Hom, restriction of scalars, and extension of scalars.

We want to define an "inverse" to the functor $M \mapsto \tilde{M}$. To do so, we use the following construction.

Definition 2.4.10 [FAC, n° 54; EGAII, Notations 2.1.1 and (2.5.10)]. Let S be an $\mathbf{N}$-graded ring and let $X = \operatorname{Proj}(S)$. Let $n \in \mathbf{Z}$.

[Har77, p. 117]

(*i*) Let M be a $\mathbf{Z}$-graded S-module. The $\mathbf{Z}$-graded S-module $M(n)$ satisfies [Har77, p. 50]

$$M(n)_d = M_{n+d}.$$

(*ii*) For every $n \in \mathbf{Z}$, *the n-th twisting sheaf of Serre* is

$$\mathscr{O}_X(n) := \widetilde{S(n)}.$$

(*iii*) The *n-th twist* of an $\mathscr{O}_X$-module $\mathscr{F}$ is

$$\mathscr{F}(n) := \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n).$$

Here are some important properties of the twisting sheaf.

[Har77, Prop. II.5.12(*a*),(*b*)]

Proposition 2.4.11 [FAC, n$^{\text{o}}$8, Proposition 4; EGAII, Proposition 2.5.7, Corollaire 2.5.8, Corollaire 2.5.9, Corollaire 2.5.14, and Corollaire 2.5.15]. *Let S be an $\mathbf{N}$-graded ring and let $X = \operatorname{Proj}(S)$.*

(*i*) *Let $d > 0$ be an integer and let $f \in S_d$. Then, for all $n \in \mathbf{Z}$, we have*

$$\bigl(S(nd)\bigr)^{\sim}\big|_{D_+(f)} \cong \mathscr{O}_X|_{D_+(f)}.$$

In particular, if S is generated by S_1 as an S_0-algebra, then $\mathscr{O}_X(n)$ is an invertible sheaf on X for every $n \in \mathbf{Z}$.

(*ii*) *Let M and N be $\mathbf{Z}$-graded S-module.* ***Assume that S is generated by S_1 as an S_0-algebra.*** *Then,*

$$\tilde{M} \otimes_{\mathscr{O}_X} \tilde{N} \cong (M \otimes_S N)^{\sim}.$$

In particular, $\tilde{M}(n) \cong (M(n))^{\sim}$ and

$$\mathscr{O}_X(n) \otimes_{\mathscr{O}_X} \mathscr{O}_X(m) \cong \mathscr{O}_X(n+m).$$

If M has a graded *finite presentation, then*

$$\Bigl({}^{*}\mathrm{Hom}_S(M,N)\Bigr)^{\sim} \cong \mathscr{H}\!om_{\mathscr{O}_X}(\tilde{M},\tilde{N}),$$

where

$$\begin{aligned} {}^{*}\mathrm{Hom}_S(M,N) &:= \bigoplus_{n\in\mathbf{Z}} \bigl\{\varphi \in \mathrm{Hom}_S(M,N) \bigm| \varphi(M_d) \subseteq N_{d+n}\bigr\} \\ &\subseteq \mathrm{Hom}_S(M,N). \end{aligned}$$

Caution 2.4.12. The notation $\mathrm{Hom}_S(M,N)$ from [EGAII, (2.1.2)] is what we call ${}^{*}\mathrm{Hom}_S(M,N)$. The notation ${}^{*}\mathrm{Hom}_S(M,N)$ is from [BH98, p. 33]. The inclusion

$${}^{*}\mathrm{Hom}_S(M,N) \subseteq \mathrm{Hom}_S(M,N)$$

is an equality when M is finitely generated [EGAII, (2.1.2)], but is not an equality in general [Jim13].

Proof of Proposition 2.4.11. (*i*). This follows from the bijection

$$\begin{aligned} (S_f)_0 &\longleftrightarrow (S_f)_{nd} \\ \frac{s}{f^m} &\longmapsto \frac{sf^n}{f^m} \\ \frac{s}{f^{n+m}} &\longleftarrow\!\shortmid \frac{s}{f^m}. \end{aligned}$$

The "in particular" statement holds since

$$X - \bigcup_{f \in S_1} D_+(f) = \bigcap_{f \in S_1} V_+(f) = V_+(S_1) = V_+(S_+) = \emptyset$$

and hence the $D_+(f)$ as f ranges over all $f \in S_1$ forms an open cover of X. [EGAII, Cor. 2.3.14]

(*ii*). We first construct the maps. Note this construction does not require the assumption that S is generated by S_1. For every $f \in S_d$ with $d > 0$, we construct a $S_{(f)}$-linear map [EGAII, (2.5.11), (2.5.12)]

$$\lambda_f \colon M_{(f)} \otimes_{S_{(f)}} N_{(f)} \longrightarrow (M \otimes_S N)_{(f)}$$

functorial in M and N as the composition

$$M_{(f)} \otimes_{S_{(f)}} N_{(f)} \hookrightarrow M_f \otimes_{S_f} N_f \xrightarrow{\sim} (M \otimes_S N)_f$$

and then noting that homogeneous elements of degree 0 on the left-hand side map to homogeneous elements of degree 0 on the right-hand side:

$$\frac{x}{f^m} \otimes \frac{y}{f^n} \longmapsto \frac{(x \otimes y)}{f^{m+n}}.$$

With this definition, for every $g \in S_e$ for $e > 0$, we have the commutative diagram

$$\begin{tikzcd}
M_{(f)} \otimes_{S_{(f)}} N_{(f)} \arrow[r, "\lambda_f"] \arrow[d] & (M \otimes_S N)_{(f)} \arrow[d] \\
M_{(fg)} \otimes_{S_{(fg)}} N_{(fg)} \arrow[r, "\lambda_{fg}"] & (M \otimes_S N)_{(fg)}.
\end{tikzcd}$$

We therefore obtain the morphism

$$\lambda \colon \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N} \longrightarrow (M \otimes_S N)^{\sim}$$

natural in M and N. Similarly, we construct the $S_{(f)}$-linear map

$$\begin{aligned}
\mu_f \colon \Big({}^*\mathrm{Hom}_S(M,N)\Big)_{(f)} &\longrightarrow \mathrm{Hom}_{S_{(f)}}\big(M_{(f)}, N_{(f)}\big) \\
\frac{u}{f^n} &\longmapsto \left(\frac{x}{f^m} \mapsto \frac{u(x)}{f^{m+n}}\right)
\end{aligned}$$

where u is a map of degree nd and x is an element of degree md. For every $g \in S_e$ for $e > 0$, we have the commutative diagram

$$\begin{tikzcd}
\Big({}^*\mathrm{Hom}_S(M,N)\Big)_{(f)} \arrow[r, "\mu_f"] \arrow[d] & \mathrm{Hom}_{S_{(f)}}\big(M_{(f)}, N_{(f)}\big) \arrow[d] \\
\Big({}^*\mathrm{Hom}_S(M,N)\Big)_{(fg)} \arrow[r, "\mu_{fg}"] & \mathrm{Hom}_{S_{(fg)}}\big(M_{(fg)}, N_{(fg)}\big).
\end{tikzcd}$$

We therefore obtain the morphism

$$\mu \colon \Big({}^*\mathrm{Hom}_S(M,N)\Big)^{\sim} \longrightarrow \mathscr{H}om_{\mathcal{O}_X}(\tilde{M}, \tilde{N})$$

natural in M and N.

It remains to show that, assuming that S is generated by S_1, the maps λ, μ are isomorphisms. For every $f \in S_1$, we want to show that

$$\begin{aligned} M_{(f)} \otimes_{S_{(f)}} N_{(f)} &\longrightarrow (M \otimes_S N)_{(f)} \\ \frac{x}{f^m} \otimes \frac{y}{f^n} &\longmapsto \frac{x \otimes y}{f^{m+n}} \end{aligned}$$

is an isomorphism. Consider the S_0-bilinear map

$$\begin{aligned} M_m \times N_n &\longrightarrow M_{(f)} \otimes_{S_{(f)}} N_{(f)} \\ (x, y) &\longmapsto \frac{x}{f^m} \otimes \frac{y}{f^n}. \end{aligned}$$

By the universal property of tensor products, we have the dashed map in the commutative diagram

$$\begin{tikzcd} M_m \times N_n \arrow[r] \arrow[dr] & M_m \otimes_{S_0} N_n \arrow[d, dashed, "\exists!"] \\ & M_{(f)} \otimes_{S_{(f)}} N_{(f)}. \end{tikzcd}$$

Now let $s \in S_q$. Then, as m, n varies on the left-hand side these morphisms are compatible with multiplication by s in the sense that

$$(sx) \otimes y \longmapsto \frac{sx}{f^{q+m}} \otimes \frac{y}{f^n} = \frac{s}{f^q}\left(\frac{x}{f^m} \otimes \frac{y}{f^n}\right).$$

Thus, passing to the quotient

$$\bigoplus_{d \in \mathbf{Z}} \bigoplus_{m+n=d} M_m \otimes_{S_0} N_n \twoheadrightarrow M \otimes_S N$$

as in [EGAII, (2.1.2)], we obtain a di-homomorphism of modules

$$\gamma_f \colon M \otimes_S N \longrightarrow M_{(f)} \otimes_{S(f)} N_{(f)}$$

over the map $S \to S_{(f)}$ mapping $s \mapsto s/f^q$ on homogeneous elements $s \in S_q$. Since the image 1 of f acts as a unit on the image of $M \otimes_S N$, the map γ_f factors through the localization at f to yield a map

$$(M \otimes_S N)_f \longrightarrow M_{(f)} \otimes_{S(f)} N_{(f)}$$

compatible with the map $S_f \to S_{(f)}$. The map $S_f \to S_{(f)}$ mapping $s/f^m \mapsto s/f^q$ restricts to the identity on $S_{(f)}$. Thus, we can restrict to degree 0 parts to obtain the $S_{(f)}$-linear map

$$\begin{aligned} \lambda'_f \colon (M \otimes_S N)_{(f)} &\longrightarrow M_{(f)} \otimes_{S(f)} N_{(f)} \\ \frac{x \otimes y}{f^{m+n}} &\longmapsto \frac{x}{f^m} \otimes \frac{y}{f^n}. \end{aligned}$$

Comparing the descriptions of λ_f and λ'_f, we see they are mutually inverse.

For μ, suppose that M has a graded finite presentation, i.e., there exists a right exact sequence

$$\bigoplus_{j=1}^{n} S(-d_j) \xrightarrow{A} \bigoplus_{i=1}^{m} S(-d_i) \longrightarrow M \longrightarrow 0$$

of graded S-modules, where $d_i, d_j \in \mathbf{Z}$ and where $A = (a_{ij})$ is an $m \times n$ matrix with $a_{ij} \in S_{d_i - d_j}$. Since ${}^*\mathrm{Hom}$ and $\mathscr{H}\!om$ are left exact, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\left({}^*\mathrm{Hom}_S(M,N)\right)^{\sim} \arrow[r, "\mu"] \arrow[d] & \mathscr{H}\!om_{\mathcal{O}_X}(\tilde{M}, \tilde{N}) \arrow[d] \\
\bigoplus_{i=1}^{m} \left({}^*\mathrm{Hom}_S\big(S(-d_i), N\big)\right)^{\sim} \arrow[r, "\mu"] \arrow[d] & \bigoplus_{i=1}^{m} \mathscr{H}\!om_{\mathcal{O}_X}\big(\widetilde{S(-d_i)}, \tilde{N}\big) \arrow[d] \\
\bigoplus_{j=1}^{n} \left({}^*\mathrm{Hom}_S\big(S(-d_j), N\big)\right)^{\sim} \arrow[r, "\mu"] & \bigoplus_{j=n}^{m} \mathscr{H}\!om_{\mathcal{O}_X}\big(\widetilde{S(-d_j)}, \tilde{N}\big)
\end{tikzcd}$$

with exact columns. Since isomorphic morphisms have isomorphic kernels, it suffices to show that the bottom two horizontal maps are isomorphism. Changing notation, it suffices to prove the case when $M = S(d)$, i.e., it suffices to show that

$$\mu\colon \left({}^*\mathrm{Hom}_S\big(S(d), N\big)\right)^{\sim} \longrightarrow \mathscr{H}\!om_{\mathcal{O}_X}\big(\widetilde{S(d)}, \tilde{N}\big)$$

is an isomorphism. As before, since the open sets $D_+(f)$ for $f \in S_1$ form an open cover of X, it suffices to show that

$$\mu_f\colon {}^*\mathrm{Hom}_S\big(S(d), N\big)_{(f)} \longrightarrow \mathrm{Hom}_{S_{(f)}}\big(S(d)_{(f)}, N_{(f)}\big)$$

is an isomorphism for every $f \in S_1$. We claim we have a commutative diagram

$$\begin{tikzcd}
{}^*\mathrm{Hom}_S\big(S(d), N\big)_{(f)} \arrow[rr, "\mu_f"] & & \mathrm{Hom}_{S_{(f)}}\big(S(d)_{(f)}, N_{(f)}\big) \\
& \big(N(-d)\big)_{(f)} \arrow[ul, "\sim", "\eta_f"'] \arrow[ur, "\sim"', "\eta'_f"] &
\end{tikzcd}$$

where η_f, η'_f are isomorphisms. Let $z \in N(-d)$ Then, we have a homomorphism $S(d) \to N$ such that $u_z(1) = z$. The resulting map $N(-d) \to \mathrm{Hom}_S(S(d), N)$ is an isomorphism. We therefore obtain an isomorphism η_f. Likewise, if $z' \in N_{(f)}$, then we define the map η'_f as

$$\begin{aligned}
\eta_{f'}\colon \big(N(-d)\big)_{(f)} &\longrightarrow \mathrm{Hom}_{S_{(f)}}\big(S(d)_{(f)}, N_{(f)}\big) \\
\frac{z'}{f^m} &\longmapsto \left(\frac{s}{f^k} \mapsto \frac{sz'}{f^{m+d+k}}\right)
\end{aligned}$$

where on the left-hand side, $z' \in (N(-d))_m = N_{m-d}$ and on the right-hand side, $s \in (S(d))_k = S_{d+k}$. This map is an isomorphism as well. ☐

Caution 2.4.13. The assumption that S is generated by S_1 as an S_0-algebra is very important! Without this assumption, these and many other facts about $\mathrm{Proj}(S)$ are false. See [Dol82, §1.5].

We also have the analogue of Corollary 2.2.13(*iii*) for Proj.

[Har77, Prop. II.5.12(c)]

PROPOSITION 2.4.14 [EGAII, Propositions 2.8.7 and 2.8.8]. *Let* $\varphi\colon S \to T$ *be a graded map of* $\mathbf{N}$*-graded rings. Consider the associated morphism*

$$\Phi\colon U \longrightarrow \operatorname{Proj}(S)$$

of schemes, where

$$U \coloneqq \bigl\{\mathfrak{p} \in \operatorname{Proj}(T) \bigm| \mathfrak{p} \not\supseteq \varphi(S_+)\bigr\}.$$

(i) *Let* N *be a graded* T*-module. There exists an isomorphism*

$$\bigl(N_{[\varphi]}\bigr)^{\sim} \xrightarrow{\sim} \Phi_*\bigl(\tilde{N}|_U\bigr)$$

of $\mathcal{O}_{\operatorname{Proj}(S)}$*-modules natural in* N*.*

(ii) *Let* M *be a graded* S*-module. There exists a morphism*

$$\nu\colon \Phi^*\tilde{M} \longrightarrow (M \otimes_S T)^{\sim}\big|_U$$

of $\mathcal{O}_U$*-modules natural in* M*. If* S *is generated by* S_1 *over* S_0*, then* ν *is an isomorphism.*

Proof. (i). Let $f \in S_+$ be a homogeneous element and let $f' = \varphi(f)$. Then, there is an isomorphism

$$\bigl(N_{[\varphi]}\bigr)_{(f)} \xrightarrow{\sim} \bigl(N_{(f')}\bigr)_{[\varphi_{(f')}]}.$$

These isomorphisms correspond to the isomorphisms

$$\bigl(N_{[\varphi]}\bigr)^{\sim}\big|_{D_+(f)} \xrightarrow{\sim} (\Phi_{f'})_*\bigl(\tilde{N}|_{D_+(f')}\bigr)$$

under Serre's equivalence for affine schemes by Corollary 2.2.13(*iii*) and Proposition 2.4.9(*i*). Moreover, for homogeneous $g \in S_+$ with image $g' = \varphi(g)$, the diagrams

$$\begin{tikzcd}
\bigl(N_{[\varphi]}\bigr)^{\sim}\big|_{D_+(f)} \arrow[r, "\sim"] \arrow[d] & (\Phi_{f'})_*\bigl(\tilde{N}|_{D_+(f')}\bigr) \arrow[d] \\
\bigl(N_{[\varphi]}\bigr)^{\sim}\big|_{D_+(fg)} \arrow[r, "\sim"] & (\Phi_{f'g'})_*\bigl(\tilde{N}|_{D_+(f'g')}\bigr)
\end{tikzcd}$$

commute by the way the isomorphism in Corollary 2.2.13(*iii*) is defined. Thus, these isomorphisms glue to yield the isomorphism we wanted. The isomorphism is natural since the isomorphisms used to construct it on distinguished open subsets, Serre's equivalence for affine schemes (Corollary 2.2.12), and the commutative diagram above are natural in N.

(ii) (Sketch). Let $f \in S_d$ for $d > 0$, and set $f' = \varphi(f)$. We can then define a natural morphism

$$\nu_f\colon M_{(f)} \otimes_{S_{(f)}} T_{(f')} \longrightarrow (M \otimes_S T)_{(f')}$$

of $T_{(f')}$-modules as the map obtained by corestricting the composition

$$M_{(f)} \otimes_{S_{(f)}} T_{(f')} \lhook\joinrel\longrightarrow M_f \otimes_{S_f} T_{f'} \xrightarrow{\sim} (M \otimes_S T)_{f'}$$

to degree 0 components. One can check that ν_f fits into the commutative diagram

$$\begin{tikzcd}
M_{(f)} \otimes_{S_{(f)}} T_{(f')} \arrow[r, "\nu_f"] \arrow[d] & (M \otimes_S T)_{(f')} \arrow[d] \\
M_{(fg)} \otimes_{S_{(fg)}} T_{(f'g')} \arrow[r, "\nu_{fg}"] & (M \otimes_S T)_{(f'g')}
\end{tikzcd}$$

and hence we obtain the morphism ν. For the isomorphism statement, it suffices to show that ν_f is an isomorphism for all $f \in S_1$ since U is covered by the open sets $D_+(f)$ as f ranges over all elements of S_1. We write down an inverse for ν_f as follows. We have an S_0-bilinear map

$$\begin{aligned} M_m \times S_n &\longrightarrow M_{(f)} \otimes_{S_{(f)}} T_{(f')} \\ (x, s) &\longmapsto \frac{x}{f^m} \otimes \frac{s}{f^n}. \end{aligned}$$

As in the proof of Proposition 2.4.11(*ii*), we obtain a di-homomorphism

$$\eta_f \colon M \otimes_S T \longrightarrow M_{(f)} \otimes_{S_{(f)}} T_{(f')}$$

with respect to the ring map $S \to S_{(f)}$ mapping $s \mapsto s/f^q$ on homogeneous elements $s \in S_q$. One then checks that the maps ν_f and η_f are mutually inverse. □

2.4.4. Graded modules associated to sheaves. We now define an "inverse" functor to $M \mapsto \tilde{M}$. $\boxed{2/20}$

Definition 2.4.15 [FAC, n° 59; EGAII, (2.6.1) and (2.6.2)]. Let S be an $\mathbf{N}$-graded ring generated by S_1 as an S_0-algebra. Let $X = \mathrm{Proj}(S)$ and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. We then have a graded ring map [Har77, p. 118]

$$S \longrightarrow \Gamma_*(\mathcal{O}_X) \coloneqq \bigoplus_{n \in \mathbf{Z}} \Gamma\bigl(X, \mathcal{O}_X(n)\bigr)$$

where the $\mathbf{N}$-grading on the right is obtained from the multiplication maps

$$\Gamma\bigl(X, \mathcal{O}_X(n)\bigr) \otimes_{\Gamma(X,\mathcal{O}_X)} \Gamma\bigl(X, \mathcal{O}_X(d)\bigr) \longrightarrow \Gamma\bigl(X, \mathcal{O}_X(n+d)\bigr).$$

The *graded* $\Gamma_*(\mathcal{O}_X)$*-module associated to* $\mathscr{F}$ is

$$\Gamma_*(\mathscr{F}) \coloneqq \bigoplus_{n \in \mathbf{Z}} \Gamma\bigl(X, \mathscr{F}(n)\bigr).$$

This is a graded $\Gamma_*(\mathcal{O}_X)$-module as follows. Consider a section $s \in \Gamma(X, \mathcal{O}_X(d))$. Then, for any $t \in \Gamma(X, \mathscr{F}(n))$, the product $s \cdot t \in \Gamma(X, \mathscr{F}(n+d))$ is defined by taking the tensor product

$$s \otimes t \in \Gamma\bigl(X, \mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathscr{F}(n)\bigr)$$

and using the isomorphism

$$\mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathscr{F}(n) \cong \mathscr{F}(n+d).$$

Since $\mathcal{O}_X(d)$ is invertible, $\Gamma_*(\,\cdot\,)$ is a left exact functor.

If M is a graded S-module, we obtain a morphism [Har77, Exer. II.5.9(*a*)]

$$\alpha \colon M \longrightarrow \Gamma_*(\tilde{M})$$

of graded S-modules natural in M since an element $m \in M_n$ determines a section of $\tilde{M}(n)$ by working on the distinguished open sets $D_+(f)$.

We calculate one example of the graded module associated to an $\mathcal{O}_X$-module.

[Har77, Prop. II.5.13] Proposition 2.4.16 [FAC, n° 62, Proposition 2(a); EGAIII$_1$, Proposition 2.1.12(ii)]. *Let A be a ring, let $S = A[x_0, x_1, \ldots, x_r]$ for $r \geq 1$, and let $X = \mathbf{P}^r_A = \operatorname{Proj}(S)$. Then,*

$$S \longrightarrow \Gamma_*(\mathcal{O}_X)$$

is an isomorphism.

Proof. We want to calculate the kernel in the exact sequence

$$0 \longrightarrow \Gamma\bigl(X, \mathcal{O}_X(n)\bigr) \longrightarrow \prod_{i=0}^{r} \Gamma\bigl(D_+(x_i), \mathcal{O}_X(n)\bigr) \longrightarrow \prod_{i,j} \Gamma\bigl(D_+(x_i x_j), \mathcal{O}_X(n)\bigr).$$

Taking the sum over all n, we want to compute the kernel in the exact sequence

$$0 \longrightarrow \Gamma_*(\mathcal{O}_X) \longrightarrow \prod_{i=0}^{r} S_{x_i} \longrightarrow \prod_{i,j} S_{x_i x_j}$$

where K consists of all tuples $(t_0, t_1, \ldots, t_r)$ such that the images of t_i and t_j in $S_{x_i x_j}$ are the same. The x_i are nonzerodivisors on S, so the localization maps $S \to S_{x_i}$ and $S_{x_i} \to S_{x_i x_j}$ are all injective. Moreover, we can think of all rings involved as subrings of $S' = S_{x_0 x_1 \cdots x_r}$. We therefore see that

$$\Gamma_*(\mathcal{O}_X) = \bigcap_{i=0}^{r} S_{x_i} \subseteq S'.$$

Any homogeneous element of S' can be written as

$$x_0^{i_0} x_1^{i_1} \cdots x_r^{i_r} \, f(x_0, x_1, \ldots, x_r)$$

where $i_j \in \mathbf{Z}$ and f is a homogeneous polynomial not divisible by any x_i. This element is in S_{x_i} if and only if $i_j \geq 0$ for all $j \neq i$. Thus, the intersection of all the S_{x_i} is just S. ☐

Caution 2.4.17. It is not true in general that $S \to \Gamma_*(\mathcal{O}_X)$ is an isomorphism [Har77, Exercise II.5.14].

2.4.5. Graded local cohomology and the kernel and cokernel of α. We can use local cohomology to understand Proposition 2.4.16 and the morphism α better. We will in fact show the following:

[Eis05, Cor. A1.12] Theorem 2.4.18 [EGAIII$_1$, Proposition 2.1.5]. *Let S be a Noetherian $\mathbf{N}$-graded ring generated by S_1 as an S_0-algebra, let M be a $\mathbf{Z}$-graded S-module, and let $X = \operatorname{Proj}(S)$. We have the exact sequence*

$$0 \longrightarrow H^0_{S_+}(M) \longrightarrow M \xrightarrow{\alpha} \Gamma_*(\tilde{M}) \longrightarrow H^1_{S_+}(M) \longrightarrow 0$$

of graded S-modules and the isomorphisms

$$\bigoplus_{n \in \mathbf{Z}} H^{i-1}\bigl(X, \tilde{M}(n)\bigr) \xrightarrow{\sim} H^i_{S_+}(M)$$

of graded S-modules for every $i \geq 2$.

Thus, $H^0_{S_+}(M)$ and $H^1_{S_+}(M)$ measure how far $\alpha\colon M \to \Gamma_*(M)$ is from being an isomorphism.

Since we are making statements about local cohomology in the setting of graded modules, we need to make sense of local cohomology as a functor on ${}^*\mathsf{Mod}(S)$. This is allowed by the following:

Lemma 2.4.19. *Let S be an $\mathbf{N}$-graded ring. Let $\mathfrak{a} \subseteq S$ be a homogeneous ideal.*

(*i*) *A graded S-module M is injective in ${}^*\mathsf{Mod}(S)$ if and only if*

$$ {}^*\mathrm{Hom}(\,\cdot\,, M)\colon {}^*\mathsf{Mod}(S) \longrightarrow {}^*\mathsf{Mod}(S) $$

is exact.

[BH98, p. 136]
[BS13, Lem. 13.2.5]

(*ii*) *${}^*\mathsf{Mod}(S)$ is a Grothendieck Abelian category with a generator. Thus, ${}^*\mathsf{Mod}(S)$ has enough injective objects.*

[BH98, Exer. 1.5.19, Thm. 3.6.2]
[BS13, 13.1.7(*i*), Thm. 13.2.4(*i*)]

(*iii*) *We have a commutative diagram*

$$\begin{tikzcd}
{}^*\mathsf{Mod}(S) \arrow[r, "\mathsf{Forget}"] \arrow[d, "\Gamma_{\mathfrak{a}}"'] & \mathsf{Mod}(S) \arrow[d, "\Gamma_{\mathfrak{a}}"] \\
{}^*\mathsf{Mod}(S) \arrow[r, "\mathsf{Forget}"] & \mathsf{Mod}(S).
\end{tikzcd}$$

[BH98, p. 143]
[BS13, Ex. 13.3.3(*ii*)]

(*iv*) *The forgetful functor*

$$ \mathsf{Forget}\colon {}^*\mathsf{Mod}(S) \longrightarrow \mathsf{Mod}(S) $$

maps injective objects in ${}^*\mathsf{Mod}(S)$ to $\Gamma_{\mathfrak{a}}$-acyclic objects in $\mathsf{Mod}(S)$. Thus, the forgetful functor defines a natural transformation of δ-functors such that the diagram

$$\begin{tikzcd}
{}^*\mathsf{Mod}(S) \arrow[r, "\mathsf{Forget}"] \arrow[d, "H^i_{\mathfrak{a}}"'] & \mathsf{Mod}(S) \arrow[d, "H^i_{\mathfrak{a}}"] \\
{}^*\mathsf{Mod}(S) \arrow[r, "\mathsf{Forget}"] & \mathsf{Mod}(S)
\end{tikzcd}$$

commutes for every i. Here, we compute the right vertical functor $H^i_{\mathfrak{a}}$ using injective resolutions on $\mathsf{Mod}(S)$.

[BS13, Prop. 13.2.6, Thm. 13.4.2]

Proof. (*i*). We have

$$ \begin{aligned} {}^*\mathrm{Hom}(\,\cdot\,, M) &= \bigoplus_{n \in \mathbf{Z}} \mathrm{Hom}_{{}^*\mathsf{Mod}(S)}\bigl(\,\cdot\,, M(n)\bigr) \\ &\cong \bigoplus_{n \in \mathbf{Z}} \mathrm{Hom}_{{}^*\mathsf{Mod}(S)}\bigl(\,\cdot\,(-n), M\bigr) \end{aligned} $$

which is exact if and only if M is injective.

(*ii*). The category ${}^*\mathsf{Mod}(S)$ is Abelian because kernels and cokernels are computed degree-wise, where on each degree, they are computed as kernels and cokernels of morphisms of S_0-modules. Arbitrary direct sums exist and filtered direct limits are exact since direct sums and direct limits are computed degree-wise. Finally, a generator is given by

$$ \bigoplus_{n \in \mathbf{Z}} S(n). $$

Thus, ${}^*\mathsf{Mod}(S)$ has enough injective objects by Theorem 1.3.35.

(*iii*). Let M be a graded S-module. Then, $m \in \Gamma_{\mathfrak{a}}(M)$ if and only if $\mathfrak{a}^n m = 0$ for some $n \geq 0$, which holds if and only if each graded component of $\mathfrak{a}^n m$ is equal to 0. Since $\mathfrak{a}$ and m are graded, this hold if and only if each component of m is in $\Gamma_{\mathfrak{a}}(M)$, and hence $\Gamma_{\mathfrak{a}}(M)$ is a graded submodule of M.

(*iv*). By dimension shifting (Proposition 1.4.21), it suffices to show that injective objects in ${}^*\mathsf{Mod}(S)$ are $\Gamma_{\mathfrak{a}}$-acyclic. To prove this, note that the composition [BS13, 13.1.8]

$$\Gamma_{\mathfrak{a}}(\,\cdot\,) \cong \varinjlim_{n \geq 0} {}^*\mathrm{Hom}_S(S/\mathfrak{a}^n,\,\cdot\,) \hookrightarrow \varinjlim_{n \geq 0} \mathrm{Hom}_S(S/\mathfrak{a}^n,\,\cdot\,) \cong \Gamma_{\mathfrak{a}}(\,\cdot\,)$$

is the identity (on the right-hand side, we think of $\Gamma_{\mathfrak{a}}$ as a functor on $\mathsf{Mod}(S)$) and hence the middle "inclusion" is an isomorphism. Thus, we can compute $H^i_{\mathfrak{a}}$ as [BS13, Rem. 13.4.6]

$$H^i_{\mathfrak{a}}(\,\cdot\,) \cong \varinjlim_{n \geq 0} {}^*\mathrm{Ext}^i_S(S/\mathfrak{a}^n,\,\cdot\,) \cong \varinjlim_{n \geq 0} \mathrm{Ext}^i_S(S/\mathfrak{a}^n,\,\cdot\,)$$

which is 0 for injective objects in ${}^*\mathsf{Mod}(S)$ by (*i*). □

To prove Theorem 2.4.18, the idea is to compute local cohomology on the affine cone over $\mathrm{Proj}(S)$. Affine cones are defined as follows.

Definition 2.4.20 [EGAII, (8.3.1) and Corollaire 8.3.6]. Let S be an $\mathbf{N}$-graded ring. The *affine cone* over $\mathrm{Proj}(S)$ is

$$C\bigl(\mathrm{Proj}(S)\bigr) \coloneqq \mathrm{Spec}(S).$$

We can define the *canonical projection morphism*

$$\pi\colon \mathrm{Spec}(S) - V(S_+) \longrightarrow \mathrm{Proj}(S) \tag{2.4.21}$$

as follows. Over each homogeneous element $f \in S$, we have a ring map

$$S_f \longleftarrow S_{(f)}$$

which yields the morphism

$$D(f) \longrightarrow D_+(f)$$

by applying Spec. These morphisms glue to give the morphism (2.4.21) since the diagram

$$\begin{tikzcd}
S_f \arrow[d] & S_{(f)} \arrow[l, hook'] \arrow[d] \\
S_{fg} & S_{(fg)} \arrow[l, hook'] \\
S_g \arrow[u] & S_{(g)} \arrow[l, hook'] \arrow[u]
\end{tikzcd}$$

commutes for all homogeneous $f, g \in S_+$. Note that π is surjective since, for every prime ideal $\mathfrak{p} \subseteq S_{(f)}$, there exists a prime ideal in S_f lying over $\mathfrak{p}$ because

$$0 \neq \frac{S_{(f)}}{\mathfrak{p}} \hookrightarrow \frac{S_f}{\mathfrak{p}S_f}.$$

Lemma 2.4.22 [EGAII, (2.2.1) and Corollaire 8.3.6]. *Let S be an $\mathbf{N}$-graded ring and let $f \in S_d$ for $d > 0$. The monomials $(f/1)^h \in S_f$ form a free system of generators for $(S^{(d)})_f$ over $S_{(f)}$, and hence*

$$(S^{(d)})_f \cong S^{(d)}_{(f)}[T, T^{-1}] \cong S^{(d)}_{(f)} \otimes_{\mathbf{Z}} \mathbf{Z}[T, T^{-1}].$$

Thus, if

$$\pi\colon C\bigl(\operatorname{Proj}(S)\bigr) - V(S_+) \longrightarrow \operatorname{Proj}(S)$$

is the canonical projection morphism from the affine cone $C(\operatorname{Proj}(S))$, then for all nonozero $f \in S_1$, we have

$$\pi^{-1}\bigl(D_+(f)\bigr) \cong \operatorname{Spec}\bigl(S_{(f)}[T, T^{-1}]\bigr).$$

Proof. For the first statement, first note that since we are working in $(S^{(d)})_f$, the element f is a unit. Thus, a relation of the form

$$\sum_{h=-a}^{b} z_h\,(f/1)^h = 0$$

holds where $z_h = x_h/f^m$ and $x_h \in S_{md}$ if and only if a relation of the form

$$\sum_{h=-a}^{b} f^{h+k} x_h = 0$$

holds for some k. Since the terms in this equation are all in diferent degrees, a relation of this form holds if and only if $f^{h+k}x_h = 0$ for all h, which is equivalent to $z_h = 0$ for all h, again using the fact that f is a unit in $(S^{(d)})_f$.

The last statement of the lemma holds by setting $d = 1$ and looking at the definition of (2.4.21) on distinguished open sets. ☐

We can now prove Theorem 2.4.18.

Proof of Theorem 2.4.18. Let $C(X) = \operatorname{Spec}(S)$ be the affine cone over $X = \operatorname{Proj}(S)$. Let $E = C(X) - V(S_+)$. Since $C(X)$ is affine, Cartan's theorem B (Theorem 2.2.15) implies that the long exact sequence on local cohomology from [Har77, Exercise III.2.3(e)] (Homework 2, Problem 3(e)) has the form

[Dol82, §1.4]

$$0 \longrightarrow H^0_{V(S_+)}\bigl(C(X), \tilde{M}\bigr) \longrightarrow M \longrightarrow H^0\bigl(E, \tilde{M}|_E\bigr) \longrightarrow H^1_{V(S_+)}\bigl(C(X), \tilde{M}\bigr) \longrightarrow 0$$

and contains the isomorphisms

$$H^{i-1}\bigl(E, \tilde{M}|_E\bigr) \xrightarrow{\sim} H^i_{V(S_+)}\bigl(C(X), \tilde{M}\bigr)$$

for every $i \geq 2$. Since S is Noetherian, [Har77, Exercise III.3.3] (Homework 4, Problem 5(e)) implies that

$$H^i_{V(S_+)}\bigl(C(X), \tilde{M}\bigr) \cong H^i_{S_+}(M)$$

for every i. Finally, we claim we have

$$\begin{aligned} H^i\bigl(E, \tilde{M}|_E\bigr) &\cong H^i\bigl(X, \pi_*\tilde{M}|_E\bigr) \\ &\cong H^i\Bigl(X, \bigoplus_{n\in\mathbf{Z}} \tilde{M}(n)\Bigr) \\ &\cong \bigoplus_{n\in\mathbf{Z}} H^i\bigl(X, \tilde{M}(n)\bigr). \end{aligned}$$

For those who are curious, it suffices here to assume that S_+ is finitely generated as an ideal by [SGA2$_{\text{new}}$, Exp. II, Prop. 5]. Another option is to assume that S_+ is generated by a *weakly proregular sequence* in the sense of [Sch03].

The first isomorphism holds since π is affine by Lemma 2.4.22 and using [Har77, Exercise III.4.1] (Homework 4, Problem 4(c)). To show the second isomorphism, note that

$$\Gamma\bigl(D_+(f),\pi_*\tilde{M}|_E\bigr)=\Gamma\bigl(D(f),\tilde{M}\bigr)=M_f\cong\bigoplus_{n\in\mathbf{Z}}\bigl(M(n)\bigr)_{(f)}$$

for every $f\in S_d$, and that the associated isomorphisms of sheaves on $D_+(f)$ under Serre's equivalence for affine schemes (Corollary 2.2.12) are compatible with restriction to distinguished open sets of the form $D_+(fg)$ for $g\in S_e$. ☐

2.4.6. Serre's equivalence for quasi-coherent sheaves on Proj. We have now constructed two functors

$$\begin{tikzcd}
{}^*\mathsf{Mod}(S) \arrow[r, shift left] & \mathsf{QCoh}\bigl(\mathscr{O}_{\operatorname{Proj}(S)}\bigr) \arrow[l, shift left] \\
M \arrow[r, mapsto] & \tilde{M} \\
\Gamma_*(\mathscr{F}) & \mathscr{F}. \arrow[l, mapsto]
\end{tikzcd}$$

We want to understand to what extent the compositions are isomorphic to the identity functor.

Proposition 2.4.23 [FAC, n° 59, Propositions 7 and 8; EGAII, (2.6.4) and Proposition 2.6.5]. *Let S be an $\mathbf{N}$-graded ring that is generated by S_1 as an S_0-algebra and let M be a graded S-module. Let $X=\operatorname{Proj}(S)$ and let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Then, there is a morphism*

$$\beta\colon\Gamma_*(\mathscr{F})^\sim\longrightarrow\mathscr{F}$$

natural in $\mathscr{F}$ such that the compositions

$$\tilde{M}\xrightarrow{\tilde{\alpha}}\Gamma_*(\tilde{M})^\sim\xrightarrow{\beta}\tilde{M}\tag{2.4.24}$$

$$\Gamma_*(\mathscr{F})\xrightarrow{\alpha}\Gamma_*\bigl(\Gamma_*(\mathscr{F})^\sim\bigr)\xrightarrow{\Gamma_*(\beta)}\Gamma_*(\mathscr{F})\tag{2.4.25}$$

are the identity maps.

Proof. We first define β. Let $f\in S_d$ for $d>0$ and $g\in S_e$ for $e>0$. Set $M=\Gamma_*(\mathscr{F})$. We then have the commutative diagrams

$$\begin{tikzcd}
M_{(f)} \arrow[r, "\beta_f"] \arrow[d] & \Gamma\bigl(D_+(f),\mathscr{F}\bigr) \arrow[d] \\
M_{(fg)} \arrow[r, "\beta_{fg}"] & \Gamma\bigl(D_+(fg),\mathscr{F}\bigr).
\end{tikzcd}$$

Since $\Gamma(D_+(f),\tilde{M})=M_{(f)}$ by Proposition 2.4.9(i), these maps glue together to obtain the morphism β.

We now show (2.4.24). It suffices to show the composition is the identity locally. On each distinguished open $D_+(f)$, the morphism corresponds to

$$\Gamma\bigl(D_+(f),\tilde{M}\bigr)\xrightarrow{\Gamma(D_+(f),\tilde{\alpha})}M_{(f)}\xrightarrow{\beta_f}\Gamma\bigl(D_+(f),\tilde{M}\bigr)$$

which is the identity.

To show (2.4.25), we work one degree at a time. Set $M=\Gamma_*(\mathscr{F})$. Then, $M_n=\Gamma(X,\mathscr{F}(n))$ and

$$\bigl(\Gamma_*(\tilde{M})\bigr)_n=\Gamma\bigl(X,\tilde{M}(n)\bigr)=\Gamma\bigl(X,\widetilde{M(n)}\bigr).$$

If $f \in S_1$ and $z \in M_n$, then over each principal open set $D_+(f)$,

$$\alpha_n(z)|_{D_+(f)} = \frac{z}{1} = \left(\frac{f}{1}\right)^n \frac{z}{f^n} \in \big(M(n)\big)_{(f)}$$

which maps via β_f to the section

$$\Big(\alpha_1(f)^n|_{D_+(f)}\Big) \cdot \Big(z|_{D_+(f)}\Big) \cdot \Big(\alpha_1(f)^n|_{D_+(f)}\Big)^{-1} = z|_{D_+(f)}.$$ □

Theorem 2.4.26 [FAC, n° 65, Proposition 6; EGAII, Théorème 2.7.5 and Corollaire 2.7.7]. *Let S be an $\mathbf{N}$-graded ring that is* finitely *generated by S_1 as an S_0-algebra and let $X = \operatorname{Proj}(S)$. Then, for every quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$, the morphism* [Har77, Prop. II.5.15]

$$\beta\colon \Gamma_*(\mathscr{F})^\sim \longrightarrow \mathscr{F}$$

is an isomorphism. As a consequence, the functor $M \mapsto \tilde{M}$ is essentially surjective.

Proof. First, S_+ is generated by finitely many elements $f_i \in S_1$, and hence we can write

$$X = \bigcup_i \operatorname{Spec}\big(S_{(f_i)}\big)$$

as a finite union of quasi-compact open subsets whose intersections are quasi-compact. That is, X is qcqs. By the qcqs lemma (Theorem 2.3.40), we see that for every $f \in S_d$ for $d > 0$, the morphisms

$$\big(\Gamma_*(\mathscr{F})\big)_{\alpha_d(f)} \xrightarrow{\sim} \Gamma\big(D_+(f), \mathscr{F}\big)$$

are isomorphisms, where we consider f as a section in $\Gamma(X, \mathcal{O}(d))$. On the other hand, the left-hand side is equal to $(\Gamma_*(\mathscr{F}))_{(f)}$ by definition, and the resulting map is β_f. □

As a consequence, we have Serre's equivalence for quasi-coherent sheaves on $\operatorname{Proj}(S)$.

Corollary 2.4.27 (Serre's equivalence for $\mathsf{QCoh}(\operatorname{Proj}(S))$). *Let S be an $\mathbf{N}$-graded ring that is* finitely *generated by S_1 as an S_0-algebra and let $X = \operatorname{Proj}(S)$. We then have an equivalence of categories*

$$\left\{\begin{matrix} \textit{graded } S\textit{-modules } M \textit{ such that} \\ \alpha\colon M \to \Gamma_*(\tilde{M}) \textit{ is an isomorphism} \end{matrix}\right\} \underset{\Gamma_*}{\overset{\widetilde{(\cdot)}}{\rightleftarrows}} \mathsf{QCoh}(\mathcal{O}_X).$$

Proof. This follows from Proposition 2.4.23 and Theorem 2.4.26. □

2.4.7. Finiteness conditions for graded modules. To understand the left-hand side of the equivalence in Corollary 2.4.27, we will restrict to quasi-coherent sheaves of finite type. On the module side, we introduce the following two finiteness conditions:

Definition 2.4.28 [FAC, n° 56; EGAII, (2.7.2)]. Let S be an $\mathbf{N}$-graded ring and let M be a graded S-module. We consider the following two finiteness conditions on M.

(TF) There exists an integer n such that the sub-module [Har77, Exer. II.5.9(c)]

$$\bigoplus_{k \geq n} M_k$$

is a finitely generated S-module.

(TN) There exists an integer n such that $M_k = 0$ for all $k \geq n$.

Note that if M satisfies (TN), then $M_{(f)} = 0$ for every homogeneous element $f \in S_+$, and hence $\tilde{M} = 0$.

PROPOSITION 2.4.29 [FAC, n° 58, Proposition 5; EGAII, Proposition 2.7.3]. *Let S be an $\mathbf{N}$-graded ring such that S_+ is a finitely generated ideal and let $X = \operatorname{Proj}(S)$. Let M be a graded S-module.*

(*i*) *If M satisfies* (TF)*, then $\tilde{M}$ is of finite type.*

(*ii*) *Suppose that M satisfies* (TF)*. Then, $\tilde{M} = 0$ if and only if M satisfies* (TN)*.*

Proof. We have already seen $\Leftarrow$ in (*ii*).

(*i*). Let $M' = \bigoplus_{k \geq n} M_k$ be the sub-module that is finitely generated. The short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M/M' \longrightarrow 0$$

induces the short exact sequence

$$0 \longrightarrow \widetilde{M'} \xrightarrow{\sim} \tilde{M} \longrightarrow \widetilde{M/M'} \longrightarrow 0$$

where $\widetilde{M/M'} = 0$ by the direction $\Leftarrow$ in (*ii*). Replacing M by M', it suffices to consider the case when M itself is finitely generated. Since being of finite type is a local condition, it suffices to show that $M_{(f)}$ is finitely generated as an $S_{(f)}$-module for every homogeneous element $f \in S_d$. Since M is finitely generated, there is a least common multiple e of the absolute values of the degrees of its generators. Then, $M^{(de)}$ is a finitely generated $S^{(de)}$-module, and we have surjection

[EGAII, Prop. 2.2.5]

$$M^{(de)} \twoheadrightarrow M_{(f^e)} \cong M_{(f)}$$

sending $z \in M_{den}$ to $z/(f^e)^n$.

It remains to show $\Rightarrow$ in (*ii*). With the same notation as above, we have $\widetilde{M'} = 0$, and the condition (TN) for M' holds if and only if it holds for M. Replacing M by M', it suffices to consider the case when M itself is finitely generated by homogeneous elements $\{x_i\}_{1 \leq i \leq p}$. Let $\{f_j\}_{1 \leq j \leq q}$ be a finite homogeneous set of generators for S_+. By hypothesis, we know that $M_{(f_j)} = 0$ for all j. Since both sets are finite, there exists an integer n such that

$$f_j^n x_i = 0$$

for all i, j. Set $n_j = \deg(f_j)$ and let $m = \max\{\sum_j r_j n_j\}$ where the r_j range over all tuples such that $\sum_j r_j \leq nq$. Then, for every $k > m$, we have $S_k x_i = 0$ for all i. If $h = \max\{\deg(x_i)\}$, then $M_k = 0$ for all $k > h + m$. ☐

[Har77, Prop. II.5.15]

COROLLARY 2.4.30 [FAC, n° 60, Théorème 2; EGAII, Corollaire 2.7.8]. *Let S be an $\mathbf{N}$-graded ring that is* finitely *generated by S_1 as an S_0-algebra and let $X = \operatorname{Proj}(S)$. Then, every quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$ of finite type is of the form $\tilde{N}$ for a graded S-module of finite type.*

Proof. By Theorem 2.4.26, we can write $\mathscr{F} = \tilde{M}$ for a graded S-module M. Write M as a direct limit of its finitely generated graded submodules M_λ. Let $f_j \in S_1$ be a finite set of generators for S over S_0. For each distinguished open set $D_+(f_j)$, a large enough submodule M_{λ_j} satisfies

$$\left(M_{\lambda_j}\right)_{(f_j)} = M_{(f_j)}$$

by the same argument as in Proposition 2.4.29(i). Now letting $\lambda' = \max\{\lambda_j\}$, we see that $\widetilde{M_{\lambda'}} = \tilde{M}$. $\square$

2.4.8. Finiteness and vanishing theorems for graded local cohomology. Corollary 2.4.30 tells us that (under the same assumptions) we can corestrict the equivalence in Corollary 2.4.27 to obtain an essentially surjective functor

$\boxed{2/25}$

$$\widetilde{(\,\cdot\,)}\colon {}^{*}\mathsf{Mod}_{\mathsf{ft}}(S) \longrightarrow \mathsf{QCoh}_{\mathsf{ft}}(\mathcal{O}_X).$$

However, this does not give an equivalence of categories since, for example, any graded S-module satisfying (TN) will map to 0, and hence the composition $M \mapsto \tilde{M} \to \Gamma_*(\tilde{M})$ may not be isomorphic to the identity.

To prove the rest of Serre's equivalence, the key is the following:

Theorem 2.4.31 [FAC, n° 63, Proposition 3; EGAIII$_1$, Théorème 2.2.1]. *Let S be a Noetherian $\mathbf{N}$-graded ring. Let M be a finitely generated graded S-module.*

[BS13, Thm. 16.1.5]
[Har77, Thm. II.5.19, Thm. III.5.2]

- (i) *For all $i \geq 0$ and all $n \in \mathbf{Z}$, the S_0-module $H^i_{S_+}(M)_n$ is finitely generated.*
- (ii) *For every $i \geq 0$, there exists $r \in \mathbf{Z}$ such that $H^i_{S_+}(M)_n = 0$ for all $n \geq r$.*

In fact, it turns out that r can be chosen independently of i in (ii). Proving this requires knowing that $H^i_{S_+}(\,\cdot\,)$ vanishes past a certain degree, which we will see later using Čech cohomology.

To prove Theorem 2.4.31, we need to prove one more result on local cohomology modules, namely, that S_+-power-torsion modules are Γ_{S_+}-acyclic. To prove this, we will use the following result. The Noetherian property is used to apply the Artin–Rees lemma [Mur24ca, Theorem 10.4.2].

Proposition 2.4.32. *Let R be a Noetherian ring and let I be an injective R-module. Consider an ideal $\mathfrak{a} \subseteq R$. Then, $\Gamma_{\mathfrak{a}}(I)$ is injective.*

[BS13, Prop. 2.1.4]

Proof. By Baer's criterion (Lemma 1.3.38), it suffices to show that for every ideal $\mathfrak{b} \subseteq R$, every morphism $\mathfrak{b} \to \Gamma_{\mathfrak{a}}(I)$ can be extended to a morphism $R \to \Gamma_{\mathfrak{a}}(I)$:

$$\begin{tikzcd}
\mathfrak{b} \arrow[r, hook] \arrow[d] & R \arrow[dl, dashed] \\
\Gamma_{\mathfrak{a}}(I). &
\end{tikzcd}$$

We put I into the diagram to obtain an extension of $\mathfrak{b} \to I$ to a morphism $R \to I$:

$$\begin{tikzcd}
\mathfrak{b} \arrow[r, hook] \arrow[d, "h"'] & R \arrow[dl, dashed] \arrow[d, "1\mapsto w"] \\
\Gamma_{\mathfrak{a}}(I) \arrow[r, hook] & I.
\end{tikzcd}$$

Since R is Noetherian, $\mathfrak{b}$ is finitely generated, and hence $\mathfrak{a}^t\, h(\mathfrak{b}) = 0$ for some t. By the Artin–Rees lemma [Mur24ca, Theorem 10.4.2] for the inclusion $h(\mathfrak{b}) \subseteq Rw$ as submodules of I, there exists $c \geq 0$ such that, for all $n \geq c$,

$$\mathfrak{a}^n(Rw) \cap h(\mathfrak{b}) = \mathfrak{a}^{n-c}\big(\mathfrak{a}^c(Rw) \cap h(\mathfrak{b})\big).$$

Thus,

$$\mathfrak{a}^{t+c}(Rw) \cap H(\mathfrak{b}) \subseteq \mathfrak{a}^t\, h(\mathfrak{b}) = 0.$$

We can therefore define an R-module map

$$\begin{aligned} \tilde{h}\colon \mathfrak{a}^{t+c} + \mathfrak{b} &\longrightarrow \Gamma_{\mathfrak{a}}(I) \\ s + r &\longmapsto rw \end{aligned}$$

which is well-defined since if $r_1 - r_2 \in \mathfrak{a}^{t+c}$, then

$$r_1 w - r_2 w = (r_1 - r_2)w \in \mathfrak{a}^{t+c}(Rw) \cap h(\mathfrak{b}) = 0.$$

This map $\tilde{h}$ fits into the commutative diagram

$$\begin{tikzcd}
\mathfrak{b} \arrow[r, hook] \arrow[dr, "h"'] & \mathfrak{a}^{t+c} + \mathfrak{b} \arrow[r, hook] \arrow[d, "\tilde{h}"] & R \arrow[dl, dashed] \arrow[d, "1 \mapsto m"] \\
& \Gamma_{\mathfrak{a}}(I) \arrow[r, hook] & I
\end{tikzcd}$$

where we use the injectivity of I again to extend $\tilde{h}$ to $1 \mapsto m$. It remains to show that $m \in \Gamma_{\mathfrak{a}}(I)$. For all $s \in \mathfrak{a}^{t+c}$, we have

$$sm = \tilde{h}(s) = \tilde{h}(s+0) = 0 \cdot w = 0. \qquad \square$$

As a consequence, we have:

[BS13, Cor. 2.1.6, Cor. 2.1.7(i)]

Corollary 2.4.33. *Let R be a Noetherian ring. Consider an ideal $\mathfrak{a} \subseteq R$ and let M be an $\mathfrak{a}$-power-torsion R-module. Then, M has an injective resolution where each term is an $\mathfrak{a}$-power-torsion R-module. Moreover, $H^i_{\mathfrak{a}}(M) = 0$ for all $i > 0$.*

Proof. Since $\mathsf{Mod}(R)$ has enough injectives, there exists an injective module I'^0 and an inclusion $M \hookrightarrow I'^0$. Since $\Gamma_{\mathfrak{a}}(\,\cdot\,)$ is left exact, applying $\Gamma_{\mathfrak{a}}$ yields an inclusion $M \hookrightarrow I^0$ where $I^0 := \Gamma_{\mathfrak{a}}(I'^0)$ is injective by Proposition 2.4.32.

Now suppose that we have an exact sequence

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^n$$

of $\mathfrak{a}$-power-torsion R-modules such that I^j is injective for all j. Applying the previous paragraph to $\operatorname{coker}(I^{n-1} \to I^n)$, we can find an $\mathfrak{a}$-power-torsion R-module I^{n+1} such that

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^n \longrightarrow I^{n+1}$$

is exact.

It remains to show that $H^i_{\mathfrak{a}}(M) = 0$ for all $i > 0$. Let $M \to I^\bullet$ be an injective resolution where I^j is $\mathfrak{a}$-power-torsion for all j as constructed above. Then, the functor $\Gamma_{\mathfrak{a}}(\,\cdot\,)$ has no effect on the resolution $I^\bullet$. Computing cohomology objects, we see that $H^i_{\mathfrak{a}}(M) = 0$ for all $i > 0$. $\square$

We can now prove Theorem 2.4.31.

Proof of Theorem 2.4.31. We induce on $i \geq -1$. If $i = -1$, then $H^{-1}_{S_+}(M) = 0$ and there is nothing to show. Now suppose that $i \geq 0$. We proceed by dévissage for finitely generated graded modules over Noetherian graded rings (see [Mur24ag, §1.7.3]). Since M is finitely generated, there exists a filtration

$$0 = M^0 \subseteq M^1 \subseteq \cdots \subseteq M^\ell = M$$

by graded S-modules such that for every i, we have

$$\frac{M^i}{M^{i-1}} \cong \left(\frac{S}{\mathfrak{p}_i}\right)(d_i)$$

where $\mathfrak{p}_i$ is a homogeneous prime ideal of S and $d_i \in \mathbf{Z}$ [Mur24ag, Proposition 1.7.7]. If $\ell = 0$, then $M = 0$ and there is nothing to show. For the inductive case, the long exact sequence associated to the short exact sequence

$$0 \longrightarrow M^{\ell-1} \longrightarrow M^{\ell} \longrightarrow \left(\frac{S}{\mathfrak{p}_\ell}\right)(d_\ell) \longrightarrow 0$$

together with the inductive hypothesis implies it suffices to show the claim for $(S/\mathfrak{p}_\ell)(d_\ell)$. Set $\mathfrak{p} \coloneqq \mathfrak{p}_\ell$ and $d \coloneqq d_\ell$.

Suppose first that $S/\mathfrak{p}$ is concentrated in degree 0. Then, $S/\mathfrak{p}$ is S_+-torsion, and hence $H^i_{S_+}(S/\mathfrak{p}) = 0$ for all $i > 0$ by Corollary 2.4.33. Moreover, since $S/\mathfrak{p}$ is a quotient of S_0, we see that $H^0_{S_+}(S/\mathfrak{p}) = S/\mathfrak{p}$ is finitely generated over S_0.

It remains to consider the case when $S/\mathfrak{p}$ has elements of positive degree. Choose a homogeneous element

$$s \in \frac{S}{\mathfrak{p}} - S_+\left(\frac{S}{\mathfrak{p}}\right)$$

of degree $t > 0$. The exact sequence

$$0 \longrightarrow \frac{S}{\mathfrak{p}} \xrightarrow{s} \left(\frac{S}{\mathfrak{p}}\right)(t) \longrightarrow \frac{S/\mathfrak{p}}{s(S/\mathfrak{p})}(t) \longrightarrow 0$$

and the corresponding long exact sequence of (graded) local cohomology modules yields the exact sequence

$$H^{i-1}_{S_+}\left(\frac{S/\mathfrak{p}}{s(S/\mathfrak{p})}\right)_{n+t} \longrightarrow H^i_{S_+}\left(\frac{S}{\mathfrak{p}}\right)_n \xrightarrow{s} H^i_{S_+}\left(\frac{S}{\mathfrak{p}}\right)_{n+t}$$

of S_0-modules for all $n \in \mathbf{Z}$. By the inductive hypothesis, the module on the left is 0 for $n \geq r$, and hence multiplication by s is injective. However, $H^i_{S_+}(S/\mathfrak{p})_n$ is S_+-power-torsion by [Har77, Exercise III.3.3(b)] (Homework 4, Problem 5(d)), and hence we must have $H^i_{S_+}(S/\mathfrak{p})_n = 0$ for all $n \geq r - t$.

The inductive hypothesis also says that the module on the left has finitely generated graded components. Fix n and choose $k \geq 0$ such that $n + kt \geq r - t$, in which case

$$H^i_{S_+}(S/\mathfrak{p})_{n+kt} = 0$$

by the previous paragraph. For each $j \in \{0, 1, \ldots, k-1\}$, there is an exact sequence

$$H^{i-1}_{S_+}\left(\frac{(S/\mathfrak{p})}{s(S/\mathfrak{p})}\right)_{n+(j+1)t} \longrightarrow H^i_{S_+}(S/\mathfrak{p})_{n+jt} \xrightarrow{s} H^i_{S_+}(S/\mathfrak{p})_{n+(j+1)t}.$$

By descending induction on j, we see that $H^i_{S_+}(S/\mathfrak{p})_n$ is finitely generated over S_0. □

Remark 2.4.34. An alternative method to prove Theorem 2.4.31 is to use the short exact sequence

$$0 \longrightarrow \Gamma_{S_+}(M) \longrightarrow M \longrightarrow \frac{M}{\Gamma_{S_+}(M)} \longrightarrow 0,$$

the Γ_{S_+}-acyclicity of S_+-power-torsion modules, and homogeneous prime avoidance (Homework 2, Problem 6(c) in MA557) on $M/\Gamma_{S_+}(M)$. This replaces the role of dévissage for graded modules and is the approach taken in [BS13].

2.4.9. Serre's equivalence for coherent sheaves on Proj. We are now ready to show:

[Har77, Exer. II.5.9(b)]

Theorem 2.4.35 [FAC, n° 65, Proposition 5; EGAIII$_1$, Théorème 2.3.1]. *Let S be a Noetherian $\mathbf{N}$-graded ring that is generated by S_1 as an S_0-algebra and let $X = \operatorname{Proj}(S)$. Let M be a graded S-module satisfying* (TF). *Then, there exists an integer N such that*

$$\alpha_n \colon M_n \longrightarrow \Gamma\bigl(X, \tilde{M}(n)\bigr)$$

is an isomorphism for all $n \geq N$. In other words, the kernel and cokernel of the morphism

$$\alpha \colon M \longrightarrow \Gamma_*(M)$$

satisfy (TN).

Proof. Let $M' = \bigoplus_{k \geq n} M_k$ be finitely generated such that $\widetilde{M'} = \tilde{M}$. We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r] \arrow[d, hook] & M' \arrow[r, "\alpha"] \arrow[d, hook] & \Gamma_*\bigl(\widetilde{M'}\bigr) \arrow[r] \arrow[d, equal] & Q' \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & K \arrow[r] & M \arrow[r, "\alpha"] & \Gamma_*(\tilde{M}) \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

with exact rows. By construction, we have $Q' \twoheadrightarrow Q$. The left hand side of the diagram yields the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r] \arrow[d, hook] & M' \arrow[r, "\alpha"] \arrow[d, hook] & \operatorname{im}(\alpha) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & K \arrow[r] & M \arrow[r, "\alpha"] & \Gamma_*(\tilde{M}) &
\end{tikzcd}$$

with exact rows. The snake lemma [KS06, Lemma 12.1.1] implies that $K/K' \hookrightarrow M/M'$. Since $K/K' \hookrightarrow M/M'$ and $Q' \twoheadrightarrow Q$, we may replace M by M' to assume that M itself is finitely generated. By Theorem 2.4.18, it suffices to show that

$$H^0_{S_+}(M)_n = H^1_{S_+}(M)_n = 0$$

for sufficiently large n. This holds by Theorem 2.4.31! □

Corollary 2.4.36 [EGAII, Corollaire 2.3.2]. *Let S_0 be a Noetherian ring and let S be a $\mathbf{N}$-graded ring finitely generated by S_1 as an S_0-algebra. Let $X = \operatorname{Proj}(S)$. Let $\mathscr{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Then, $\Gamma_*(\mathscr{F})$ satisfies* (TF).

Proof. By Corollary 2.4.30, we know that $\mathscr{F} \cong \tilde{M}$ for some finitely generated graded S-module M. The morphism $M \to \Gamma_*(\mathscr{F})$ is an isomorphism in sufficiently large degree by Theorem 2.4.35. □

We therefore obtain:

[Har77, Exer. II.5.9(c)]

Theorem 2.4.37 (Serre's equivalence for $\mathsf{Coh}(\operatorname{Proj}(S))$ [FAC, n° 65, Propositions 5 and 6; EGAIII$_1$, Scholie 2.3.3]). *Let S_0 be a Noetherian ring. Let S be a* $\mathbf{N}$-

graded ring finitely generated by S_1 *an* S_0*-algebra. Let* $X = \operatorname{Proj}(S)$*. We have an equivalence of categories*

$$\frac{\left\{\begin{matrix}\textit{graded } S\textit{-modules}\\ \textit{satisfying } (\mathrm{TF})\end{matrix}\right\}}{\left\{\begin{matrix}\textit{graded } S\textit{-modules}\\ \textit{satisfying } (\mathrm{TN})\end{matrix}\right\}} \rightleftarrows \mathsf{Coh}(\mathcal{O}_X)$$

$$\begin{aligned} M &\longmapsto \widetilde{M}\\ \Gamma_*(\mathscr{F}) &\longleftarrow\!\shortmid\ \mathscr{F}. \end{aligned}$$

To give a careful proof of this, we would need to discuss more preliminaries from homological algebra and category theory, namely the notion of a quotient category of an Abelian category by a Serre subcategory. Instead, we give an indication of the proof with some references.

Idea of proof. The subcategory of ${}^*\mathsf{Mod}(S)$ consisting of graded modules satisfying (TN) forms what is now called a *Serre subcategory* (or *thick* subcategory [TohokuI, p. 138]), which was first defined in [Ser53, (I) on p. 259]. The quotient category on the left-hand side is defined as in [TohokuI, §1.11]: The objects are graded S-modules satisfying (TF) and the quotient category is obtained by formally inverting inclusions (resp. quotient maps) whose cokernel (resp. kernel) satisfy (TN). We can now show that $\mathsf{Coh}(\mathcal{O}_X)$ together with the functor $M \mapsto \widetilde{M}$ from graded S-modules satisfying (TF) satisfies the universal property of quotient categories of Abelian categories as stated in [Gab62, Chapitre III, §1, Proposition 1, Corollaire 2]. $\square$

Example 2.4.38 [Smi04, Example 4.6]. Serre's equivalence does not hold without "generated by S_1" hypotheses. Consider the weighted polynomial ring $S = k[x, y]$ where x has degree 1 and y has degree 2. Let $M = S/(x)$ and consider the graded module $M(1)$. Note that $M(1)$ is not isomorphic to 0 in the category on the left-hand side of Theorem 2.4.37.

We claim $(M(1))^\sim = 0$, and hence the functor $N \mapsto \widetilde{N}$ is not faithful. It suffices to show that

$$M(1)_{(x)} = M(1)_{(y)} = 0.$$

First, $M(1)_{(x)} = 0$ since $x \in \operatorname{Ann}_S(M)$. Next, to show $M(1)_{(y)} = 0$, we note that

$$M(1)_y = \left\{ \frac{f(y)}{y^n} \;\middle|\; f(y) \in M(1),\ n \in \mathbf{Z}_{\geq 0} \right\}$$

has no nonzero degree 0 elements since the numerators $f(y)$ have odd degree, while the denominators y^n have even degree. Thus, $M(1)_{(y)} = 0$.

2.4.10. Serre's finiteness, vanishing, and global generation theorems for Proj.

Three more important consequences of Theorem 2.4.31 and Serre's equivalence Theorem 2.4.37 are the following finiteness and vanishing theorems for Proj. For projective varieties (i.e., the case $S_0 = k$ for an algebraically closed field k), these results are due to Serre [FAC].

Theorem 2.4.39 [FAC, n° 66, Théorème 1 and Théorème 2; EGAII, Corollaire 2.7.9; EGAIII$_1$, Théorème 2.2.1]. *Let* S_0 *be a Noetherian ring and let* S *be an* $\mathbf{N}$*-graded ring finitely generated by* S_1 *as an* S_0*-algebra. Let* $X = \operatorname{Proj}(S)$ *and let* $\mathscr{F}$ *be a coherent* $\mathcal{O}_X$*-module.*

[Har77, Thm. II.5.19, Thm. III.5.2]

(*i*) (Finiteness) *For all $i \geq 0$ and all $n \in \mathbf{Z}$, the S_0-module $H^i(X, \mathscr{F}(n))$ is finitely generated.*

(*ii*) (Vanishing) *There exists $r \in \mathbf{Z}$ such that $H^i(X, \mathscr{F}(n)) = 0$ for all $i > 0$ and all $n \geq r$.*

[Har77, Thm. II.5.17]

(*iii*) (Global generation) *There exists $r \in \mathbf{Z}$ such that $\mathscr{F}(n)$ is globally generated for all $n \geq r$.*

Proof. By Corollary 2.4.36, there exists a finitely generated graded S-module M such that $\tilde{M} \cong \mathscr{F}$. We then have the exact sequence

$$0 \longrightarrow H^0_{S_+}(M) \longrightarrow M \longrightarrow \bigoplus_{n \in \mathbf{Z}} H^0\bigl(X, \mathscr{F}(n)\bigr) \longrightarrow H^1_{S_+}(M) \longrightarrow 0 \tag{2.4.40}$$

and the isomorphisms

$$\bigoplus_{n \in \mathbf{Z}} H^i\bigl(X, \mathscr{F}(n)\bigr) \xrightarrow{\sim} H^{i+1}_{S_+}(M)$$

for all $i \geq 1$ by Theorem 2.4.18. The isomorphisms for $i \geq 1$ imply (*i*) and (*ii*) for $i \geq 1$ by Theorem 2.4.31.

We show (*i*) for $i = 0$. The exact sequence (2.4.40) restricts to the exact sequence

$$0 \longrightarrow H^0_{S_+}(M)_n \longrightarrow M_n \longrightarrow H^0\bigl(X, \mathscr{F}(n)\bigr) \longrightarrow H^1_{S_+}(M)_n \longrightarrow 0$$

in each degree n. Since all of the terms are finitely generated S_0-modules by Theorem 2.4.31 and the fact that M is finitely generated over S, we see that $H^0(X, \mathscr{F}(n))$ is finitely generated as an S_0-module.

[EGAII, Lem. 2.7.9.1]

It remains to show (*iii*). Since M is finitely generated as a graded S-module, we can find a surjection

$$\bigoplus_i S(n_i) \twoheadrightarrow M$$

for finitely many direct summands $S(n_i)$ and integers n_i. Setting $r = -\min\{n_i\}$, we see that

$$\bigoplus_i S(n + n_i) \twoheadrightarrow M(n)$$

is surjective and each $n + n_i \geq r + n_i \geq 0$ for all i. Since S is generated in degree 1, by choosing a set of generators for $S(n + n_i)_0 = S_{n+n_i}$ as an S_0-module, we obtain a surjection $S_0^{(I_i)} \twoheadrightarrow S(n + n_i)_0$ for finite indexing sets I_i. We therefore obtain the surjective composition

$$\bigoplus_i S^{(I_i)} \twoheadrightarrow \bigoplus_i S(n + n_i) \twoheadrightarrow M(n).$$

Taking associated sheaves, we are done by the discussion in Definition 2.1.3. □

2.5. Fiber products, separation axioms, and Čech cohomology

So far, we have developed enough tools to write down some interesting examples of Spec and Proj schemes and to say something about their sheaf cohomology, at least in theory. However, when we work with concrete geometric objects, at some point we will need to *compute* something. Our definition of sheaf cohomology via derived functors is not well-suited for computations since injective (or even flasque) resolutions are hard to get your hands on.

Our goal will be to define and study the necessary separation axioms for schemes that will allow us to compute sheaf cohomology easily using what is known as the *Čech complex* associated to a chosen affine open cover.

2.5.1. Fiber products of schemes.

We start with the general notion of a fiber product.

Definition 2.5.1 [EGAI$_{\text{new}}$, Chapitre 0, (1.2.2)]. Let $\mathscr{C}$ be a category. Let S be an object of $\mathscr{C}$ and let X and Y be objects equipped with morphisms to S. The *fiber product* of X and Y over S, if it exists, is an object $X \times_S Y$ of $\mathscr{C}$ together with *projection morphisms* $p_1 \colon X \times_S Y \to X$ and $p_2 \colon X \times_S Y \to Y$ over S satisfying the following universal property: For every object Z of $\mathscr{C}$ and every solid commutative diagram

$$\begin{tikzcd}
Z \arrow[rrd, bend left, "g"] \arrow[rd, dashed, "\theta"] \arrow[rdd, bend right, "f"'] & & \\
& X \times_S Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

there exists a unique dashed morphism $\theta \colon Z \to X \times_S Y$ making the diagram commute. We call a square of the form above (where the object in the top left is the fiber product of the three objects in the rest of the square) a *Cartesian square* or a *pullback square*.

Example 2.5.2. Consider the category of quasi-projective varieties Var_k over an algebraically closed field k from last semester. Last semester, we showed that Var_k has products [Mur24ag, Theorem 1.3.37]. Products in Var_k are special cases of fiber products where $S = \{*\}$. More generally, if $\mathscr{C}$ is a category with a terminal object $\{*\}$, then $X \times_{\{*\}} Y$ is the product of X and Y.

We want to show that fiber products exist in the category Sch of schemes. In fact, we will show that the functor $\mathsf{Sch} \hookrightarrow \mathsf{LRS}$ preserves fiber products.

Theorem 2.5.3 [EGAI, Théorème 3.2.6; EGAI$_{\text{new}}$, Théorème 3.2.1]. *Let S be a scheme and let X and Y be schemes over S. Then, the fiber product $X \times_S Y$ exists in* Sch. *Moreover, $X \times_S Y$ is a fiber product for X and Y over S in* LRS. [Har77, Thm. II.3.3]

Remark 2.5.4. It turns out that fiber products exist in LRS. This is a fairly recent result attributed to Becker, and published in [Gil11, Corollary 5].

We construct fiber products "by hand" following [EGAI; Har77] instead of following [EGAI$_{\text{new}}$] or [Gil11]. As a preliminary result (used multiple times throughout the proof), we show the following.

Lemma 2.5.5 [Stacks, Tag 01HI]. *Let $f \colon X \to Y$ be a morphism of ringed spaces. Let $U \hookrightarrow X$ and $V \hookrightarrow Y$ be open ringed subspaces such that $f(\mathrm{sp}(U)) \subseteq \mathrm{sp}(V)$.*

Then, there exists a unique morphism $f|_U \colon U \to V$ *making the diagram*

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d, dashed, "f|_U"'] & X \arrow[d, "f"] \\
V \arrow[r, hook] & Y
\end{tikzcd}$$

commute. If f *is a morphism of locally ringed spaces, then* $f|_U$ *is also a morphism of locally ringed spaces.*

Proof. The map $\mathrm{sp}(f|_U)$ on underlying topological spaces exists and is unique. Since any open subset $W \subseteq V$ has open inverse image $f|_U^{-1}(W) = f^{-1}(W) \cap U \subseteq X$, we see that $\mathrm{sp}(f|_U)$ is continuous.

It remains to define the action of $f|_U$ on structure sheaves. Let $W \subseteq V$ be an open subset. We have the commutative diagram

$$\begin{tikzcd}
\Gamma(W, \mathcal{O}_Y) \arrow[r] \arrow[d, equal] & \Gamma\bigl(f^{-1}(W), \mathcal{O}_X\bigr) \arrow[d] \\
\Gamma(W, \mathcal{O}_V) \arrow[r, dashed] & \Gamma\bigl((f|_U)^{-1}(W), \mathcal{O}_U\bigr).
\end{tikzcd}$$

This diagram determines the behavior of $f|_U$ on structure sheaves uniquely since the behavior must be compatible with restriction maps for f. ☐

We now show Theorem 2.5.3.

Proof of Theorem 2.5.3. The idea is to first construct the fiber product for affine schemes and then to glue using the universal property on overlaps. We proceed in five steps.

Step 1 [EGAI, Proposition 3.2.2]. The case when $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, and $S = \mathrm{Spec}(R)$ are all affine.

By the universal property of tensor products of algebras [Mur24ca, Theorem 7.11.2], we have the commutative diagram

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\
A \arrow[r] \arrow[drr, bend right] & A \otimes_R B \arrow[dr, dashed, "\exists!"] & \\
& & \Gamma(Z, \mathcal{O}_Z)
\end{tikzcd}$$

By [Mur24ag, Proposition 2.2.17], giving a morphism $Z \to \mathrm{Spec}(A \otimes_R B)$ is equivalent to giving a morphism $A \otimes_R B \to \Gamma(Z, \mathcal{O}_Z)$, and similarly for $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$. Applying Spec in the top left square and applying [Mur24ag, Proposition 2.2.17] three times to each of the morphisms involving Z, we are done.

Step 2 [EGAI, Lemme 3.2.6.1]. If X and Y are schemes over a scheme S, if $U \subseteq X$ is an open subset, and if the product $X \times_S Y$ exists, then $p_1^{-1}(U) \subseteq X \times_S Y$ (with the induced open subscheme structure) is a product for U and Y over S.

Consider the commutative diagram

$$\begin{tikzcd}
Z \arrow[drr, dashed, "\theta", "\exists!"'] \arrow[drrr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & & \\
& p_1^{-1}(U) \arrow[r, hook] \arrow[d] & X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& U \arrow[r, hook] & X \arrow[r] & S
\end{tikzcd}$$

where the restricted morphism $p_1^{-1}(U) \to U$ is constructed using Lemma 2.5.5. Since $X \times_S Y$ is a fiber product, the dashed map $Z \dashrightarrow X \times_S Y$ exists. We need to show that θ factors uniquely through $p_1^{-1}(U)$. This follows from Lemma 2.5.5 applied to the diagram

$$\begin{tikzcd}
Z \arrow[r, equal] \arrow[d, dashed, "\exists!"'] & Z \arrow[d, "\theta"] \\
p_1^{-1}(U) \arrow[r, hook] & X \times_S Y.
\end{tikzcd}$$

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Step 3 [EGAI, Lemme 3.2.6.2, (3.2.6.3)]. Let X, Y be schemes over S. Suppose that $\{X_i\}$ is an open cover of X and suppose that $X_i \times_S Y$ exists for every i. Then, $X \times_S Y$ exists.

For each i, j, let

$$U_{ij} \coloneqq p_1^{-1}(X_{ij}) \subseteq X_i \times_S Y.$$

By Step 2, U_{ij} is a product for X_{ij} and Y over S. By the uniqueness of fiber products, there are unique isomorphisms

$$\varphi_{ij} \colon U_{ij} \xrightarrow{\sim} U_{ji}$$

for every i, j compatible with the projection morphisms. Moreover, these isomorphisms are compatible with each other for each i, j, k in the sense of [Mur24ag, Lemma 2.2.22] by the uniqueness part of the universal property of fiber products. By the gluing lemma [Mur24ag, Lemma 2.2.22], we therefore obtain a scheme $X \times_S Y$ which we claim is the fiber product for X and Y over S. The projection morphisms p_1, p_2 are defined by gluing the projection maps from the pieces $X_i \times_S Y$. For each i, set $Z_i = f^{-1}(X_i)$, and consider the commutative diagram

$$\begin{tikzcd}
Z_i \arrow[rr, hook] \arrow[dr, "\theta_i", "\exists!"'] \arrow[ddr, bend right, "f_i"'] & & Z \arrow[dr, dashed, "\theta", "\exists!"'] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& X_i \times_S Y \arrow[rr] \arrow[d] & & X \times_S Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d] \\
& X_i \arrow[rr] & & X \arrow[r] & S.
\end{tikzcd}$$

Then, the two compositions

$$\begin{aligned}
Z_i \cap Z_j &\xrightarrow{\theta_i} X_i \times_S Y \longrightarrow X \times_S Y \\
Z_i \cap Z_j &\xrightarrow{\theta_j} X_j \times_S Y \longrightarrow X \times_S Y
\end{aligned} \tag{2.5.6}$$

both factor through $X_{ij} \times_S Y$ by Lemma 2.5.5, and hence the two compositions (2.5.6) are equal by the universal property for $X_{ij} \times_S Y$. Thus, the morphisms

(2.5.6) glue to give a map θ, which is unique by post-composing with the inclusion into $X_i \times_S Y$ and applying the universal property for each $X_i \times_S Y$.

Step 4 [EGAI, (3.2.6.5))]. The case when S is affine.

By Step 1, $X \times_S Y$ exists when X, Y, S are all affine. By Step 3, $X \times_S Y$ exists when X is arbitrary but Y, S are affine. By interchanging the roles of X and Y and applying Step 3 again, $X \times_S Y$ exists when X, Y are arbitrary but S is affine.

Step 5 [EGAI, (3.2.6.4), (3.2.6.5))]. Conclusion of proof.

Given arbitrary X, Y, S, let $q\colon X \to S$ and $r\colon Y \to S$ be the given structure morphisms. Let $\{S_i\}$ be an affine open cover of S. Let $X_i = q^{-1}(S_i)$ and $Y_i = r^{-1}(S_i)$. Then, by Step 4, $X_i \times_{S_i} Y_i$ exists. The same scheme is a fiber product for X_i and Y over S because in the commutative diagram

$$\begin{tikzcd}
Z \arrow[dr, dashed] \arrow[drrr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & & \\
& X_i \times_{S_i} Y_i \arrow[r] \arrow[d] & Y_i \arrow[r, hook] \arrow[d] & Y \arrow[d] \\
& X_i \arrow[r] & S_i \arrow[r, hook] & S
\end{tikzcd}$$

the morphism g factors through Y_i by Lemma 2.5.5. Thus, $X_i \times_S Y$ exists for every i. Applying Step 3 again, $X \times_S Y$ exists.

Note that we did not use anywhere that Z is a scheme, only that it is a locally ringed space. Thus, $X \times_S Y$ is a fiber product for X and Y over S in the category LRS. In other words, the inclusion functor $\mathsf{Sch} \hookrightarrow \mathsf{LRS}$ preserves fiber products. $\square$

[Har77, Exer. II.3.9(a)]

Caution 2.5.7. Let k be a field. The underlying topological space of $X \times_{\operatorname{Spec}(k)} Y$ is not the product of X and Y, even if k is algebraically closed! This is because

$$\mathbf{A}^1_k \times_{\operatorname{Spec}(k)} \mathbf{A}^1_k = \operatorname{Spec}\bigl(k[s] \otimes_k k[t]\bigr) \cong \operatorname{Spec}\bigl(k[s,t]\bigr) = \mathbf{A}^2_k,$$

which has many non-closed points corresponding to prime ideals of height 1 that are not of the form $((0), (t-b))$ or $((s-a), (0))$ by Theorem 2.2.3.

As one example of fiber products, we can define a scheme structure on fibers of morphisms.

[Har77, p. 89]

Definition 2.5.8 [EGAI$_{\text{new}}$, (3.4.7)]. Let $f\colon X \to Y$ be a morphism of schemes and let $y \in Y$ be a point. Let $k(y) \coloneqq \mathcal{O}_{Y,y}/\mathfrak{m}_y$ be the residue field of Y at y and let $\operatorname{Spec}(k(y)) \to Y$ be the morphism obtained as the composition

$$\operatorname{Spec}\bigl(k(y)\bigr) \longrightarrow \operatorname{Spec}(\mathcal{O}_{Y,y}) \longrightarrow Y.$$

See [Har77, Exercise II.2.7] (Homework 5, Problem 1(a)). The *fiber* of the morphism f over the point y is

$$X_y \coloneqq X \times_Y \operatorname{Spec}\bigl(k(y)\bigr).$$

[EGAI$_{\text{new}}$, Prop. 3.4.6]

The fiber X_y is a scheme over $k(y)$. The underlying topological space $\operatorname{sp}(X_y)$ homeomorphic to $f^{-1}(y)$ with the subspace topology: The construction of fiber products (Theorem 2.5.3) says that fiber products can be constructed affine-locally, and for a ring map $\varphi\colon A \to B$, the spectrum

$$\operatorname{Spec}(B \otimes_A A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})$$

is homeomorphic to the fiber $\operatorname{Spec}(\varphi)^{-1}(\mathfrak{p})$ by [Mur24ca, Propositions 1.3.12 and 3.2.10(*ii*)].

This construction occurs throughout algebraic geometry: You can regard a morphism as a family of schemes, in which case the morphism encodes different deformations or degenerations of a scheme. The case when $Y = \operatorname{Spec}(\mathbf{Z})$ gives rise to the technique of *reduction modulo p*.

Example 2.5.9 [Har77, Examples II.3.3.1 and II.3.3.2]. Let k be an algebraically closed field.

(*i*) Consider

$$X = \operatorname{Spec}\biggl(\frac{k[x,y,t]}{(ty-x^2)}\biggr) \longrightarrow \operatorname{Spec}\bigl(k[t]\bigr).$$

This is a family of parabolas $V(ty - x^2) \subseteq \mathbf{A}^2_k$ parametrizsed by t. The fiber over $t \neq 0$ is an honest parabola, but over $t = 0$ we get the "double line" $\operatorname{Spec}(k[x,y]/(x^2))$, which is a non-reduced scheme. In this case, we say that the family of parabolas *degenerates* to a double line or that the double line *deforms* to a parabola.

Note that the generic fiber is

$$X_\eta = \operatorname{Spec}\biggl(\frac{k(t)[x,y]}{(ty-x^2)}\biggr) \cong \operatorname{Spec}\biggl(\frac{k(t)[x,y]}{(y-x^2)}\biggr),$$

which is a parabola over $k(t)$. Thus, the generic fiber describes the "generic behavior" of the family.

(*ii*) Consider

$$X = \operatorname{Spec}\biggl(\frac{k[x,y,t]}{(xy-t)}\biggr) \longrightarrow \operatorname{Spec}\bigl(k[t]\bigr).$$

This is a family of hyperbolas $V(xy - t) \subseteq \mathbf{A}^2_k$ parametrized by t. The fiber over $t \neq 0$ is an honest hyperbola, but over $t = 0$ we get the union of two axes $\operatorname{Spec}(k[x,y]/(xy))$, which is a reducible scheme. In this case, we say that the family of hyperbolas *degenerates* to the union of two lines or that the union of two lines *deforms* to a parabola.

The generic fiber in this case is

$$X_\eta = \operatorname{Spec}\biggl(\frac{k(t)[x,y]}{(xy-t)}\biggr) \cong \operatorname{Spec}\biggl(\frac{k(t)[x,y]}{(xy-1)}\biggr),$$

which is a hyperbola over $k(t)$.

2.5.2. Base change. The construction of fiber products gives rise to the notion of base change, a special case of which includes the operation of taking fibers. Base change is a very useful way of replacing the base scheme that you are working over: If $f\colon X \to X'$ is a morphism over S and $g\colon S' \to S$ is a morphism, taking fiber products yields a morphism

$$\begin{tikzcd}
X \times_S S' \arrow[rr, "f \times \mathrm{id}_{S'}"] \arrow[dr] & & X' \times_S S' \arrow[dl] \\
& S' &
\end{tikzcd}$$

over S' by looking at the commutative diagram

$$\begin{tikzcd}
X \times_S S' \arrow[dd, "p_1"'] \arrow[dr, dashed, "f\times \mathrm{id}_{S'}", "\exists!"'] \arrow[drr, bend left, "p_2"] & & \\
& X' \times_S S' \arrow[r] \arrow[d] & S' \arrow[d] \\
X \arrow[r, "f"] & X' \arrow[r] & S
\end{tikzcd}$$

where the square is Cartesian. A common example of this is when $S = \mathrm{Spec}(k)$ and $S' = \mathrm{Spec}(k')$ for a field extension $k \subseteq k'$. We can even take the product of two morphisms by looking at the commutative diagram

$$\begin{tikzcd}
X \times_S Y \arrow[dd, "p_1"'] \arrow[dr, dashed, "f\times g", "\exists!"'] \arrow[rr, "p_2"] & & Y \arrow[d, "g"] \\
& X' \times_S Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\
X \arrow[r, "f"] & X' \arrow[r] & S.
\end{tikzcd}$$

Many properties of morphisms we have seen so far are "stable under base change." Let us define two more, important classes of morphisms that we have not yet seen in this class.

[Har77, p. 84]

Definition 2.5.10 [EGAII, Définition 6.1.1]. Let $f\colon X \to Y$ be a morphism of schemes. We say that f is *integral* (resp. *finite*) if there exists an affine open cover $Y = \bigcup_i \mathrm{Spec}(B_i)$ such that

$$f^{-1}\bigl(\mathrm{Spec}(B_i)\bigr) = \mathrm{Spec}(A_i)$$

is an affine open subset of X with A_i an integral (resp. module-finite) B_i-algebra. Note that we do not assume that $B_i \to A_i$ is injective, and hence this is not the same as saying that A_i is an integral or module-finite *extension* of B_i.

We have the implications

$$\text{finite} \implies \text{integral} \implies \text{affine}.$$

However, morphisms (locally) of finite type are not necessarily affine.

By [Mur24ca, Theorem 4.2.6] and [Har77, Exercise II.3.3(c)] (Homework 5, Problem 4(b)), we know that

$$\text{integral} + \text{locally of finite type} \iff \text{finite}.$$

As usual, by Nike's trick [Vak, Proposition 5.3.1] (Homework 4, Problem 1), you can show that f is integral (resp. finite) if and only if the inverse image of *every* affine open $\mathrm{Spec}(B)$ is of the form $\mathrm{Spec}(A)$ where A is module-finite over B.

Proposition 2.5.12 below says that many properties of morphisms are stable under base change. Here are two examples of properties of morphisms that are *not* stable under base change.

[Har77, Exer. II.3.15(c)]

Example 2.5.11. Let $f\colon X \to \mathrm{Spec}(k)$ for a field k. Following [$\mathrm{EGAI}_{\mathrm{new}}$, (3.2.2)], we denote $X \otimes_k k' \coloneqq X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$ for a field extension $k \subseteq k'$.

(i) The property "the fibers of f are irreducible" is not stable under base change. Let $k = \mathbf{R}$ and $X = \operatorname{Spec}(\mathbf{R}[x]/(x^2+1))$. Since x^2+1 is irreducible over $\mathbf{R}$, the scheme X is integral, and in particular, irreducible. However,

$$X \otimes_{\mathbf{R}} \mathbf{C} = \operatorname{Spec}\left(\frac{\mathbf{C}[x]}{(x+i)(x-i)}\right)$$

is a disjoint union of two points, which is reducible.

(ii) The property "the fibers of f are reduced" is not stable under base change. Let $k = \mathbf{F}_p(t)$ and $X = \operatorname{Spec}(\mathbf{F}_p(t)[x]/(x^p - t))$. Since $x^p - t$ is irreducible over $\mathbf{F}_p(t)$, the scheme X is integral, and in particular, reduced. However,

$$X \otimes_{\mathbf{F}_p(t)} \mathbf{F}_p(t^{1/p}) = \operatorname{Spec}\left(\frac{\mathbf{F}_p(t^{1/p})[x]}{(x - t^{1/p})^p}\right)$$

is non-reduced.

On Homework 7, Problem 1, you will investigate "geometric" versions of these properties. We say that a k-scheme X is *geometrically irreducible* (resp. *geometrically reduced*, *geometrically integral*) if $X \otimes_k k'$ is irreducible (resp. reduced, integral) for every finite field extension $k \subseteq k'$.

We want to show that various properties of morphisms are stable under base change. $\boxed{3/4}$

Proposition 2.5.12. *Let S be a scheme. Let $f\colon X \to X'$ and $g\colon Y \to Y'$ be morphisms of schemes over S satisfying one of the following properties $\mathscr{P}$:*

(i) [EGAI$_{\text{new}}$, Proposition 6.1.5(*iv*)] *quasi-compact.*
(ii) [EGAI$_{\text{new}}$, Proposition 6.2.3(*iv*)] *locally of finite type.*
(iii) [EGAI$_{\text{new}}$, Proposition 6.3.4(*iv*)] *finite type.* [Har77, Exer. II.3.13(*d*)]
(iv) [EGAI$_{\text{new}}$, Proposition 4.3.6(*iii*)] *closed immersion.*
(v) [EGAI$_{\text{new}}$, Proposition 4.3.6(*iii*)] *open immersion.* [Har77, Exer.
(vi) [EGAI$_{\text{new}}$, Proposition 4.3.6(*iii*)] *immersion.* II.3.11(*a*)]
(vii) [EGAII, Proposition 1.6.2(*iv*)] *affine.* [EGAI, Prop. 4.3.1]
(viii) [EGAII, Proposition 6.1.5(*iv*)] *integral.*
(ix) [EGAII, Proposition 6.1.5(*iv*)] *finite.*

Then, the product

$$f \times g\colon X \times_S Y \longrightarrow X' \times_S Y'$$

has the same property $\mathscr{P}$. In particular, setting $g = \mathrm{id}_Y$, the properties $\mathscr{P}$ are stable under base change.

Proof. Let $\operatorname{Spec}(A' \otimes_R B') \subseteq X' \times_S Y'$ be an affine open subset. The inverse image of this affine open subset is

$$(f \times g)^{-1}\bigl(\operatorname{Spec}(A' \otimes_R B')\bigr) \xrightarrow{\sim} f^{-1}\bigl(\operatorname{Spec}(A')\bigr) \times_{\operatorname{Spec}(R)} g^{-1}\bigl(\operatorname{Spec}(B')\bigr)$$

by applying Step 2 of Theorem 2.5.3 to each factor, and then using the argument in Step 5 of Theorem 2.5.3 to replace the base scheme S in the fiber product with $\operatorname{Spec}(R)$.

For affine, integral, and finite morphisms, write

$$f^{-1}\bigl(\operatorname{Spec}(A')\bigr) = \operatorname{Spec}(A)$$
$$g^{-1}\bigl(\operatorname{Spec}(B')\bigr) = \operatorname{Spec}(B)$$

for some R-algebras A and B. We then have

$$f^{-1}\big(\mathrm{Spec}(A')\big) \times_{\mathrm{Spec}(R)} g^{-1}\big(\mathrm{Spec}(B')\big) \xleftarrow{\sim} \mathrm{Spec}(A \otimes_R B),$$

and hence we are done in the affine case. The integral (resp. finite) cases follow from the fact that

$$A' \otimes_R B' \longrightarrow A' \otimes_R B \longrightarrow A \otimes_R B$$

is integral (resp. finite) if $A' \to A$ and $B' \to B$ are. For the finite case, module-finiteness is preserved under base change [Mur24ca, Corollary 7.5.4] and composition [Mur24ca, Lemma 4.2.8]. For the integral case, write the maps $A' \to A$ and $B' \to B$ as a direct limit of module-finite maps. The finite case implies that the base changes of these maps are direct limits of module-finite maps, and hence are integral.

See [BouCA, V.1.1, Props. 5, 6] for a textbook reference.

For quasi-compactness (resp. locally of finite type), write

$$f^{-1}\big(\mathrm{Spec}(A')\big) = \bigcup_i \mathrm{Spec}(A_i)$$
$$g^{-1}\big(\mathrm{Spec}(B')\big) = \bigcup_j \mathrm{Spec}(B_j)$$

as finite unions (resp. unions) of affine open subsets. Then,

$$(f \times g)^{-1}\big(\mathrm{Spec}(A' \otimes_R B')\big) = \bigcup_{i,j} \mathrm{Spec}(A_i \otimes_R B_j)$$

is a finite union (resp. union) of affine open subsets (resp. affine open subsets of finite type over $A' \otimes_R B'$). The statement for finite type follows from combining the statements for quasi-compactness and locally of finite type.

For open immersions, we decompose $f \times g$ as

$$X \times_S Y \xrightarrow{f \times \mathrm{id}_Y} X' \times_S Y \xrightarrow{\mathrm{id}_{X'} \times g} X' \times_S Y'.$$

For each individual morphism, the statement for open immersions follows from Step 2 of Theorem 2.5.3. Thus, the composition is also an open immersion by definition of an open immersion.

For closed immersions, replace the closed immersions with the canonical inclusions of closed subschemes. We can therefore consider the quasi-coherent ideal sheaves $\mathscr{I}_1 \subseteq \mathcal{O}_{X'}$ and $\mathscr{I}_2 \subseteq \mathcal{O}_{Y'}$ defining the closed subschemes $X \subseteq X'$ and $Y \subseteq Y'$ as in (2.3.23). We claim that

$$p_1^{-1}\mathscr{I}_1 \cdot \mathcal{O}_{X' \times_S Y'} + p_2^{-1}\mathscr{I}_2 \cdot \mathcal{O}_{X' \times_S Y'}$$

is quasi-coherent and defines a closed subscheme W that is the fiber product $X \times_S Y$. This sheaf is quasi-coherent since pullback preserve quasi-coherence, and since this sheaf is the image of the morphism

$$p_1^*\mathscr{I}_1 \oplus p_2^*\mathscr{I}_2 \xrightarrow{[1\ \ 1]} \mathcal{O}_{X' \times_S Y'}.$$

Here, we recall that images of quasi-coherent sheaves are quasi-coherent by Corollary 2.3.9. On affine open subsets $U = \mathrm{Spec}(A' \otimes_R B')$, the closed subscheme $W \cap U$ is defined by

$$I_1(A' \otimes_R B') + I_2(A' \otimes_R B')$$

where $I_1 = \Gamma(\mathrm{Spec}(A'), \mathscr{I}_1)$ and $I_2 = \Gamma(\mathrm{Spec}(B'), \mathscr{I}_2)$. We are now done by using the isomorphism

$$\frac{A' \otimes_R B'}{I_1(A' \otimes_R B') + I_2(A' \otimes_R B')} \xleftarrow{\sim} (A'/I_1) \otimes_R (B'/I_2)$$

from [Mur24ca, Corollary 7.2.4].

The statement for immersions follows by decomposing the immersions f, g as closed immersions followed by open immersions. □

Before we move on, we want to prove something that we omitted a long time ago (see Remark 2.3.28). You may have noticed that our definition of a closed immersion is *not* the one in [Har77, p. 85]. Statement (*ii*) below shows that our definition of a closed immersion in Definition 2.3.25 is equivalent to the definition in [Har77].

[Har77, Exer. II.3.11(*b*)]

Proposition 2.5.13 [EGAI$_{\text{new}}$, Proposition 4.2.2]. *Let $f\colon X \to Y$ be a morphism of schemes. For every $x \in X$, consider the local ring map*

$$f_x^\#\colon \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}.$$

(*i*) *The morphism f is an open immersion if and only if* $\mathrm{sp}(f)$ *is a homeomorphism of X onto an open subset of Y and $f_x^\#$ is an isomorphism for every $x \in X$.*

(*ii*) *The morphism f is an immersion (resp. a closed immersion) if and only if* $\mathrm{sp}(f)$ *is a homeomorphism of X onto a locally closed (resp. closed) subset of Y and $f_x^\#$ is surjective for every $x \in X$.*

Proof. The implications $\Rightarrow$ are true by definition in Definition 2.3.25.

(*i*). The condition in (*i*) implies $f^\#$ is an isomorphism $\mathcal{O}_X \xrightarrow{\sim} f^{-1}\mathcal{O}_Y$. Under the homeomorphism $X \to f(X)$, the sheaf $f^{-1}\mathcal{O}_Y$ is identified with $\mathcal{O}_Y|_{f(X)}$.

(*ii*). Let U_0 be the largest open subset of Y such that $f(X)$ is closed in U_0. Replacing Y by the open subscheme U_0, it suffices to show the case when $Z = f(X)$ is closed in Y, in which case we want to show that f is a closed immersion. Since $f(X)$ has the subspace topology, the morphism f is qcqs. Thus, $f_*\mathcal{O}_X$ is quasi-coherent by [EGAI$_{\text{new}}$, Proposition 6.7.1] (Homework 4, Problem 3(*b*)). We now consider the short exact sequence

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X \longrightarrow 0.$$

Since $\mathcal{O}_Y$ and $f_*\mathcal{O}_X$ are quasi-coherent, we know that $\mathscr{I}$ is quasi-coherent by "2 out of 3" for quasi-coherent sheaves on schemes (Corollary 2.3.9(*i*)), and we can factor the surjection $\mathcal{O}_Y \to f_*\mathcal{O}_X$ as

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{I} \arrow[r] & \mathcal{O}_Y \arrow[rr] \arrow[dr, two heads] & & f_*\mathcal{O}_X \arrow[r] & 0 \\
& & & \mathcal{O}_Y/\mathscr{I} \arrow[ur, "\sim", "\omega"'] & &
\end{tikzcd}$$

Now setting $\mathcal{O}_Z \coloneqq (\mathcal{O}_Y/\mathscr{I})|_Z$, the scheme $(Z, \mathcal{O}_Z)$ is a closed subscheme of $(X, \mathcal{O}_X)$ and f factors as

$$(X, \mathcal{O}_X) \xrightarrow[\sim]{(f, \omega|_Z)} (Z, \mathcal{O}_Z) \hookrightarrow (Y, \mathcal{O}_Y).$$

Thus, f is a closed immersion. □

Remark 2.5.14. The advantage of the characterization of a closed immersion in Proposition 2.5.13(*ii*) is that it is more obviosuly a local condition on the target. However, it has the disadvantage of not obviously being affine, making it difficult (for example) to show that closed immersions are stable under base change. See, e.g., [Mur18a]. Note that the key result connecting the two definitions is that the pushforward of a quasi-coherent sheaf by a qcqcs morphism is quasi-coherent [EGAI_{new}, Proposition 6.7.1] (Homework 4, Problem 3(*b*)). In [EGAI_{new}], qcqs morphisms do not appear until later in the book, so Grothendieck and Dieudonné proceed by proving the special case of [EGAI_{new}, Proposition 6.7.1] for affine morphisms [EGAI_{new}, Lemme 4.2.21 and Lemme 4.2.22].

2.5.3. The diagonal morphism and separated morphisms. Now that we have constructed fiber products, we can define the diagonal morphism.

[Har77, p. 96]

Definition 2.5.15 [EGAI_{new}, Chapitre 0, (1.4.3)]. Let $f\colon X \to S$ be a morphism of schemes. The *diagonal morphism* is the unique morphism

$$\Delta_{X/S} \coloneqq \Delta_f \colon X \longrightarrow X \times_S X$$

fitting into the commutative diagram

$$\begin{tikzcd}
X \arrow[rrd, equal, bend left] \arrow[rd, dashed, "\Delta_{X/S}"] \arrow[rdd, equal, bend right] & & \\
& X \times_S X \arrow[r] \arrow[d] & X \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

where the square is Cartesian. We sometimes denote the diagonal morphism by Δ_X or Δ to simplify notation.

To motivate separated morphisms, we recall the following result from topology.

Proposition 2.5.16 [BouGT, Chapter I, §8, no. 1, Proposition 1]. *Let X be a topological space. Then, X is Hausdorff if and only if the image of the diagonal map*

$$\begin{aligned} \Delta_X \colon X &\longrightarrow X \times X \\ x &\longmapsto (x, x) \end{aligned}$$

is closed in $X \times X$.

Since closed immersions are the "scheme-theoretic" version of a continuous injective map with closed image, the following definition gives a scheme-theoretic version of the Hausdorff condition for schemes. Because of this categorical formulation of the definition, this definition also makes sense in the relative setting, i.e., for morphisms of schemes.

[Har77, p. 96]

Definition 2.5.17 [EGAI_{new}, Définition 5.2.1]. Let $f\colon X \to S$ be a morphism of schemes. We say that f is *separated* if $\Delta_{X/S}$ is a closed immersion. We say that X is *separated* if $X \to \operatorname{Spec}(\mathbf{Z})$ is separated.

We record some important properties of the diagonal morphism. We start with the following:

Lemma 2.5.18 [EGAI$_{\text{new}}$, Corollaire 3.2.3]. *Consider the Cartesian square of schemes*

$$\begin{tikzcd}
X \times_S Y \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "\psi"] \\
X \arrow[r, "\varphi"] & S.
\end{tikzcd}$$

Let $U \subseteq S$ be an open subset. Let $V \subseteq \varphi^{-1}(U) \subseteq X$ and $W \subseteq \psi^{-1}(U) \subseteq Y$ be open subsets. Then, the dashed morphism j induced by the universal property of $X \times_S Y$ in the commutative diagram

$$\begin{tikzcd}
V \times_U W \arrow[rr, "q"] \arrow[dr, dashed, "\exists!", "j"'] \arrow[dd, "p"'] & & W \arrow[d, hook] \\
& X \times_S Y \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "\psi"] \\
V \arrow[r, hook] & X \arrow[r, "\varphi"] & S
\end{tikzcd}$$

is an open immersion. The morphism j induces an isomorphism

$$V \times_U W \xrightarrow{\sim} p^{-1}(V) \cap q^{-1}(W) \subseteq X \times_S Y. \tag{2.5.19}$$

Proof. Proposition 2.5.12(v) implies j is an open immersion. Since the image of j is contained in $p^{-1}(V) \cap q^{-1}(W)$, Lemma 2.5.5 implies j factors uniquely through $p^{-1}(V) \cap q^{-1}(W)$. Since the square in the commutative diagram

$$\begin{tikzcd}
V \times_U W \arrow[drr, bend left] \arrow[dr, dashed, "\exists!", "j"'] \arrow[ddr, bend right] & & \\
& p^{-1}(V) \cap q^{-1}(W) \arrow[r, "q"'] \arrow[d, "p"'] & W \arrow[d, "\psi"] \\
& V \arrow[r, "\varphi"] & U
\end{tikzcd}$$

is Cartesian by applying Step 2 of Theorem 2.5.3 to each factor, and since j fits into the commmutative diagram, we see that j induces an isomorphism (2.5.19) as claimed. ☐

To prove statements about separated morphisms, the following general result is surprisingly useful. The Cartesian square below is called the *diagonal-base-change diagram* in [Vak, Exercise 1.2.S]. Previous versions of [Vak] called this diagram the *magic diagram*.

Proposition 2.5.20 (Diagonal-base-change diagram [EGAI, Proposition 5.3.5; EGAI$_{\text{new}}$, Chapitre 0, Proposition 1.4.8]). *Let $\mathscr{C}$ be a category with arbitrary fiber products. Let $f_1 \colon X_1 \to Y$, $f_2 \colon X_2 \to Y$, and $g \colon Y \to Z$ be morphisms in $\mathscr{C}$. Then, the following diagram commutes and is a Cartesian square:*

[Har77, Exer. II.4.4]

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[r, "(p_1, p_2)_Z"] \arrow[d] & X_1 \times_Z X_2 \arrow[d, "f_1 \times_Z f_2"] \\
Y \arrow[r, "\Delta_{Y/Z}"] & Y \times_Z Y.
\end{tikzcd} \tag{2.5.21}$$

Here, p_1, p_2 are the projection morphisms from $X_1 \times_Z X_2$ to X_1, X_2.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & & & & \\
& X_1 \times_Z X_2 \arrow[r, "p_2"] \arrow[d, "p_1"] & X_2 \arrow[d] \arrow[ddr, bend left, "f_2"] \arrow[dddrrr, bend left, "f_2"] & & & \\
S \arrow[uu, dashed] \arrow[ur, dashed, red] \arrow[drrr, dashed, red] & X_1 \arrow[r] \arrow[drr, bend right, "f_1"'] \arrow[dddrrr, bend right, "f_1"'] & Z & & & \\
& & & Y \arrow[ul] \arrow[drr, bend left, "id"] \arrow[dr, dashed, "\Delta_{Y/Z}"] \arrow[ddr, bend right, "id"'] & & \\
& & & & Y \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& & & & Y \arrow[r] & Z
\end{tikzcd}$$

where the dashed arrows are induced by the universal property of fiber products. The two compositions $X_1 \times_Y X_2 \to Y \times_Z Y$ are the same by the commutativity of the diagram above and the fact that $X_1 \times_Y X_2 \to Y \times_Z Y$ is induced by the universal property of the fiber product in the bottom right square and taking the outermost morphisms $X_1 \times_Y X_2 \to Y$.

Now suppose we have a morphism $S \to Y$ and a morphism $S \to X_1 \times_Z X_2$ such that after composing with $Y \times_Z Y$, the square (2.5.21) commutes. By definition of our map $X_1 \times_Z X_2 \to Y \times_Z Y$, drawing in our morphisms from S in the diagram above (as done with the dashed red arrows), we get that the entire diagram including S above commutes. The maps

$$S \longrightarrow X_1 \times_Z X_2 \longrightarrow X_i \xrightarrow{f_i} Y$$

for $i = 1, 2$ are then the same by the commutativity of the diagram, hence we get an induced morphism $S \to X_1 \times_Y X_2$ by the universal property of $X_1 \times_Y X_2$. Thus, there exists a unique dashed arrow $S \to X_1 \times_Y X_2$ making the diagram commute. □

[Har77, Prop. II.4.1]

PROPOSITION 2.5.22 [EGAI_{new}, Proposition 5.1.2 and Proposition 5.2.2]. *Let $f\colon X \to S$ be a morphism of schemes. Then, $\Delta_{X/S}\colon X \to X \times_S X$ is an immersion. If X and S are affine, then $\Delta_{X/S}$ is a* closed *immersion. As a consequence, affine schemes are always separated (over* $\operatorname{Spec}(\mathbf{Z})$*).*

Proof. Let $x \in X$. Let $V \subseteq S$ be an affine open neighborhood of $f(x)$ and let $U \subseteq X$ be an affine open neighborhood of x such that $f(U) \subseteq V$. By Proposition 2.5.13(*ii*), it suffices to show that for every x, V, U as chosen above,

$$\Delta_{U/V}\colon U \longrightarrow U \times_V U$$

is a *closed* immersion, since

$$U \times_V U \xrightarrow{\sim} U \times_S U \hookrightarrow X \times_S X$$

is an open immersion by Proposition 2.5.12(v), and since the square

$$\begin{tikzcd}
U \arrow[r, "\Delta_{U/V}"] \arrow[d, hook, "\text{open imm.}"'] & U \times_V U \arrow[d, hook, "\text{open imm.}"] \\
X \arrow[r, "\Delta_{X/S}"] & X \times_S X
\end{tikzcd}$$

is Cartesian by Proposition 2.5.20. Since $\Delta_{U/V}$ is a morphism of affine schemes, it suffices by Remark 2.3.27 to note that

$$\begin{aligned}
A \otimes_R A &\longrightarrow A \\
\sum_i a_i \otimes a'_i &\longmapsto \sum_i a_i a'_i
\end{aligned}$$

is surjective. ☐

Corollary 2.5.23 [EGAI$_{\text{new}}$, p. 277]. *Let $f\colon X \to S$ be a morphism of schemes. Then, f is separated if and only if the set-theoretic image of $\Delta_{X/S}$ is closed in $X \times_S X$.* [Har77, Cor. II.4.2]

Proof. Since $\Delta_{X/S}$ is an immersion by Proposition 2.5.22, this is a consequence of Proposition 2.5.13(ii). ☐

Corollary 2.5.24 [EGAI$_{\text{new}}$, Proposition 5.3.10]. *Let X be a separated scheme and let U, V be affine open subsets of X. Then, $U \cap V$ is an affine open subset.* [Har77, Exer. II.4.3]

Proof. Set $S = \operatorname{Spec}(\mathbf{Z})$. By Lemma 2.5.18, we see that the square

$$\begin{tikzcd}
U \cap V \arrow[r] \arrow[d, hook] & U \times_S V \arrow[d, hook] \\
X \arrow[r, hook, "\Delta_{X/S}", "\text{closed imm.}"'] & X \times_S X
\end{tikzcd}$$

is Cartesian. The top horizontal morphism is a closed immersion by the definition of separatedness and by base change (Proposition 2.5.12(iv)). We are done since closed subschemes of affine schemes are affine (see Remark 2.3.28). ☐

Proposition 2.5.25. *The following statements hold in* Sch. [Har77, Cor. II.4.6]

(a) [EGAI$_{\text{new}}$, Proposition 5.3.1(i)] *Open and closed immersions are separated.*

(b) [EGAI$_{\text{new}}$, Proposition 5.3.1(ii)] *Compositions of separated morphisms are separated.*

(c) [EGAI$_{\text{new}}$, Proposition 5.3.1(iii)] *If $f\colon X \to X'$ and $g\colon Y \to Y'$ are separated morphisms over a base scheme S, then*

$$f \times_S g\colon X \times_S Y \longrightarrow X' \times_S Y'$$

is separated.

(d) [EGAI$_{\text{new}}$, Proposition 5.3.1(iv)] *Separated morphisms are stable under base change.*

(e) [EGAI$_{\text{new}}$, Proposition 5.3.1(v)] *If $f\colon X \to Y$ and $g\colon Y \to Z$ are two morphisms such that $g \circ f$ is separated and g is separated, then f is separated.*

(f) [EGAI$_{\text{new}}$, Proposition 5.3.5] (Local on the target) *A morphism $f\colon X \to Y$ is separated if and only if Y can be covered by open subsets V_i such that $f^{-1}(V_i) \to V_i$ is separated for every i.*

Proof. (*a*). Let $X \hookrightarrow Y$ be an open or closed immersion. We claim that

[EGAI, Prop. 5.3.8] [EGAI$_{\text{new}}$, Prop. 3.7.1(c'')]

$$\Delta_{X/Y}\colon X \longrightarrow X \times_Y X$$

is an isomorphism. It suffices to show that the commutative square in the diagram

$$\begin{tikzcd}
Z \arrow[dr, dashed] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& X \arrow[r, equal] \arrow[d, equal] & X \arrow[d, hook] \\
& X \arrow[r, hook] & Y
\end{tikzcd}$$

is Cartesian. If $f, g\colon Z \to X$ make the diagram commute, then the dashed morphism must equal $\mathrm{sp}(f)$ and $\mathrm{sp}(g)$ on sets since $X \hookrightarrow Y$ are injective. Let $U \subseteq Y$ be an affine open subset. The commutativity of the diagram implies that in the diagram

$$\begin{tikzcd}
& & \Gamma\big(f^{-1}(U \cap X), \mathcal{O}_Z\big) \arrow[dd, equal] \\
\Gamma(U, \mathcal{O}_Y) \arrow[r, two heads] \arrow[urr, bend left] \arrow[drr, bend right] & \Gamma(U \cap X, \mathcal{O}_X) \arrow[ur, "f^\sharp(U\cap X)"'] \arrow[dr, "g^\sharp(U\cap X)"] & \\
& & \Gamma\big(g^{-1}(U \cap X), \mathcal{O}_Z\big)
\end{tikzcd}$$

the two compositions from $\Gamma(U, \mathcal{O}_Y)$ to the right column are equal.

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(*b*). Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two morphisms. The square in the commutative diagram

$$\begin{tikzcd}
X \arrow[d, "\Delta_{X/Y}"'] \arrow[dr, bend left, "\Delta_{X/Z}"] & \\
X \times_Y X \arrow[r, "j"] \arrow[d] & X \times_Z X \arrow[d, "f \times f"] \\
Y \arrow[r, "\Delta_{Y/Z}"] & Y \times_Z Y
\end{tikzcd}$$

is Cartesian by Proposition 2.5.20, and the composition in the top row is $\Delta_{X/Z}$ by the universal property of $X \times_Z X$. Since closed immersions are stable under base change (Proposition 2.5.12(*iv*)) and composition, $\Delta_{X/Z} = j \circ \Delta_{X/Y}$ is a closed immersion.

(c). We have the commutative diagram

$$\begin{tikzcd}
(X \times_S Y) \times_{X' \times_S Y'} (X \times_S Y) \arrow[rr] \arrow[dd] \arrow[drrrrr, dashed] \arrow[dddddr, dashed] & & X \times_S Y \arrow[dddd] \arrow[dddr] \arrow[ddrrrr, bend left] & & & & \\
& & & & & Y \times_{Y'} Y \arrow[dl] \arrow[dr] & \\
X \times_S Y \arrow[rrrr] \arrow[drrr] \arrow[ddddrr, bend right] & & & & Y \arrow[dr] & & Y \arrow[dl] \\
& & & X' \times_S Y' \arrow[rr] \arrow[dd] & & Y' \arrow[dd] & \\
& & X \arrow[dr] & & & & \\
& X \times_{X'} X \arrow[ur] \arrow[dr] & & X' \arrow[rr] & & S & \\
& & X \arrow[ur] & & & &
\end{tikzcd}$$

where the red squares are Cartesian. We claim that $(X \times_S Y) \times_{X' \times_S Y'} (X \times_S Y)$ satisfies the universal property for $(X \times_{X'} X) \times_S (Y \times_{Y'} Y)$, which implies they are naturally isomorphic. The projection morphisms from $(X \times_S Y) \times_{X' \times_S Y'} (X \times_S Y)$ are defined to be the dashed morphisms in the commutative diagram above, induced by the universal properties of $X \times_{X'} X$ and $Y \times_{Y'} Y$. Let $Z \to X \times_{X'} X$ and $Z \to Y \times_{Y'} Y$ be morphisms whose compositions to S are equal. By the commutativity of the diagram and the universal property of fiber products, we obtain unique morphisms $Z \to X \times_S Y$ over $X' \times_S Y'$. Thus, there is a unique morphism $Z \to (X \times_S Y) \times_{X' \times_S Y'} (X \times_S Y)$ as claimed. We may therefore form the commutative diagram

$$\begin{tikzcd}
X \times_S Y \arrow[rr] \arrow[dr, dashed, "\Delta_{f \times g}"] \arrow[dd] & & Y \arrow[d, "\Delta_g"] \\
& (X \times_S Y) \times_{X' \times_S Y'} (X \times_S Y) \arrow[r, "p_2 \times p_2"] \arrow[d, "p_1 \times p_1"'] & Y \times_{Y'} Y \arrow[d] \\
X \arrow[r, "\Delta_f"] & X \times_{X'} X \arrow[r] & S
\end{tikzcd}$$

where the square is Cartesian. This identifies $\Delta_{f \times g} \cong \Delta_f \times_S \Delta_g$ as a product of closed immersions. Thus, $\Delta_{f \times g}$ is a closed immersion by Proposition 2.5.12(iv).

(d) is the special case of (c) where $g = \mathrm{id}_Y$.

(e). By Proposition 2.5.20, we have the commutative diagram

$$\begin{tikzcd}
& X \arrow[r, "g \circ f"] & Z \\
X \arrow[r, "\Gamma_f"] \arrow[d] & X \times_Z Y \arrow[r, "p_2"] \arrow[u, "p_1"] \arrow[d, "f \times \mathrm{id}_Y"] & Y \arrow[u, "g"'] \\
Y \arrow[r, "\Delta_{Y/Z}"] & Y \times_Z Y &
\end{tikzcd}$$

with Cartesian squares, where $f = p_2 \circ \Gamma_f$. The left square is Cartesian by Proposition 2.5.20. Sinec Γ_f and p_2 are separated by base change (d), we see that f is separated by (b).

(f). Since $Y = \bigcup_i V_i$, we have

$$X \times_Y X = \bigcup_i f^{-1}(V_i) \times_{V_i} f^{-1}(V_i)$$

by construction in Theorem 2.5.3, Step 5. We are done since the property of being a closed immersion is local on the target by Proposition 2.5.13(ii). ☐

As one example application of separatedness, we have the following:

[Har77, Exer. II.4.2]

Proposition 2.5.26 [EGAI$_{\text{new}}$, Propositions 5.4.1(d) and 5.4.3]. *Let S be a scheme. Let X be a scheme over S and let $Y \to S$ be a separated morphism of schemes. Let $f, g\colon X \to Y$ be morphisms over S for which there exists a scheme-theoretically dominant morphism $U \to X$ such that the compositions*

$$U \longrightarrow X \overset{f}{\underset{g}{\rightrightarrows}} Y$$

are equal. Then, $f = g$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
U \arrow[rrd, hook, bend left] \arrow[rd, dashed, "\exists!"] \arrow[rdd, bend right, "f|_U = g|_U"'] & & \\
& Z \arrow[r, "\text{cl. imm.}"] \arrow[d] & X \arrow[d, "(f,g)_S"] \\
& Y \arrow[r, "\Delta_{Y/S}"] & Y \times_S Y
\end{tikzcd}$$

where the square is Cartesian. Since $U \to X$ is scheme-theoretically dominant, we know that $Z \to X$ is an isomorphism [EGAI$_{\text{new}}$, Proposition 5.4.3] (Homework 5, Problem 5(b)). By the commutativity of the diagram, we have the commutative diagram

$$\begin{tikzcd}
& & & & Y \arrow[rd] & \\
X \arrow[rrrru, bend left, "f"] \arrow[rrrrd, bend right, "g"'] & Z \arrow[l, "\sim"'] \arrow[r] & Y \arrow[r, "\Delta_{Y/S}" description] \arrow[rru, equal, bend left] \arrow[rrd, equal, bend right] & Y \times_S Y \arrow[ru, "p_1"] \arrow[rd, "p_2"'] & & S. \\
& & & & Y \arrow[ru] &
\end{tikzcd}$$

Thus, $f = g$. ☐

Example 2.5.27. Proposition 2.5.26 is false when X is nonreduced or when Y is not separated.

(i) When X is nonreduced, let

$$X = Y = \operatorname{Spec}\left(\frac{k[x,y]}{(xy, y^2)}\right)$$

and let f be given by the identity and g be given by the ring map

$$\begin{array}{ccccc} \dfrac{k[x,y]}{(xy,y^2)} & \twoheadrightarrow & k[x] & \hookrightarrow & \dfrac{k[x,y]}{(xy,y^2)} \\ y & \longmapsto & 0. & & \end{array}$$

In the complement of the origin $x = y = 0$, both maps are the identity map and are induced by identity map on the ring $k[x, x^{-1}]$.

(ii) When Y is not separated, let $X = \mathbf{A}^1_k$, and Y be the affine line with the origin doubled (Example 2.3.5), which is not separated over k because it is not affine as we showed in Example 2.3.5. Then, consider the two inclusions f, g one of which maps to one of the origins in Y and the other of which maps to the other origin in Y. Note that f, g come from the construction of Y and are equal on the complement of the origin in X, but $f \neq g$.

2.5.4. Čech cohomology. We are now ready to prove the following important result: Sheaf cohomology on separated schemes can be computed combinatorially using a Čech complex.

DEFINITION 2.5.28 [God73, Chapitre II, §5.1]. Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of X where I is totally ordered. Consider an $\mathcal{O}_X$-module $\mathscr{F}$. The *Čech complex $\check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F})$ for $\mathscr{F}$ with respect to the covering* $\mathfrak{U}$ is defined as follows.

[Har77, pp. 219–220]
[EGAIII$_1$, (1.2.2)]

- The terms in the complex are
$$\check{\mathscr{C}}^p(\mathfrak{U}, \mathscr{F}) = \prod_{i_0 < i_1 < \cdots < i_p} {}_{U_{i_0 i_1 \cdots i_p}}\mathscr{F}$$
where $U_{i_0 i_1 \cdots i_p} := \bigcap_{j=0}^p U_{i_j}$ and the subscript on the left denotes $j_* j^{-1}$.
- The coboundary maps
$$d \colon \check{\mathscr{C}}^p(\mathfrak{U}, \mathscr{F}) \longrightarrow \check{\mathscr{C}}^{p+1}(\mathfrak{U}, \mathscr{F})$$
are given on each open subset $V \subseteq X$ by
$$(d\alpha)_{i_0, i_1, \ldots, i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \bigl(\alpha_{i_0, \ldots, \hat{\imath}_k, \ldots, i_{p+1}}\bigr)\big|_{V \cap U_{i_0 i_1 \cdots i_{p+1}}}.$$

These definitions yield an actual complex $\check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F})$ since

$$\begin{aligned} (d^2\alpha)_{i_0, i_1, \ldots, i_{p+2}} &= \sum_{k=0}^{p+2} (-1)^k \bigl((d\alpha)_{i_0, \ldots, \hat{\imath}_k, \ldots, i_{p+2}}\bigr)\big|_{V \cap U_{i_0 i_1 \cdots i_{p+2}}} \\ &= \sum_{k=0}^{p+2} (-1)^k \left(\sum_{\ell=0}^{k-1} (-1)^\ell \bigl(\alpha_{i_0, \ldots, \hat{\imath}_\ell, \hat{\imath}_k, \ldots, i_{p+2}}\bigr)\big|_{V \cap U_{i_0 i_1 \cdots i_{p+2}}} \right. \\ &\qquad\qquad \left. + \sum_{\ell=k+1}^{p+2} (-1)^{\ell-1} \bigl(\alpha_{i_0, \ldots, \hat{\imath}_k, \hat{\imath}_\ell, \ldots, i_{p+2}}\bigr)\big|_{V \cap U_{i_0 i_1 \cdots i_{p+2}}} \right) \\ &= 0. \end{aligned}$$

For each integer $i \geq 0$, the *i-th Čech cohomology module of $\mathscr{F}$ with respect to the covering* $\mathfrak{U}$ is

$$\check{H}^i(\mathfrak{U},\mathscr{F}) \coloneqq \mathbf{h}^i\Big(\Gamma\big(X,\check{\mathscr{C}}^\bullet(\mathfrak{U},\mathscr{F})\big)\Big).$$

[Har77, Caution III.4.0.2]

Without assumptions on X or $\mathfrak{U}$, Čech cohomology modules do not form a δ-functor. For example, take $\mathfrak{U} = \{X\}$, and observe that taking global sections is not exact. Nevertheless, for separated schemes, Čech cohomology with respect to a finite affine cover computes sheaf cohomology, and is therefore a δ-functor.

[Kem80, Thm. 13] [Har77, Lem. III.4.1, Lem. III.4.2, Prop. III.4.3, Lem. III.4.4, Thm. III.4.5]

THEOREM 2.5.29 [God73, Chapitre II, Théorèmes 5.2.1, 5.2.2, and 5.2.3(a); EGAIII_1, Proposition 1.4.1 and Remarque 1.4.2]. *Let X be a scheme. Let $\mathfrak{U} = (U_i)_{i\in I}$ be a finite open covering of X indexed by a totally ordered set I consisting of affine open subsets where all intersections of subfamilies of $\mathfrak{U}$ are affine. If $\mathscr{F}$ is a quasi-coherent $\mathcal{O}_X$-module, then*

$$0 \longrightarrow \mathscr{F} \longrightarrow \check{\mathscr{C}}^\bullet(\mathfrak{U},\mathscr{F}) \tag{2.5.30}$$

gives a $\Gamma(X,\,\cdot\,)$-acyclic resolution of $\mathscr{F}$ by quasi-coherent $\mathcal{O}_X$-modules. Thus, there are isomorphisms

$$\check{H}^i(\mathfrak{U},\mathscr{F}) \xrightarrow{\sim} H^i(X,\mathscr{F}) \tag{2.5.31}$$

of $\Gamma(X,\mathcal{O}_X)$-modules natural in $\mathscr{F}$ for all i.

Proof. We proceed in steps.

STEP 1. $\check{\mathscr{C}}^\bullet(\mathfrak{U},\mathscr{F})$ is a resolution for $\mathscr{F}$.

The sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \prod_{i} {}_{U_i}\mathscr{F} \longrightarrow \prod_{i<j} {}_{U_{ij}}\mathscr{F}$$

is exact at $\mathscr{F}$ by the sheaf axiom (3) and at $p = 0$ by the sheaf axiom (4).

For $p \geq 1$, it suffices to check exactness at stalks $x \in X$. We may therefore replace X by $\operatorname{Spec}(\mathcal{O}_{X,x})$ and the cover $\mathfrak{U} = \{U_i\}_{i\in I}$ by their intersections with $\operatorname{Spec}(\mathcal{O}_{X,x})$.

For each $p \geq 1$, we define a homotopy operator

$$h\colon \mathscr{C}^p(\mathfrak{U},\mathscr{F}) \longrightarrow \mathscr{C}^{p-1}(\mathfrak{U},\mathscr{F})$$

as follows. Choose $j \in I$ such that $x \in U_j$, in which case $U_j = X$. Then, consider

$$(h\alpha)_{i_0,i_1,\ldots,i_{p-1}} = \alpha_{j,i_0,i_1,\ldots,i_{p-1}} \coloneqq (-1)^\sigma \alpha_{\sigma(j),\sigma(i_0),\sigma(i_1),\ldots,\sigma(i_{p-1})}$$

where σ is the permutation for which $\sigma(j) < \sigma(i_0) < \cdots < \sigma(i_{p-1})$. We then have

$$\begin{aligned}(dh+hd)(\alpha)_{i_0,i_1,\ldots,i_p} &= \sum_{k=0}^{p}(-1)^k\big(\alpha_{j,i_0,\ldots,\hat{\imath}_k,\ldots,i_p}\big)\big|_{U_{i_0 i_1\cdots i_p}} \\ &\qquad + \alpha_{i_0,i_1,\ldots,i_p} + \sum_{k=0}^{p}(-1)^{k+1}\big(\alpha_{j,i_0,\ldots,\hat{\imath}_k,\ldots,i_p}\big)\big|_{U_{i_0 i_1\cdots i_p}} \\ &= \alpha_{i_0,i_1,\ldots,i_p}\end{aligned}$$

and hence h is indeed a homotopy operator. This shows that the identity map on $\check{\mathscr{C}}^\bullet(\mathfrak{U},\mathscr{F})$ is homotopic to the zero map, and hence $\mathbf{h}^i(\check{\mathscr{C}}^\bullet(\mathfrak{U},\mathscr{F})) = 0$ for all $i \geq 1$ by [Wei94, Lemma 1.4.5] (see Definition 1.4.7).

Step 2. Conclusion of proof.

Let $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ be an injective resolution. We have a morphism

$$\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{I}^\bullet$$

by injectivity of the $\mathscr{I}^j$. Note that in (1.4.9), we did not need the source complex to consist of injectives. This morphism, which is unique up to homotopy equivalence, induces the map $\check{H}^i(\mathfrak{U}, \mathscr{F}) \to H^i(X, \mathscr{F})$.

By assumption, every intersection $V = U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_n}$ is affine and $V \hookrightarrow X$ is an affine morphism. By [Har77, Exercise III.4.1] (Homework 4, Problem 4(c)),

$$H^i(X, {}_V\mathscr{F}) \cong H^i(V, \mathscr{F}),$$

which vanishes for $i > 0$ by Cartan's Theorem B (Theorem 2.2.15). Thus, ${}_V\mathscr{F}$ is acyclic for $\Gamma(X, \cdot\,)$ and $0 \to \mathscr{F} \to \mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$ is an acyclic resolution. We are now done since right derived functors can be computed using acyclic resolutions (Proposition 1.4.21). □

We can now compute examples!

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[Har77, Exer. III.4.7]

Example 2.5.32 (Arithmetic genus of a plane curve). Let k be a field and let $X = V_+(f) \subseteq \mathbf{P}^2_k$ be the plane curve defined by a homogeneous polynomial

$$f(x_0, x_1, x_2) \in k[x_0, x_1, x_2]_d$$

of degree $d > 0$. Suppose that

$$\bigl\{[1 : 0 : 0]\bigr\} = V_+(x_1, x_2) \not\subseteq X.$$

Then, $U = D_+(x_1) \cap X$ and $V = D_+(x_2) \cap X$ form an affine open cover of X since

$$X - \bigl(D_+(x_1) \cup D_+(x_2)\bigr) = X \cap V_+(x_1, x_2) = \emptyset.$$

By Theorem 2.5.29, we can compute sheaf cohomology using the Čech complex

$$0 \longrightarrow {}_U\mathcal{O}_X \oplus {}_V\mathcal{O}_X \longrightarrow {}_{U\cap V}\mathcal{O}_X \longrightarrow 0.$$

Write

$$U \cap V = \operatorname{Spec}\left(\frac{k\left[\dfrac{x_0}{x_2}, \dfrac{x_1}{x_2}, \dfrac{x_2}{x_1}\right]}{f\left(\dfrac{x_0}{x_2}, \dfrac{x_1}{x_2}, 1\right)}\right)$$

and set $u = x_0/x_2$ and $v = x_1/x_2$. Taking global sections, we have the chain complex

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(U,\mathcal{O}_X)\oplus\Gamma(V,\mathcal{O}_X) \arrow[r, "{[1\ -1]}"] \arrow[d, equal] & \Gamma(U\cap V,\mathcal{O}_X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \dfrac{k\left[\dfrac{x_0}{x_1},\dfrac{x_2}{x_1}\right]}{f\left(\dfrac{x_0}{x_1},1,\dfrac{x_2}{x_1}\right)} \oplus \dfrac{k\left[\dfrac{x_0}{x_2},\dfrac{x_1}{x_2}\right]}{f\left(\dfrac{x_0}{x_2},\dfrac{x_1}{x_2},1\right)} \arrow[r, "{[1\ -1]}"] \arrow[d, equal] & \Gamma(U\cap V,\mathcal{O}_X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \dfrac{k\left[\dfrac{u}{v},v^{-1}\right]}{f\left(\dfrac{u}{v},1,v^{-1}\right)} \oplus \dfrac{k[u,v]}{f(u,v,1)} \arrow[r, "{[1\ -1]}"] & \dfrac{k\left[u,v^{\pm 1}\right]}{f(u,v,1)} \arrow[r] & 0.
\end{tikzcd}$$

To compute the cohomology modules of this sequence, write $f = a_0x_0^d + f'$ for $a_0 \in k^\times$ which is possible since $[1:0:0] \notin X$. After replacing f by $a_0^{-1}x_0^d$, we may assume that $f = x_0^d + f'$. As k-vector spaces, the complex above is

$$\left(\bigoplus_{i=0}^{d-1} k[v^{-1}]\cdot\left(\frac{u}{v}\right)^i\right) \oplus \left(\bigoplus_{i=0}^{d-1} k[v]\cdot u^i\right) \xrightarrow{[1\ -1]} \bigoplus_{i=0}^{d-1} k[v^{\pm}]\cdot u^i.$$

This complex further decomposes into d complexes

$$k[v^{-1}]\cdot\left(\frac{u}{v}\right)^i \oplus k[v]\cdot u^i \xrightarrow{[1\ -1]} k[v^{\pm}]\cdot u^i \tag{2.5.33}$$

for each $0 \le i \le d-1$.

We now calculate the cohomology of the map (2.5.33). The kernel of (2.5.33) is 0 for all $i \neq 0$ by looking at the exponents on v. When $i = 0$, the map is

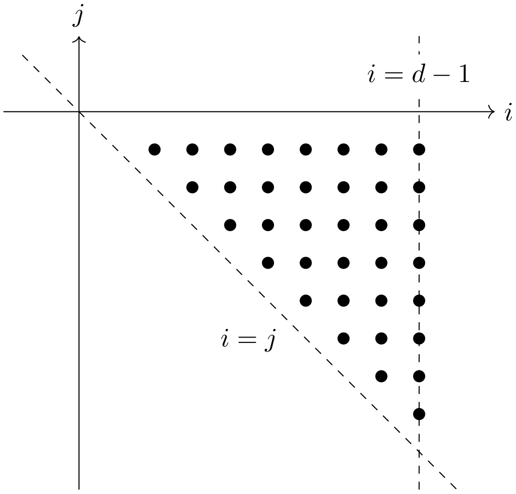
$$k[v^{-1}] \oplus k[v] \xrightarrow{[1\ -1]} k[v^{\pm}],$$

which has kernel $\{(a,a) \mid a \in k\}$. Thus, $\dim_k(H^0(X,\mathcal{O}_X)) = 1$.

It remains to compute $\dim_k(H^1(X,\mathcal{O}_X))$. Every monomial of the form

$$v^{-j}\left(\frac{u}{v}\right)^i = \frac{u^i}{v^{i+j}} \quad \text{for } 0 \le i \le d-1,\ j \ge 0$$

is in the image of the $k[v^{-1}]\cdot(u/v)^i$ and every monomial of the form u^iv^j for i, j as above is in the image of $k[v]\cdot u^i$. The missed monomials are those pictured below in the (i, j) plane:



This is a triangle with height $d - 2$ and width $d - 1$, hence contains

$$\frac{1}{2}(d-1)(d-2)$$

dots representing monomials in $H^1(X, \mathcal{O}_X)$. Thus,

$$\dim_k\bigl(H^1(X, \mathcal{O}_X)\bigr) = \frac{1}{2}(d-1)(d-2).$$

2.5.5. Cohomology of projective space. We can now compute the cohomology of projective space. This one computation is very important—it is critical for many general results about the cohomology of coherent sheaves on projective varieties.

Theorem 2.5.34 [FAC, n° 62, Proposition 2; EGAIII$_1$, Proposition 2.1.12]. *Let A be a ring and let $X = \mathbf{P}^r_A = \operatorname{Proj}(A[x_0, x_1, \ldots, x_r])$ for $r \geq 1$. We have the following:* [Har77, Thm. III.5.1]

(a) *The natural map*

$$S \longrightarrow \Gamma_*(\mathcal{O}_X) = \bigoplus_{n \in \mathbf{Z}} H^0\bigl(X, \mathcal{O}_X(n)\bigr)$$

is an isomorphism of graded S-modules.

(b) *$H^i(X, \mathcal{O}_X(n)) = 0$ for all $i \notin \{0, r\}$ and all $n \in \mathbf{Z}$.*

(c) *We have the isomorphism*

$$\bigoplus_{n \in \mathbf{Z}} H^r\bigl(X, \mathcal{O}_X(n)\bigr) \cong \bigoplus_{n \in \mathbf{Z}} \bigoplus_{\substack{p_i > 0 \\ -\sum p_i = n}} A \cdot \{\xi_{p_0, \ldots, p_r}\}$$

of graded S-modules where $x_i \cdot \xi_{p_0, \ldots, p_r} = \xi_{p_0, \ldots, p_i - 1, \ldots, p_r}$.

(d) *There is a perfect pairing*

$$\begin{aligned} &H^0\bigl(X, \mathcal{O}_X(n)\bigr) \times H^r\bigl(X, \mathcal{O}_X(-n-r-1)\bigr) \\ &\qquad \longrightarrow H^r\bigl(X, \mathcal{O}_X(-r-1)\bigr) \cong A \end{aligned}$$

of finitely generated free A-modules for each $n \in \mathbf{Z}$.

The following proof works without assuming that A is Noetherian.

Proof of Theorem 2.5.34. Let $\mathscr{F} = \bigoplus_{n\in\mathbf{Z}} \mathcal{O}_X(n)$. By Theorem 2.5.29, it suffices to compute the Čech cohomology of $\mathscr{F}$ relative to the standard affine open covering $\mathfrak{U} = (D_+(x_i))_{i=0}^r$ and sort out direct summands at the end.

Taking global sections of the Čech complex $\check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F})$ and using the description of sections over $D_+(f)$ in Proposition 2.4.9(*i*), we obtain the complex

$$\begin{aligned} 0 \longrightarrow \prod_{i=0}^r S_{x_i} \longrightarrow \prod_{i<j} S_{x_ix_j} \longrightarrow \prod_{i<j<k} S_{x_ix_jx_k} \\ \longrightarrow \cdots \longrightarrow \prod_{i=0}^r S_{x_0\cdots\hat{x}_i\cdots x_r} \longrightarrow S_{x_0x_1\cdots x_r} \longrightarrow 0 \end{aligned}$$

where the direct sum defining $\mathscr{F}$ corresponds to the $\mathbf{Z}$-grading on the terms and maps appearing in this complex. We have already seen the computation (*a*) in Proposition 2.4.16.

For (*c*), we think of $S_{x_0\cdots x_r}$ as a $\mathbf{Z}^{r+1}$-graded A-module

$$S_{x_0\cdots x_r} = \bigoplus_{(l_0,\ldots,l_r)\in\mathbf{Z}^{r+1}} A\cdot\{x_0^{l_0}\cdots x_r^{l_r}\}$$

where each graded component is a free A-module of rank 1. Then, d^{r-1} is $\mathbf{Z}^{r+1}$-graded, and one can recover the original $\mathbf{Z}$-grading by setting $n = \sum_i l_i$. The image of d^{r-1} is the $\mathbf{Z}^{r+1}$-graded submodule consisting of the direct summands for which $l_i \geq 0$ for some i. Thus,

$$\begin{aligned} \bigoplus_{n\in\mathbf{Z}} H^r\bigl(X, \mathcal{O}_X(n)\bigr) &\cong \bigoplus_{\substack{(l_0,\ldots,l_r)\in\mathbf{Z}^{r+1} \\ l_i<0 \text{ for all } i}} A\cdot\{x_0^{l_0}\cdots x_r^{l_r}\} \\ &\cong \bigoplus_{n\in\mathbf{Z}} \bigoplus_{\substack{(l_0,\ldots,l_r)\in\mathbf{Z}^{r+1} \\ \sum_i l_i = n \\ l_i<0 \text{ for all } i}} A\cdot\{x_0^{l_0}\cdots x_r^{l_r}\} \end{aligned}$$

where $x_0^{l_0}\cdots x_r^{l_r}$ is in degree $\sum_i l_i$.

For (*d*), the pairing is given by

$$\bigl(x_0^{m_0}\cdots x_r^{m_r}\bigr)\cdot\bigl(x_0^{l_0}\cdots x_r^{l_r}\bigr) = \begin{cases} x_0^{m_0+l_0}\cdots x_r^{m_r+l_r} & \text{if } m_i+l_i<0 \text{ for all } i \\ 0 & \text{if } m_i+l_i\geq 0 \text{ for some } i. \end{cases}$$

for $(m_0,\ldots,m_r)\in\mathbf{Z}^{r+1}$ such that $m_i\geq 0$ for all i and $(l_0,\ldots,l_r)\in\mathbf{Z}^{r+1}$ such that $l_i<0$ for all i. This restricts to a perfect pairing where the dual basis for $\{x_0^{m_0}\cdots x_r^{m_r}\}$ is $\{x_0^{-m_0-1}\cdots x_r^{-m_r-1}\}$.

It remains to prove (*b*). We proceed by induction on r. For $r=1$, there is nothing to prove. For $r>1$, we have the isomorphism of complexes

$$\Gamma\bigl(X, \check{\mathscr{C}}^\bullet(\mathfrak{U},\mathscr{F})\bigr)_{x_i} \cong \Gamma\Bigl(U_r, \check{\mathscr{C}}^\bullet\bigl(\mathfrak{U}\cap U_r, \mathscr{F}|_{U_r}\bigr)\Bigr)$$

where $U_r = D_+(x_r)$. Since U_r is affine, Cartan's Theorem B (Theorem 2.2.15) and Theorem 2.5.29 imply this complex has no higher cohomology modules. Thus, $H^i(X,\mathscr{F})$ is x_r-power-torsion for all $i>0$. It therefore suffices to show that for all $0<i<r$, multiplication by x_r on $H^i(X,\mathscr{F})$ is injective.

Consider the exact sequence

$$0 \longrightarrow S(-1) \xrightarrow{x_r \cdot -} S \longrightarrow S/(x_r) \longrightarrow 0 \tag{2.5.35}$$

Taking associated sheaves, we have the short exact sequence

$$0 \longrightarrow \mathscr{O}_X(-1) \xrightarrow{x_r \cdot -} \mathscr{O}_X \longrightarrow \mathscr{O}_H \longrightarrow 0 \tag{2.5.36}$$

where H is the hyperplane $\{x_r = 0\}$ and we think of $\mathscr{O}_H$ as a sheaf on X via pushforward. Twisting by all $n \in \mathbf{Z}$ and taking the direct sum, we have

$$0 \longrightarrow \mathscr{F}(-1) \xrightarrow{x_r \cdot -} \mathscr{F} \longrightarrow \mathscr{F}_H \longrightarrow 0, \tag{2.5.37}$$

where $\mathscr{F}_H = \bigoplus_{n \in \mathbf{Z}} \mathscr{O}_H(n)$, again thought of as a sheaf on X via pushforward. Taking cohomology, we have the long exact sequence

$$\cdots \longrightarrow H^i\bigl(X, \mathscr{F}(-1)\bigr) \xrightarrow{x_r \cdot -} H^i(X, \mathscr{F}) \longrightarrow H^i(X, \mathscr{F}_H) \longrightarrow \cdots.$$

Considered as graded S-modules, we have $H^i(X, \mathscr{F}(-1)) = H^i(X, \mathscr{F})(-1)$. By the inductive hypothesis, we know that $H^i(X, \mathscr{F}_H) = 0$ for all $0 < i < r-1$. We therefore see that $H^i(X, \mathscr{F}) = 0$ for all $1 < i < r$. Finally, for $i = 0$ the long exact sequence becomes (2.5.35), and hence multiplication by x_r on $H^1(X, \mathscr{F})$ is also injective. □

If we allow ourselves to assume that A is Noetherian, we can use our work so far relating local cohomology with sheaf cohomology to prove (*b*).

Alternative proof of Theorem 2.5.34(b) when A is Noetherian. By Theorem 2.4.18, we have

$$\bigoplus_{n \in \mathbf{Z}} H^i\bigl(X, \mathscr{O}_X(n)\bigr) \xrightarrow{\sim} H^{i+1}_{S_+}(S)$$

for all $i \geq 1$. We know that $H^j_{S_+}(S) = 0$ for all $r+1 > j > 0$ since $x_0, \ldots, x_r$ is a regular sequence on S [Har77, Exercise III.3.4(*b*)] (Homework 6, Problem 5(*b*)), proving (*b*). □

Remark 2.5.38. The second proof of Theorem 2.5.34(*b*) also works in the non-Noetherian case with some modifications. For the proof of Theorem 2.4.18, as mentioned in the note on p. 95, the hypothesis that S is Noetherian can be replaced by the assumption that S_+ is finitely generated as an ideal by [SGA2$_{\text{new}}$, Exposé II, Proposition 5]. Let us remind ourselves how the Noetherian hypotheses were used.

(*a*) In the proof of Theorem 2.4.18, the hypothesis that S is Noetherian was used in [Har77, Exercise III.3.3] (Homework 4, Problem 5(*e*)) to relate local cohomology with cohomology with supports.

(*b*) In [Har77, Exercise III.3.3], the hypothesis that A is Noetherian was used to say that (1) $\mathfrak{a}$ is finitely generated, and (2) $\tilde{I}$ is flasque for an injective A-module [Har77, Proposition III.3.4 or Exercise III.3.7(*b*)] (Homework 6, Problem 7(*b*)).

Instead, [SGA2$_{\text{new}}$, Exposé II, Proposition 5] is proved by relating $H^i(X,\mathscr{F})$ with the Čech-like complex

$$\begin{aligned} 0 \longrightarrow \prod_{i=0}^{r} M_{f_i} \longrightarrow \prod_{i<j} M_{f_if_j} \longrightarrow \prod_{i<j<k} M_{f_if_jf_k} \\ \longrightarrow \cdots \longrightarrow \prod_{i=0}^{r} M_{f_0\cdots\hat{f}_i\cdots f_r} \longrightarrow M_{f_0f_1\cdots f_r} \longrightarrow 0 \end{aligned}$$

where $M = \Gamma(X,\mathscr{F})$ and then using the qcqs lemma (Theorem 2.3.40). See [EGAIII$_1$, Proposition 1.2.3 and Corollaire 1.4.3].

2.5.6. Cohomological dimension. We end the first half of this course with some applications. We start with a definition.

[Har77, Exer. III.4.8(a)]

Definition 2.5.39 [Har68, p. 404]. Let X be a scheme. The *cohomological dimension of* X is

$$\operatorname{cd}(X) \coloneqq \inf\left\{ n \in \mathbf{Z} \;\middle|\; \begin{array}{c} H^i(X,\mathscr{F}) = 0 \text{ for all quasi-coherent} \\ \text{sheaves } \mathscr{F} \text{ and all } i > n \end{array} \right\}.$$

Serre's criterion for affineness (Theorem 2.3.35) says that $\operatorname{cd}(X) = 0$ if and only if X is affine. Grothendieck's vanishing Theorem 1.4.27 says that if $\operatorname{sp}(X)$ is Noetherian, then $\operatorname{cd}(X) \le \dim(X)$.

[Har77, Exer. III.4.8(c)]

Lemma 2.5.40 [Har68, p. 408]. *Let X be a scheme that has an open covering by $r+1$ affine open subsets U_i such that finite intersections of the U_i are also affine. Then,* $\operatorname{cd}(X) \le r$.

Proof. By Theorem 2.5.29, it suffices to note that the Čech complex $\check{\mathscr{C}}^\bullet(\mathfrak{U},\mathscr{F})$ is concentrated in degrees $[0,r]$. □

We can also give a geometric proof that $\operatorname{cd}(X) \le \dim(X)$ for locally closed subschemes of $\mathbf{P}^n_k$.

[Har77, Exer. III.4.8(d)]

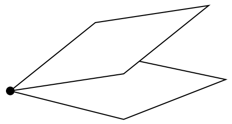
Proposition 2.5.41 [FAC, n° 62, Lemme 1]. *Let k be a field and let $X \subseteq \mathbf{P}^n_k$ be a locally closed subscheme. Then, X can be covered by* $\dim(X)+1$ *affine open subsets of the form $D_+(f) \cap X$. As a consequence,* $\operatorname{cd}(X) \le \dim(X)$.

Proof. We proceed by induction on $\dim(X)$. If $\dim(X) < 0$, there is nothing to show. Let $P_1, P_2, \ldots, P_s$ be the prime ideals corresponding to the irreducible components of $\bar{X}$. Let $J \subseteq S$ be the saturated ideal corresponding to $\bar{X} - X$. By homogeneous prime avoidance (Homework 2, Problem 6 in MA557), since $J \not\subseteq P_j$ for all j,ℓ, we have $J \not\subseteq P_j$, and hence there exists a homogeneous element $g \in J$ such that $g \notin \bigcup_j P_j$. By Krull's principal ideal theorem [Mur24ca, Theorem 8.12.1] applied to the domains S/P_j, we see that

$$\dim\bigl(X \cap V_+(g)\bigr) < \dim(X).$$

By inductive hypothesis, $X \cap V_+(g)$ can be covered by $\le \dim(X)$ open subsets of the form $D_+(f_i) \cap X \cap V_+(g)$ and moreover, $V_+(g)$ does not contain any irreducible component of $\bar{X} - X$. Finally, X can be covered by the open subsets $D_+(f_i) \cap X \cap V_+(g)$ and $D_+(g) \cap X$, the latter of which is an affine open subset since $V_+(g) \supseteq \bar{X} - X$. The inequality $\operatorname{cd}(X) \le \dim(X)$ follows from Lemma 2.5.40. □

We now recall the following:



$V\big((x_1,x_2)\cap(x_3,x_4)\big)$

FIGURE 2.6. The union of two planes in $\mathbf{A}^4_k$.

[Har77, Exer. I.2.17]

DEFINITION 2.5.42 [Har68, p. 408]. Let k be a field. We say that a closed subset $Y\subseteq \operatorname{sp}(\mathbf{P}^n_k)$, respectively $Y\subseteq \operatorname{sp}(\mathbf{A}^n_k)$, of dimension $n-r$ is a *set-theoretic complete intersection* in $\mathbf{P}^n_k$, respectively $\mathbf{A}^n_k$, if

$$Y=\operatorname{sp}\big(V_+(f_1,f_2,\ldots,f_r)\big)$$

for homogeneous polynomials $f_i\in k[x_0,x_1,\ldots,x_n]$, respectivel

$$Y=\operatorname{sp}\big(V(f_1,f_2,\ldots,f_r)\big)$$

for polynomials $f_i\in k[x_1,x_2,\ldots,x_n]$.

One of Hartshorne's motivations for introducing cohomological dimension is the following:

QUESTION 2.5.43 [Per41/43; Har68, p. 404; Har77, Exercise I.2.17(d)]. *Is every irreducible curve in $\mathbf{P}^3_k$ a set-theoretic complete intersection?*

The following result shows that cohomological dimension gives an obstruction for a closed subscheme to be a set-theoretic complete intersection.

COROLLARY 2.5.44 [Har68, p. 408]. *Let k be a field and let $X=\mathbf{P}^n_k$. Consider a closed subscheme $Y\subseteq\mathbf{P}^n_k$ of dimension $n-r$ such that $\operatorname{sp}(Y)$ is a set-theoretic complete intersection. Then, $\operatorname{cd}(X-Y)\le r-1$. This holds also if Y is a closed subscheme of $\mathbf{A}^n_k$.*

[Har77, Exer. III.4.8(e), III.4.9]

Proof. This follows from Lemma 2.5.40 since $X-Y$ is covered by r affine open subsets $D_+(f_1),\ldots,D_+(f_r)$. □

EXAMPLE 2.5.45 [Har62, Example 3.4.1; Har68, Example 3] (asked by Gunning). Let k be a field and consider the ideal

[Har77, Exer. III.4.9]

$$I=(x_1,x_2)\cap(x_3,x_4)\subseteq k[x_1,x_2,x_3,x_4].$$

Note that $V_+(I)\subseteq\mathbf{P}^3_k$ is the union of two skew lines and that $V(I)\subseteq\mathbf{A}^4_k$ is the union of two planes intersecting at a point (see Figure 2.6).

We claim that neither $V_+(I)$, $V(I)$, nor the projective completion $\overline{V(I)}\subseteq\mathbf{P}^4_k$ are set-theoretic complete intersections. It suffices to show the affine case. Set $X=\mathbf{A}^4_k$ and $Y=V(I)$. By Corollary 2.5.44, it suffices to show that $\operatorname{cd}(X-Y)\ge 2$. Set $U=X-Y$. We will show that $H^2(U,\mathcal{O}_U)\neq 0$.

By [Har77, Exercise III.2.3(e)] (Homework 2, Problem 3(e)), there is an exact sequence

$$\cdots\longrightarrow H^2(X,\mathcal{O}_X)\longrightarrow H^2(U,\mathcal{O}_U)\xrightarrow{\sim} H^3_Y(X,\mathcal{O}_X)\longrightarrow H^3(X,\mathcal{O}_X)\longrightarrow\cdots$$

where the map in the middle is an isomorphism by Cartan's Theorem B (Theorem 2.2.15). Applying the Mayer-Vietoris sequence [Har77, Exercise III.2.4] (Homework 2, Problem 4), to the decomposition $Y = Y_1 \cup Y_2$ where $Y_1 = V(x_1, x_2)$ and $Y_2 = V(x_3, x_4)$, the sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H^3_{Y_1 \cap Y_2}(X, \mathcal{O}_X) \arrow[r] & H^3_{Y_1}(X, \mathcal{O}_X) \oplus H^3_{Y_2}(X, \mathcal{O}_X) \arrow[r] & H^3_Y(X, \mathcal{O}_X) \arrow[dll, out=0, in=180] & \\
& H^4_{Y_1 \cap Y_2}(X, \mathcal{O}_X) \arrow[r] & H^4_{Y_1}(X, \mathcal{O}_X) \oplus H^3_{Y_2}(X, \mathcal{O}_X) \arrow[r] & H^4_Y(X, \mathcal{O}_X) \arrow[r] & \cdots
\end{tikzcd}$$

is exact. Since $H^3_{Y_i}(X, \mathcal{O}_X) \cong H^2(X - Y_i, \mathcal{O}_X) = 0$ for $i \in \{1, 2\}$ by Lemma 2.5.40 and similarly for $H^4_{Y_i}(X, \mathcal{O}_X)$, we have an isomorphism

$$H^3_Y(X, \mathcal{O}_X) \xrightarrow{\sim} H^4_{Y_1 \cap Y_2}(X, \mathcal{O}_X).$$

Now let $V = X - (Y_1 \cap Y_2)$. By [Har77, Exercise III.2.3(e)] (Homework 2, Problem 3(e)) again, we have

$$\cdots \longrightarrow H^3(X, \mathcal{O}_X) \longrightarrow H^3(V, \mathcal{O}_V) \longrightarrow H^4_{Y_1 \cap Y_2}(X, \mathcal{O}_X) \longrightarrow H^4(X, \mathcal{O}_X) \longrightarrow \cdots$$

and as before, $H^3(V, \mathcal{O}_V) \cong H^3_{Y_1 \cap Y_2}(X, \mathcal{O}_X)$. We compute $H^3(V, \mathcal{O}_V)$ using the Čech complex and Theorem 2.5.29. We have that

$$V = D(x_1) \cup D(x_2) \cup D(x_3) \cup D(x_4).$$

After taking global sections, the Čech complex around $p = 3$ looks like

$$\begin{tikzcd}
A[x_2^{-1}, x_3^{-1}, x_4^{-1}] \arrow[drrr, "1"] & & & & \\
\oplus & & & & \\
A[x_1^{-1}, x_3^{-1}, x_4^{-1}] \arrow[drrr, "-1"] & & & & \\
\oplus & & & A[x_1^{-1}, x_2^{-1}, x_3^{-1}, x_4^{-1}] \arrow[r] & 0. \\
A[x_1^{-1}, x_2^{-1}, x_4^{-1}] \arrow[urrr, "1"] & & & & \\
\oplus & & & & \\
A[x_1^{-1}, x_2^{-1}, x_3^{-1}] \arrow[uuurrr, "-1"] & & & &
\end{tikzcd}$$

Thus, we have

$$H^3(V, \mathcal{O}_V) \cong k \cdot \left\{ x_1^a x_2^b x_3^c x_4^d \,\middle|\, a, b, c, d < 0 \right\} \neq 0.$$

Finally, we have

$$H^2(U, \mathcal{O}_U) \cong H^3_Y(X, \mathcal{O}_X) \cong H^4_{Y_1 \cap Y_2}(X, \mathcal{O}_X) \cong H^3(V, \mathcal{O}_V) \neq 0,$$

and hence Y is not a set-theoretic complete intersection.

Its projective closure $\bar{Y}$ is then also not a set-theoretic complete intersection, for if it were, then $Y = \bar{Y} \cap D_+(x_0)$ would also be a set-theoretic complete intersection by taking the hypersurfaces defining Y and intersecting them with $D_+(x_0)$.

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2.5.7. Quasi-separated morphisms in terms of the diagonal. Recall the following definitions from Homework 4, Problem 3.

Definition 2.5.46 [EGAI$_{\text{new}}$, Proposition 6.1.11 and Proposition 6.1.12]. Let X be a scheme. We say that X is *quasi-separated* if, for every pair U, V of affine open subsets of X, the intersection $U \cap V$ is quasi-compact. We say that a morphism $f\colon X \to S$ of schemes is *quasi-separated* if, for every affine open subset $W \subseteq S$, the inverse image $f^{-1}(W)$ is quasi-separated.

The following result says that our definition for quasi-separated morphisms is the same as the definition in [EGAI$_{\text{new}}$, Définition 6.1.3].

Proposition 2.5.47 [EGAI$_{\text{new}}$, Proposition 6.1.11]. *Let $f\colon X \to S$ be a morphism. Then, f is quasi-separated if and only if $\Delta_{X/S}$ is quasi-compact.*

Proof. Let $W \subseteq S$ be an affine open subset. Setting $X_W := f^{-1}(W)$, we have that f is quasi-separated if and only if X_W is quasi-separated for every W. This holds if and only if, for every W and for every pair U, V of affine open subsets of X_W, the intersection $U \cap V$ is quasi-compact. By considering the open subschemes

$$U \times_W V \hookrightarrow X_W \times_W X_W$$

which have inverse images $U \cap V$ under $\Delta_{X_W/W}$ by Lemma 2.5.18, this holds if and only if

$$\Delta_{X_W/W}\colon X_W \longrightarrow X_W \times_W X_W$$

is quasi-compact for every W. By construction of fiber products (Theorem 2.5.3, especially Step 2 and Step 5), the $X_W \times_W X_W$ cover $X \times_S X$. Thus, the morphisms $\Delta_{X_W/W}$ are quasi-compact for every W if and only if $\Delta_{X/S}$ is quasi-compact. $\square$

2.5.8. The valuative criterion of separatedness. Theorem 2.5.49 below characterizes separatedness by another analogue of the Hausdorff condition: the uniqueness of limits [BouGT, Chapter I, §8, no. 1, Proposition 1] (see also [Mun00, Theorem 17.10 and Exercise 17.13]). For this statement, recall the following.

Lemma 2.5.48 [BouCA, Chapter VI, §1, no. 2, Theorem 1]. *Let R be a valuation* [AK21, Exer. 26.16] *ring and consider ideals $\mathfrak{a}, \mathfrak{b} \subseteq R$. Then, either $\mathfrak{a} \subseteq \mathfrak{b}$ or $\mathfrak{b} \subseteq \mathfrak{a}$. In particular, if $\dim(R) = d$, then the prime ideals in R are totally ordered:*

$$(0) = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_d.$$

Proof. Suppose $\mathfrak{a} \not\subseteq \mathfrak{b}$. Let $y \in \mathfrak{b}$. We want to show that $y \in \mathfrak{a}$. Choose $x \in \mathfrak{a} - \mathfrak{b}$. We must have $x/y \notin R$, for otherwise $x = (x/y)y \in \mathfrak{b}$. By the definition of a valuation ring, we therefore have $y/x \in R$, and hence $y = (y/x)x \in \mathfrak{a}$. $\square$

We now state the valuative criterion of separatedness. By the description of spectra of valuation rings above, the valuative criterion says that sequences "parametrized" by the spectrum of a valuation ring have at most one lift to X. Setting $Y = \operatorname{Spec}(\mathbf{Z})$ recovers the separatedness condition in Chevalley's definition of a scheme in [SCC55/56, Exposé 5, §3, Définition].

Theorem 2.5.49 (Valuative criterion of separatedness [EGAI$_{\text{new}}$, Proposition 5.5.4]). *Let $f\colon X \to Y$ be a morphism of schemes. The following conditions are* [Har77, Thm. II.4.3] *equivalent.*

(*a*) *f is separated.*

(*b*) *f is quasi-separated and for every valuation ring R with fraction field K, there exists* ***at most one*** *dashed morphism making the diagram*

$$\begin{tikzcd}
\operatorname{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec}(R) \arrow[r] \arrow[ur, dashed, "!"] & Y
\end{tikzcd}$$

commute.

Let us see an example of how to apply the valuative criterion.

[Vak, Fig. 13.7] Example 2.5.50 (The affine line with two origins II). Let k be a field. Consider the affine line with two origins X over k from Example 2.3.5. We give yet another proof that is is not affine. More strongly, we show that X is not even separated over k. Recall that affine schemes are separated over $\mathbf{Z}$ by Proposition 2.5.22. If

$$X \longrightarrow \operatorname{Spec}(k) \longrightarrow \operatorname{Spec}(\mathbf{Z})$$

were separated, then $X \to \operatorname{Spec}(k)$ would be separated by Proposition 2.5.25(*e*).

Let $R = k[x]_{(x)}$ be the DVR corresponding to the origin in $\mathbf{A}^1_k$, and let $K = \operatorname{Frac}(R)$. Then, there are two distinct morphisms fitting into the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec}(R) \arrow[r] \arrow[ur, dashed, shift left] \arrow[ur, dashed, shift right] & \operatorname{Spec}(k)
\end{tikzcd}$$

corresponding to the two origins in X.

The key ingredient for proving Theorem 2.5.49 is the following.

[Har77, Lem. II.4.4] Proposition 2.5.51 [EGAI$_{\text{new}}$, Proposition 5.5.2]. *Let Y be a scheme and let $f\colon X \to Y$ be a morphism. Let $x \in X$ be a point, let $y = f(x)$, and let y' be a specialization of y distinct from y. There exists a valuation ring R and a morphism $g\colon \operatorname{Spec}(R) \to Y$ such that $g(\mathfrak{m}) = y'$ and $g((0)) = y$, and such that there exists a commutative diagram*

$$\begin{tikzcd}
\kappa(x) \arrow[rr, "\sim"] & & \operatorname{Frac}(R) \\
& \kappa(y) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

of field extensions where the extension $\kappa(x) \to \operatorname{Frac}(R)$ is an isomorphism.

Proof. Consider $Y_1 = \overline{\{y\}}$ with the reduced closed subscheme structure (Proposition 2.3.34) and let $X_1 = f^{-1}(Y_1)$, which is a closed subscheme in X. Since $y' \in \overline{\{y\}}$ by assumption, we may replace Y by Y_1 and X by X_1 to assume that Y is integral with generic point y. Then, $\mathcal{O}_{Y,y'}$ is a local domain that is not a field with fraction field $\mathcal{O}_{Y,y} = \kappa(y)$, and $\kappa(x)$ is an extension of $\kappa(y)$. By [BouCA, Chapter VI, §1, no. 2, Corollary of Theorem 2] or [AK21, Proposition 26.6] (stated as [Mur24ag, Theorem 1.6.10] last seemster), we can find a valuation ring R with fraction field $\kappa(x)$ dominating $\mathcal{O}_{Y,y'}$. □

We also need the following result.

Proposition 2.5.52 ("Specializations lift along closed morphisms" or "Going Up for closed morphisms" [EGAI_{new}, Proposition 6.1.8; Stacks, Tag 0066]). *Let $f\colon X \to Y$ be a morphism of schemes. Suppose that f is closed, that is, the set-theoretic image of every closed subset in X is closed in Y. Then, for every $x \in X$ and every specialization y' of $y = f(x)$, distinct from y, there exists a specialization x' of x such that $f(x') = y'$. The converse holds if f is quasi-compact.*

[Har77, Lem. II.4.5]

In particular, a quasi-compact immersion $f\colon X \to Y$ is a closed immersion if and only if the set-theoretic image of X is stable under specialization.

Proof. We first show that if f is closed, then specializations lift along f. Consider the closed subset $T = \overline{\{x\}}$. Then, $f(T) \subseteq Y$ is a closed subset such that $y \in f(T)$. Thus, $y' \in f(T)$ also.

Conversely, suppose that f is quasi-compact and that specializations lift along f. Let $X' \subseteq X$ be a closed subset and let $Y' = \overline{f(X')}$. We want to show that $Y' = f(X')$. Consider X' and Y' with their reduced closed subscheme structures (Proposition 2.3.34). We then obtain the commutative diagram

$$\begin{tikzcd}
X' \arrow[r, hook] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, hook] & Y
\end{tikzcd}$$

by taking the fiber product and noting that $X' \subseteq f^{-1}(Y')$. Replacing f by f', we may assume that f is quasi-compact and dominant, in which case we need to show that $Y = f(X)$. Let $y' \in Y$ and let y be the generic point of an irreducible component containing Y. By the hypothesis, it suffices to show that $f^{-1}(y)$ is nonempty.

Let $U \ni y$ be an affine open subset. Since f is quasi-compact, the inverse image $f^{-1}(U)$ is quasi-compact, and hence is the finite union of affine open subsets $V_i \subseteq X$. Since f is dominant, $y \in f(V_i)$ for some i. Replacing X by V_i and Y by $\overline{f(V_i)} \cap U$ with the reduced closed subscheme structure (Proposition 2.3.34), we have now reduced to showing that if $A \hookrightarrow B$ is an extension of reduced rings, then there exists a prime ideal in B lying over every minimal $\mathfrak{p} \subseteq A$. This holds since we have

[EGAI_{new}, Prop. 6.1.7(i)]

$$0 \neq A_{\mathfrak{p}} \subseteq B_{\mathfrak{p}}$$

and hence any minimal prime of $B_{\mathfrak{p}}$ (which exists by [AK21, Exercise 3.16]) contracts to $\mathfrak{p}A_{\mathfrak{p}}$. See [BouCA, Chapter II, §2, no. 6, Proposition 16]. □

We can now prove Theorem 2.5.49.

Proof of Theorem 2.5.49. (a) ⇒ (b). Separated morphisms are quasi-separated by Proposition 2.5.47. For the second part of (b), we know that $\operatorname{Spec}(K) \to \operatorname{Spec}(R)$ is dominant. Since R is reduced, we see that $\operatorname{Spec}(K) \to \operatorname{Spec}(R)$ is scheme-theoretically dominant by [EGAI_{new}, Proposition 5.4.3] (Homework 5, Problem 5(b)). Thus, by Proposition 2.5.26[2], we know there can be at most one morphism $\operatorname{Spec}(R) \dashrightarrow X$ making the diagram in (b) commute.

(b) ⇒ (a). We want to show that the diagonal morphism

$$\Delta_{X/Y}\colon X \longrightarrow X \times_Y X$$

---

[2] When proving Proposition 2.5.26 in class, I assumed that U is an open dense subset of a reduced scheme X. The proof still works if $U \to X$ is scheme-theoretically dominant and X is not necessarily reduced. I have edited Proposition 2.5.26 accordingly.

is closed. By Proposition 2.5.47, we know that $\Delta_{X/Y}$ is quasi-compact. Thus, by Proposition 2.5.52, it suffices to show that $\Delta_{X/Y}(X)$ is stable under specialization. Let $z \in \Delta_{X/Y}(X)$ and let $z \rightsquigarrow z'$ be a specialization distinct from z. By Proposition 2.5.51, there exists a valuation ring R and a morphism

$$g\colon \operatorname{Spec}(R) \longrightarrow X \times_Y X$$

such that $g(\mathfrak{m}) = z'$ and $g((0)) = z$. Let $h_1, h_2\colon \operatorname{Spec}(R) \to X$ be the morphisms obtained by composing g with the two projections $p_1, p_2\colon X \times_Y X \to X$. By the hypothesis in ($b$) applied to

$$\begin{tikzcd}
\operatorname{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec}(R) \arrow[r] \arrow[ur, dashed, shift left, "h_1"] \arrow[ur, dashed, shift right, "h_2"'] & Y,
\end{tikzcd}$$

since f is separated, we have $h \coloneqq h_1 = h_2$. We can then consider the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}(R) \arrow[dr, "h"] \arrow[drr, bend left, "g"] & & & & \\
& X \arrow[dr, dashed, "\Delta_{X/Y}"] \arrow[drr, red, bend left] \arrow[ddr, red, bend right] & & & \\
& & X \times_Y X \arrow[r, blue] \arrow[d, blue] & X \arrow[d, red, equal] \arrow[ddr, blue, bend left] & \\
& & X \arrow[r, red, equal] \arrow[drr, blue, bend right] & X \arrow[dr, "f"] & \\
& & & & Y
\end{tikzcd}$$

where the red and blue squares are Cartesian. By the universal property of $X \times_Y X$, we see that $g = \Delta_{X/Y} \circ h$. Thus, we see that $z' \in \Delta_{X/Y}(X)$. □

2.5.9. The valuative criteria of universally closedness and properness. In the valuative criterion of separatedness (Theorem 2.5.49), one thing you may have noticed is that $\operatorname{Spec}(R) \to Y$ may not lift to X. In other words, using our intuition that valuation rings parametrize sequences in a scheme, we see that X is "missing" points.

To fix this, we will define two important classes of morphisms in algebraic geometry. These are two possible analogues of proper morphisms in topology, defined as follows.

Definition 2.5.53 [BouGT, Chapter I, §10, no. 1, Definition 1]. Let $f\colon X \to Y$ be a map of topological spaces. We say that f is *proper* if f is continuous and if the map

$$f \times \mathrm{id}_Z\colon X \times Z \longrightarrow Y \times Z$$

is closed for every topological space Z.

Taking $Z = \{*\}$, we see that every proper map is closed. While this may not be the definition of a proper map you are used to, proper maps can be characterized as follows. The condition (b) is often used as the definition of proper maps, and is called *perfect* in [Mun00, Exercise 26.12].

Theorem 2.5.54 [BouGT, Chapter I, §10, no. 2, Theorem 1 and Remark on pp. 103–104]. *Let $f\colon X \to Y$ be a continuous map. The following conditions are equivalent.*

(*a*) *f is proper.*
(*b*) *f is closed and $f^{-1}(y)$ is quasi-compact for every $y \in Y$.*
(*c*) *If $\mathfrak{F}$ is a filter on X and if $y \in Y$ is a cluster point of $f(\mathfrak{F})$, then there is a cluster point x of $\mathfrak{F}$ such that $f(x) = y$.*

Moreover, suppose that X and Y are Hausdorff. Then, these conditions are also equivalent to:

(*d*) *If $\mathfrak{F}$ is a filter on X such that $f(\mathfrak{F})$ has a (unique) cluster point, then $\mathfrak{F}$ has a (unique) cluster point.*

In algebraic geometry, we want to have an analogue of proper maps. The naïve analogue of Definition 2.5.53 will be called *universally closed*. The terminology *proper* will be reserved for morphisms that are closer to condition (*d*) above.

Definition 2.5.55 [EGAI$_{\text{new}}$, Définition 3.8.1]. Let $f\colon X \to Y$ be a morphism of schemes. We say that f is *universally closed* if the base change

$$f \times \mathrm{id}_{Y'}\colon X \times_Y Y' \longrightarrow Y'$$

is closed for every morphism $Y' \to Y$.

Definition 2.5.56 [EGAII, Définition 5.4.1]. Let $f\colon X \to Y$ be a morphism of schemes. We say that f is *proper* if it is separated, of finite type, and universally closed. In this case, we also say that X is *proper over Y*.

You can think of properness over $\mathrm{Spec}(\mathbf{Z})$ or $\mathrm{Spec}(k)$ as the analogue of compactness for topological spaces. To illustrate this, here is an example.

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Example 2.5.57 [Har77, Example II.4.6.1]. Let k be a field and let $X = \mathbf{A}^1_k$ be the affine line over k. We claim that X is not proper over k. Consider the base change

$$\mathbf{A}^2_k = X \times_k X \longrightarrow X$$

of $X \to \mathrm{Spec}(k)$. Then, the hyperbola $\{xy = 1\}$ is closed in $\mathbf{A}^2_k$, but its image in X is $X - \{0\}$, which is not closed.

In terms of valuation rings, what is going on is that we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}\big(k(y)\big) \arrow[r] \arrow[d] & \{xy=1\} \arrow[r, "p_1"] & X \arrow[d] \\
\mathrm{Spec}\big(k[y]_{(y)}\big) \arrow[rr] \arrow[rru, dashed] & & \mathrm{Spec}(k)
\end{tikzcd}$$

corresponding to the commutative diagram

$$\begin{tikzcd}
k(y) & k[x] \arrow[l] \arrow[dl, dashed, "\nexists"'] \\
k[y]_{(y)} \arrow[u] & k \arrow[l] \arrow[u]
\end{tikzcd}$$

of k-algebras, and there is no dashed map making the diagram commute.

The valuative criterion of universal closedness and the valuative criterion of properness correct for this.

[Har77, Thm. II.4.7]

Theorem 2.5.58 (Valuative criterion of universal closedness [EGAI$_{\text{new}}$, Propositions 5.5.7 and 5.5.8]). *Let $f\colon X \to Y$ be a quasi-compact morphism of schemes. The following conditions are equivalent.*

(*a*) *f is universally closed.*

(*b*) *For every valuation ring R with fraction field K, **there exists** a dashed morphism making the diagram*

$$\begin{tikzcd}
\operatorname{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec}(R) \arrow[r] \arrow[ur, dashed, "\exists"] & Y
\end{tikzcd}$$

commute.

Proof. (*a*) ⇒ (*b*). Consider the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}(K) \arrow[r] & X_R \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
& \operatorname{Spec}(R) \arrow[r] & Y
\end{tikzcd}$$

where the square is Cartesian. Let $x \in X_R$ be the image of the unique point in $\operatorname{Spec}(K)$. Since f' is closed, the specialization $(0) \rightsquigarrow \mathfrak{m}$ in $\operatorname{Spec}(R)$ lifts to a specialization $x \rightsquigarrow x'$ in X_R by Proposition 2.5.52. The diagram of specializations

$$\begin{tikzcd}
x \arrow[r, rightsquigarrow] \arrow[d, mapsto] & x' \arrow[d, mapsto] \\
(0) \arrow[r, rightsquigarrow] & \mathfrak{m}
\end{tikzcd}$$

corresponds to the commutative diagram

$$\begin{tikzcd}
K & \mathcal{O}_{X_R,x} \arrow[l] & \mathcal{O}_{X_R,x'} \arrow[l] \\
& K \arrow[u] & R \arrow[l] \arrow[u]
\end{tikzcd}$$

where the vertical maps are local and the horizontal maps are localization maps. Since the map $R \to \mathcal{O}_{X_R,x'}$ is local, the image of $\mathcal{O}_{X_R,x'}$ in K dominates R. By [Mur24ag, Theorem 1.6.10], we know that valuation rings are maximal with respect to domination, and hence $R \to \mathcal{O}_{X_R,x'}$ is an isomorphism. Thus, there is a section $\operatorname{Spec}(R) \to X_R$ of f', which induces a morphism $\operatorname{Spec}(R) \to X$ by composition.

(*b*) ⇒ (*a*). We first note that the criterion in (*b*) holds for every base change of f along $Y' \to Y$, since in the diagram

$$\begin{tikzcd}
\operatorname{Spec}(K) \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
\operatorname{Spec}(R) \arrow[r] \arrow[ur, dashed] \arrow[urr, dashed] & Y' \arrow[r] & Y
\end{tikzcd}$$

where the right square is Cartesian, if the morphism $\operatorname{Spec}(R) \to X$ exists, then the morphism $\operatorname{Spec}(R) \to X'$ exists by the universal property of X'. By Proposition 2.5.52, it suffices to show that specializations lift along f'. Let $y \rightsquigarrow y'$ be a specialization where y' is distinct from y. Let $x \in X'$ such that $f'(x) = y$. By Proposition 2.5.51, there exists a valuation ring R and a morphism

$$g\colon \operatorname{Spec}(R) \longrightarrow Y'$$

such that $g((0)) = y$ and $g(\mathfrak{m}) = y'$ such that

$$K = \operatorname{Frac}(R) = \kappa(x).$$

We can now apply (*b*) to obtain the morphism $\operatorname{Spec}(R) \to X'$. □

Remark 2.5.59. It turns out that universally closed morphisms are quasi-compact. This is a fairly recent result due to Bjorn Poonen [Poo10; Stacks, Tag 04XU]. As a consequence, the statement of Theorem 2.5.58 can be modified to include quasi-compactness in (*b*) instead. To summarize, we have analogous valuative criteria for separatedness and universal closedness:

$$\text{separated} \Longleftrightarrow \text{quasi-separated} + \begin{array}{c}\text{uniqueness part of}\\ \text{the valuative criterion,}\end{array}$$

$$\text{universally closed} \Longleftrightarrow \text{quasi-compact} + \begin{array}{c}\text{existence part of}\\ \text{the valuative criterion.}\end{array}$$

Finally, we can state the valuative criterion of properness. I have seen this result attributed to Serre, and indeed, it seems that Serre told Grothendieck about this result in a letter [GS01, p. 103]. The valuative criterion connects the definition of properness in [EGAII] with the definition in Chevalley's theory of schemes [SCC55/56, Exposé 5, §4] mentioned in [Har77, Exercise II.4.5(*b*)].

Theorem 2.5.60 (Valuative criterion of properness [EGAII, Théorème 7.3.8]). *Let $f\colon X \to Y$ be a finite type, quasi-separated morphism of schemes. The following conditions are equivalent.* [Har77, Thm. II.4.7]

(*a*) *f is proper.*

(*b*) *For every valuation ring R with fraction field K,* ***there exists a unique*** *dashed morphism making the diagram*

$$\begin{tikzcd}
\operatorname{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec}(R) \arrow[r] \arrow[ur, dashed, "\exists!"] & Y
\end{tikzcd}$$

commute.

Proof. Combine Theorems 2.5.49 and 2.5.58. □

2.6. Projective geometry via schemes

Now that we understand the Proj construction relatively well and have developed some machinery relating to separated and proper morphisms, we want to apply our understanding of Proj and $\mathbf{P}^n_A$ to study other schemes. In other words, we want to construct morphisms to $\operatorname{Proj}(S)$, with the eventual goal of using the geometry of $\mathbf{P}^n_A$ to study more general schemes. For example, as you saw on Homework 6, $\operatorname{Proj}(S)$ comes with a specific invertible sheaf $\mathcal{O}(1)$ that encodes a lot of information about the geometry of $\operatorname{Proj}(S)$, for example whether it can be embedded in $\mathbf{P}^r_A$ for a ring

A. This means that we somehow want to find an invertible sheaf on a given scheme X that looks like it could be the pullback of the $\mathcal{O}(1)$ on the Proj of some ring.

In this section, we will develop the necessary language to make sense of schemes and morphisms to $\operatorname{Proj}(S)$. Using this, we will be able to say a lot about a large class of schemes of geometric interest. In particular, we will be able to apply what we have proved about separated and proper morphisms to examples of interest, namely, projective and quasi-projective morphisms.

2.6.1. Relative Proj. To work with and construct interesting examples of projective morphisms, we will need a relative version of the Proj construction, which we call relative **Proj**. We start with the sheaf-theoretic analogue of an $\mathbf{N}$-graded ring.

[Har77, p. 160]

Definition 2.6.1 [EGAII, (3.1.1)]. Let Y be a scheme. An $\mathbf{N}$*-graded sheaf of* $\mathcal{O}_Y$*-algebras* is a sheaf $\mathscr{S}$ of $\mathcal{O}_Y$-algebras equipped with a decomposition

$$\mathscr{S} = \bigoplus_{n \in \mathbf{N}} \mathscr{S}_n$$

as a direct sum of sheaves of $\mathcal{O}_Y$-modules such that

$$\mathscr{S}_n \cdot \mathscr{S}_m \subseteq \mathscr{S}_{n+m}$$

for all $n, m \in \mathbf{N}$.

To construct relative **Proj** using graded sheaves of algebras, we first prove the following result on the behavior of Proj under base change.

Proposition 2.6.2 [EGAII, (2.8.9) and Proposition 2.8.10]. *Let $A \to A'$ be a ring map and let S be an $\mathbf{N}$-graded A-algebra. The tensor product $S' \coloneqq S \otimes_A A'$ is an $\mathbf{N}$-graded A'-algebra with graded components $S'_n \coloneqq S_n \otimes_A A'$. The co-Cartesian square*

$$\begin{tikzcd}
S' & S \arrow[l, "\varphi"'] \\
A' \arrow[u] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

of A-algebras induces the Cartesian square

$$\begin{tikzcd}
\operatorname{Proj}(S') \arrow[r, "\Phi"] \arrow[d] & \operatorname{Proj}(S) \arrow[d] \\
\operatorname{Spec}(A') \arrow[r] & \operatorname{Spec}(A)
\end{tikzcd}$$

of schemes such that for every $\mathbf{Z}$-graded S-module M with extension of scalars $M' \coloneqq M \otimes_S S' \cong M \otimes_A A'$, the morphism

$$\nu \colon \Phi^*(\tilde{M}') \longrightarrow \tilde{M}$$

from Proposition 2.4.14(ii) is an isomorphism.

Proof. Note that $\varphi(S_+) \subseteq S'_+$, and hence the morphism Φ is defined on all of $\operatorname{Proj}(S')$ by [Har77, Exercise II.2.14] (Homework 5, Problem 2). Working locally on distinguished open subsets of $\mathrm{Proj}(S)$, it therefore suffices to show that for every $f \in S_+$, the square

$$\begin{tikzcd}
\mathrm{Spec}\big(S'_{(\varphi(f))}\big) \arrow[r] \arrow[d] & \mathrm{Spec}\big(S_{(f)}\big) \arrow[d] \\
\mathrm{Spec}(A') \arrow[r] & \mathrm{Spec}(A)
\end{tikzcd}$$

is Cartesian. This is true since the isomorphism

$$S'_{\varphi(f)} \xrightarrow{\sim} S_f \otimes_A A'$$

is graded. For the statement about ν, it suffices to note that the isomorphism

$$M'_{\varphi(f)} \xrightarrow{\sim} M_f \otimes_A A'$$

is graded. □

We can now construct relative **Proj**.

Proposition 2.6.3 [EGAII, Proposition 3.1.2]. *Let Y be a scheme and let $\mathscr{S}$ be an $\mathbf{N}$-graded quasi-coherent sheaf of $\mathcal{O}_Y$-algebras. There exists a scheme morphism $f\colon X \to Y$, unique up to Y-isomorphism, satisfying the following property: For every affine open subset $U \subseteq Y$, there exists a U-isomorphism* [Har77, p. 160]

$$\eta_U\colon f^{-1}(U) \xrightarrow{\sim} X_U \coloneqq \mathrm{Proj}\big(\Gamma(U,\mathscr{S})\big)$$

such that for every inclusion $V \subseteq U$ of affine open subsets of Y, the diagram

$$\begin{tikzcd}
f^{-1}(V) \arrow[r, "\eta_V", "\sim"'] \arrow[d, hook] & X_V \arrow[d, hook, "(\rho^U_V)^*"] \\
f^{-1}(U) \arrow[r, "\eta_U", "\sim"'] & X_U
\end{tikzcd} \tag{2.6.4}$$

commutes.

Proof. For every affine open subset $U \subseteq Y$, the $\mathbf{N}$-graded $\mathcal{O}_Y|_U$-algebra $\mathscr{S}|_U$ is isomorphic to $\tilde{S}$ where $S = \Gamma(U,\mathscr{S})$ is an $\mathbf{N}$-graded $\Gamma(U,\mathcal{O}_Y)$-algebra. Consider the morphism

$$f_U\colon X_U \coloneqq \mathrm{Proj}\big(\Gamma(U,\mathscr{S})\big) \longrightarrow U.$$

If $j\colon U' \hookrightarrow U$ is the open inclusion of a second affine open subset, we have a U'-isomorphism

$$\sigma_{U',U}\colon f_U^{-1}(U') \xrightarrow{\sim} X_{U'}$$

by Proposition 2.6.2.

We need to show that the f_U glue together. For each pair U, V of affine open subsets of Y, let

$$X_{U,V} \coloneqq f_U^{-1}(U \cap V) \subseteq X_U.$$

We define a Y-isomorphism

$$\theta_{U,V}\colon X_{V,U} \xrightarrow{\sim} X_{U,V}$$

as follows. Let $W' \subseteq W$ be an inclusion of affine open subsets in $U \cap V$. We then have the commutative diagram

$$\begin{tikzcd}
\tau_{W'}\colon f_U^{-1}(W') \arrow[r, "\sigma_{W',U}", "\sim"'] \arrow[d, hook] & X_{W'} \arrow[r, "\sigma^{-1}_{W',V}", "\sim"'] \arrow[d, hook] & f_V^{-1}(W') \arrow[d, hook] \\
\tau_{W}\colon f_U^{-1}(W) \arrow[r, "\sigma_{W,U}", "\sim"'] & X_{W} \arrow[r, "\sigma^{-1}_{W,V}", "\sim"'] & f_V^{-1}(W)
\end{tikzcd}$$

with Cartesian squares by Proposition 2.6.2. The τ_W therefore define a Y-isomorphism $\theta_{V,U}$.

To show that the $\theta_{U,V}$ yield a gluing $f\colon X \to Y$ of the morphisms $f_U\colon X_U \to U$, it suffices by [Mur24ag, Lemma 2.2.22] to show that for every triple U, V, W of affine open subsets in Y, letting $\theta'_{U,V}$, $\theta'_{V,W}$, and $\theta'_{U,W}$ be the restrictions of $\theta_{U,V}$, $\theta_{V,W}$, and $\theta_{U,W}$ to the inverse images of $U \cap V \cap W$ in X_V, X_W, and X_W, respectively, we have

$$\theta'_{U,V} \circ \theta'_{V,W} = \theta'_{U,W}.$$

This holds by definition of the isomorphisms σ. Uniqueness up to Y-isomorphism follows by concatenating the two instances of the commutative diagram (2.6.4) for two candidates X_1, X_2 for f. $\square$

[Har77, p. 160]

Definition 2.6.5 [EGAII, (3.1.3)]. Let Y be a scheme and let $\mathscr{S}$ be an $\mathbf{N}$-graded quasi-coherent sheaf of $\mathcal{O}_Y$-algebras. The *relative* **Proj**

$$\mathbf{Proj}_Y(\mathscr{S}) \longrightarrow Y$$

of $\mathscr{S}$ is the morphism $f\colon X \to Y$ constructed in Proposition 2.6.3. If $\mathscr{M}$ is a $\mathbf{Z}$-graded quasi-coherent sheaf of $\mathscr{S}$-modules, then we denote by $\tilde{\mathscr{M}}$ the quasi-coherent sheaf obtained by gluing the sheaves $\Gamma(U, \mathscr{M})^\sim$ on each affine open subset $U \subseteq X$ using [Har77, Exercise II.1.22] (Homework 10, Problem 5) and Proposition 2.6.2. By Proposition 2.6.2, we can define Serre twisting sheaves $\mathcal{O}(n)$ for every $n \in \mathbf{Z}$, which are invertible if $\mathscr{S}$ is generated by $\mathscr{S}_1$ as an $\mathscr{S}_0$-algebra.

$\boxed{4/1}$
[Har77, Ex. II.7.8.7]

Example 2.6.6 [EGAII, Définition 4.1.1]. Let Y be a scheme and let

$$\mathscr{S} = \mathcal{O}_Y[T_0, T_1, \ldots, T_n] \coloneqq \operatorname{Sym}^\bullet_{\mathcal{O}_Y}(\mathcal{O}_Y^{n+1}).$$

Then, $\mathbf{P}^n_Y \coloneqq \mathbf{Proj}_Y(\mathscr{S})$ is the *relative projective n-space* over Y. More generally, let $\mathscr{E}$ be a quasi-coherent $\mathcal{O}_Y$-module, and consider the symmetric algebra

$$\operatorname{Sym}^\bullet_{\mathcal{O}_Y}(\mathscr{E}),$$

which is quasi-coherent. The *projective bundle* associated to $\mathscr{E}$ over Y is

$$\mathbf{P}_Y(\mathscr{E}) \coloneqq \mathbf{Proj}_Y\bigl(\operatorname{Sym}^\bullet_{\mathcal{O}_Y}(\mathscr{E})\bigr).$$

Caution 2.6.7. Some authors (see, e.g., [Ful98, §B.5.5]) use the convention

$$P_Y(E) \coloneqq \mathbf{P}_Y(\mathscr{E}^\vee) = \mathbf{Proj}_Y\bigl(\operatorname{Sym}^\bullet_{\mathcal{O}_Y}(\mathscr{E}^\vee)\bigr).$$

This is especially common when $\mathscr{E}$ is locally free and one thinks of $\mathscr{E}$ as a geometric vector bundle E.

Before defining rational maps to **Proj**, we want to understand how twisting a graded algebra (or in the projective bundle case, $\mathscr{E}$) by an invertible sheaf $\mathscr{L}$ affects the **Proj**. In short, while you get isomorphic schemes over Y, the corresponding Serre twisting sheaf $\mathcal{O}(1)$ gets twisted by $\mathscr{L}$.

[Har77, Lem. II.7.9, Prop. II.7.12]

Proposition 2.6.8 [EGAII, Propositions 3.1.8(*iii*) and 3.2.10]. *Let Y be a scheme and let $\mathscr{S}$ be an $\mathbf{N}$-graded quasi-coherent sheaf of $\mathcal{O}_Y$-algebras. Let $\mathscr{L}$ be an invertible $\mathcal{O}_Y$-module and let*

$$\mathscr{S}_{(\mathscr{L})} \coloneqq \bigoplus_{n\geq 0} \bigl(\mathscr{S}_n \otimes_{\mathcal{O}_Y} \mathscr{L}^{\otimes n}\bigr).$$

Then, there is an isomorphism

$$\begin{tikzcd}
\mathbf{Proj}_Y(\mathscr{S}_{(\mathscr{L})}) \arrow[rr, "g", "\sim"'] \arrow[dr, "f_{(\mathscr{L})}"'] & & \mathbf{Proj}_Y(\mathscr{S}) \arrow[dl, "f"] \\
& Y &
\end{tikzcd}$$

such that $g_*(\mathcal{O}(n)) \cong \mathcal{O}(n) \otimes f^*\mathscr{L}^{\otimes n}$.

Proof. Let $\operatorname{Spec}(A)$ be an affine open subset of Y such that $\mathscr{L}|_{\operatorname{Spec}(A)} = \tilde{L}$ for a free rank one module L with generator c. We then have the isomorphism

$$\begin{aligned} S_n &\xrightarrow{\sim} S_n \otimes_A L^{\otimes n} \\ x_n &\longmapsto x_n \otimes c^{\otimes n} \end{aligned}$$

of A-module isomorphisms, which give the isomorphism

$$\varphi_c \colon S \longrightarrow S_{(L)} \coloneqq \bigoplus_{n\geq 0} S_n \otimes_A L^{\otimes n}.$$

Let $f \in S_d$ be homogeneous of degree $d > 0$. For every $x \in S_{nd}$, we have

$$\frac{x \otimes c^{nd}}{(f \otimes c^d)^n} = \frac{x \otimes (\varepsilon c)^{nd}}{(f \otimes (\varepsilon c)^d)^n}$$

for every invertible element $\varepsilon \in A$. This shows that the isomorphisms

$$\begin{aligned} (\varphi_c)_{(f)} \colon \bigl(S(e)\bigr)_{(f)} &\xrightarrow{\sim} \bigl(S_{(L)}(e)\bigr)_{(f\otimes c^d)} \\ \frac{x}{f^k} \otimes c^e &\longmapsto \frac{x \otimes c^{e+nd}}{(f \otimes c^d)^n} \end{aligned}$$

are independent of the generator c of L chosen. Moreover, these isomorphisms are compatible with restriction $D_+(f) \to D_+(fg)$ (by definition) and with restriction on Y (by Proposition 2.6.2). ☐

2.6.2. Rational maps and rational maps to relative Proj. Instead of keeping track of the open set U where a morphism on Proj is defined, we define the following terminology.

Definition 2.6.9 [EGAI$_{\text{new}}$, (8.1.1) and Définition 8.1.2]. Let X, Y be two schemes, let U, V be dense open subsets of X, and let $f\colon U \to Y$ and $g\colon V \to U$ be two morphisms. We say that f and g are *equivalent* if they coincide on a dense open subset of $U \cap V$. Since finite intersections of dense open subsets of X are dense open subsets of X, this notion defines an equivalence relation. [Har77, p. 24]

A *rational map*

$$X \dashrightarrow Y$$

is an equivalence class of morphisms from dense open subsets of X to Y. If X and Y are S-schemes, then a *rational map over S* is a rational map that has a representative that is an S-morphism. We say that a rational map *is a morphism* if it can be represented by a morphism defined on the entire domain.

We explain how to define rational maps to relative **Proj**.

[Har77, Ex. II.7.1.2, Prop. II.7.12]
[Ful98, p. 83]

Definition 2.6.10 [EGAII, (3.5.6) and (3.7.1)]. Let $q\colon X \to Y$ be a morphism of schemes. Let $\mathscr{L}$ be an invertible $\mathcal{O}_X$-module and let $\mathscr{S}$ be an $\mathbf{N}$-graded quasi-coherent $\mathcal{O}_Y$-algebra. Then, $q^*\mathscr{S}$ is an $\mathbf{N}$-graded quasi-coherent $\mathcal{O}_X$-algebra.

Consider the $\mathbf{N}$-graded quasi-coherent $\mathcal{O}_X$-algebra

$$\mathscr{S}' = \bigoplus_{n \geq 0} \mathscr{L}^{\otimes n}$$

and suppose we are given a graded $\mathcal{O}_X$-algebra homomorphism

$$\psi\colon q^*\mathscr{S} \longrightarrow \mathscr{S}',$$

which induces a rational map

$$\mathbf{Proj}_X(\psi)\colon X \cong \mathbf{Proj}_X(\mathscr{S}') \dashrightarrow \mathbf{Proj}_X(q^*\mathscr{S})$$

by the functoriality of $\mathbf{Proj}_X$ [Har77, Exercise II.2.14] (Homework 5, Problem 2). By applying adjunction [Har77, p. 110] (Homework 1, Problem 5) to ψ, we also have the morphism $\psi^\flat$ of $\mathbf{N}$-graded quasi-coherent $\mathcal{O}_Y$-algebras below which factors as

$$\begin{tikzcd}
\mathscr{S} \arrow[rr, "\psi^\flat"] \arrow[dr, "\eta"'] & & q_*\mathscr{S}' \\
& q_*q^*\mathscr{S} \arrow[ur, "q_*\psi"'] &
\end{tikzcd}$$

where η is the unit of the adjunction $q^* \dashv q_*$.

We have the commutative diagram

$$\begin{tikzcd}
& \mathbf{Proj}_X(q^*\mathscr{S}) \arrow[dr, "\pi"] & \\
X \arrow[ur, dashed, "\mathbf{Proj}_X(\psi)"] \arrow[rr, dashed, "r_{\mathscr{L},\psi}"] & & \mathbf{Proj}_Y(\mathscr{S})
\end{tikzcd}$$

where π is the first projection

$$\mathbf{Proj}_X(q^*\mathscr{S}) \cong \mathbf{Proj}_Y(\mathscr{S}) \times_Y X \longrightarrow \mathbf{Proj}_Y(\mathscr{S}).$$

The isomorphism holds by Proposition 2.6.2 (see [EGAII, Proposition 3.5.3]). Note that by the way π and $\mathbf{Proj}_X(\psi)$ are defined, the pullback of $\mathcal{O}(1)$ on $\mathbf{Proj}_Y(\mathscr{S})$ is $\mathscr{L}$.

[Har77, p. 157]

Example 2.6.11 (Linear systems). Let k be a field and let X be a k-scheme with structure morphism $q\colon X \to \operatorname{Spec}(k)$. Let $\mathscr{L}$ be an invertible $\mathcal{O}_X$-module. Suppose we are given a k-vector subspace

$$V \subseteq H^0(X, \mathscr{L}),$$

which is an example of a *linear system*. The map

$$\psi\colon q^* \operatorname{Sym}_k^\bullet(V) \cong \operatorname{Sym}_{\mathcal{O}_X}^\bullet(q^*V) \longrightarrow \operatorname{Sym}_{\mathcal{O}_X}^\bullet(\mathscr{L})$$

obtained from evaluation induces the map

$$X \overset{|V|}{\dashrightarrow} \mathbf{P}_k(V).$$

For some concrete examples, the subspace

$$k \cdot \{s^3, s^2t, st^2, s^3\} \subseteq H^0\bigl(\mathbf{P}^1_k, \mathscr{O}_{\mathbf{P}^1_k}(3)\bigr)$$

defined the embedding of the twisted cubic in $\mathbf{P}^3_k$. More generally, the Veronese embedding ν_d defined in this way is such that the pullback of $\mathscr{O}(1)$ is $\mathscr{O}(d)$.

The subspace

$$k \cdot \{s^4, s^3t, st^3, s^4\} \subseteq H^0\bigl(\mathbf{P}^1_k, \mathscr{O}_{\mathbf{P}^1_k}(4)\bigr)$$

has image equal to the *Macaulay curve* [Mac94, p. 98; Mur24ca, Example 8.16.8(*ii*)] in $\mathbf{P}^3_k$. The subspace

$$k \cdot \{xy, xz, yz\} \subseteq H^0\bigl(\mathbf{P}^2_k, \mathscr{O}_{\mathbf{P}^2_k}(2)\bigr)$$

defines the *Cremona transformation* [Mur24ag, Example 1.4.29].

We want to describe when $r_{\mathscr{L},\psi}$ is a morphism. For example, the twisted cubic and the Macaulay curve are defined by morphisms, but the Cremona transformation is not a morphism. First, unraveling the definitions when constructing rational maps associated to graded ring maps, we have:

[Har77, Proof of Thm. II.7.1]

PROPOSITION 2.6.12 [EGAII, Proposition 3.7.3]. *With notation as in Definition 2.6.10, suppose that $Y = \operatorname{Spec}(A)$ is affine and write $\mathscr{S} = \tilde{S}$ for an $\mathbf{N}$-graded A-algebra S. For every $f \in S_d = H^0(Y, \mathscr{S}_d)$, we have*

$$r^{-1}_{\mathscr{L},\psi}\bigl(D_+(f)\bigr) = X_{\psi^\flat(f)}$$

where we consider $\psi^\flat(f)$ as a section in $H^0(X, \mathscr{L}^{\otimes d})$, and the restriction

$$r_{\mathscr{L},\psi}|_{X_{\psi^\flat(f)}} \colon X_{\psi^\flat(f)} \longrightarrow D_+(f) = \operatorname{Spec}\bigl(S_{(f)}\bigr)$$

corresponds to the A-algebra map

$$\begin{aligned} \psi^\flat_{(f)} \colon S_{(f)} &\longrightarrow \Gamma\bigl(X_{\psi^\flat(f)}, \mathscr{O}_X\bigr) \\ \frac{s}{f^n} &\longmapsto \bigl(\psi^\flat(s)|_{X_{\psi^\flat(f)}}\bigr)\bigl(\psi^\flat(f)|_{X_{\psi^\flat(f)}}\bigr)^{-n}. \end{aligned}$$

REMARK 2.6.13. Using this description, we see that the examples in Example 2.6.11 are the same as the rational maps we defined using coordinates in [Har77, Chapter I; Mur24ag].

We therefore obtain the following characterization for when $r_{\mathscr{L},\psi}$ is a morphism. The condition on the existence of s holds in particular if ψ is (TN)-surjective.

[Har77, Thm. II.7.1]

COROLLARY 2.6.14 [EGAII, Corollaire 3.7.4]. *With notation as in Definition 2.6.10, suppose that $Y = \operatorname{Spec}(A)$ is affine and $\mathscr{S} = \tilde{S}$ for an $\mathbf{N}$-graded A-algebra S. Then, $r_{\mathscr{L},\psi}$ is a morphism if and only if for every $x \in X$, there exist an integer $n > 0$ and a section*

$$s \in H^0(Y, \mathscr{S}_n) = S_n$$

such that setting

$$t = \psi^\flat(s) \in H^0(X, \mathscr{L}^{\otimes n}),$$

we have $t(x) \neq 0$.

The previous corollary suggests the following definition, which is obtained by considering the graded ring map

$$f^* \operatorname{Sym}^{\bullet}_{\mathcal{O}_Y}\bigl(f_*\mathscr{L}\bigr) \longrightarrow \operatorname{Sym}^{\bullet}_{\mathcal{O}_X}(\mathscr{L}) \tag{2.6.15}$$

in degree 1.

[Har77, p. 121]
The terminology "f-generated" is from [KMM87, Def. 0-1-4].
The terminology "f-free" is from [CT20, §2.1.1].

Definition 2.6.16 [KMM87, Definition 0-1-4]. Let $f\colon X \to Y$ be a morphism of schemes and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. We say that $\mathscr{F}$ is *f-generated* or *f-free* if the counit morphism

$$f^* f_* \mathscr{F} \longrightarrow \mathscr{F}$$

for the adjunction $f^* \dashv f_*$ is surjective. We say that $\mathscr{F}$ is *globally generated* or *generated by global sections* if $\mathscr{F}$ is f-generated relative to the unique morphism $f\colon X \to \operatorname{Spec}(\mathbf{Z})$.

Since $f\colon X \to \operatorname{Spec}(\mathbf{Z})$ factors as

$$X \xrightarrow{g} \operatorname{Spec}\bigl(\Gamma(X,\mathcal{O}_X)\bigr) \xrightarrow{\rho} \operatorname{Spec}(\mathbf{Z})$$

by [Mur24ag, Proposition 2.2.17] and the adjunction morphism for f factors as

$$f^*f_*\mathscr{F} = g^*\rho^*\rho_*g_*\mathscr{F} \twoheadrightarrow g^*g_*\mathscr{F} \longrightarrow \mathscr{F}$$

since restriction of scalars ρ_* is exact for the affine morphism ρ, it is equivalent to check global generation using the morphism $X \to \operatorname{Spec}(\mathbf{Z})$ or $X \to \operatorname{Spec}(\Gamma(X,\mathcal{O}_X))$. As we mentioned in Definition 2.1.3, $\mathscr{F}$ is globally generated if and only if there is a family of global sections $(s_i)_{i\in I} \subseteq \Gamma(X,\mathscr{F})$ such that the morphism

$$\mathcal{O}_X^{(I)} \longrightarrow \mathscr{F}$$

determined by the s_i is surjective.

Now suppose that $\mathscr{L}$ is an invertible $\mathcal{O}_X$-module. We say that $\mathscr{L}$ is *f-semiample* if $\mathscr{L}^{\otimes m}$ is f-generated for some $m > 0$. By Corollary 2.6.14, $\mathscr{L}$ is f-generated if and only if

$$\begin{tikzcd} X \arrow[r, dashed, "|\mathscr{L}|"] & \mathbf{P}_Y\bigl(f_*\mathscr{L}\bigr) \end{tikzcd}$$

is a morphism, which holds if and only if (2.6.15) is surjective. Moreover, $\mathscr{L}$ is f-semiample if and only if

$$\begin{tikzcd} X \arrow[r, dashed, "|\mathscr{L}^{\otimes m}|"] & \mathbf{P}_Y\bigl(f_*\mathscr{L}^{\otimes m}\bigr) \end{tikzcd}$$

is a morphism for some $m > 0$.

We also obtain the following characterization for when $r_{\mathscr{L},\psi}$ is an immersion.

[Har77, Prop. II.7.2]

Proposition 2.6.17 [EGAII, Proposition 3.8.2]. *With notation as in Definition 2.6.10, suppose that $Y = \operatorname{Spec}(A)$ is affine and $\mathscr{S} = \tilde{S}$ for an $\mathbf{N}$-graded A-algebra S. Then, $r_{\mathscr{L},\psi}$ is an everywhere defined immersion if and only if there exists a family of sections $s_\alpha \in S_{n_\alpha}$ for $n_\alpha > 0$ such that, setting $f_\alpha = \psi^\flat(s_\alpha)$, the following conditions hold.*

- *(i) The X_{f_α} cover X.*
- *(ii) The X_{f_α} are affine open subsets of X.*
- *(iii) For every α and every $t \in H^0(X_{f_\alpha},\mathcal{O}_X)$, there exists an integer $m > 0$ and a section $s \in S_{mn_\alpha}$ such that*

$$t = \bigl(\psi^\flat(s)|_{X_{f_\alpha}}\bigr)\bigl(\psi^\flat(f_\alpha)|_{X_{f_\alpha}}\bigr)^{-m}.$$

The map $r_{\mathscr{L},\psi}$ *is an everywhere defined* open *immersion if and only if there exists a family of sections* $\{s_\alpha\}$ *as above satisfying* (*i*), (*ii*), (*iii*), *and*

(*iv*) *For every* $n > 0$ *and every* $s \in S_{nn_\alpha}$ *such that* $\psi^\flat(s)|_{X_{f_\alpha}} = 0$, *there exists an integer* $k > 0$ *such that* $s_\alpha^k s = 0$.

The map $r_{\mathscr{L},\psi}$ *is an everywhere defined* closed *immersion if and only if there exists a family of sections* $\{s_\alpha\}$ *as above satisfying* (*i*), (*ii*), (*iii*), *and*

(*v*) *The* $D_+(s_\alpha)$ *cover* $\mathrm{Proj}(S)$.

Proof. The condition (*i*) holds if and only if $r_{\mathscr{L},\psi}$ is everywhere defined. The conditions (*ii*) and (*iii*) are equivalent to saying that $r_{\mathscr{L},\psi}$ is a closed immersion into $\bigcup_\alpha D_+(s_\alpha)$. This proves the statements for immersions and closed immersions.

For open immersions, (*iv*) says that $\psi^\flat_{(s_\alpha)}$ is bijective, and hence (*i*)–(*iv*) hold if and only if $r_{\mathscr{L},\psi}$ is induces an isomorphism

$$X_{f_\alpha} \xrightarrow{\sim} D_+(s_\alpha)$$

for every α. □

To capture when $r_{\mathscr{L},\psi}$ is an immersion, we make the following definition.

Definition 2.6.18 [EGAII, Définition 4.4.2, Définition 4.5.3, Définition 4.6.1]. Let X be a scheme and consider an invertible $\mathcal{O}_X$-module $\mathscr{L}$ on X.

(*i*) Let $f\colon X \to Y$ be a morphism of schemes. We say that $\mathscr{L}$ is f-*very ample* or *very ample over* Y if there exists a quasi-coherent $\mathcal{O}_Y$-module $\mathscr{E}$ and a factorization [Har77, p. 120]

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathbf{P}_Y(\mathscr{E}) \arrow[d] \\ & Y \end{tikzcd}$$

of f where i is an immersion, such that $\mathscr{L} \cong i^*(\mathcal{O}(1))$.

(*ii*) Suppose that X is quasi-compact. We say that $\mathscr{L}$ is *ample* if setting [EGAII, Thm. 4.5.2(*b*)] [Har77, p. 153]

$$S = \bigoplus_{n\geq 0} H^0(X, \mathscr{L}^{\otimes n})$$

and considering the canonical morphism $p\colon X \to \mathrm{Spec}(\mathbf{Z})$, the map

$$\varepsilon\colon p^*(\tilde{S}) \longrightarrow \bigoplus_{n\geq 0} \mathscr{L}^{\otimes n}$$

induces an everywhere defined dominant open immersion

$$r_{\mathscr{L},\varepsilon}\colon X \hookrightarrow \mathrm{Proj}(S).$$

(*iii*) Suppose that f is quasi-compact. We say that $\mathscr{L}$ is f-*ample* or *(relatively) ample over* Y if there exists an affine open covering $Y = \bigcup_\alpha U_\alpha$ such that setting $X_\alpha = f^{-1}(U_\alpha)$, each restriction $\mathscr{L}|_{X_\alpha}$ is an ample $\mathcal{O}_{X_\alpha}$-module for every α.

As written, it is not obvious how ampleness and very ampleness are related. We will connect the two notions later.

Caution 2.6.19. The definition of very ample in [Har77, p. 120] is not equivalent to the one above. The reason is that without extra assumptions, the immersion i in our definition may not be an H-immersion in the sense of Caution 2.3.26. I am not sure if the result below (for example) is true withou Hartshorne's definition of an immersion because it is not immediate (and may be false in examples) that the composition (2.6.21) is an H-immersion.

For ampleness, the definition above is equivalent to that in [EGAII] if X is quasi-compact and quasi-separated [EGAII, Proposition 4.5.5; EGAIV$_1$, (1.7.14)].

[Har77, Exer. II.5.12(a)]

Proposition 2.6.20 [EGAII, Corollaire 4.4.9(ii)]. *Let $f\colon X \to Y$ be a morphism. If $\mathscr{L}$ and $\mathscr{L}'$ are f-very ample invertible $\mathcal{O}_X$-modules, then $\mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{L}'$ is f-very ample.*

Proof. Consider the two immersions $i\colon X \hookrightarrow \mathbf{P}_Y(\mathscr{E})$ and $i'\colon X \hookrightarrow \mathbf{P}_Y(\mathscr{E}')$ obtained from the definition of very ampleness. Consider the composition

$$X \xhookrightarrow{\Delta_{X/Y}} X \times_Y X \xhookrightarrow{i \times_Y i'} \mathbf{P}_Y(\mathscr{E}) \times_Y \mathbf{P}_Y(\mathscr{E}') \xhookrightarrow{\sigma} \mathbf{P}_Y(\mathscr{E} \otimes_{\mathcal{O}_Y} \mathscr{E}') \tag{2.6.21}$$

where σ is the Segre embedding [EGAII, §4.3] (the version where Y is affine was constructed in Homework 7, Problem 7 or [Har77, Exercise II.5.11]). The diagonal $\Delta_{X/Y}$ is a closed immersion since X is separated over Y: It factors as an immersion followed by the projection from a projective bundle. Next, $i \times_Y i'$ is an immersion since immersions are stable under products (Proposition 2.5.12(vi)). Finally, σ is a closed immersion by [EGAII, Proposition 4.3.3]. Alternatively, whether σ is a closed immersion can be checked locally on Y. When Y is affine, your solution to Homework 7, Problem 7 or [Har77, Exercise II.5.11] constructed as a closed immersion.

Since the composition of immersions is an immersion, we see that the composition

$$X \hookrightarrow \mathbf{P}_Y(\mathscr{E} \otimes_{\mathcal{O}_Y} \mathscr{E}')$$

is an immersion. The pullback of $\mathcal{O}(1)$ under this immersion is $\mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{L}'$ by [EGAII, (4.3.2.1)] (when Y is affine, one can use Homework 7, Problem 7 or [Har77, Exercise II.5.11]). $\square$

$\boxed{4/3}$

Remark 2.6.22 (The Segre embedding). Using our construction of rational maps to **Proj**, we can give another way to think about the Segre embedding from [EGAII, §4.3] (the case when Y is affine is Homework 7, Problem 7 or [Har77, Exercise II.5.11]). Let Y be a scheme and let $\mathscr{E}, \mathscr{E}'$ be two quasi-coherent $\mathcal{O}_Y$-modules. The *Segre embedding*

$$\sigma\colon \mathbf{P}_Y(\mathscr{E}) \times_Y \mathbf{P}_Y(\mathscr{E}') \hookrightarrow \mathbf{P}_Y(\mathscr{E} \otimes_{\mathcal{O}_Y} \mathscr{E}')$$

is defined as follows. Set $X \coloneqq \mathbf{P}_Y(\mathscr{E}) \times_Y \mathbf{P}_Y(\mathscr{E}')$ and denote by p_1, p_2 the two projection morphisms. Denote by $q\colon X \to Y$ the composition of p_1 or p_2 to Y. A fact about the projective bundle $\mathbf{P}_Y(\mathscr{E})$ is that there is a surjection

$$\pi^*\mathscr{E} \twoheadrightarrow \mathcal{O}(1)$$

and similarly for $\mathbf{P}_Y(\mathscr{E}')$; see [EGAII, Proposition 4.1.6]. Pulling back to X and tensoring the maps for $\mathscr{E}$ and $\mathscr{E}'$ together, we obtain the surjection

$$q^*\mathscr{E} \otimes_{\mathcal{O}_X} q^*\mathscr{E}' \twoheadrightarrow p_1^*\mathcal{O}(1) \otimes_{\mathcal{O}_X} p_2^*\mathcal{O}(1).$$

Taking symmetric algebras, we have the surjection

$$\psi \colon q^* \operatorname{Sym}^\bullet_{\mathcal{O}_Y}(\mathscr{E} \otimes_{\mathcal{O}_Y} \mathscr{E}') \twoheadrightarrow \operatorname{Sym}^\bullet_{\mathcal{O}_X}\bigl(p_1^*\mathcal{O}(1) \otimes_{\mathcal{O}_X} p_2^*\mathcal{O}(1)\bigr).$$

Now the associated rational map $r_{p_1^*\mathcal{O}(1)\otimes p_2^*\mathcal{O}(1),\psi}$ is the Segre embedding σ. Checking that σ is indeed an embedding can be done using the local criteria for rational maps to Proj to be a closed immersion (Proposition 2.6.17).

2.6.3. Projective and quasi-projective morphisms. We now define different versions of projective morphisms.

Definition 2.6.23. Let $f \colon X \to Y$ be a morphism of schemes.

(1) [EGAII, Définition 5.5.2] We say that f is *projective* if f factors as

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & \mathbf{P}_Y(\mathscr{E}) \arrow[d] \\ & Y \end{tikzcd}$$

where $X \hookrightarrow \mathbf{P}_Y(\mathscr{E})$ is a closed immersion and $\mathscr{E}$ is a quasi-coherent $\mathcal{O}_Y$-module of finite type.

(2) [Har77, p. 103] We say that f is *H-projective* if there exists an integer n for which f factors as

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & \mathbf{P}^n_Y \arrow[d] \\ & Y \end{tikzcd}$$

where $X \hookrightarrow \mathbf{P}^n_Y$ is a closed immersion.

(3) [Stacks, Tag 01W8(3)] We say that f is *locally projective* if there exists an open covering $Y = \bigcup_j V_j$ such that $f^{-1}(V_j) \to V_j$ is projective.

(4) [EGAII, Définition 5.3.1] We say that f is *quasi-projective* or that X is *quasi-projective over* Y if f is of finite type and there exists an f-ample invertible $\mathcal{O}_X$-module.

(5) [Stacks, Tag 01VW(2)] We say that f is *H-quasi-projective* if there exists a quasi-compact immersion $X \hookrightarrow \mathbf{P}^n_Y$ over Y.

(6) [Stacks, Tag 01VW(3)] We say that f is *locally quasi-projective* if there exists an open covering $Y = \bigcup_j V_j$ such that $f^{-1}(V_j) \to V_j$ is quasi-projective.

Note that

$$\begin{tikzcd} \text{H-projective} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{projective} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{locally projective} \arrow[d, Rightarrow] \\ \text{H-quasi-projective} \arrow[r, Rightarrow] & \text{quasi-projective} \arrow[r, Rightarrow] & \text{locally quasi-projective} \arrow[d, Rightarrow] \\ & & \text{separated.} \end{tikzcd}$$

On Homework 8, you will show some properties of projective morphisms.

2.6.4. Abstract varieties. We can finally say which schemes correspond to the varieties from last semester [Mur24ag]. We will not prove this result in this class.

THEOREM 2.6.24 [Har77, Propositions II.2.6 and II.4.10]. *Let k be an algebraically closed field. There is a natural fully faithful functor*

$$t\colon \mathsf{Var}_k \longrightarrow \mathsf{Sch}_k$$

from the category of varieties over k (in the sense of [Har77, Chapter I]*) to the category of schemes over k satisfying the following properties.*

- *The topological space of a variety V is homeomorphic to the set of closed points of $\operatorname{sp}(t(V))$.*
- *The sheaf of regular functions on V is obtained by restricting the structure sheaf of $t(V)$ to the set of closed points of $\operatorname{sp}(t(V))$.*
- *The essential image is the subcategory consisting of integral schemes that are quasi-projective over k.*
- *The essential image of the subcategory of projective varieties is the subcategory consisting of integral schemes that are projective over k.*

In particular, for any variety V, $t(V)$ is an integral separated scheme of finite type over k.

This result suggests the following definition.

[Har77, p. 105]
[Ful98, §B.2.4]

DEFINITION 2.6.25 (cf. [Wei62, Chapter VII, §3; FAC, n° 32]). Let k be a field. An *(abstract) variety over k* is an integral separated scheme X of finite type over k. We say that X is *complete* (resp. *projective*, *quasi-projective*) if it is proper (resp. projective, quasi-projective) over k.

REMARK 2.6.26. Hartshorne assumes that k is algebraically closed. To make it possible to talk about varieties over non-algebraically closed fields, we will not assume that k is algebraically closed.

[Har77, Rem. II.4.10.2]

Historically, quasi-projective varieties were defined before abstract varieties. André Weil introduced abstract varieties in [Wei62] (first edition published in 1946) while proving the Riemann hypothesis for curves over finite fields in the 1940s [Wei48]. During the proof, Weil constructs what is called the *Jacobian variety* associated to a curve. The construction of the Jacobian variety does not make it obvious whether or not the Jacobian variety is quasi-projective.

2.6.5. Elimination theory via schemes. We now come to one of the strengths of modern algebraic geometry. Last semester, we proved that the image of a projective variety is closed using resultants and elimination theory [Mur24ag, Theorem 1.8.3]. Now that we have the valuative criterion and the notion of projective morphisms, we can prove the same result as a consequence of the following:

[Har77, Thm. II.4.9]

THEOREM 2.6.27 [EGAII, Théorème 5.5.3]. *Let $f\colon X \to Y$ be a morphism of schemes.*

(i) *If f is projective, then f is quasi-projective and proper.*
(ii) *Conversely, if Y is quasi-compact and quasi-separated and f is quasi-projective and proper, then f is projective.*

REMARK 2.6.28. Dieudonné famously said that "elimination theory must be eliminated" (« Il faut éliminer la théorie de l'élimination » [Dem12, p. 336]). This remark has also been attributed to Weil; see [Wei62, Footnote 2 on p. 31]. On the other hand, Abhyankar (who was at Purdue) wrote in a poem [Abh70]:

> Eliminate, eliminate, eliminate
> Eliminate the eliminators of elimination theory.

We will not prove (*ii*). It follows from the fact that "the image of a proper scheme is proper" (Homework 8, Problem 3(*d*)) [EGAII, Corollaire 5.4.3(*ii*)] plus some tricks to replace the $\mathrm{Proj}(S)$ appearing in the definition of ampleness with a projective bundle $\mathbf{P}_Y(\mathscr{E})$ appearing in the definition of projective morphisms. See [Stacks, Tag 0BCL].

Proof of Theorem 2.6.27(i). By definition, if $f\colon X \to Y$ is projective, then it is both of finite type and quasi-projective. Since quasi-projective morphisms are separated (locally on Y, f can be factored as an open immersion followed by the projection morphism from Proj), it remains to show that f is universally closed.

Consider the factorization

[EGAII, Rem. 7.3.9(*ii*)]
[Stacks, Tag 01WC]

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr, "f"'] & \mathbf{P}_Y(\mathscr{E}) \arrow[d] \\
& Y
\end{tikzcd}$$

where $X \hookrightarrow \mathbf{P}_Y(\mathscr{E})$ is a closed immersion and $\mathscr{E}$ is a quasi-coherent $\mathcal{O}_Y$-module of finite type. Since the question is local on Y, we may replace Y by its affine open subsets $\mathrm{Spec}(A)$. By Serre's equivalence, write $\mathscr{E} = \tilde{M}$ for a locally free A-module M generated by finitely many elements $m_0, m_1, \ldots, m_n$. Consider the surjective graded map

$$\begin{aligned} A[x_0, x_1, \ldots, x_n] &\longrightarrow \mathrm{Sym}_A^\bullet(M) \\ X_i &\longmapsto m_i. \end{aligned}$$

By the functoriality of Proj and the description of closed subschemes of Proj in Homework 5, Problem 2 [Har77, Exercises II.2.14 and II.3.12(*a*)], we can enlarge the commutative diagram above as

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr, "f"'] & \mathbf{P}_Y(\mathscr{E}) \arrow[r, hook] \arrow[d] & \mathbf{P}^n_Y \arrow[dl] \\
& Y &
\end{tikzcd}$$

where $\mathbf{P}_Y(\mathscr{E}) \hookrightarrow \mathbf{P}^n_Y$ is a closed immersion. It therefore suffices to show that $\mathbf{P}^n_Y \to Y$ is universally closed. Since it is the base change of $\mathbf{P}^n_{\mathbf{Z}} \to \mathrm{Spec}(\mathbf{Z})$, it suffices to show that $\mathbf{P}^n_{\mathbf{Z}} \to \mathrm{Spec}(\mathbf{Z})$ is universally closed. By the valuative criterion for universal closedness (Theorem 2.5.58), it suffices to show that for every valuation ring R with fraction field K, a solid diagram of the form below can be filled in with a dashed morphism making the diagram commute:

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathbf{P}^n_{\mathbf{Z}} \arrow[d] \\
\mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed, "\exists"] & \mathrm{Spec}(\mathbf{Z}).
\end{tikzcd}$$

We induce on n. If $n = 0$, then $\mathbf{P}^n_{\mathbf{Z}} \to \mathrm{Spec}(\mathbf{Z})$ is an isomorphism, and hence there is nothing to show. Suppose that $n > 0$. By the inductive hypothesis, we may assume that the image ξ_1 of $\operatorname{Spec}(K)$ is not contained in any of the coordinate hyperplanes $V_+(x_i) \cong \mathbf{P}^{n-1}_{\mathbf{Z}}$. Thus, $\xi_1 \in \bigcap_i D_+(x_i)$, and hence all the x_i/x_j are invertible in $\mathcal{O}_{\mathbf{P}^n_{\mathbf{Z}},\xi_1}$.

We have an inclusion $k(\xi_1) \subseteq K$ induced by the morphism $\operatorname{Spec}(K) \to X$. Let $f_{ij} \in K$ be the images of the x_i/x_j. Then, the f_{ij} are nonzero elements in K such that

$$f_{ik} = f_{ij} \cdot f_{jk}$$

for all i, j, k. Let $v \colon K \to G$ be the valuation associated to the valuation ring R. Let $g_i = v(f_{i0})$ for $i \in \{0, 1, \ldots, n\}$. Choose k such that g_k is minimal among the set $\{g_0, g_1, \ldots, g_n\}$ for the ordering on G. Then, for every i we have

$$v(f_{ik}) = v(f_{i0}) - v(f_{k0}) = g_i - g_k \geq 0,$$

and hence $f_{ik} \in R$ for all $i \in \{0, 1, \ldots, n\}$. We can therefore define a morphism $\mathbf{Z}[x_0/x_k, \ldots, x_n/x_k] \to R$ fitting into the diagram below:

$$\begin{tikzcd}
K & k(\xi_1) \arrow[l, hook'] & \mathcal{O}_{\mathbf{P}^n_{\mathbf{Z}},\xi_1} \arrow[l, hook'] & \mathbf{Z}\left[\dfrac{x_0}{x_k}, \dfrac{x_1}{x_k}, \ldots, \dfrac{x_n}{x_k}\right] \arrow[l, hook'] \arrow[dlll, dashed] \\
R \arrow[u, hook] & & & \mathbf{Z}. \arrow[lll] \arrow[u, hook]
\end{tikzcd}$$

The dashed morphism yields the morphism $\operatorname{Spec}(R) \to \mathbf{P}^n_{\mathbf{Z}}$ needed. ☐

Remark 2.6.29. The elimination theory-type result in Theorem 2.6.27(*i*) is very useful! For one example, it is a critical ingredient in Mori's "bend and break" theorem [Mor79], which gives a very useful sufficient criterion for a projective variety to contain a rational curve, that is, a curve birational to $\mathbf{P}^1_k$. The proof uses Theorem 2.6.27(*i*) to reduce modulo p. See [KM98, p. 15]. There is no proof known of Mori's theorem that does not use reduction modulo p.

2.6.6. The curve-to-projective extension theorem.

We can now prove the promised generalization of the curve-to-projective extension theorem. The name for the theorem is from [Vak, Theorem 15.3.1]. Codimension for non-closed sets is defined as

$$\operatorname{codim}_X(Z) \coloneqq \inf_{z \in Z} \bigl\{\dim(\mathcal{O}_{Z,z})\bigr\}.$$

See [EGAIV$_2$, p. 87].

[Har77, Prop. I.6.8]

Theorem 2.6.30 (Curve-to-projective extension theorem [EGAII, Corollaire 7.3.6; Stacks, Tag 0BX7]). *Let S be a scheme and let $Y \to S$ be a proper morphism. Let X be a reduced locally Noetherian scheme such that*

$$\operatorname{codim}_X\bigl(\operatorname{Sing}(X)\bigr) \geq 2$$

where

$$\operatorname{Sing}(X) \coloneqq \bigl\{x \in X \bigm| \mathcal{O}_{X,x} \text{ is not regular}\bigr\}.$$

Let $f \colon X \dashrightarrow Y$ be a rational map over S. Then, the complement X' of the domain of definition of f satisfies $\operatorname{codim}_X(X') \geq 2$.

Proof. It suffices to show that f can be extended to a morphism over all points $z \in X$ such that $\dim(\mathcal{O}_{X,z}) \leq 1$. If $\dim(\mathcal{O}_{X,z}) = 0$, then z is a generic point of an irreducible component of X. Since rational maps are defined on open *dense* subsets of the domain, we know that f is already defined at z.

Suppose that $\dim(\mathcal{O}_{X,z}) = 1$. By assumption on X, we know that $\mathcal{O}_{X,z}$ is a DVR. By the previous paragraph, we know that f is defined on the generic point of $\operatorname{Spec}(\mathcal{O}_{X,z})$, that is, we have the solid diagram

$$\begin{tikzcd}
\operatorname{Spec}\bigl(\operatorname{Frac}(\mathcal{O}_{X,z})\bigr) \arrow[r, "f|_{\operatorname{Spec}(\operatorname{Frac}(\mathcal{O}_{X,z}))}"] \arrow[d] & Y \arrow[d] \\
\operatorname{Spec}(\mathcal{O}_{X,z}) \arrow[r] \arrow[ur, dashed, "\exists!", "g"'] & S.
\end{tikzcd}$$

By the valuative criterion of properness (Theorem 2.5.60), the dashed morphism exists and is unique. Let $y \in Y$ be the image of Y, choose an affine open neighborhood $\operatorname{Spec}(C)$ of the image of y in S, and choose an affine open neighborhood $\operatorname{Spec}(B)$ of Y contained in the inverse image of $\operatorname{Spec}(C)$ in Y. Then, the dashed morphism g is induced by a C-algebra map [EGAI_{new}, Prop. 6.6.1(ii)]

$$B \longrightarrow \mathcal{O}_{X,z}.$$

We claim there exists an open neighborhood $\operatorname{Spec}(A) \subseteq X$ and a morphism $g'\colon \operatorname{Spec}(A) \to \operatorname{Spec}(B)$ restricting to g. Since Y is of finite type over S, it suffices to find an affine open neighborhood of z such that the finitely many generators x_i of B over C lie in $\Gamma(X, \operatorname{Spec}(A)) \subseteq \operatorname{Frac}(A)$. Note that we need

$$\Gamma\bigl(X, \operatorname{Spec}(A)\bigr) \lhook\joinrel\longrightarrow \operatorname{Frac}(A)$$

to be injective for this argument to work. This injectivity holds because X is reduced.

It remains to show that the morphism $g'\colon \operatorname{Spec}(A) \to \operatorname{Spec}(B)$ can be glued to the morphism f. We have the commutative diagram [EGAI_{new}, Prop. 6.6.1(i)]

$$\begin{tikzcd}
 & A \arrow[dr] & \\
B \arrow[ur, "f^{\#}"] \arrow[dr, "g^{\#}"'] & & \mathcal{O}_{X,z} \\
 & A \arrow[ur] &
\end{tikzcd}$$

where the maps $A \to \mathcal{O}_{X,z}$ are the localization maps. Since B is a C-algebra of finite type and $\Gamma(X, \operatorname{Spec}(A)) \to \operatorname{Frac}(A)$ is injective, there exists an element $a \in A$ such that the images of the generators x_i of B map to the same elements in A_a, i.e., we have the commutative diagram

$$\begin{tikzcd}
 & A \arrow[d] \arrow[dr] & \\
B \arrow[ur, "f^{\#}"] \arrow[dr, "g^{\#}"'] \arrow[r] & A_a \arrow[r] & \mathcal{O}_{X,z} \\
 & A. \arrow[u] \arrow[ur] &
\end{tikzcd}$$

□

2.6.7. Linear projections. An important class of examples of rational maps defined using sections of an invertible sheaf are linear projections. To say something interesting about them, we prove the following, more local criterion for when a rational map is a closed immersion. $\boxed{4/8}$

Proposition 2.6.31. *Let k be an algebraically closed field. Let X be a projective scheme over k and let $\mathscr{L}$ be an invertible $\mathcal{O}_X$-module. Consider sections* [Har77, Prop. II.7.3]

$$s_0, s_1, \ldots, s_n \in H^0(X, \mathscr{L})$$

inducing a morphism $\varphi \colon X \to \mathbf{P}(V)$ *where*

$$V = \operatorname{span}_k\{s_0, s_1, \ldots, s_n\} \subseteq H^0(X, \mathscr{L}).$$

Then, φ is a closed immersion if and only if

(1) *Elements of V separate points, i.e., for any two distinct closed points $P, Q \in X$, there exists $s \in V$ such that $s \in \mathfrak{m}_P \mathscr{L}_P$ but $s \notin \mathfrak{m}_Q \mathscr{L}_Q$, or vice versa. In other words, for any two distinct closed points $P, Q \in X$, the surjection*

$$\mathscr{L} \twoheadrightarrow \frac{\mathscr{L}_P}{\mathfrak{m}_P \mathscr{L}_P} \oplus \frac{\mathscr{L}_Q}{\mathfrak{m}_Q \mathscr{L}_Q}$$

induces a surjection

$$V \twoheadrightarrow \frac{\mathscr{L}_P}{\mathfrak{m}_P \mathscr{L}_P} \oplus \frac{\mathscr{L}_Q}{\mathfrak{m}_Q \mathscr{L}_Q}.$$

(2) *Elements of V generate tangent vectors, i.e., for each closed point $P \in X$, the set*

$$\left\{ s \in V \;\middle|\; s_P \in \mathfrak{m}_P \mathscr{L}_P \right\}$$

spans the k-vector space $\mathfrak{m}_P \mathscr{L}_P / \mathfrak{m}_P^2 \mathscr{L}_P$. In other words, the surjection

$$\mathfrak{m}_P \mathscr{L}_P \twoheadrightarrow \frac{\mathfrak{m}_P \mathscr{L}_P}{\mathfrak{m}_P^2 \mathscr{L}_P}$$

induces a surjection

$$V \cap H^0(X, \mathfrak{m}_P \mathscr{L}_P) \twoheadrightarrow \frac{\mathfrak{m}_P \mathscr{L}_P}{\mathfrak{m}_P^2 \mathscr{L}_P}.$$

Proof. $\Rightarrow$. We think of X as a closed subscheme of $\mathbf{P}_k(V)$, in which case $\mathscr{L} = \mathcal{O}_X(1)$ and V is spanned by the images of

$$x_0, x_1, \ldots, x_n \in H^0\big(\mathbf{P}_k(V), \mathcal{O}(1)\big).$$

Choosing a global section defining a hyperplane containing P but not Q, we see that (1) holds. For (2), the hyperplanes passing through P are defined by global sections of $\mathcal{O}(1)$ which generate $\mathfrak{m}_P \mathscr{L}_P / \mathfrak{m}_P^2 / \mathscr{L}_P$. Changing our basis on V so that

$$P = [1 : 0 : 0 : \cdots : 0],$$

in the standard open set U_0 with local coordinates $y_1, y_2, \ldots, y_n$, we see that all vectors in $\mathfrak{m}_P \mathscr{L}_P / \mathfrak{m}_P^2 / \mathscr{L}_P$ are images of global sections under restriction.

$\Leftarrow$. (1) implies φ is injective as a map of sets. Since X is projective over k, its image $\varphi(X)$ in $\mathbf{P}_k(V)$ is closed by Homework 8, Problem 3(d) [EGAII, Corollaire 5.4.3(ii)]. Thus, φ is a homeomorphism onto is image. To show that φ is a closed immersion, by Proposition 2.5.13(ii) it therefore suffices to show that

$$\varphi_P^\# \colon \mathcal{O}_{\mathbf{P}_k(V), P} \longrightarrow \varphi_* \mathcal{O}_{X,P}$$

is surjective for every closed point $P \in X$. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m}_{\mathbf{P}_k(V),P} \arrow[r] \arrow[d, two heads] & \mathscr{O}_{\mathbf{P}_k(V),P} \arrow[r] \arrow[d] & \dfrac{\mathscr{O}_{\mathbf{P}_k(V),P}}{\mathfrak{m}_{\mathbf{P}_k(V),P}} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \dfrac{\mathfrak{m}_{X,P}}{\mathfrak{m}^2_{X,P}} \arrow[r] & \dfrac{\mathscr{O}_{X,P}}{\mathfrak{m}^2_{X,P}} \arrow[r] & \dfrac{\mathscr{O}_{X,P}}{\mathfrak{m}_{X,P}} \arrow[r] & 0.
\end{tikzcd} \tag{2.6.32}$$

The hypothesis (2) implies the left vertical map is surjective. The right vertical map is an isomorphism since they are both isomorphic to k. By the snake lemma [KS06, Lemma 12.1.1], the middle vertical map is also surjective, and hence

$$\mathscr{O}_{\mathbf{P}_k(V),P} \twoheadrightarrow \frac{\mathscr{O}_{X,P}}{\mathfrak{m}_{X,P}}$$

is surjective. Since $\varphi_*\mathscr{O}_{X,P}$ is coherent (apply Serre's finiteness Theorem 2.4.39(*i*) locally), we see that

$$\mathscr{O}_{\mathbf{P}_k(V),P} \twoheadrightarrow \mathscr{O}_{X,P}$$

is surjective by NAK [Mur24ca, Lemma 2.3.8]. $\square$

Remark 2.6.33. Let k be an algebraically closed field, let X be a projective scheme over k, and let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. The rational map

$$\varphi \colon X \dashrightarrow \mathbf{P}_k\big(H^0(X, \mathscr{L})\big)$$

is a morphism if and only if for every closed point $P \in X$, the map

$$H^0(X, \mathscr{L}) \longrightarrow H^0\left(X, \mathscr{L} \otimes_{\mathscr{O}_X} \frac{\mathscr{O}_X}{\mathfrak{m}_P}\right)$$

is surjective. Using (2.6.32), the rational map φ is a closed immersion if and only if for every pair of (possibly coinciding) closed points $P, Q \in X$, the map

$$H^0(X, \mathscr{L}) \longrightarrow H^0\left(X, \mathscr{L} \otimes_{\mathscr{O}_X} \frac{\mathscr{O}_X}{\mathfrak{m}_P\mathfrak{m}_Q}\right)$$

is surjective. More generally, we say that $\mathscr{L}$ is *k-jet ample* [BS93, p. 358] if, for every tuple of integers k_i such that $\sum_i k_i = k + 1$ and closed points $P_i \in X$, the map

$$H^0(X, \mathscr{L}) \longrightarrow H^0\left(X, \mathscr{L} \otimes_{\mathscr{O}_X} \frac{\mathscr{O}_X}{\prod_i \mathfrak{m}_{P_i}^{k_i}}\right).$$

We apply this criterion to show that any projective curve can be embedded in $\mathbf{P}^3_k$.

Definition 2.6.34. Let k be an algebraically closed field. Let $X \subseteq \mathbf{P}^n_k$ be a [Har77, p. 309] closed subscheme. If $P, Q \in X$ are closed points, the *secant line* determined by P and Q is the line in $\mathbf{P}^n_k$ joining P and Q. If $P \in X$ is a closed point (i.e., $\mathscr{O}_{X,P}$ is a regular local ring), the *(projective) tangent space* of X at P is the linear subspace $L \subseteq \mathbf{P}^n_k$ passing through P whose tangent space $T_P(L)$ is equal to $T_P(X)$ as a subspace of $T_P(\mathbf{P}^n_k)$.

[Har77, Prop. IV.3.4]

Proposition 2.6.35. *Let k be an algebraically closed field and let $X \subseteq \mathbf{P}^n_k$ be a regular one-dimensional closed subscheme. Let O be a closed point not on X and let $\varphi\colon X \to \mathbf{P}^{n-1}_k$ be the linear projection away from O. Then, φ is a closed immersion if and only if*

(1) *O is not on any secant line of X.*
(2) *O is not on any tangent line of X.*

Proof. The morphism φ corresponds to the subspace of $H^0(X, \mathcal{O}(1))$ defining hyperplanes passing through O. The two conditions correspond to the two conditions in Proposition 2.6.31. □

[Har77, Prop. IV.3.5]

Proposition 2.6.36. *Let k be an algebraically closed field and let $X \subseteq \mathbf{P}^n_k$ be a regular one-dimensional closed subscheme with $n \geq 4$. Then, there exists a closed point $O \notin X$ such that the linear projection from O gives a closed immersion of X into $\mathbf{P}^{n-1}_k$.*

Proof. Let $\operatorname{Sec}(X)$ be the closure of the union of all secant lines of X. This is called the *secant variety* of X. It is a locally closed subset of $\mathbf{P}^n_k$ of dimension ≤ 3 since (in suitable local coordinates) it can be defined as the image of the morphism

$$\begin{aligned}(X \times_k X - \Delta) \times \mathbf{P}^1_k &\longrightarrow \mathbf{P}^n_k\\ (P, Q, t) &\longmapsto \text{the point } t \text{ on } \overline{PQ}.\end{aligned}$$

Similarly, let $\operatorname{Tan}(X)$ be the closure of the union of all tangent lines of X. This is called the *tangent variety* of X. It is a locally closed subset of $\mathbf{P}^n_k$ of dimension ≤ 2 since (in suitable local coordinates) it can be defined as the image of the morphism

$$\begin{aligned}X \times \mathbf{P}^1_k &\longrightarrow \mathbf{P}^n_k\\ (P, t) &\longmapsto \text{the point } t \text{ on } T_P(X).\end{aligned}$$

Since $n \geq 4$, we have

$$\operatorname{Sec}(X) \cup \operatorname{Tan}(X) \subsetneq \mathbf{P}^n_k,$$

and hence there exists a point $O \in \mathbf{P}^n_k$ not lying on any secant line or tangent line of X. We are done by Proposition 2.6.35. □

Example 2.6.37. It is not always possible to project further to $\mathbf{P}^2_k$ without introducing nodes. For example, consider the twisted cubic $X \subseteq \mathbf{P}^3_k$, which satisfies

$$\mathcal{O}_{\mathbf{P}^3_k}(1)|_X \cong \mathcal{O}_{\mathbf{P}^1_k}(3)$$

since X is the image of the Veronese embedding defined by the linear system

$$\{s^3, s^2t, st^2, t^3\} \subseteq H^0\bigl(\mathbf{P}^1_k, \mathcal{O}(3)\bigr).$$

Since $\dim(\operatorname{Sec}(X)) = 3$, the proof of Proposition 2.6.36 does not apply. More strongly, we claim there cannot possibly be a linear projection

$$\mathbf{P}^3_k - \{O\} \longrightarrow \mathbf{P}^2_k$$

that induces a closed immersion $\varphi\colon X \hookrightarrow \mathbf{P}^2_k$. Suppose such a closed immersion were to exist. Since $X \cong \mathbf{P}^1_k$, we see that the arithmetic genus of $\varphi(X)$ must be 0. For plane curves, Example 2.5.32 shows that

$$p_a\bigl(\varphi(X)\bigr) = \frac{1}{2}(d-1)(d-2),$$

and hence $\varphi(X)$ is a plane curve defined by a homogeneous polynomial of degree 1 or 2. This implies $\mathcal{O}_{\mathbf{P}^2_k}(1)|_{\varphi(X)} \cong \mathcal{O}_{\mathbf{P}^1_k}(1)$ or $\mathcal{O}_{\mathbf{P}^2_k}(1)|_{\varphi(X)} \cong \mathcal{O}_{\mathbf{P}^1_k}(2)$. But the linear projection pull backs $\mathcal{O}(1)$ to $\mathcal{O}_{\mathbf{P}^1_k}(3)$ by the previous paragraph, a contradiction.

For the twisted cubic, we can still choose to project from a point contained in a secant line, which introduces a node in the projection. Is this the worst possible?

Remark 2.6.38. Over an algebraically closed field, every regular projective curve projects birationally to a plane curve with at worst nodes as singularities [Sam66, Appendix to Chapter II, Theorem; Har77, Theorems IV.3.9 and IV.3.10]. For surfaces, every regular projective surface projects birationally to a surface in $\mathbf{P}^3_k$ with at worst *ordinary singularities* [GH78, pp. 611–618].

These are useful results! For example, they allow one to prove the Riemann–Roch theorem for curves by reducing to the plane curve case [ACGH85, Appendix A], and Noether's formula for surfaces by reducing to the hypersurface case [GH78, Chapter 4, §6].

In higher dimensions, Roberts [Rob71] proved that generic projections mapping birationally to a hypersurface are generically 1:1, 2:1 along a codimension 1 subset, 3:1 along a codimension 2 subset, etc.. In dimensions ≤ 5, Roberts [Rob75] gave explicit equations for the images of these generic projections hypersurfaces. Still in dimensions ≤ 5, these generic projections hypersurfaces are known to have at worst semi-log canonical singularities in characteristic zero [Doh08] and F-pure singularities in characteristic $p > 0$ [DM24]. There are examples of 30-dimensional projective varieties whose generic projection hypersurfaces have worse than semi-log canonical or F-pure singularities [Doh08; DM24]. In general, the best known result is that generic projection hypersurfaces are weakly normal [GT80, Theorem 3.7; ZN84, Theorem 3.2; RZN84, Theorem 1.1; CM81, Theorem 3.8; CGM87, Osservazione 2.4(v)]. As far as I am aware, the following questions are open:

1. What is the largest dimension for which generic projection hypersurfaces are semi-log canonical or F-pure?
2. In some (low) dimensions > 5, is it possible to classify all singularities that appear as singularities on generic projection hypersurfaces?

2.6.8. Divisors.

We now come to the theory of divisors. These give a way to probe a scheme by using codimension 1 integral subschemes. There will be a close connection with the notion of linear systems from last time, since one can obtain many codimension 1 subschemes by taking the zero locus of a global section of an invertible sheaf.

To define divisors, we will use the following definition of the sheaf of rational functions on a scheme.

Definition 2.6.39 [Kle79, p. 204]. Let X be a ringed space. The *sheaf of rational* or *meromorphic functions* is the sheaf $\mathscr{K}_X$ associated to the presheaf

$$U \longmapsto \Gamma(U, \mathcal{O}_X)\bigl[S(U)^{-1}\bigr],$$

where $S(U)$ is the set of elements of $\Gamma(U, \mathcal{O}_X)$ whose germs are nonzerodivisors in the stalks $\mathcal{O}_{X,x}$ for all $x \in X$.

[Har77, pp. 140–141] The definition in [EGAIV$_4$, (20.1.3)] is incorrect according to [Kle79].

There are two ways to get integral subschemes of codimension 1. The first is named after Weil.

[Har77, pp. 130–131]
[Ful98, p. 10]

Definition 2.6.40 [EGAIV$_4$, (21.6.2)]. Let X be a Noetherian scheme. A *prime Weil divisor* on X is a closed integral subscheme $Y \subseteq X$ of codimension 1, that is, if η is the generic point of Y, then $\dim(\mathcal{O}_{X,\eta}) = 1$. A *Weil divisor* is an element of the free Abelian group $\operatorname{WDiv}(X)$ generated by prime divisors. We write

$$D = \sum_i n_i Y_i$$

where the Y_i are prime divisors, the n_i are integers, and only finitely many n_i are different from zero. If all the n_i are non-negative, we say that D is *effective*. The monoid of effective Weil divisors is denoted $\operatorname{WDiv}_{\geq 0}(X)$.

The second is named after Cartier. These are divisors defined locally by one equation.

[Har77, p. 141]
[Ful98, p. 29]

Definition 2.6.41 [EGAIV$_4$, Définition 21.1.2, Définition 21.1.6, and (21.2.12); Kle79]. Let X be a ringed space and let $\mathscr{K}_X$ be the sheaf of rational functions on X. Let $\mathscr{K}_X^*$ be the sheaf of invertible elements in the sheaf $\mathscr{K}_X$, which is an Abelian sheaf under multiplication. Similarly, let $\mathcal{O}_X^*$ be the sheaf of invertible elements in $\mathcal{O}_X$. A *Cartier divisor* is a global section D of the sheaf $\mathscr{K}_X^*/\mathcal{O}_X^*$. In other words, by properties of quotient sheaves, a Cartier divisor D on X can be defined by data of an open cover $\{U_i\}$ of X and for each i an element $f_i \in \Gamma(U_i, \mathscr{K}_X^*)$ called a *local equation* of D such that for all i, j, we have

$$\frac{f_i}{f_j} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*).$$

A Cartier divisor is *effective* if its stalks are in

$$S_x \coloneqq \varinjlim_{U \ni x} S(U)$$

for all $x \in X$.

While the group structure on $\Gamma(X, \mathscr{K}_X^*/\mathcal{O}_X^*)$ is multiplicative, we denote the group additively as $\operatorname{CDiv}(X)$. The monoid of effective Cartier divisors is denoted $\operatorname{CDiv}_{\geq 0}(X)$. If $D \in \operatorname{CDiv}_{\geq 0}(X)$, then its associated closed subscheme is $\mathsf{Y}(D)$.

Here is an example of a Weil divisor that is not Cartier.

[Har77, Ex. II.6.5.2]

Example 2.6.42 (Homework 12, Problem 3 in [Mur24ca]). Let k be a field of characteristic $\neq 2$ and consider

$$R = \frac{k[x, y, z]}{(xy - z^2)}.$$

This is a normal domain. This is the coordinate ring of an affine quadric cone in k^2. If $k = \mathbf{R}$, you can visualize this as a cone in $\mathbf{R}^2$ in (almost) the usual sense. See Figure 2.7.

Consider a ruling D of the cone given by the line $y = z = 0$, corresponding to the prime ideal $\mathfrak{p} = (y, z)$. The ideal $\mathfrak{p}$ is not locally principal at the origin at the origin $\mathfrak{m} = (x, y, z)$ since the vector space $\mathfrak{m}/\mathfrak{m}^2$ is three-dimensional over k, and the image of $\mathfrak{p} = (y, z)$ in $\mathfrak{m}/\mathfrak{m}^2$ is at least two-dimensional. We claim that $2D$ is Cartier, and in fact, is the principal divisor defined by the local equation y. We

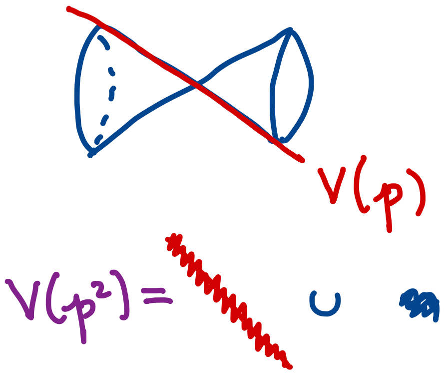

FIGURE 2.7. The ruling on a quadric cone is not Cartier. See also [Har77, Figure 8].

compute a primary decomposition for the second ordinary power of $\mathfrak{p}$:

$$\begin{aligned}\mathfrak{p}^2 &= (y^2, yz, z^2)\\ &= (y^2, yz, xy)\\ &= (y) \cap (x, z, y^2).\end{aligned}$$

The only $\mathfrak{p}$-primary component in this primary decomposition is (y), and hence we see that $\mathfrak{p}^{(2)} = (y)$. In terms of Weil divisors, this says that $2D$ is the Cartier divisor defined by y.

Cartier divisors often give rise to Weil divisors. $\boxed{4/10}$

THEOREM 2.6.43 [EGAIV$_4$, Proposition 21.6.6 and Théorème 21.6.9]. *Let X be a Noetherian scheme. There is an exact sequence* [Mum66, p. 66]

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathscr{K}_X^* \longrightarrow \sum_{\dim(\mathcal{O}_{X,x})=1} \mathbf{Z}_X$$

where the second map is defined on affine open subsets $U = \operatorname{Spec}(A) \subseteq X$ by

$$\begin{aligned}\mathscr{K}^*(U) &\longrightarrow \sum_{\dim(\mathcal{O}_{U,x})=1} \mathbf{Z}_U\\ \frac{f}{g} &\longmapsto \sum_{x\in U} \operatorname{ord}_{A_{\mathfrak{p}_x}}\left(\frac{f}{g}\right)\cdot \overline{\{x\}}\end{aligned} \tag{2.6.44}$$

where

$$\operatorname{ord}_{A_{\mathfrak{p}_x}}\left(\frac{f}{g}\right) := \operatorname{length}\big(A_{\mathfrak{p}_x}/f\big) - \operatorname{length}\big(A_{\mathfrak{p}_x}/g\big). \tag{2.6.45}$$

If X is normal, we have the following.

- *(i)* cyc *is injective and the image consists of locally principal Weil divisors.*
- *(ii) The following conditions are equivalent:* [Har77, Prop. II.6.2, Prop. II.6.11]
  - *(a)* cyc *is bijective.*
  - *(b) Every Weil divisor is locally principal.*

(c) *For every $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a UFD, in which case we say that X is locally factorial.*

Proof. The assignment (2.6.44) is a map of sheaves since in the definition (2.6.45) for $\mathrm{ord}_{A_{\mathfrak{p}_x}}(f/g)$, the difference on the right-hand side does not depend on the choice of representative for f/g [Ful98, Lemma A.2.5 and Definition A.3].

We now show that (2.6.44) is additive. Working with each term in the definition of $\mathrm{ord}_{A_{\mathfrak{p}_x}}(f/g)$ independently, it suffices to show that if $\mathcal{O}_{X,x}$ is a 1-dimensional local ring of X and $t, t' \in \mathcal{O}_{X,x}$ are nonzerodivisors, then

$$\mathrm{length}(A/tt'A) = \mathrm{length}(A/tA) + \mathrm{length}(A/t'A).$$

This follows from [Ful98, Lemma A.2.5] (which is a version of the additivity of length in short exact sequences [Mur24ca, Lemma 8.11.3]).

We now assume that X is normal. It suffices to show that

$$\mathrm{cyc}^{-1}\bigl(\mathrm{WDiv}_{\geq 0}(X)\bigr) = \mathrm{CDiv}_{\geq 0}(X)$$

since

$$\mathrm{WDiv}_{\geq 0}(X) \cap \bigl(-\mathrm{WDiv}_{\geq 0}(X)\bigr) = 0$$
$$\mathrm{CDiv}_{\geq 0}(X) \cap \ \bigl(-\mathrm{CDiv}_{\geq 0}(X)\bigr) = 0.$$

[EGAIV$_4$, Prop. 21.1.8]

We want to show that if $\mathrm{mult}_x(D) \geq 0$ for all $x \in X$ such that $\dim(\mathcal{O}_{X,x}) = 1$, then $D \geq 0$. This holds by the version of the algebraic Hartog's lemma from commutative algebra last semester [Mur24ca, Theorem 9.4.1], which says that for a Noetherian normal domain R, we have

$$R = \bigcap_{\mathrm{ht}(P)=1} R_P$$

as a subset of $\mathrm{Frac}(R)$. The characterization for when cyc is bijective is also from commutative algebra last semester [Mur24ca, Theorem 9.3.4(*iii*)]. ☐

We want to do some fun geometry with divisors.

[Har77, p. 131, p. 141]

Definition 2.6.46 [EGAIV$_4$, (21.3.1) and (21.6.7)]. Let X be a ringed space. We say that a Cartier divisor is *principal* if it lies in the image of the natural map

$$\Gamma(X, \mathscr{K}_X^*) \longrightarrow \Gamma(X, \mathscr{K}_X^*/\mathcal{O}_X^*).$$

The subgroup of $\mathrm{CDiv}(X)$ consisting of principal Cartier divisors is denoted by $\mathrm{CDiv}_{\mathrm{princ}}(X)$.

Now suppose that X is a Noetherian scheme. We say that a Weil divisor is *principal* if it lies in the image of $\mathrm{CDiv}_{\mathrm{princ}}(X)$ under cyc. The subgroup of $\mathrm{WDiv}(X)$ consisting of principal Weil divisors is denoted by $\mathrm{WDiv}_{\mathrm{princ}}(X)$.

We say that $D, D' \in \mathrm{CDiv}(X)$ (resp. $\mathrm{WDiv}(X)$) are *linearly equivalent* and write $D \sim D'$ if $D - D'$ is principal. The *divisor class group of X* is

$$\mathrm{Cl}(X) \coloneqq \mathrm{WDiv}(X)/{\sim}$$

for Noetherian schemes X. Similarly, we set

$$\mathrm{CaCl}(R) \coloneqq \mathrm{CDiv}(X)/{\sim}$$

for ringed spaces X. By definition and Theorem 2.6.43, the cycle map descends to the cycle class map

$$\mathrm{cyc}\colon\ \mathrm{CaCl}(X) \longrightarrow \mathrm{Cl}(X)$$

for Noetherian schemes X, and this map is injective if X is normal.

The divisor class group is a very interesting invariant, but is difficult to calculate. We already saw that it can detect when a scheme is locally factorial in Theorem 2.6.43(*ii*). Let us compute more examples.

Proposition 2.6.47 [Har77, Proposition II.6.4]. *Let $X = \mathbf{P}^n_k$ for a field k. For any divisor $D = \sum_i n_i Y_i$, define the degree by $\deg(D) = \sum_i n_i \deg(Y_i)$ where $\deg(Y_i)$ is the degree of the hypersurface Y_i. Let $H = V_+(x_0)$. We then have the following.*

(*a*) *If D is a divisor of degree d, then $D \sim dH$.*
(*b*) *For any $f \in K(X)^*$, we have $\deg(\operatorname{div}(f)) = 0$.*
(*c*) *The degree function gives an isomorphism*

$$\deg\colon \operatorname{Cl}(X) \xrightarrow{\sim} \mathbf{Z}.$$

Proof. Let $S = k[x_0, x_1, \ldots, x_n]$ be the homogeneous coordinate ring of X. Let $f = g/h \in K$ be a rational function and factor the homogeneous polynommials g and h as

$$g = g_1^{m_1} g_2^{m_2} \cdots g_r^{m_r} \qquad \text{and} \qquad h = h_1^{n_1} h_2^{n_2} \cdots h_s^{n_s}.$$

Then, each g_i defines a hypersurface Y_i of degree $d_i = \deg(g_i)$ and each h_j defines a hypersurface Z_j of degree $d_j = \deg(h_j)$. The *divisor of zeros of f* is [EGAIV$_4$, (21.6.7)]

$$Z^+(f) := \sum_i m_i Y_i$$

and the *divisor of poles of f* is

$$Z^-(f) := \sum_j n_j Z_j.$$

If $f \in K(X)^*$, then $\deg(g) = \deg(h)$, and hence

$$\deg\bigl(\operatorname{div}(f)\bigr) = \deg\bigl(Z^+(f)\bigr) - \deg\bigl(Z^-(f)\bigr) = 0,$$

proving (*b*).

To prove (*a*), write $D = D_1 - D_2$ as a difference of effective divisors such that $\deg(D_1) - \deg(D_2) = d$. We then have $D_1 = Z^+(g_1)$ and $D_2 = Z^+(g_2)$ for some homogeneous $g_1, g_2 \in S$ since homogeneous prime ideals of height 1 in S are principal. We then have $D - dH = \operatorname{div}(f)$ where $f = g_1/x_0^d g_2$, proving (*a*).

Finally, (*c*) holds by combining (*a*), (*b*), and the fact that $\deg(H) = 1$. □

We also have the scheme-theoretic version of Homework 12, Problem 2 in MA557 last semester.

Proposition 2.6.48 [Har77, Proposition II.6.5]. *Let X be a Noetherian scheme. Let $Z \subsetneq X$ be a proper closed subset with dense open complement U. We then have the following:*

(*a*) *There is a surjective homomorphism*

$$\begin{tikzcd} \operatorname{Cl}(X) \arrow[r] & \operatorname{Cl}(U) \\ \displaystyle\sum_i n_i Y_i \arrow[r, mapsto] & \displaystyle\sum_{\{i \mid Y_i \cap U \neq \emptyset\}} n_i (Y_i \cap U). \end{tikzcd}$$

(*b*) *If $\operatorname{codim}_X(Z) \geq 2$, then $\operatorname{Cl}(X) \to \operatorname{Cl}(U)$ is an isomorphism.*

(c) If Z is an irreducible subset of codimenion 1, then there is an exact sequence

$$\begin{aligned} \mathbf{Z} &\longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0\\ 1 &\longmapsto 1\cdot[Z]. \end{aligned}$$

Proof. (*a*). If Y is a prime divisor on X, then $Y \cap U$ is either empty or prime. If $f \in K(X)^*$ and $\mathrm{div}(f) = \sum_i n_i Y_i$, then considering f as a rational function on U, we have

$$\mathrm{div}(f)|_U = \sum_{\{i \mid Y_i \cap U \neq \emptyset\}} n_i(Y_i \cap U).$$

This map is surjective since every prime divisor on U is the restriction of its closure in X.

(*b*). Both $\mathrm{WDiv}(X)$ and $\mathrm{Cl}(X)$ depend only the points in X of codimension 1. Thus, removing a closed subset Z of codimension ≥ 2 does not change anything.

(*c*). The kernel of $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ consists of divisors whose support is contained in Z. If Z is irreducible, the kernel is just the subgroup of $\mathrm{Cl}(X)$ generated by $1 \cdot [Z]$. □

[Har77, Ex. II.6.5.1] EXAMPLE 2.6.49. Let $Y \subseteq \mathbf{P}^2_k$ be an irreducible curve of degree d. By Propositions 2.6.47 and 2.6.48, we see that

$$\mathrm{Cl}\bigl(\mathbf{P}^2_k - Y\bigr) = \mathbf{Z}/d\mathbf{Z}.$$

[Har77, Ex. II.6.5.2] EXAMPLE 2.6.50. Let k be a field of characteristic $\neq 2$ and consider

$$R = \frac{k[x,y,z]}{(xy - z^2)}$$

from last time. Let Y be a ruling of the cone given by the line $y = z = 0$. We then have the right exact sequence

$$\begin{aligned} \mathbf{Z} &\longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(X - Y) \longrightarrow 0\\ 1 &\longmapsto 1\cdot[Y] \end{aligned}$$

by Proposition 2.6.48. Since $\mathrm{sp}(Y) = \mathrm{sp}(V(y))$ as we saw last time, we have $X - Y = \mathrm{Spec}(R_y)$. Since

$$R_y \cong \frac{k[x,y^{\pm},z]}{(x - yz^2)} \cong k[y^{\pm}, z]$$

is a UFD, we have $\mathrm{Cl}(X - Y) = 0$. By the right exact sequence above, this shows that $\mathrm{Cl}(X)$ is generated by Y and that $2 \cdot Y = 0$. Finally, we saw last time that Y is not locally principal at the origin $\mathfrak{m} = (x,y,z)$, and hence

$$\mathrm{Cl}(X) \cong \mathbf{Z}/2\mathbf{Z}.$$

We also investigate what happens to class groups when adjoining a variable.

PROPOSITION 2.6.51 [Har77, Proposition II.6.6]. *Let X be an integral Noetherian scheme. Then, we have*

$$\mathrm{Cl}(X) \cong \mathrm{Cl}\bigl(X \times_{\mathbf{Z}} \mathbf{A}^1_{\mathbf{Z}}\bigr).$$

Proof. Denote by $\pi\colon X \times_{\mathbf{Z}} \mathbf{A}^1_{\mathbf{Z}} \to X$ the first projection map. We want to define a map

$$\begin{aligned} \pi^*\colon\ \mathrm{Cl}(X) &\longrightarrow \mathrm{Cl}\big(X \times_{\mathbf{Z}} \mathbf{A}^1_{\mathbf{Z}}\big) \\ \sum_i n_i[Y_i] &\longmapsto \sum_i n_i\big[\pi^{-1}(Y_i)\big]. \end{aligned}$$

This map is well-defined on WDiv. If $f \in K(X)^*$, then

$$\pi^*\big(\mathrm{div}(f)\big) = \mathrm{div}(\pi^{\#} f),$$

where we think of $\pi^{\#} f$ as an element of $K(X)(t)$, the function field of $X \times_{\mathbf{Z}} \mathbf{A}^1_{\mathbf{Z}}$. This shows the map descends to Cl.

We show that π^* is injective. Suppose $D \in \mathrm{WDiv}(X)$ is such that $\pi^* D = \mathrm{div}(f)$ for $f \in K(X)(t)$. Since every prime divisor appearing in $\pi^* D$ maps to a divisor in X, we know that f must be in $K(X)$. Otherwise, we could write $f = g/h$ for $g, h \in K(X)[t]$ relatively prime that involve a prime divisor mapping surjectively onto X.

To show that π^* is surjective, it suffices to show that any prime divisor in $X \times_{\mathbf{Z}} \mathbf{A}^1_{\mathbf{Z}}$ is linearly equivalent to a linear combination of prime divisors mapping to divisors in X. Let $Z \subseteq X \times_{\mathbf{Z}} \mathbf{A}^1_{\mathbf{Z}}$ be a prime divisor surjecting onto X. Localizing at the generic point of X, we obtain a prime divisor in $\mathrm{Spec}(K(X)[t])$, which corresponds to a prime ideal $\mathfrak{p} \subseteq K(X)[t]$. The ideal $\mathfrak{p}$ is principal with generator f since $K(X)[t]$ is a PID. Then, $f \in K(X)(t)$, and

$$\mathrm{div}(f) = Z + \text{divisors mapping to divisors in } X. \qquad \square$$

Example 2.6.52. Let k be a field. Let [Har77, Ex. II.6.6.1]

$$Q = V_+(xy - zw) \subseteq \mathbf{P}^3_k.$$

We will show that

$$\mathrm{Cl}(Q) \cong \mathbf{Z} \oplus \mathbf{Z}.$$

We know that $Q \cong \mathbf{P}^1_k \times_k \mathbf{P}^1_k$ from last semester. Let $p_1, p_2\colon Q \to \mathbf{P}^1_k$ be the projection maps. Let $Y = \{*\} \times_k \mathbf{P}^1_k$. Then,

$$Q - Y = \mathbf{A}^1_k \times_k \mathbf{P}^1_k$$

and the composition

$$\mathrm{Cl}(\mathbf{P}^1_k) \xrightarrow{p_2^*} \mathrm{Cl}(Q) \longrightarrow \mathrm{Cl}(\mathbf{A}^1_k \times_k \mathbf{P}^1_k)$$

is the isomorphism from Proposition 2.6.51. Thus, p_2^* is injective, and similarly for p_1^*. We now consider the exact sequence from Proposition 2.6.48:

$$\begin{aligned} \mathbf{Z} &\longrightarrow \mathrm{Cl}(Q) \longrightarrow \mathrm{Cl}(\mathbf{A}^1_k \times_k \mathbf{P}^1_k) \longrightarrow 0 \\ 1 &\longmapsto 1 \cdot [Y]. \end{aligned}$$

Identifying $\mathrm{Cl}(\mathbf{P}^1_k)$ with $\mathbf{Z}$ by letting 1 be the class of a closed point the first map can be identified with p_1^*, and is therefore injective. Since the image of p_2^* maps isomorphically to $\mathrm{Cl}(\mathbf{A}^1_k \times_k \mathbf{P}^1_k)$, we see that this exact sequence is split exact, and hence

$$\mathrm{Cl}(Q) \cong \mathbf{Z} \oplus \mathbf{Z}.$$

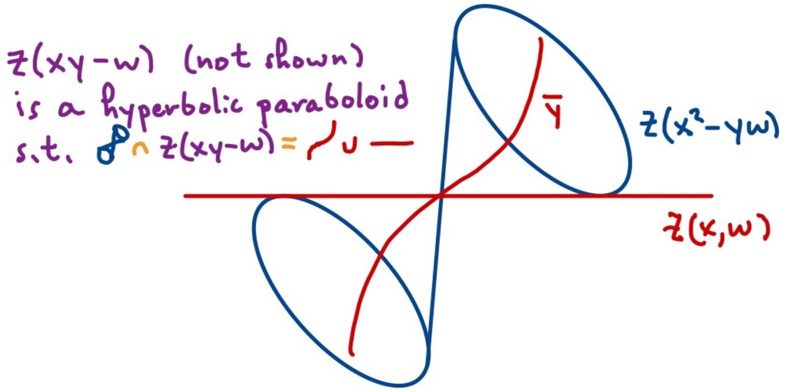

FIGURE 2.8. The twisted cubic curve is an irreducible component of an intersection of two quadrics. What is shown is the affine chart $z \neq 0$.

If D is any divisor on Q, let (a, b) be the ordered pair of integers in $\mathbf{Z} \oplus \mathbf{Z}$ corresponding to the class of D under this isomorphism. Then, we say that D is *of type* (a, b) on Q.

$\boxed{4/15}$ EXAMPLE 2.6.53. Let k be a field. Let us continue with the example

[Har77, Ex. II.6.6.2]

$$Q = V_+(xy - zw) \subseteq \mathbf{P}^3_k.$$

Then, restriction defines a homomorphism

$$\begin{aligned} \mathrm{Cl}(\mathbf{P}^3_k) &\longrightarrow \mathrm{Cl}(Q) \\ [H] &\longmapsto (1,1) \end{aligned}$$

as follows. One way to see this is that an irreducible hypersurface $Y \subseteq \mathbf{P}^3_k$ not containing Q is defined by a homogeneous polynomial f, and we can take the divisor of zeros after restricting f to Q. Since linearly equivalent divisors restrict to linearly equivalent divisors and any divisor on $\mathbf{P}^3_k$ is linearly equivalent to a prime divisor not containing Q (use Proposition 2.6.47 and homogeneous prime avoidance), we obtain a homomorphism of the form above.

Finally, we want to determine the image of H. The hyperplane $V_+(w)$ is a representative for the class of H, and

$$H|_Q = V_+(x, w) + V_+(y, w),$$

which is of type $(1, 1)$. Note these two lines correspond to the two rulings on Q.

[Har77, Ex. II.6.6.3] EXAMPLE 2.6.54. Let k be a field. Let $C \subseteq \mathbf{P}^3_k$ be the twisted cubic curve with $w = s^3, x = s^2t, y = st^2, z = t^3$, which lies on Q. If Y is the quadric cone $x^2 - yw$, then we claim $Y \cap Q = C \cup L$ where $L = V_+(x, w)$ is a line. We showed this last semester [Mur24ag, Example 1.2.19] at least as varieties.

To show $Y \cap Q = C \cup L$ as schemes, we need to show that

$$(x^2 - yw, xy - zw) = (x^2 - yw, xy - zw, y^2 - xz) \cap (x, w)$$

as ideals. The inclusion $\subseteq$ holds by checking on generators. For $\supseteq$, consider an element

$$\begin{aligned} &a \cdot (x^2 - yw) + b \cdot (xy - zw) + c \cdot (y^2 - xz) \\ &\qquad \in (x^2 - yw, xy - zw, y^2 - xz) \cap (x, w). \end{aligned}$$

Reducing modulo (x, w), we see that $c \cdot y^2 \in (x, w)$. Since $k[w, x, y, z]$ is a UFD, we see that $c = c_1 x + c_2 w$. Thus,

$$\begin{aligned} c\,(y^2 - xz) &= (c_1 x + c_2 w)(y^2 - xz) \\ &= c_1(xy^2 - x^2 z) + c_2(wy^2 - wxz) \\ &\equiv c_1(yzw - x^2 z) + c_2(x^2 y - wxz) \\ &= \ -c_1 z(x^2 - yw) + c_2 x(xy - zw) \end{aligned}$$

modulo $(x^2 - yw, xy - zw)$.

Since $Y \sim 2H$ on $\mathbf{P}^3_k$, we see that $Y \cap Q$ is of type $(2, 2)$. Since L has type $(1, 0)$, we see that C has type $(1, 2)$. This shows there cannot be a surface $Y \subseteq \mathbf{P}^3_k$ not containing Q such that $Y \cap Q = C$, even set-theoretically! For then, we would need to have $Y \cap Q = rC$ for some integer $r > 0$, which would be of type $(r, 2r)$ in $\mathrm{Cl}(Q)$. But if Y is of degree d, then $Y \cap Q$ is of type (d, d), which cannot equal $(r, 2r)$.

2.6.9. Divisors on curves. We want to apply the theory of divisors to curves.

DEFINITION 2.6.55. Let k be an algebraically closed field. A *curve* over k is an [Har77, p. 136]
integral separated scheme X of finite type over k of dimension 1. If all local rings of X are regular local rings, we say that X is *nonsingular*.

We prove some preliminaries about curves.

PROPOSITION 2.6.56 [Har77, Proposition II.6.7]. *Let k be an algebraically closed field. Let X be a nonsingular curve over k with function field K. Then, the following conditions are equivalent:*

- *(i) X is projective.*
- *(ii) X is complete.*
- *(iii) $X \cong t(C_K)$ where C_K is the abstract nonsingular curve of* [Mur24ag, Definition 1.6.13] *and t is the functor from varieties to schemes in Theorem 2.6.24.*

Proof. (*i*) $\Rightarrow$ (*ii*) holds by Theorem 2.6.27(*i*).

(*ii*) $\Rightarrow$ (*iii*). Recall from [Mur24ag, Definition 1.6.13] that

$$C_K \coloneqq \bigl\{\text{DVRs of } K/k\bigr\}$$

with the finite complement topology. For each point $P \in C_K$, we denote the corresponding DVR by $(R_P, \mathfrak{m}_P)$. The structure sheaf on C_K is

$$\mathcal{O}_{C_K}(U) = \bigcap_{P \in U} R_P.$$

Since X is complete, for every $P \in C_K$, there is a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(R_P) \arrow[r] \arrow[ur, dashed, "\exists!"'] & \mathrm{Spec}(k) \end{tikzcd}$$

where the dashed map exists and is unique by the valuative criterion for properness (Theorem 2.5.60). This sets up a 1-1 correspondence between the closed points in X and the points in C_K. Finally, the description of structure sheaves is compatible with this bijection, and hence $X \cong t(C_K)$.

(*iii*) $\Rightarrow$ (*i*). Since C_K is isomorphic to a projective variety by [Mur24ag, Theorem 1.6.19], we see that X is projective (as a scheme). ☐

PROPOSITION 2.6.57 [Har77, Proposition II.6.8]. *Let k be an algebraically closed field. Let X be a complete nonsingular curve over k, let Y be any curve over k, and let $f\colon X \to Y$ be a morphism. Then, either*

(1) *$f(X)$ is a point.*
(2) *$f(X) = Y$.*

In case (2), $K(X)$ is a finite extension field of $K(Y)$, f is a finite morphism, and Y is also complete.

Proof. Since X is complete, $f(X)$ is closed in Y and proper over $\operatorname{Spec}(k)$ by Homework 8, Problem 3(c) [Har77, Exercise II.4.4]. On the other hand, $f(X)$ is irreducible since it is the image of an irreducible topological space under a continuous map. Thus, either (1) or (2) holds, and in case (2), Y is complete.

It remains to show that in case (2), $K(X)$ is a finite extension field of $K(Y)$ and f is finite. Since f is dominant, it induces an inclusion $K(Y) \hookrightarrow K(X)$ of function fields. Since both fields are finitely generated extension fields of transcendence degree 1 over k, we see that $K(X)$ is a finite algebraic extension of $K(Y)$. It remains to show that f is a finite morphism. Consider the extensions

$$\mathcal{O}_Y \hookrightarrow f_*\mathcal{O}_X \hookrightarrow f_*\mathscr{K}_X$$

where $f_*\mathcal{O}_X$ is a coherent sheaf of $\mathcal{O}_Y$-algebras by applying Serre's finiteness Theorem 2.4.39(*i*) locally. We claim that for every $y \in Y$, $f_*\mathcal{O}_X$ is the integral closure of $\mathcal{O}_{Y,y}$ in K_X. This holds since $\mathcal{O}_Y \hookrightarrow f_*\mathcal{O}_X$ is integral and for every $V \subseteq Y$, the ring

$$\Gamma(V, f_*\mathcal{O}_X) = \Gamma\bigl(f^{-1}(V), \mathcal{O}_X\bigr) = \bigcap_{x \in f^{-1}(V)} \mathcal{O}_{X,x}$$

is integrally closed in $K(X)$. We therefore obtain the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \mathbf{Spec}_Y(f_*\mathcal{O}_X) \arrow[d, "\pi"] \\ & Y \end{tikzcd} \tag{2.6.58}$$

where $\mathbf{Spec}_Y(\,\cdot\,)$ is the relative Spec construction from Homework 7, Problem 3 [Har77, Exercise II.5.17]. Thus, the morphism π is finite. We claim that

$$X \longrightarrow \mathbf{Spec}_Y(f_*\mathcal{O}_X)$$

is an isomorphism. The morphism is birational, and hence this follows from [Mur24ag, Corollary 1.6.22]. Alternatively, since the morphism is birational, there is an open subset U in the target over which this morphism is an isomorphism (work locally and use the fact that the morphism is of finite type to clear denominators). By the curve-to-projective extension theorem (Theorem 2.6.30), the inverse rational map extends to all of $\mathbf{Spec}_Y(f_*\mathcal{O}_X)$. Since both curves are separated, the compositions in both directions must be isomorphisms (Proposition 2.5.26). ☐

REMARK 2.6.59. The factorization (2.6.58) is an example of a *Stein factorization*. See [Har77, Corollary III.11.5] for the statement of Stein factorization for projective morphisms and see [EGAIII_1, p. 131] for the original statement for schemes. The original statement for complex analytic spaces is [Ste56, Satz 12].

DEFINITION 2.6.60. Let $f\colon X \to Y$ be a finite dominant morphism of curves. The *degree* of f is [Har77, p. 137]

$$\deg(f) \coloneqq \big[K(X) : K(Y)\big].$$

We can define pullbacks of Weil divisors on nonsingular curves.

DEFINITION 2.6.61. Let $f\colon X \to Y$ be a finite dominant morphism of nonsingular curves. We define the homomorphism [Har77, p. 137]

$$f^*\colon \operatorname{WDiv}(Y) \longrightarrow \operatorname{WDiv}(X)$$

as follows. For any closed point $Q \in Y$, let $t \in \mathcal{O}_Q$ be a *local parameter* at Q, i.e., t is a uniformizer for $\mathcal{O}_Q$. We define

$$f^*Q = \sum_{f(P)=Q} \operatorname{mult}_P(t) \cdot P.$$

One can compute $\operatorname{mult}_P(t) = \operatorname{length}(\mathcal{O}_P/t)$ as $v_P(t)$, where v_P is the discrete valuation associated to $\mathcal{O}_P$. Since f is a finite morphism, this is a finite sum, and hence we get a Weil divisor on X. We extend the definition by linearity to all Weil divisors on Y. This operation preserves linear equivaence, and hence we obtain a homomorphism

$$f^*\colon \operatorname{Cl}(Y) \longrightarrow \operatorname{Cl}(X).$$

PROPOSITION 2.6.62. *Let $f\colon X \to Y$ be a finite dominant morphism of nonsingular curves. Then, for any Weil divisor D on Y, we have* [Har77, Prop. II.6.9]

$$\deg(f^*D) = \deg(f) \cdot \deg(D).$$

Proof. It suffices to show that for every closed point $Q \in Y$, we have

$$\deg(f^*Q) = \deg(f).$$

The inclusion

$$A \coloneqq \mathcal{O}_{Y,Q} \lhook\joinrel\longrightarrow \big(f_*\mathcal{O}_X\big)_Q \eqqcolon A'$$

is module-finite and the fact that f is dominant implies A' is torsion-free. Taking stalks at the generic point, we see that $\operatorname{rank}(A') = r$. Since A is a DVR, the classification of finitely generated modules over a PID shows that A' is a free module of rank r. Now by Incomparability, we know that $f^{-1}(Q)$ is a disjoint union of points P_i corresponding to maximal ideals $\mathfrak{m}_i \subseteq A'$. Letting t be a uniformizer for A, we have

$$tA' = \bigcap_i \big(tA'_{\mathfrak{m}_i} \cap A'\big).$$

By the Chinese remainder theorem, we have

$$\frac{A'}{tA'} \cong \prod_i \frac{A'}{tA'_{\mathfrak{m}_i} \cap A'} \cong \prod_i \frac{A'_{\mathfrak{m}_i}}{tA'_{\mathfrak{m}_i}}.$$

Taking lengths, we see that

$$\deg(f^*Q) = \operatorname{length}_{A'}(A'/tA') = r = \deg(f). \qquad \square$$

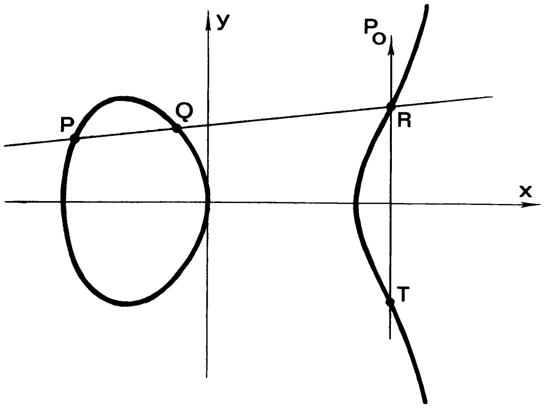

FIGURE 2.9. The group law on an elliptic curve.

[Har77, Cor. II.6.10]

COROLLARY 2.6.63. *A principal divisor on a complete nonsingular curve X has degree 0. Thus, the degree function induces a surjective homomorphism*

$$\deg\colon \operatorname{Cl}(X) \longrightarrow \mathbf{Z}.$$

Proof. Let $f \in K(X)^*$. If $f \in k$, then $\operatorname{div}(f) = 0$, and hence there is nothing to prove. if $f \notin k$, then the inclusion $k(f) \subseteq K(X)$ induces a finite morphism

$$\varphi\colon X \longrightarrow \mathbf{P}^1_k.$$

It it is a morphism by Theorem 2.6.30 and is finite by Proposition 2.6.57. We know that $\operatorname{div}(f) = \varphi^*(\{0\} - \{\infty\})$. Since $\deg(\{0\} - \{\infty\}) = 0$, we see that $\deg(\operatorname{div}(f)) = 0$. Thus, the degree of a Weil divisor on X only depends on its linear equivalence class, and we obtain a homomorphism $\operatorname{Cl}(X) \to \mathbf{Z}$. It is surjective since the degree of a closed point is 1. ☐

Let us look at two key examples of curves and divisors on them.

[Har77, Ex. II.6.10.1]

EXAMPLE 2.6.64. A complete nonsingular curve X is rational if and only if there are two distinct points $P, Q \in X$ with $P \sim Q$. Note that if X is rational, then $X \cong \mathbf{P}^1_k$ by Proposition 2.6.56.

We already know that any two points are linearly equivalent on $\mathbf{P}^1_k$ since $\operatorname{Cl}(\mathbf{P}^1_k) \cong \mathbf{Z}$. Conversely, suppose $P \neq Q$ are linearly equivalent. Then, there is a rational function $f \in K(X)$ such that $\operatorname{div}(f) = P - Q$. Consider the morphism $\varphi\colon X \to \mathbf{P}^1_k$ determined by f as in Corollary 2.6.63. We then have $\varphi^*(\{0\}) = P$, and hence φ is a morphism of degree 1. Thus, φ is birational, and hence X is rational.

EXAMPLE 2.6.65 (The group law on an elliptic curve). Suppose $\operatorname{char}(k) \neq 2$. Consider the nonsingular cubic curve

$$X := \{y^2 z = x^3 - xz^2\} \subseteq \mathbf{P}^2_k.$$

In [Mur24ag, Example 1.6.24], we saw that X is not rational. Let $\operatorname{Cl}^\circ(X) := \ker(\operatorname{Cl}(X) \to \mathbf{Z})$. The previous example shows that $\operatorname{Cl}^\circ(X) \neq 0$. We will show that tehre is a natural 1-1 correspondence between the set of closed points of X and the elements of the group $\operatorname{Cl}^\circ(X)$. This allows us to understand the group $\operatorname{Cl}^\circ(X)$ and also put a group variety structure on the closed points of X. See Figure 2.9.

Let $P_0 = [0:1:0]$. It is an inflection point, and hence the tangent line $z = 0$ meets X in the divisor $3P_0$. If L is any other line in $\mathbf{P}^2_k$ meeting X in three points P, Q, R, then since $L \sim \{z = 0\}$ in $\mathbf{P}^2_k$, we have

$$P + Q + R \sim 3P_0$$

on X.

We consider the map

$$\begin{aligned} \{\text{closed points in } X\} &\longrightarrow \mathrm{Cl}^\circ(X) \\ P &\longmapsto [P - P_0]. \end{aligned}$$

The map is injective since

$$P - P_0 \sim Q - P_0 \implies P \sim Q,$$

which would imply that X is rational by the previous example.

To show that the map from the closed points of X to $\mathrm{Cl}^\circ(X)$ is surjective, we proceed in steps. Let $D \in \mathrm{Cl}^\circ(X)$. Then, $D = \sum_i n_i P_i$ with $\sum_i n_i = 0$. Thus, we can write

$$D = \sum_i n_i (P_i - P_0).$$

Now for any point R, let the line $\overline{P_0R}$ meet X further in a point T (counting intersections with multiplicity, so if for example $R = P_0$, we take $\overline{P_0R}$ to be the tangent line at P_0, in which case the third intersection T is also P_0). Then,

$$P_0 + R + T \sim 3P_0,$$

and hence

$$R - P_0 \sim -(T - P_0).$$

If i is an index such that $n_i < 0$ in D, we take $P_i = R$. Then, replacing P_i by T, we get a linearly equivalent divisor with the i-th coefficient $-n_i > 0$. Continuing in this fashion, we may assume that D is effective. We now show by induction on $\sum_i n_i$ that $D \sim P - P_0$ for some point P. If $\sum_i n_i = 1$, there is nothing to prove. Suppose that $\sum_i n_i \geq 2$ and let P, Q be two of the points P_i which occur in D. Let the line $\overline{PQ}$ meet X in R and let the line P_0R meet X in T. Then, we have

$$P + Q + R \sim 3P_0 \qquad \text{and} \qquad P_0 + R + T \sim 3P_0$$

and hence

$$(P - P_0) + (Q - P_0) \sim (T - P_0).$$

Replacing P and Q by T, we get D linearly equivalent to another divisor of the same form whose $\sum_i n_i$ is one less. By induction, we see that

$$D \sim P - P_0$$

for some P.

We have therefore shown that $\mathrm{Cl}^\circ(X)$ is in 1-1 correspondence with the set of closed points of X. One can show directly that the addition law determines a morphism

$$X \times_k X \longrightarrow X$$

and the inverse law determines a morphism $X \to X$. Thus, X is a group variety.

See [Har77, Example IV.3.7] for a generalization using the Riemann–Roch theorem that works for any cubic plane curve with a choice of P_0.

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2.6.10. Cartier divisors, invertible sheaves, and divisorial sheaves. We want to relate Cartier divisors to invertible sheaves.

[Har77, p. 143]

Definition 2.6.66 [EGAI$_{\text{new}}$, Chapitre 0, (5.6.2)]. Let X be a ringed space. The *Picard group* $\operatorname{Pic}(X)$ *of* X is the group of isomorphism classes of invertible sheaves on X under the operation $\otimes$. By Homework 3, Problem 3 (see also [Har77, Proposition II.6.12]), $\operatorname{Pic}(X)$ is a group.

Remark 2.6.67. The isomorphism classes of invertible sheaves form a set because they can be specified (up to isomorphism) using a choice of open covering that trivializes the invertible sheaf, and a choice of transition map on intersections of members of that open covering. This is part of the content of Homework 7, Problem 10.

[Har77, Rem. II.6.12.1]

One of the ungraded problems (Homework 7, Problem 10) shows that

$$\operatorname{Pic}(X) \cong H^1(X, \mathcal{O}_X^*).$$

See [EGAI$_{\text{new}}$, Chapitre 0, (5.6.3)]. With this interpretation, the long exact sequence associated to

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathscr{K}_X^* \longrightarrow \mathscr{K}_X^*/\mathcal{O}_X^* \longrightarrow 0$$

yields a group homomorphism

$$\operatorname{CaCl}(X) \longrightarrow \operatorname{Pic}(X).$$

Concretely, this group homomorphism takes a Cartier divisor D represented by $\{U_i, f_i\}$ and maps it to the invertible sheaf $\mathcal{O}_X(D)$ defined below.

[Har77, p. 144]

Definition 2.6.68 [EGAIV$_4$, (21.2.8); Rei80, p. 282]. Let X be a ringed space and let D be a Cartier divisor on X. The *invertible sheaf* $\mathcal{O}_X(D)$ *associated to* D is the sub-$\mathcal{O}_X$-module of $\mathscr{K}_X$ consisting of sections

$$\Gamma\bigl(U, \mathcal{O}_X(D)\bigr) := \Bigl\{ f \in \mathscr{K}_X(U) \Bigm| \operatorname{div}_U(f) + D|_U \geq 0 \Bigr\}. \tag{2.6.69}$$

In other words, if D is represented by $\{U_i, f_i\}$ on each U_i, then

$$\mathcal{O}_X(D)\big|_{U_i} = \mathcal{O}_U \cdot f_i^{-1} \subseteq \mathscr{K}_U.$$

Let X be a normal Noetherian scheme and let D be a Weil divisor on X. The *divisorial sheaf* $\mathcal{O}_X(D)$ *associated to* D is defined using the same formula (2.6.69).

[Har77, Prop. II.6.13]
[Laz04, §1.1]
[Mur18b]

Remark 2.6.70. By definition, there is a 1-1 correspondence between Cartier divisors on X and invertible subsheaves of $\mathscr{K}_X$ and

$$\mathcal{O}_X(D_1 - D_2) \cong \mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2)^{-1}.$$

Moreover, $D_1 \sim D_2$ if and only if $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$ as abstract invertible sheaves since if $\operatorname{div}(f) = D_1 - D_2$, then

$$\mathcal{O}_X(D_1 - D_2) \cong f^{-1} \cdot \mathcal{O}_X \cong \mathcal{O}_X.$$

We therefore obtain an injective map

$$\begin{aligned} l\colon \operatorname{CaCl}(X) &\longrightarrow \operatorname{Pic}(X) \\ D &\longmapsto \mathcal{O}_X(D). \end{aligned}$$

There are non-reduced schemes for which the map l is not surjective [Kle00, §2; Sch00, §2]. The following proposition gives a sufficient condition for l to be surjective. The map l is surjective also when X is a projective scheme over an infinite field [Nak63, Theorem 4] or more generally, a Noetherian scheme with an ample invertible sheaf [EGAIV_4, Corollaire 21.3.5].

Proposition 2.6.71 [EGAIV_4, Proposition 21.3.4(b)]. *Let X be an integral scheme. Then, the map l is surjective, and hence $l\colon \mathrm{CaCl}(X) \to \mathrm{Pic}(X)$ is an isomorphism.*

[Har77, Prop. II.6.15]

Proof. We need to show that every invertible sheaf $\mathscr{L}$ is isomorphic to a subsheaf of $\mathscr{K}_X$. Let U be a dense open subset such that $\mathscr{L}|_U \cong \mathcal{O}_U$. On U, there is a section $s \in H^0(U, \mathscr{L}|_U)$ corresponding to 1 under this isomorphism. We may therefore define the injection

$$\begin{tikzcd}
\mathscr{L} \arrow[r, hook] & \mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{K}_X \arrow[r, "\sim"] & j_*\bigl(\mathscr{L}|_U \otimes_{\mathcal{O}_U} \mathscr{K}_U\bigr) & \mathscr{K}_X \arrow[l, "\sim"'] \\
 & & s \otimes 1 & 1 \arrow[l, mapsto]
\end{tikzcd}$$

since $\mathscr{K}_X$ is isomorphic to the skyscraper sheaf with value $K(X)$ at the generic point of X, and using the projection formula. ☐

2.6.11. Automorphisms of projective space. An important corollary of our work so far on divisors is the following.

Corollary 2.6.72. *Let k be a field and let $X = \mathbf{P}^n_k$. Then, every invertible sheaf on X is isomorphic to $\mathcal{O}(l)$ for some $l \in \mathbf{Z}$.*

[Har77, Cor. II.6.17]

Proof. We know that

$$\begin{tikzcd}
\mathbf{Z} & \mathrm{Cl}(X) \arrow[l, "\sim"'] & \mathrm{CaCl}(X) \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathrm{Pic}(X) \\
1 & {[H]} \arrow[l, mapsto] & {[H]} \arrow[l, mapsto] \arrow[r, mapsto] & \mathcal{O}_X(1)
\end{tikzcd}$$

by Proposition 2.6.47(c), Theorem 2.6.43(ii), and Proposition 2.6.71. ☐

We can use this to compute the automorphisms of $\mathbf{P}^n_k$.

Example 2.6.73. Let k be a field. We claim that

[Har77, Ex. II.7.1.1]
[GIT, §0.5, (b)]

$$\mathrm{Aut}_k(\mathbf{P}^n_k) \cong \mathrm{PGL}_k(n) := \mathrm{GL}_k(n+1)/k^*.$$

Every element of $\mathrm{PGL}_k(n)$ gives an automorphism. Conversely, we know that $\mathrm{Pic}(\mathbf{P}^n_k) \cong \mathbf{Z}$, and hence any automorphism φ satisfies

$$\varphi^*\mathcal{O}(1) \in \bigl\{\mathcal{O}(1), \mathcal{O}(-1)\bigr\}.$$

Only the former possibility can happen since $\mathcal{O}(-1)$ has no global sections. We see that φ induces a k-vector space automorphism of $H^0(\mathbf{P}^n_k, \mathcal{O}(1))$. Since φ is determined by the images of the basis elements $x_i \in H^0(\mathbf{P}^n_k, \mathcal{O}(1))$, we conclude that φ comes from an element in $\mathrm{PGL}_k(n)$.

2.7. Ample invertible sheaves

2.7.1. Characterization of ampleness via global generation. Recall the definitions of very ample, ample, and relatively ample invertible sheaves from Definition 2.6.18. We want to relate these three notions. We start by proving the following characterization of ampleness.

Proposition 2.7.1 [EGAII, Proposition 4.5.5; EGAIV$_1$, (1.7.14)]. *Let X be a quasi-compact quasi-separated scheme. Let $\mathscr{L}$ be an invertible $\mathcal{O}_X$-module and set*

$$S = \bigoplus_{n \geq 0} H^0(X, \mathscr{L}^{\otimes n}).$$

The following are equivalent.

[Har77, Thm. II.7.6] (a) *The open subsets X_f for homogeneous $f \in S$ form a basis for X.*

(b) *$\mathscr{L}$ is ample.*

[Har77, p. 153] (c) *For every quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$ of finite type, there exists an integer n_0 such that*

$$\mathscr{F}(n) := \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n}$$

is globally generated for all $n \geq n_0$.

Proof. (a) $\Leftrightarrow$ (b) holds by Proposition 2.6.17 and the qcqs lemma (Theorem 2.3.40).

(a) $\Rightarrow$ (c). We have a finite affine open cover X_{f_i} where

$$f_i \in H^0(X, \mathscr{L}^{\otimes n_i})$$

since if $U \subseteq X$ is an affine open subset and $X_f \subseteq U$, then $X_f = D(f) \subseteq U$ is affine. Replacing the f_i by suitable powers, which does not change the X_{f_i}, we may assume that the n_i are all equal to a fixed integer m. Consider $\mathscr{F}|_{X_{f_i}}$ for every i. By Serre's equivalence for affine schemes, we know that each $\mathscr{F}|_{X_{f_i}}$ is finitely generated by sections h_{ij} over X_{f_i}. By the qcqs lemma, there exists an integer k_0 such that

$$h_{ij} \otimes f_i^{\otimes k_0}$$

extends to a global section of $\mathscr{F}(k_0 m)$ for all i, j. Taking larger powers of f_i, the sections

$$h_{ij} \otimes f_i^{\otimes k}$$

extend to global sections of $\mathscr{F}(km)$ for all $k \geq k_0$, and for these values of k, the sheaf $\mathscr{F}(km)$ is globally generated. Now for every p such that $0 < p < m$, we know that $\mathscr{F}(p)$ is of finite type, and hence there exists an integer k_p such that

$$\mathscr{F}(p)(km) = \mathscr{F}(p + km)$$

is generated by global sections for all $k \geq k_p$. Taking

$$n_0 := \max_p \{k_p m\}$$

works, since any $n \geq n_0$ can be written as $n = km + p$ for $k \geq k_p$ and $0 \leq p < m$.

(c) $\Rightarrow$ (a). Let $x \in X$ be a point and let U be an open neighborhood of x. Let $\mathscr{I}$ be a quasi-coherent ideal sheaf defining the complement of U with reduced scheme structure (Proposition 2.3.34). Let $\mathscr{I}' \subseteq \mathscr{I}$ be a quasi-coherent ideal sheaf of finite type contained in $\mathscr{I}$, which exists by [EGAI$_{\text{new}}$, Corollaire 6.9.9] (Homework 8, Problem 1). By hypothesis, there exists an integer n such that $\mathscr{I}'(n)$ is globally generated. Letting f be such a global section not vanishing at x, we see that $X_f \subseteq U$. $\square$

2.7.2. Cohomological criterion of ampleness. The following result characterizes ampleness in terms of sheaf cohomology.

Proposition 2.7.2 (Serre vanishing for ample invertible sheaves [EGAIII$_1$, Proposition 2.6.1]). *Let A be a Noetherian ring and let X be a proper scheme over* $\operatorname{Spec}(A)$*. Let $\mathscr{L}$ be an invertible $\mathcal{O}_X$-module. Then, the following conditions are equivalent.*

[Har77, Prop. III.5.3]
[Laz04, Thm. 1.2.6]

- (*i*) *$\mathscr{L}$ is ample.*
- (*ii*) (Serre vanishing for ample invertible sheaves) *For every coherent $\mathcal{O}_X$-module $\mathscr{F}$, there exists an integer n_0, depending on $\mathscr{F}$, such that for each $i > 0$ and for every $n \geq n_0$, we have*

$$H^i\bigl(X, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n}\bigr) = 0.$$

Proof. (*i*) ⇒ (*ii*). If $\mathscr{L}$ is ample, then $\mathscr{L}^{\otimes m}$ is very ample over $\operatorname{Spec}(A)$ for some $m > 0$ by Proposition 2.7.3(*ii*) below. Since $f\colon X \to \operatorname{Spec}(A)$ is proper and $\mathscr{L}^{\otimes m}$ is very ample over $\operatorname{Spec}(A)$, the morphism $f\colon X \to \operatorname{Spec}(A)$ is projective (the immersion in Definition 2.6.18(*i*) is a closed immersion). By Homework 8, Problem 2(*b*) [Stacks, Tag 087S(1)], we know that f is H-projective. Applying Serre's vanishing theorem for Proj (Theorem 2.4.39(*ii*)) to the sheaves

$$\mathscr{F}, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes 2}, \ldots, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes (m-1)}$$

we obtain (*ii*).

(*ii*) ⇒ (*i*). We will show there exists an integer n_0 such that $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n}$ is globally generated for all $n \geq n_0$. This suffices by Proposition 2.7.1.

Let $P \in X$ be a closed point and let $\mathscr{I}_P$ be the coherent ideal sheaf of the closed subset $\{P\}$ with reduced scheme structure. Then, there is a short exact sequence

$$0 \longrightarrow \mathscr{I}_P \cdot \mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \mathscr{F} \otimes_{\mathcal{O}_X} k(P) \longrightarrow 0$$

where $k(P)$ is the skyscraper sheaf $\mathcal{O}_X/\mathscr{I}_P$. Tensoring with $\mathscr{L}^{\otimes n}$, we get the short exact sequence

$$0 \longrightarrow \mathscr{I}_P \cdot \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n} \longrightarrow \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n} \longrightarrow \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n} \otimes_{\mathcal{O}_X} k(P) \longrightarrow 0.$$

By (*ii*), there is an integer n_0 such that

$$H^1\bigl(X, \mathscr{I}_P \cdot \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n}\bigr) = 0$$

for all $n \geq n_0$. By the long exact sequence on cohomology, the restriction map

$$H^0\bigl(X, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n}\bigr) \twoheadrightarrow H^0\bigl(X, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n} \otimes_{\mathcal{O}_X} k(P)\bigr)$$

is surjective for all $n \geq n_0$. By NAK [Mur24ca, Lemma 2.3.8] applied to the local ring $\mathcal{O}_{X,P}$, the generating set for $H^0(X, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n} \otimes_{\mathcal{O}_X} k(P))$ coming from $H^0(X, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n})$ also generates the stalk $\mathscr{F}_P \otimes_{\mathcal{O}_{X,P}} \mathscr{L}_P^{\otimes n}$ as an $\mathcal{O}_{X,P}$-module. Thus, the evaluation map

$$H^0\bigl(X, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n}\bigr) \otimes_A \mathcal{O}_{X,P} \twoheadrightarrow \mathscr{F}_P \otimes_{\mathcal{O}_{X,P}} \mathscr{L}_P^{\otimes n}$$

is also surjective. Since the support of a coherent sheaf is closed, there exists an open neighborhood $U \ni P$ such that the evaluation map

$$H^0\bigl(X, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n}\bigr) \otimes_A \mathcal{O}_X \longrightarrow \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n}$$

is surjective over U. We refer to this surjectivity property as "globally generated over U."

We now take $\mathscr{F} = \mathcal{O}_X$ to find an integer $n_1 > 0$ and an open neighborhood V of P such that $\mathscr{L}^{n_1}$ is globally generated over V. Applying the previous paragraph to each $r \in \{0, 1, \ldots, n_1 - 1\}$, there exists a neighborhood U_r of P such that $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes(n_0+r)}$ is globally generated over U_r. Now let

$$U_P = V \cap U_0 \cap \cdots \cap U_{n_1-1}.$$

Then, all of the sheaves $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n}$ are globally generated over U_P for all $n \geq n_0$ since

$$\bigl(\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes(n_0+r)}\bigr) \otimes_{\mathcal{O}_X} \bigl(\mathscr{L}^{\otimes n_1}\bigr)^{\otimes m}$$

for suitable $0 \leq r < n_1$ and $m \geq 0$.

Finally, we cover X by a finite number of open sets U_P for various closed points P, and let the new n_0 be the maximum of the n_0 corresponding to those points P. Then, $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n}$ is globally generated on all of X for all $n \geq n_0$. □

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Serre's global generation Theorem 2.4.39(*iii*) says that when $X \subseteq \mathbf{P}^n_A$ is a closed subscheme over a Noetherian ring A, the restriction of $\mathcal{O}(1)$ is ample. More generally, we have:

Proposition 2.7.3 [EGAII, Proposition 4.5.10]. *Let Y be an affine scheme, let $\pi\colon X \to Y$ be a quasi-compact quasi-separated morphism, and let $\mathscr{L}$ be an invertible $\mathcal{O}_X$-module.*

(*i*) *If $\mathscr{L}$ is π-very ample, then $\mathscr{L}$ is ample.*

[Har77, Thm. II.7.6] [Har77, Exer. II.7.5(e)]

(*ii*) *Suppose that π is of finite type and that Y is Noetherian. The following conditions are equivalent.*
  - (*a*) *$\mathscr{L}$ is ample.*
  - (*b*) *There exists an integer $n_0 > 0$ such that $\mathscr{L}^{\otimes n}$ is π-very ample for all $n \geq n_0$.*
  - (*c*) *There exists an integer $n > 0$ such that $\mathscr{L}^{\otimes n}$ is π-very ample.*

Proof. (*i*). Write $Y = \operatorname{Spec}(A)$. Then, there exists an A-module E and a surjective map

$$\psi\colon \pi^*\Bigl(\bigl(\operatorname{Sym}^\bullet_A(E)\bigr)^\sim\Bigr) \twoheadrightarrow \bigoplus_{n \geq 0} \mathscr{L}^{\otimes n}$$

such that $i = r_{\mathscr{L},\psi}$ is an everywhere defined immersion $X \to \mathbf{P}_Y(\tilde{E})$ such that $\mathscr{L} = i^*\mathcal{O}(1)$. Since the $D_+(f)$ for f homogeneous form a basis for the topology of $\mathbf{P}_Y(\tilde{E})$ and

$$i^{-1}\bigl(D_+(f)\bigr) = X_{\psi^\flat(f)}$$

we see that Proposition 2.7.1(*a*) is satisfied.

(*ii*). (*a*) ⇒ (*b*). By Proposition 2.7.1 and the quasi-compactness of X, there exist finitely many sections

$$f_i \in H^0(X, \mathscr{L}^{\otimes m_i})$$

for some $m_i > 0$ such that the X_{f_i} form an affine open cover of X. Replacing the m_i by their least common multiple and the f_i by suitable powers, we may assume the m_i are all equal to a common value m. For each i, set

$$B_i = H^0(X_{f_i}, \mathcal{O}_{X_i}).$$

Since X is of finite type over A, each B_i is an A-algebra of finite type by Homework 5, Problem 4(*a*) [Har77, Exercise II.3.3]. Let $\{b_{ij}\}_{j=1}^{k_i}$ be a set of generators for B_i

as an A-algebra. By the qcqs lemma (Theorem 2.3.40), for each i, j, there exists an integer n_{ij} such that $f_i^{n_{ij}} b_{ij}$ extends to a global section in $H^0(X, \mathscr{L}^{\otimes n_{ij}})$. Taking the least common multiple of the n_{ij} and m, we may choose a common integer n such that the $f_i^n b_{ij}$ extend to global sections

$$c_{ij} \in H^0(X, \mathscr{L}^{\otimes n}).$$

We now take the sections

$$\big\{f_i^{n/m}\big\}_i \cup \{c_{ij}\}_{i,j} \subseteq H^0(X, \mathscr{L}^{\otimes n}).$$

Since the $f_i^{n/m}$ generate $\mathscr{L}^{\otimes n}$, these sections generate $\mathscr{L}^{\otimes n}$, and therefore define a morphism

$$\varphi \colon X \longrightarrow \mathbf{P}_A^N.$$

We claim that φ is an immersion. Over each X_{f_i}, the map on coordinate rings is

$$\begin{tikzcd}
A\big[\{y_i\},\{y_{ij}\}\big] \arrow[r, two heads] & B_i \\
y_i \arrow[r, mapsto] & f_i^{n/m} \\
y_{ij} \arrow[r, mapsto] & c_{ij}.
\end{tikzcd}$$

Thus, the map $X_{f_i} \to D_+(y_i)$ is an immersion. This shows that $\mathscr{L}^{\otimes n}$ is π-very ample.

[Har77, Exer. II.7.5(d)]

Now let $n_1 > 0$ be an integer such that $\mathscr{L}^{\otimes n'}$ is globally generated for all $n' \geq n_1$, which exists by Proposition 2.7.1. Since X is quasi-compact and $\mathscr{L}^{\otimes n'}$ is coherent, there exist finitely many global sections of $\mathscr{L}^{\otimes n'}$ generating $\mathscr{L}^{\otimes n'}$. Taking the morphism $X \to \mathbf{P}_A^{N'}$ corresponding to these finitely many sections, we obtain the immersion

$$X \lhook\joinrel\longrightarrow \mathbf{P}_A^N \times_A \mathbf{P}_A^{N'} \overset{\sigma}{\lhook\joinrel\longrightarrow} \mathbf{P}_A^M$$

which shows that $\mathscr{L}^{\otimes(n+n')}$ is π-very ample for all $n' \geq n_1$.

(b) $\Rightarrow$ (c) is clear by setting $n = n_0$.

(c) $\Rightarrow$ (a). Consider the immersion

$$X \lhook\joinrel\longrightarrow \mathbf{P}_A(\mathscr{E}) \lhook\joinrel\longrightarrow \mathbf{P}_A^N$$

which exists by the definition of π-very ample. Denote by $\bar{X}$ the scheme-theoretic closure of X in $\mathbf{P}_A^N$. Let $\mathscr{F}$ be a coherent $\mathcal{O}_X$-module. By Homework 8, Problem 1(d) [EGAI$_{\text{new}}$, Théorème 6.9.7], there exists a coherent $\mathcal{O}_{\bar{X}}$-module $\bar{\mathscr{F}}$ such that $\bar{\mathscr{F}}|_X \cong \mathscr{F}$. By Serre's global generation Theorem 2.4.39(iii) (we use the Noetherianity of A here), there exists an integer n_0 such that $\bar{\mathscr{F}}(n)$ is globally generated for all $n \geq n_0$. Thus,

$$\bar{\mathscr{F}}(n)\big|_X \cong \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n}$$

is globally generated for all $n \geq n_0$. By Proposition 2.7.1, $\mathscr{L}$ is ample. □

2.7.3. Linear systems.

We setup the language of linear systems.

Definition 2.7.4 [Har94, p. 304]. Let X be a Noetherian scheme and let $\mathscr{L}$ be a torsion-free coherent $\mathcal{O}_X$-module of rank 1. A global section $s \in H^0(X, \mathscr{L})$ is *nondegenerate* if it generates $\mathscr{L}_\eta$ for every generic point of an irreducible component $\eta \in X$.

Linear systems will arise as sets that parametrize effective Cartier or Weil divisors. For Weil divisors, we are using the characterization of sheaves associated to Weil divisors as reflexive sheaves of rank 1 on normal Noetherian schemes. See [Rei80, Appendix to §1, Proposition 2; BouCA, Chapter VII, §4, no. 2, Theorem 2].

Proposition 2.7.5 [Mum66, Proposition on p. 64; Har77, Proposition II.7.7; Har94, Proposition 2.9]. *Let X be a Noetherian scheme and let $\mathscr{L}$ be a torsion-free coherent $\mathcal{O}_X$-module of rank 1. If $\mathscr{L}$ is invertible, there is a 1-1 correspondence*

$$\left\{\begin{matrix}\text{effective Cartier divisors}\\ D \text{ such that } \mathcal{O}_X(D) \cong \mathscr{L}\end{matrix}\right\} \overset{\text{1-1}}{\longleftrightarrow} \left\{\begin{matrix}\text{nondegenerate sections } s \in H^0(X,\mathscr{L})\\ \text{modulo } s \sim \alpha \cdot s \text{ for } \alpha \in H^0(X,\mathcal{O}_X^*)\end{matrix}\right\}.$$

If X is normal, there is a 1-1 correspondence

$$\left\{\begin{matrix}\text{effective Weil divisors}\\ D \text{ such that } \mathcal{O}_X(D) \cong \mathscr{L}\end{matrix}\right\} \overset{\text{1-1}}{\longleftrightarrow} \left\{\begin{matrix}\text{nondegenerate sections } s \in H^0(X,\mathscr{L})\\ \text{modulo } s \sim \alpha \cdot s \text{ for } \alpha \in H^0(X,\mathcal{O}_X^*)\end{matrix}\right\}.$$

Sketch of proof. We explain the maps in both directions.

Let a nondegenerate section s be given. We get an injective map

$$\begin{aligned} \mathcal{O}_X &\hookrightarrow \mathscr{L}\\ 1 &\longmapsto s. \end{aligned}$$

Dualizing gives an injective map $\mathscr{L}^\vee \hookrightarrow \mathcal{O}_X$ which identifies $\mathscr{L}^\vee$ with an ideal sheaf that, in the Cartier case, is locally principal.

Conversely, given an effective divisor D, consider its ideal sheaf $\mathcal{O}_X(-D) \subseteq \mathcal{O}_X$. Dualizing gives a map $\mathcal{O}_X \subseteq \mathcal{O}_X(D)$. The image of $1 \in H^0(X,\mathcal{O}_X)$ gives a nondegenerate section $s \in H^0(X,\mathcal{O}_X(D))$. □

To define linear systems, we need to know that coherent sheaves on complete schemes have finite dimensional global sections. This follows from Serre's finiteness Theorem 2.4.39(*i*) and Chow's lemma (Homework 9, Problem 1 [Cho57, Lemma 1; EGAII, Théorème 5.6.1]) which provides a birational projective morphism $\pi\colon \tilde{X} \to X$ such that $\tilde{X}$ is projective over k. Note that $\mathcal{O}_X \hookrightarrow \pi_*\mathcal{O}_{\tilde{X}}$, and hence

$$H^0(X,\mathcal{O}_X) \longhookrightarrow H^0(\tilde{X},\mathcal{O}_{\tilde{X}})$$

is finite-dimensional by Serre's finiteness Theorem 2.4.39(*i*). See Theorem 2.8.5 for a more general statement due to Cartan and Serre.

Definition 2.7.6 [Har77, p. 157]. Let X be a complete variety over a field k and let $\mathscr{L}$ be an invertible $\mathcal{O}_X$-module. The *complete linear system* associated to $\mathscr{L}$ is

$$|\mathscr{L}| := \frac{H^0(X,\mathscr{L}) - \{0\}}{k^*}.$$

Oftentimes we use the notation $|\mathscr{L}|$ to designate that a rational map is defined using the sections $H^0(X,\mathscr{L})$.

Let X be a variety over a field k and let $\mathscr{L}$ be an invertible $\mathcal{O}_X$-module. A *linear system* is the projective space

$$|V| := \frac{V - \{0\}}{k^*}$$

associated to a finite-dimensional k-vector subspace $V \subseteq H^0(X,\mathscr{L})$.

2.7.4. Base schemes and blowups. We want a way to measure how far an invertible sheaf is from being globally generated and a way to modify a rational map to become a morphism.

Definition 2.7.7 [Kür13, Definition 2.1]. Let X be a scheme and let $\mathscr{L}$ be an invertible $\mathcal{O}_X$-module. The *base ideal* of $\mathscr{L}$ is

$$\mathfrak{b}(\mathscr{L}) \coloneqq \operatorname{im}\Bigl(H^0(X,\mathscr{L}) \otimes_{\mathbf{Z}} \mathscr{L}^{-1} \xrightarrow{\text{eval}} \mathcal{O}_X\Bigr).$$

The associated closed subscheme is the *base scheme* $\operatorname{Bs}(\mathscr{L})$ of $\mathscr{L}$, whose underlying topological space is the *base locus* $\operatorname{Bs}(\mathscr{L})_{\text{red}}$ of $\mathscr{L}$.

If $V \subseteq H^0(X,\mathscr{L})$ is a subgroup, we define

$$\mathfrak{b}\bigl(|V|\bigr) \coloneqq \operatorname{im}\Bigl(V \otimes_{\mathbf{Z}} \mathscr{L}^{-1} \xrightarrow{\text{eval}} \mathcal{O}_X\Bigr).$$

The associated closed subscheme is the *base scheme* $\operatorname{Bs}|V|$ of $|V|$, whose underlying topological space is the *base locus* $\operatorname{Bs}|V|_{\text{red}}$ of $|V|$.

We now consider a hypothetical situation where X is a complete scheme over a field k and we have an invertible sheaf $\mathscr{L}$ with base ideal $\mathfrak{b} \coloneqq \mathfrak{b}(\mathscr{L})$. The associated morphism of sheaves is

$$H^0(X,\mathscr{L}) \otimes_k \mathscr{L}^{-1} \twoheadrightarrow \mathfrak{b} \subseteq \mathcal{O}_X.$$

Taking symmetric algebras and $\mathbf{Proj}_X$, we obtain the commutative diagram

$$\begin{tikzcd}
& \mathbf{Proj}_X\Bigl(\bigoplus_{n\ge 0} \mathfrak{b}^n\Bigr) \arrow[rr, hook] \arrow[dl, "\pi"'] \arrow[dr] & & \mathbf{P}_k\bigl(H^0(X,\mathscr{L})\bigr) \times_k X \arrow[dl, "\mathrm{pr}_1"] \\
X \arrow[rr, dashed, "|\mathscr{L}|"] & & \mathbf{P}_k\bigl(H^0(X,\mathscr{L})\bigr) &
\end{tikzcd}$$

such that $\mathcal{O}(1)$ pulls back to $\mathscr{L}$ on X and to $\pi^*\mathscr{L} \otimes \mathcal{O}(1)$ on $\mathbf{Proj}_X(\bigoplus_{n\ge 0} \mathfrak{b}^n)$.

This motivates the definition of a blowup.

Definition 2.7.8 [EGAII, Définition 8.1.3]. Let X be a scheme and let $\mathscr{I} \subseteq \mathcal{O}_X$ be a quasi-coherent ideal sheaf. The *blowup* of X along $\mathscr{I}$ is [Har77, p. 163]

$$\operatorname{Bl}_{\mathscr{I}}(X) \coloneqq \mathbf{Proj}_X\Bigl(\bigoplus_{n\ge 0} \mathscr{I}^n\Bigr).$$

If Y is the closed scheme defined by $\mathscr{I}$, we set $\operatorname{Bl}_Y(X) \coloneqq \operatorname{Bl}_{\mathscr{I}}(X)$.

Example 2.7.9. Let $X = \mathbf{A}^n_k$ with coordinates x_i for a field k. Let [Har77, Ex. II.7.12.1]

$$I = (x_1,\ldots,x_n) \subseteq k[x_1,\ldots,x_n] = A$$

be the ideal defining the origin $\mathbf{0}$. The blowup is $\operatorname{Proj}(S)$ where $\bigoplus_{n\ge 0} I^n$. Then, there is a surjective map of graded rings

$$\varphi\colon A[y_1,\ldots,y_n] \twoheadrightarrow \bigoplus_{n\ge 0} I^n$$

sending y_i to the copy of x_i in degree 1. Thus, $\operatorname{Bl}_{\mathbf{0}}(X)$ is the closed subscheme of $\mathbf{A}^n_k \times_k \mathbf{P}^{n-1}_k$ defined by the ideal

$$\ker(\varphi) = \Bigl\{x_iy_j - x_jy_i \Bigm| i,j \in \{1,2,\ldots,n\}\Bigr\}$$

where the coordinates on $\mathbf{P}_k^{n-1}$ are $y_1, y_2, \dots, y_n$.

Remark 2.7.10. The $\mathcal{O}_X$-algebra used to define the blowup is an example of a *Rees algebra*. Traditionally, Rees algebras are often denoted as $A[It]$, thought of as a graded subring of $A[t]$, where the exponent on t keeps track of the $\mathbf{N}$-grading on

$$A[It] = \bigoplus_{n \geq 0} I^n t^n.$$

We prove some basic properties of blowups.

[Har77, p. 163]

Definition 2.7.11. Let $f\colon X \to Y$ be a morphism of schemes and let $\mathscr{I} \subseteq \mathcal{O}_Y$ be an ideal sheaf. The *inverse image ideal sheaf* is

$$f^{-1}\mathscr{I} \cdot \mathcal{O}_X \subseteq \mathcal{O}_X.$$

[Har77, Prop. II.7.13]

Proposition 2.7.12. *Let X be a scheme, let $\mathscr{I} \subseteq \mathcal{O}_X$ be a quasi-coherent ideal sheaf, and let $\pi\colon \tilde{X} \to X$ be the blowup along $\mathscr{I}$.*

- (*a*) *The inverse image ideal sheaf $\pi^{-1}\mathscr{I} \cdot \mathcal{O}_{\tilde{X}}$ is invertible.*
- (*b*) *If Y is the closed subscheme defined by $\mathscr{I}$ and $U = X - Y$, then $\pi^{-1}(U) \to U$ is an isomorphism.*

Proof. The first statement follows from the fact that the invertible sheaf $\mathcal{O}(1)$ coming from the definition of relative **Proj** is the sheaf associated to

$$\mathscr{I} \cdot \bigoplus_{n \geq 0} \mathscr{I}^n \subseteq \bigoplus_{n \geq 0} \mathscr{I}^n,$$

and therefore $\mathcal{O}(1) \cong \pi^{-1}\mathscr{I} \cdot \mathcal{O}_{\tilde{X}}$. The second statement follows since **Proj** is compatible with passing to open subschemes of X by definition. $\square$

[Har77, Prop. II.7.14]

Proposition 2.7.13 (Universal property of blowups [Stacks, Tag 0806]). *Let X be a scheme and let $\mathscr{I} \subseteq \mathcal{O}_X$ be a quasi-coherent ideal. The blowup $\pi\colon \tilde{X} \to X$ along $\mathscr{I}$ satisfies the following universal property: If $f\colon Z \to X$ is any morphism such that $f^{-1}\mathscr{I} \cdot \mathcal{O}_Z$ is an invertible sheaf of ideals on Z, then there exists a unique morphism $g\colon Z \to \tilde{X}$ making the diagram*

$$\begin{tikzcd} Z \arrow[r, dashed, "g"] \arrow[dr, "f"'] & \tilde{X} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

commute.

Proof. We have a surjection

$$f^*\mathscr{I} \twoheadrightarrow f^{-1}\mathscr{I} \cdot \mathcal{O}_Z$$

of sheaves which yields the commutative diagram in question by the functoriality of **Proj**. Uniqueness follows from separatedness and the fact that any two morphisms making the diagram commute match over $U = X - V(\mathscr{I})$. $\square$

[Har77, Cor. II.7.15]

Corollary 2.7.14. *Let $f\colon Y \to X$ be a morphism of schemes and let $\mathscr{I} \subseteq \mathcal{O}_X$*

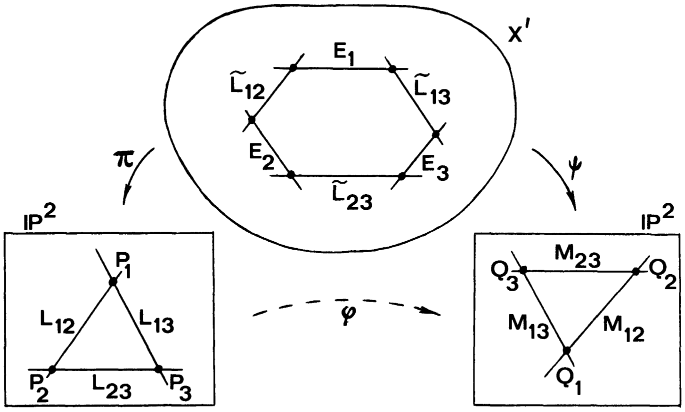

Figure 2.10. The quadratic transformation $\mathbf{P}^2_k \dashrightarrow \mathbf{P}^2_k$. From [Har77, Figure 21].

be a quasi-coherent ideal sheaf. Let $\tilde{X}$ *be the blowup of* $\mathscr{I}$ *and let* $\tilde{Y}$ *be the blowup of* $\mathscr{J} = f^{-1}\mathscr{I} \cdot \mathcal{O}_Y$*. Then, there is a unique morphism* $\tilde{f}\colon \tilde{Y} \to \tilde{X}$ *making the diagram*

$$\begin{tikzcd}
\tilde{Y} \arrow[r, dashed, "\tilde{f}"] \arrow[d] & \tilde{X} \arrow[d] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

commute. Moreover, if f *is a closed immersion, then so is* $\tilde{f}$*.*

Proof. The uniqueness and existence of $\tilde{f}$ follows from the previous proposition. To show the fact about closed immersions, we note that $\tilde{Y} \to \tilde{X}$ comes from the surjection

$$\bigoplus_{n\geq 0} \mathscr{I}^n \twoheadrightarrow \bigoplus_{n \geq 0} \mathscr{J}^n$$

of Rees algebras defining the blowups. □

Definition 2.7.15. In the situation of Corollary 2.7.14, if Y is a closed subscheme of X, we call $\tilde{Y}$ the *strict transform* of Y under the blowup $\pi\colon \tilde{X} \to X$. [Har77, p. 165]

We give one example.

Example 2.7.16 (Cremona transformations). Let k be a field and let $X = \mathbf{P}^2_k$. $\boxed{4/24}$ Consider the linear system associated to

$$V = \operatorname{span}_k\{xy, xz, yz\} \subseteq H^0\bigl(X, \mathcal{O}_X(2)\bigr).$$

The image of the evaluation morphism is

$$V \otimes_k \mathcal{O}_X(-2) \longrightarrow \mathcal{O}_X$$

which has image $(x,y) \cap (x,z) \cap (y,z)$ corresponding to points P, Q, R. We have the commutative diagram

$$\begin{tikzcd}
& \operatorname{Bl}_{\{P,Q,R\}}(X) \arrow[dl] \arrow[dr] & \\
X \arrow[rr, dashed, "|V|"] & & X.
\end{tikzcd}$$

This shows how useful the scheme-theoretic definition of the blowup is: Given a rational map defined by a linear system (or a morphism of sheaves), the blowup gives a "minimal" way to blowup X and turn the rational map into an actual morphism. See Figure 2.10 for an illustration.

There is a lot more one can say about blowups, for example:

[Har77, Thm. II.7.17]

Theorem 2.7.17 [EGAIII$_1$, Proposition 2.3.5 and Corollaire 2.3.6]. *Let $f\colon X \to Y$ be projective birational morphism between quasi-projective varieties over a field k, or more generally, integral Noetherian schemes for which there exists an ample invertible $\mathcal{O}_Y$-module on Y. Then, there exists a coherent ideal sheaf $\mathscr{I} \subseteq \mathcal{O}_Y$ such that f is isomorphic to the blowup of Y along $\mathscr{I}$.*

In the interest of time, we move on to talk about other topics.

2.8. Flat and smooth morphisms

2.8.1. Flat morphisms and flat base change. To begin, we first define the scheme-theoretic version of a flat ring map. The definition makes sense for any morphism of ringed spaces.

[Har77, p. 254]

Definition 2.8.1 [EGAI$_{\text{new}}$, Chapitre 0, (5.7.1)]. Let $f\colon X \to Y$ be a morphism of ringed spaces and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. We say that $\mathscr{F}$ is *f-flat* or *Y-flat at $x \in X$* if $\mathscr{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module via restriction of scalars along the ring map

$$f_x^\sharp\colon \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}.$$

We say that $\mathscr{F}$ is *f-flat above $y \in Y$* if $\mathscr{F}$ is f-flat at all $x \in f^{-1}(y)$. We say that $\mathscr{F}$ is *f-flat* if $\mathscr{F}$ is f-flat at all points in X.

We say that the morphism f is *flat at $x \in X$* (resp. *flat above $y \in Y$*, *flat*) if $\mathcal{O}_X$ is f-flat at x (resp. flat above y, flat). If f is flat, we say that X is *flat over Y* or *Y-flat*.

Flat morphisms are amazingly nice. For example, cohomology and higher direct images are compatible with flat base change. Before stating the theorem, we define higher direct images.

[Har77, p. 250]
[God73, §II.4.17]

Definition 2.8.2 [TohokuI, p. 173]. Let $f\colon X \to Y$ be a morphism of ringed spaces. The *higher direct image functors*

$$R^i f_*\colon \ \mathsf{Mod}(\mathcal{O}_X) \longrightarrow \mathsf{Mod}(\mathcal{O}_Y)$$

are the right derived functors of the direct image functor f_*. The definition makes sense since f_* is left exact and $\mathsf{Mod}(\mathcal{O}_X)$ has enough injectives (Proposition 1.3.41).

Higher direct images can be described in terms of sheaf cohomology of inverse images of open sets.

[Har77, Prop. III.8.1, Prop. III.8.5]

Proposition 2.8.3 [TohokuI, Lemme 3.7.2]. *Let $f\colon X \to Y$ be a morphism of ringed spaces. The sheaf*

$$\Bigl(V \longmapsto H^i\bigl(f^{-1}(V), \mathscr{F}|_{f^{-1}(V)}\bigr)\Bigr)^{\#}$$

defines a universal δ-functor that is isomorphic to the higher direct images $R^i f_*$.

If $f\colon X \to Y$ is a morphism of schemes such that X is Noetherian and $Y =$ Spec(A) is affine, then for every quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$, we have

$$R^i f_*(\mathscr{F}) \cong H^i(X, \mathscr{F})^{\sim}.$$

Proof. Following [God73, Chapitre II, §4.17], we denote the sheaf in the first statement as $\mathscr{H}^i(X,\mathscr{F})$. Since sheafification is exact, the functors $\mathscr{H}^i(X,\cdot)$ form a δ-functor from $\mathsf{Mod}(\mathcal{O}_X)$ to $\mathsf{Mod}(\mathcal{O}_Y)$. For $i=0$, we have $f_*\mathscr{F}=\mathscr{H}^0(X,\mathscr{F})$ by the definition of f_*. For an injective object $\mathscr{I}$ in $\mathsf{Mod}(\mathcal{O}_X)$, we have $\mathscr{H}^i(X,\mathscr{I})=0$ for all $i>0$ since $\mathscr{I}|_{f^{-1}}(V)$ is flasque for all open $V\subseteq Y$ by Lemma 1.4.23. Thus, $\mathscr{H}^i(X,\cdot)$ is effaceable, and hence is a universal δ-functor by Proposition 1.4.17.

For the second statement, it again suffices to show that $H^i(X,\mathscr{F})^\sim$ is effaceable. This holds since any quasi-coherent $\mathcal{O}_X$-module embeds in a flasque quasi-coherent $\mathcal{O}_X$-module by taking an affine open cover U_i of X, choosing injective modules I_i containing $\Gamma(U_i,\mathscr{F}|_{U_i})$, and then taking $\mathscr{F}\to\bigoplus_i\tilde{I}_i$, which is flasque by Homework 6, Problem 7(b) [Har77, Exercise III.3.7(b)]. See also [Har77, Proposition III.3.4 and Corollary III.3.6]. □

We can now prove that cohomology commutes with flat base change. This is extremely useful! It says for example that sheaf cohomology of a variety does not change if you extend the ground field. In practice, this means that you can often reduce to working over infinite, perfect, or algebraically closed fields, provided that you assume some geometric assumptions (geometrically irreducible, geometrically reduced, geometrically integral, geometrically normal, geometrically regular, etc.) on your variety.

Proposition 2.8.4 (Flat base change [EGAIII$_1$, Proposition 1.4.15]). *Let $f\colon X\to Y$ be a separated morphism of finite type between locally Noetherian schemes. Let $\mathscr{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $u\colon Y'\to Y$ be a morphism of schemes and consider the Cartesian diagram*

[Har77, Prop. III.9.3, Rem. III.9.3.1]
[Kem80, Cor. 9, Thm. 12]
See [Lip09, Thm. 3.10.3] for a more general statement.

$$\begin{tikzcd}
X' \arrow[r, "v"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "u"] & Y.
\end{tikzcd}$$

For every $i\geq 0$, there is a natural base change morphism

$$u^*R^if_*(\mathscr{F})\longrightarrow R^ig_*(v^*\mathscr{F}).$$

that is an isomorphism if u is flat.

Proof. We first construct the morphism. The unit of the adjunction $v^*\dashv v_*$ yields a natural morphism

$$\mathscr{F}\longrightarrow v_*\bigl(v^*(\mathscr{F})\bigr)$$

which, by functoriality, yields a morphism

$$R^if_*(\mathscr{F})\longrightarrow R^if_*\bigl(v_*\bigl(v^*(\mathscr{F})\bigr)\bigr).$$

Next, there are natural morphisms

$$R^if_*\bigl(v_*(\mathscr{G})\bigr)\longrightarrow R^i(f\circ v)_*(\mathscr{G})=R^i(u\circ g)_*(\mathscr{G})\longrightarrow u_*R^ig_*(\mathscr{G})$$

coming from the fact that a flasque resolution of $\mathscr{G}$ on X' pushes forward to a complex of flasque sheaves on X, which can be compared to an injective resolution of $v_*\mathscr{G}$. Composing these morphisms for $\mathscr{G}=v^*(\mathscr{F})$ and applying the adjunction $u^*\dashv u_*$, we obtain the natural morphism desired.

We now prove that the base change morphism is an isomorphism when u is flat. By Proposition 2.8.3, the question is local on Y and on Y'. We may therefore assume that $Y = \operatorname{Spec}(A)$ and $Y' = \operatorname{Spec}(A')$ where $A \to A'$ is a flat ring map, in which case we want to show that

$$H^i(X, \mathscr{F}) \otimes_A A' \longrightarrow H^i(X', v^*\mathscr{F})$$

is an isomorphism. Since X is separated and Noetherian and $\mathscr{F}$ is quasi-ciherent, we can calculate $H^i(X, \mathscr{F})$ using Čech cohomology (Theorem 2.5.29). Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an affine open cover of X. Then, $\mathfrak{U}' = \{v^{-1}(U_i)\}_{i \in I}$ is an affine open cover of X' because the base change of an affine morphism is affine. By Theorem 2.5.29 applied on X', it suffices to show that

$$\Gamma\bigl(X, \check{\mathscr{C}}^\bullet(\mathfrak{U}, \mathscr{F})\bigr) \otimes_A A' \longrightarrow \Gamma\bigl(X', \check{\mathscr{C}}^\bullet(\mathfrak{U}', v^*\mathscr{F})\bigr)$$

induces an isomorphism on cohomology. In this case, this map of complexes is actually an isomorphism by definition of the Čech complex and the formula for inverse images of quasi-coherent sheaves under affine morphisms (Corollary 2.2.13(*iii*)). $\square$

2.8.2. The Cartan–Serre finiteness theorem (not proved in class). To define linear systems, we needed the following generalization of Serre's finiteness Theorem 2.4.39(*i*) in the special case where $Y = \operatorname{Spec}(k)$ and $i = 0$. For completeness, we state it here.

[Har77, Thm. III.8.8(*b*)]

Theorem 2.8.5 (Cartan–Serre finiteness theorem [CS53, Théorème; EGAIII$_1$, Théorème 3.2.1]). *Let $f\colon X \to Y$ be a proper morphism of locally Noetherian schemes and let $\mathscr{F}$ be a coherent $\mathcal{O}_X$-module. Then, $R^i f_* \mathscr{F}$ is coherent for every i.*

2.8.3. Flat families. We often think of flat morphisms as families of schemes parametrized by the target scheme. For example, Hilbert polynomials are constant in flat families [Har77, Theorem III.9.9], which means that the dimension, degree, and arithmetic genus are constant in flat families [Har77, Corollary III.9.10]. While we do not have time to talk about everything, here are some sample properties of flat morphisms.

[Har77, Prop. III.9.5]

Proposition 2.8.6 [EGAIV$_2$, Corollaire 6.1.4]. *Let $f\colon X \to Y$ be a flat morphism of locally Noetherian schemes. For every point $x \in X$, let $y = f(x)$. Then,*

$$\dim_x\bigl(f^{-1}(y)\bigr) = \dim_x(X) - \dim_y(Y)$$

where $\dim_x(X) = \dim(\mathcal{O}_{X,x})$ *and similarly for $f^{-1}(y)$ and Y.*

Proof. We consider the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}(\mathcal{O}_{X,x}) \arrow[r, dashed] \arrow[dr, "f'"'] & X' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& \operatorname{Spec}(\mathcal{O}_{Y,y}) \arrow[r] & Y
\end{tikzcd}$$

where the square is Cartesian. Since flat morphisms are stable under base change and $\operatorname{Spec}(\mathcal{O}_{X,x}) \to X'$ is flat, the morphism f' is flat. The result now follows from MA557, Homework 10, Problem 1 [Hoc14, p. 2]. $\square$

The following says that over regular curves, flatness is easy to verify. We first define the scheme-theoretic version of an associated prime ideal.

[Har77, p. 257] Definition 2.8.7 [EGAIV$_2$, Définition 3.1.1]. Let X be a scheme and let $\mathscr{F}$ be a quasi-coherent $\mathcal{O}_X$-module. A point $x \in X$ is *associated to* $\mathscr{F}$ if

$$\mathfrak{m}_x \in \operatorname{Ass}_{\mathcal{O}_{X,x}}(\mathscr{F}_x).$$

The set of associated points of $\mathscr{F}$ is denoted by $\operatorname{Ass}_{\mathcal{O}_X}(\mathscr{F})$. If $\mathscr{F} = \mathcal{O}_X$, we say that the associated points of $\mathscr{F}$ are *associated to* X.

Proposition 2.8.8 [EGAIV$_2$, Corollaire 2.8.2]. *Let $f \colon X \to Y$ be a morphism of schemes such that Y is integral and regular of dimension 1. Then, f is flat if and only if every associated point $x \in X$ maps to the generic point of Y. In particular, if X is reduced, this says that every irreducible component of X dominates Y.* [Har77, Prop. III.9.7]

Proof. The question is local, and hence we may assume that Y is the spectrum of a DVR $(R,(t))$.

$\Rightarrow$. The image of t in $\mathcal{O}_{X,x}$ is a nonzerodivisor for every $x \in X$ by flatness.

$\Leftarrow$. We show the contrapositive. Since R is a PID, by [Mur24ca, Corollary 7.12.2], there exists $x \in X$ such that setting $y = f(x)$, the module $\mathcal{O}_{X,x}$ is not torsion-free over $\mathcal{O}_{Y,y}$. Then, $f^{\#}t$ is a zerodivisor in $\mathfrak{m}_x$. Thus, $f^{\#}t$ is contained in an associated prime ideal $\mathfrak{p} \in \operatorname{Ass}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x})$ by [Mur24ca, Lemma 8.6.4]. $\square$

Example 2.8.9 [Har77, Example III.9.7.1]. Without regularity, Proposition 2.8.8 is false. Let Y be a curve with a node (for example, the nodal cubic) and let $f \colon X \to Y$ be the normalization of Y. We claim f is not flat. After localizing at the node, the associated local ring ring map is $R \to R'$ where R is an integral essentially of finite type k-algebra of dimension 1 and R' is its normalization in $\operatorname{Frac}(R)$. We see that R' is torsion-free of rank 1 over R, but is not free over R since there are two maximal ideals in R' lying over the maximal ideal in R. Since a finitely generated module over a Noetherian local ring is free if and only if it is flat [Mur24ca, Corollary 7.10.3], we see that R' is not flat over R, and hence f is not flat.

Example 2.8.10 [Har77, Example III.9.7.2]. Without $\dim(Y) = 1$, Proposition 2.8.8 is false. This is because blowing up a point on $\mathbf{A}^2_k$ does not satisfy the dimension equality in Proposition 2.8.6.

Proposition 2.8.8 allows us to take limits of schemes in flat families over a punctured curve.

Proposition 2.8.11. *Let Y be a regular integral scheme of dimension 1, let $P \in Y$ be a closed point, and let $X \subseteq \mathbf{P}^n_{Y-P}$ be a closed subscheme which is flat over $Y - P$. Then, there exists a unique closed subscheme $\bar{X} \subseteq \mathbf{P}^n_Y$, flat over Y, whose restriction to $\mathbf{P}^n_{Y-P}$ is X.*

Proof. Take $\bar{X}$ to be the scheme-theoretic closure of X in $\mathbf{P}^n_Y$ (Homework 8, Problem 3 [Har77, Exercise II.3.11(d)]). Then, the associated points of $\bar{X}$ are just the associated points of X. Furthermore, $\bar{X}$ is unique because any other extension of X to $\mathbf{P}^n_Y$ would have some associated points mapping to P. $\square$

We give some examples of flat families.

Example 2.8.12. We give some examples of tensor products in action. Both are flat by Proposition 2.8.8 or [Mur24ca, Corollary 7.12.2]. [Mur24ca, Ex. 7.12.3]

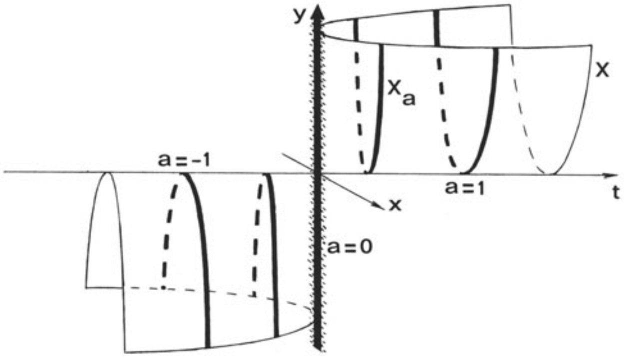

Figure 2.11. A family of parabolas parametrized by $\mathbf{A}^1_k$. From [Har77, Figure 7].

(1) Consider the morphism [Har77, Ex. II.3.3.1]

$$\operatorname{Spec}\left(\frac{k[x,y,t]}{(ty-x^2)}\right) \longrightarrow \operatorname{Spec}\big(k[t]\big).$$

We can think of this as a flat family of parabolas $\{ty - x^2 = 0\} \subseteq \mathbf{A}^2_k$ parametrized by t. The fiber over $t \neq 0$ is an honest parabola, but over $t = 0$ we get a non-reduced scheme $\operatorname{Spec}(k[x,y]/(x^2))$, the "double line" $\{x^2 = 0\}$. See Figure 2.11.

[Har77, Ex. II.3.3.2] (2) Consider the morphism

$$\operatorname{Spec}\left(\frac{k[x,y,t]}{(xy-t)}\right) \longrightarrow \operatorname{Spec}\big(k[t]\big).$$

This corresponds to a family of hyperbolas parametrized by t. When $t = 0$, you get the reducible scheme $\operatorname{Spec}(k[x,y]/(xy))$.

[Mur24ca, Ex. 7.12.4] Example 2.8.13 [Har77, Example III.9.8.4]. Consider the family

$$\begin{cases} x = t^2 - 1 \\ y = t^3 - t \\ z = at \end{cases}$$

of parametric curves in $\mathbf{A}^3_k$ parametrized by $a \in k$. This is a family of twisted cubics. We want to turn this into a *flat* family. To do so, we eliminate t from the parametric equations, and make sure that a is not a zerodivisor in $k[a,x,y,z]/I$. This yields the flat map

$$k[a] \longrightarrow \frac{k[x,y,z,a]}{(a^2(x+1) - z^2, ax(x+1) - yz, xz - ay, y^2 - x^2(x+1))}$$

corresponding to a family of twisted cubics in $\mathbf{A}^3_k$ parametrized by a when $a \neq 0$. When $a = 0$, we get the scheme

$$\operatorname{Spec}\left(\frac{k[x,y,z]}{(z^2, yz, xz, y^2 - x^2(x+1))}\right),$$

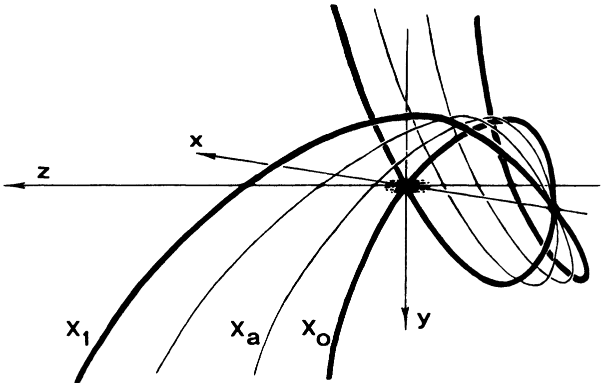

FIGURE 2.12. A family of twisted cubics parametrized by $\mathbf{A}^1_k$. From [Har77, Figure 11].

which is a nodal cubic with an embedded point at the node. The reduction of the fiber at $a = 0$ is

$$\operatorname{Spec}\left(\frac{k[x,y]}{(y^2 - x^2(x+1))}\right),$$

which has arithmetic genus 1 by Example 2.5.32, while the other fibers have arithmetic genus 0 (they are isomorphic to $\mathbf{P}^1_k$). Thus, the embedded point on the fiber at $a = 0$ "fixes" the lack of flatness at $a = 0$.

2.8.4. Modules of differentials. Flat morphisms are nice, but they are not the nicest morphisms possible. We want to define smooth morphisms, which are the analogue of submersions from differential topology (see [Lee13, p. 78]). For this subsection, let A be a ring, let B be an A-algebra, and let M be a B-module.

DEFINITION 2.8.14 [EGAIV$_1$, Chapitre 0, Définition 20.1.2]. An A-*derivation* of B into M is a map $d\colon B \to M$ such that

(1) d is additive.
(2) (Leibniz rule) $d(bb') = b\,db' + b'\,db$ for all $b, b' \in B$.
(3) $da = 0$ for all $a \in A$.

These form the B-module $\operatorname{Der}_A(B, M)$.

[Har77, p. 172]
[Mat89, p. 190]
[BLR90, p. 31]
[Kun86, Def. 1.1]

DEFINITION 2.8.15 [EGAIV$_1$, Chapitre 0, Définition 20.4.3, Définition 20.4.6]. The *module of relative differentials (of degree 1)* of B over A is a B-module $\Omega^1_{B/A}$ together with an A-derivation

$$d_{B/A}\colon B \longrightarrow \Omega^1_{B/A}$$

called the *(exterior) differential* satisfying the following universal property: For every B-module M and every A-derivation $d'\colon B \to M$, there exists a unique B-module homomorphism $f\colon \Omega^1_{B/A} \to M$ such that $d' = f \circ d$. In other words, $\operatorname{Der}_A(B, M)$ is such that

$$\begin{aligned} \operatorname{Hom}_B(\Omega^1_{B/A}, M) &\xrightarrow{\sim} \operatorname{Der}_A(B, M) \\ f &\longmapsto f \circ d \end{aligned}$$

is an isomorphism.

[Har77, p. 172]
[Mat89, p. 192]
[BLR90, p. 31]
[Kun86, Def. 1.20, Prop. 1.23]

Such a module $\Omega^1_{B/A}$ exists by taking the free B-module F generated by symbols db for $b \in B$ modulo the relations

1. $d(b+b') - db - db'$ for $b, b' \in B$,
2. $d(bb') - b\,db' - b'\,db$ for $b, b' \in B$, and
3. da for $a \in A$.

The derivation $d\colon B \to \Omega^1_{B/A}$ is defined by $b \mapsto db$.

Remark 2.8.16. Derivations in our sense match the usual notion from analysis. See [Kun86, Example 1.2].

2.8.5. Extensions of algebras. Formally smooth, formally unramified, and formally étale ring maps. We briefly discuss extensions of algebras. For this subsection, let A be a ring, let B, C be A-algebras, and let M be a B-module.

[Mat89, p. 191]

Definition 2.8.17 [EGAIV$_1$, Chapitre 0, (18.2.2)]. Let $N \subseteq C$ be an ideal such that $N^2 = 0$. Set $B = C/N$. The C-module N can be viewed as a B-module. In this situation, we say that C is an *extension* of the A-algebra B by the B-module N. We can write this extension as an exact sequence

$$0 \longrightarrow N \longrightarrow C \longrightarrow B \longrightarrow 0.$$

We say that the extension is *split* or that it is the *trivial extension* if there exists an A-algebra map $\varphi\colon B \to C$ that splits $C \to B$, in which case we can write $C = B \oplus N$ as an A-module. Conversely, starting from an A-algebra B and a B-module N, we can make the direct sum $B \oplus N$ into a trivial extension of B by N by defining the product

$$(b, x) \cdot (b', x') = (bb', bx' + b'x).$$

Remark 2.8.18. Trivial extensions are the same thing as Nagata's idealization trick. Nagata's trick first appeared in [Nag56, p. 226]. See also [Nag75, p. 2].

$\boxed{4/29}$

To extend the construction of differentials to the scheme case, we will use the following alternative description in terms of the multiplication map, which is the ring map induced by the diagonal morphism on coordinate rings.

[Har77, Prop. II.8.1A]
[Mat89, pp. 191–192]
[BLR90, p. 32]

Proposition 2.8.19 [EGAIV$_1$, Chapitre 0, Théorème 20.4.8]. *Let*

$$m\colon B \otimes_A B \longrightarrow B$$

be the multiplication map. Set $I = \ker(m)$. *Define a map*

$$\begin{aligned} d\colon B &\longrightarrow I/I^2 \\ b &\longmapsto 1 \otimes b - b \otimes 1 \pmod{I^2}. \end{aligned}$$

Then, the pair $(I/I^2, d)$ *is a module of relative differentials for* B/A.

Proof. Consider the short exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow \frac{B \otimes_A B}{I^2} \xrightarrow{\mu'} B \longrightarrow 0.$$

This is an extension of B by I/I^2. The right map has a splitting map as A-algebras by defining

$$\lambda_i\colon B \longrightarrow \frac{B \otimes_A B}{I^2}$$

where λ_i is the insertion map to the i-th factor.

We claim that $d = \lambda_2 - \lambda_1$ defines a derivation $B \to I/I^2$. First, the map

$$\lambda_2 - \lambda_1 \colon B \longrightarrow \frac{B \otimes_A B}{I^2}$$

factors through I/I^2 by the universal property of kernels. It is additive and satisfies $da = 0$ for all $a \in A$ since both λ_i are maps of A-algebras, which implies they are additive. For the Leibniz rule, we verify that [Kun86, p. 5]

$$\begin{aligned}
&d(bb') - b\,db' - b'\,db\\
&\qquad= (1 \otimes bb' - bb' \otimes 1) - (b \otimes 1)(1 \otimes b' - b' \otimes 1) - (b' \otimes 1)(1 \otimes b - b \otimes 1)\\
&\qquad= 1 \otimes bb' - bb' \otimes 1 - b \otimes b' + bb' \otimes 1 - b' \otimes b + bb' \otimes 1\\
&\qquad= 1 \otimes bb' - (b \otimes b' + b' \otimes b) + bb' \otimes 1\\
&\qquad= (1 \otimes b - b \otimes 1)(1 \otimes b' - b' \otimes 1)\\
&\qquad\in I^2.
\end{aligned} \tag{2.8.20}$$

It remains to show that $(I/I^2, d)$ is the module of relative differentials of B over A. To do so, we first show that I is generated by the elements $db = 1 \otimes b - b \otimes 1$. Suppose that $\sum_i a_i \otimes b_i \in \ker(m)$, and hence $\sum_i a_i b_i = 0$. Then,

$$\begin{aligned}
\sum_i (a_i \otimes 1)(1 \otimes b_i - b_i \otimes 1) &= \sum_i (a_i \otimes b_i - a_i b_i \otimes 1)\\
&= \sum_i a_i \otimes b_i - \Bigl(\sum_i a_i b_i\Bigr) \otimes 1\\
&= \sum_i a_i \otimes b_i.
\end{aligned}$$

We now verify that $(I/I^2, d)$ satisfies the universal property for $(\Omega^1_{B/A}, d_{B/A})$. Consider the diagram

$$\begin{tikzcd}
B \arrow[r, "d_M"] \arrow[dr, "d"'] & M \\
& I/I^2. \arrow[u, dashed, "f"', "\exists!"]
\end{tikzcd}$$

By the commutativity of the diagram, we see that

$$f(b\,db') = b\,d_M(b')$$

for all $b, b' \in B$. We need to show that this defines a well-defined map $f \colon I/I^2 \to M$. Suppose that $(1 \otimes b - b \otimes 1)db' \in I$. Then, we have

$$\begin{aligned}
f\bigl((1 \otimes b - b \otimes 1)db'\bigr) &= f\bigl(d(bb') - b\,db' - b'\,db\bigr)\\
&= d_M(bb') - b\,d_M(b') - b'\,d_M(b)\\
&= 0
\end{aligned}$$

as computed in (2.8.20). ☐

EXAMPLE 2.8.21 [EGAIV$_1$, Chapitre 0, Exemple 20.4.13(i)]. Suppose that B is generated by a set $U \subseteq B$ as an A-algebra. If $b \in B$ is an arbitary element, this [Mat89, pp. 192–193] means that we have $b = f(b_1, b_2, \dots, b_n)$ for some $b_i \in U$ and $f \in A[X_1, X_2, \dots, X_n]$. By the definition of the exterior differential, we have

$$db = \sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(b_1, b_2, \dots, b_n)\, db_i.$$

In particular, if $B = A[X_1, X_2, \dots, X_n]$ is a polynomial ring, then

$$\Omega_{B/A} = B \cdot dX_1 + B \cdot dX_2 + \cdots + B \cdot dX_n$$

and the dX_i are linearly independent over B. This follows from the fact that there are $D_i \in \operatorname{Der}_A(B, B)$ such that $D_i X_j = \delta_{ij}$, defined as the partial derivative operator with respect to the variable X_i.

To define formally smooth ring maps, we will use the following terminology.

[Mat89, p. 191]

Definition 2.8.22 [EGAIV$_1$, Définition 19.3.1]. Given a commutative diagram

$$\begin{tikzcd}
C \arrow[r, "f"] & B \\
& D \arrow[u, "g"'] \arrow[ul, "h"]
\end{tikzcd}$$

of A-algebras, where we think of f as being fixed, we say that h is a *lifting* of g to C.

Write $N = \ker(f) \subseteq C$. We note that the uniqueness of liftings corresponds exactly to the module of derivations $\operatorname{Der}_A(D, N)$ under certain assumptions. If $h' \colon D \to C$ is another lifting of g, then $h - h'$ is a map from D to N. If $N^2 = 0$ then N is an $f(C)$-module, and by means of $g \colon D \to f(C) \subseteq B$, we can consider N as a D-module. We then see that

$$h - h' \colon D \longrightarrow N$$

is an A-derivation of D to N. Conversely, if $d \in \operatorname{Der}_A(D, N)$, then $h + d$ is another lifting of g to C.

With this terminology, we can define formally smooth, formally unramified, and formally étale ring maps. One way to think of the definitions is that they are trying to capture whether and how an A-algebra map $B \to C/N$ lifts to an "infinitesimal deformation" $B \to C$.

[Mat89, p. 193]
[Har77, Exer. II.8.6]
The terminology *0-smooth* etc. is from [And74a].

Definition 2.8.23 [EGAIV$_1$, Chapitre 0, Définitions 19.3.1 and 19.10.2; And74a, Chapitre XVI, Définition 14]. We say that B is *formally smooth over A (with respect to the discrete topology)* or *0-smooth over A* if it has the following property: For every A-algebra C, every ideal $N \subseteq C$ satisfying $N^2 = 0$, and every A-algebra map $u \colon B \to C/N$, there exists a lifting $v \colon B \to C$ of u to C as an A-algebra map. In terms of diagrams, given a commutative diagram

$$\begin{tikzcd}
B \arrow[r, "u"] & C/N \\
A \arrow[u] \arrow[r] & C \arrow[u, two heads]
\end{tikzcd}$$

of A-algebras, there exists v such that

$$\begin{tikzcd} B \arrow[r, "u"] \arrow[rd, "v", "\exists"'] & C/N \\ A \arrow[u] \arrow[r] & C \arrow[u, two heads] \end{tikzcd}$$

commutes. We say that B is *formally unramified over A (with respect to the discrete topology)* or *0-unramified* if there exists at most one such v. When B is both formally smooth and formally unramified over A, that is, when for given u there exists a *unique* v, we say that A is *formally étale (with respect to the discrete topology)* or *0-étale*.

By Definition 2.8.22 and the proof of Proposition 2.8.19, we have:

Proposition 2.8.24 [EGAIV$_4$, Proposition 17.2.1]. *Let $A \to B$ be a ring map. Then, B is formally unramfied over A if and only if $\Omega^1_{B/A} = 0$.* [Mat89, p. 193]

Here are some examples.

Example 2.8.25 [EGAIV$_1$, Chapitre 0, Exemples 19.10.3]. Let $\varphi\colon A \to B$ be a ring map.

(*i*) If φ is surjective, then B is formally unramified over A. This is because in the diagram

$$\begin{tikzcd} B \arrow[r, "u"] \arrow[rd, "v", "!"'] & C/N \\ A \arrow[u, two heads, "\varphi"] \arrow[r] & C \arrow[u, two heads] \end{tikzcd}$$

the map v (if it exists) is uniquely determined by how A maps to C.

(*ii*) If φ is a localization map, then $B = S^{-1}A$ is formally étale over A. This is because in the diagram [Mat89, p. 193]

$$\begin{tikzcd} S^{-1}A \arrow[r, "u"] \arrow[rd, "v", "\exists!"'] & C/N \\ A \arrow[u, "\varphi"] \arrow[r] & C \arrow[u, two heads] \end{tikzcd}$$

the map v exists since if $s \in S$ maps to an invertible element in C/N, then its the image of s in C is invertible [EGAI$_{\text{new}}$, Chapitre 0, Corollaire 7.1.12] (see also [Mat89, Exercise 1.1]). We then apply the universal property of localization [Mur24ca, Theorem 3.2.2].

(*iii*) [EGAIV$_1$, Chapitre 0, Corollaire 19.3.3] If $B = A[X_i]_{i\in I}$ is a polynomial ring, then B is formally smooth over A. This is because in the diagram

$$\begin{tikzcd} A[X_i]_{i\in I} \arrow[r, "u"] \arrow[rd, "v", "\exists"'] & C/N \\ A \arrow[u, "\varphi"] \arrow[r] & C \arrow[u, two heads] \end{tikzcd}$$

we can choose arbitrary lifts of the elements $u(X_i)$ to C. The polynomial ring B is formally étale over A if and only if $I = \emptyset$.

Remark 2.8.26. By [EGAIV$_1$, Théorème 19.7.1], a formally smooth local map between Noetherian rings is flat and has geometrically regular closed fiber. When writing [EGAIV$_1$; EGAIV$_2$], it was an open question whether the converse holds. More specifically, Grothendieck and Dieudonné asked whether formally smooth local ring maps between Noetherian complete local rings are flat with geometrically regular fibers [EGAIV$_2$, Remarque 7.5.4(i)]. This question is a special case of what was later called *Grothendieck's localization problem* in [AF94].

André [And74b] solved Grothendieck and Dieudonné's question about formal smoothness above, which is the "regular" case of Grothendieck's localization problem. André's theorem is one of the first appliations of André–Quillen homology and simplicial methods in commutative algebra. The corresponding questions for many other properties of local rings have been studied since then. See [Mur22, Table 1] for a summary of existing results.

2.8.6. Properties of differentials. We prove some properties of modules of differentials.

[Har77, Prop. II.8.2A]
[Mat89, Exer. 25.4]

Proposition 2.8.27 [EGAIV$_4$, Chapitre 0, Propsition 20.5.5]. *Let $A \to B$ be a ring map. Let A' be an A-algebra and let $B' = B \otimes_A A'$. Then, there is a natural map*

$$\Omega^1_{B/A} \otimes_B B' \longrightarrow \Omega^1_{B'/A'}$$

that is an isomorphism.

Proof. Note that

$$\Omega^1_{B/A} \otimes_B B' \xleftarrow{\sim} \Omega^1_{B/A} \otimes_B B \otimes_A A' \xrightarrow{\sim} \Omega^1_{B/A} \otimes_A A'.$$

We want to show that $(\Omega^1_{B/A} \otimes_A A', d_{B/A} \otimes_A \mathrm{id}_{A'})$ satisfies the universal property for $(\Omega_{B'/A'}, d_{B'/A'})$, i.e., for the diagram

$$\begin{tikzcd}
B' \arrow[rr, "d'_M"] \arrow[dr, "d_{B/A}\otimes_A \mathrm{id}_{A'}"'] & & M \\
& \Omega^1_{B/A} \otimes_A A' \arrow[ur, dashed, "\exists!", "f'"'] &
\end{tikzcd}$$

where both maps from B' are A'-derivations, we want to show there By the adjunction between extension and restriction of scalars along $A \to A'$ [Mur24ca, Corollary 7.6.2], it suffices to show there exists a unique map f making the diagram

$$\begin{tikzcd}
B \arrow[rr, "d_M"] \arrow[dr, "d_{B/A}"'] & & M \\
& \Omega^1_{B/A} \arrow[ur, dashed, "\exists!", "f"'] &
\end{tikzcd}$$

commute. This is just the universal property of $(\Omega^1_{B/A}, d_{B/A})$. ☐

We now state and prove what Matsumura refers to as the "fundamental exact sequences" for modules of differentials.

[Mat89, Proof of Thm. 25.1]

Lemma 2.8.28. *Consider a sequence*

$$N' \xrightarrow{\alpha} N \xrightarrow{\beta} N'' \tag{2.8.29}$$

of R-module maps. Suppose that the induced sequence

$$\operatorname{Hom}_R(N',T) \xleftarrow{\alpha^*} \operatorname{Hom}_R(N,T) \xleftarrow{\beta^*} \operatorname{Hom}_R(N'',T)$$

is exact for every R-module T. Then, the sequence (2.8.29) *is exact.*

Proof. Taking $T = N''$, we see that

$$(\alpha^* \circ \beta^*)(\mathrm{id}_T) = 0,$$

and hence $\beta \circ \alpha = 0$. This shows that $\operatorname{im}(\alpha) \subseteq \ker(\beta)$. Conversely, taking $T = N/\operatorname{im}(\alpha)$, we see that

$$\operatorname{Hom}_R\bigl(N', N/\operatorname{im}(\alpha)\bigr) \xleftarrow{\alpha^*} \operatorname{Hom}_R\bigl(N, N/\operatorname{im}(\alpha)\bigr) \xleftarrow{\beta^*} \operatorname{Hom}_R\bigl(N'', N/\operatorname{im}(\alpha)\bigr)$$

is exact. Since the composition

$$N' \longrightarrow N \twoheadrightarrow N/\operatorname{im}(\alpha)$$

is the 0 map, the map α^* is the 0 map. Thus, β^* is surjective, which implies that the quotient map $N \to N/\operatorname{im}(\alpha)$ factors through N''. This shows that $\operatorname{im}(\alpha) \supseteq \ker(\beta)$, and hence (2.8.29) is exact. $\square$

Theorem 2.8.30 (First fundamental exact sequence [EGAIV$_1$, Chapitre 0, Théorème 20.5.7]). *Let $u\colon A \to B$ and $v\colon B \to C$ be ring maps. There exists a natural exact sequence*

[Mat89, Thm. 25.1]
[Har77, Prop. II.8.3A]
[Sin11, Prop. 15.2.6(1)]

$$\begin{tikzcd}
\Omega_{B/A} \otimes_B C \arrow[r, "\alpha"] & \Omega_{C/A} \arrow[r, "\beta"] & \Omega_{C/B} \arrow[r] & 0 \\
d_{B/A}(b) \otimes c \arrow[r, mapsto] & c\, d_{C/A}\bigl(v(b)\bigr) & & \\
& d_{C/A}(b) \arrow[r, mapsto] & d_{C/B}(b) &
\end{tikzcd}$$

of C-modules. Moreover, if C is formally smooth over B, then the sequence

$$0 \longrightarrow \Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B} \longrightarrow 0 \tag{2.8.31}$$

is a split exact sequence.

Proof. By Lemma 2.8.28 and tensor–Hom adjunction, it suffices to show that

$$\operatorname{Der}_A(B,T) \longleftarrow \operatorname{Der}_A(C,T) \longleftarrow \operatorname{Der}_B(C,T) \longleftarrow 0$$

is exact for every C-module T. Exactness at $\operatorname{Der}_B(C,T)$ holds since the map $\operatorname{Der}_B(C,T) \to \operatorname{Der}_A(C,T)$ is the inclusion of B-derivations $C \to T$ into A-derivations $C \to T$. Exactness at $\operatorname{Der}_A(C,T)$ holds since an A-derivation $C \to T$ maps B to 0 if and only if it is a B-derivation.

We now consider the case when C is formally smooth over B. By the previous paragraph, to show exactness of (2.8.31), it suffices to show that

$$0 \longleftarrow \operatorname{Der}_A(B,T) \longleftarrow \operatorname{Der}_A(C,T)$$

is exact for every C-module T. Let $D\colon B \to T$ be an A-derivation and consider the commutative diagram

$$\begin{tikzcd}
C \arrow[r, equal] \arrow[rd, dashed, "h"] & C \\
B \arrow[u, "v"] \arrow[r, "\varphi"] & C \oplus T \arrow[u, two heads, "\mathrm{pr}_1"']
\end{tikzcd}$$

where $\varphi(b) = (v(b), D(b))$. By the formal smoothness of $B \to C$, there exists a lifting $h\colon C \to C \oplus T$ which is a B-algebra map making the diagram commute.

Write $h(c) = (c, D'(c))$ where $D' \colon C \to T$ is additive and satisfies $D = D' \circ v$. We claim that D' is an A-derivation. The equation $D = D' \circ v$ shows that D' maps A to 0. The Leibniz rule holds since

$$h(cc') = \big(cc', D'(cc')\big)$$

which equals

$$h(c) \cdot h(c') = \big(c, D'(c)\big) \cdot \big(c', D'(c')\big) = \big(cc', c\, D(c') + c'\, D(c)\big).$$

It remains to show that the exact sequence (2.8.31) splits. Let $T = \Omega_{B/A} \otimes_B C$ and define $D \colon B \to T$ by

$$\begin{aligned} D \colon B &\longrightarrow \Omega_{B/A} \otimes_B C \\ b &\longmapsto d_{B/A}(b) \otimes 1. \end{aligned}$$

By the previous paragraph, we have $D = D' \circ v$. Then, D' corresponds to a C-linear map $\alpha' \colon \Omega_{C/B} \to \Omega_{B/A} \otimes_B C$, and the equality $D = D' \circ v$ corresponds to $\alpha' \circ \alpha = \mathrm{id}_T$. Thus, (2.8.31) splits. ☐

[Mat89, Thm. 25.2]
[Har77, Prop. II.8.4A, Thm. II.8.17]
[Sin11, Prop. 15.2.6(2)]

Theorem 2.8.32 (Second fundamental exact sequence or conormal exact sequence [EGAIV$_1$, Chapitre 0, Théorème 20.5.12 and Corollaire 20.5.14]). *Let $A \to B$ be a ring map. Let $I \subseteq B$ be an ideal and let $C = B/I$. Then, there is a natural exact sequence*

$$\begin{tikzcd} I/I^2 \arrow[r, "\delta"] & \Omega_{B/A} \otimes_B C \arrow[r, "\alpha"] & \Omega_{C/A} \arrow[r] & 0 \\ \bar{b} \arrow[r, mapsto] & d_{B/A}(b) \otimes 1 \end{tikzcd}$$

of C-modules. Moreover, if C is formally smooth over A, then the sequence

$$0 \longrightarrow I/I^2 \longrightarrow \Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \longrightarrow 0 \tag{2.8.33}$$

is a split exact sequence.

Proof. By Lemma 2.8.28 and tensor–Hom adjunction, it suffices to show that

$$\mathrm{Hom}_C(I/I^2, T) \xleftarrow{\delta^*} \mathrm{Der}_A(B, T) \xleftarrow{\alpha^*} \mathrm{Der}_A(C, T) \longleftarrow 0$$

is exact for every C-module T. The map α^* is injective since distinct A-derivations $C \to T$ give rise to distinct A-derivations $B \to T$ via precomposition with α. For exactness at $\mathrm{Der}_A(B, T)$, we have $\delta^*(D) = 0$ if and only if $D(I) = 0$, which holds if and only if D factors through $C = B/I$.

We now consider the case when C is formally smooth over A. By the previous paragraph, to show exactness of (2.8.33), it suffices to show that

$$0 \longleftarrow \mathrm{Hom}_C(I/I^2, T) \xleftarrow{\delta^*} \mathrm{Der}_A(B, T)$$

is exact for every C-module T. If C is formally smooth over A, then considering the commutative diagram

$$\begin{tikzcd}
 & 0 \\
C \arrow[r, equal] \arrow[dr, dashed, "h"] & C \arrow[u] \\
A \arrow[u] \arrow[r] & B/I^2 \arrow[u, "g"'] \\
 & I/I^2 \arrow[u] \\
 & 0 \arrow[u]
\end{tikzcd}$$

the formal smoothness of C implies there exists a lifting $h\colon C \to B/I^2$ which is a A-algebra map making the diagram commute. By the commutativity of the diagram, h is a splitting for the extension in the right column, i.e., $gh = \mathrm{id}_C$. Now $hg\colon B/I^2 \to B/I^2$ is an A-algebra map vanishing on I/I^2 and $g(1-hg) = 0$. Thus, if we set $D = 1-hg$, then $D\colon B/I^2 \to I/I^2$ is an A-derivation. If $\psi \in \mathrm{Hom}_C(I/I^2, T)$, then the composition

$$D'\colon B \longrightarrow B/I^2 \xrightarrow{D} I/I^2 \xrightarrow{\psi} T$$

is an element of $\mathrm{Der}_A(B,T)$ satisfying $\delta^*(D') = \psi$, since for $x \in I$, letting $\bar{x} = x \pmod{I}^2$, then

$$D'(x) = \psi\bigl(D(\bar{x})\bigr) = \psi\bigl(\bar{x} - hg(\bar{x})\bigr) = \psi(\bar{x}).$$

Thus, δ^* is surjective and (2.8.33) is therefore exact. Setting $T = I/I^2$, we see that (2.8.33) is split exact. $\square$

2.8.7. Sheaves of differentials. The description in Proposition 2.8.19 allows us to define sheaves of differentials for morphisms of schemes.

Definition 2.8.34 [EGAIV$_4$, Définition 16.1.2]. Let $i\colon Z \to X$ be an immersion of schemes. Identifying Z as the closed subscheme of an open subscheme $W \subseteq X$, let $\mathscr{I}$ be the quasi-coherent $\mathcal{O}_W$-module defining Z in W. The *conormal sheaf of Z in X* or the *conormal sheaf of i* is the quasi-coherent $\mathcal{O}_Z$-module

$$\mathscr{C}_{Z/X} \coloneqq i^{-1}\mathscr{I}/\mathscr{I}^2.$$

[Har77, p. 182] Our notation is from [Stacks, Tag 01R2]. The sheaf is denoted by $\mathscr{N}_{Z/X}$ in [EGAIV$_4$], which we avoid since $\mathscr{N}_{Z/X}$ is often used to denote the *normal* sheaf, e.g. in [Har77].

Definition 2.8.35 [EGAIV$_4$, Définition 16.3.1]. Let $f\colon X \to S$ be a morphism of schemes and let $\Delta_{X/S}\colon X \to X \times_S X$ be the diagonal morphism. Note that $\Delta_{X/S}$ is an immersion by Proposition 2.5.22. The *sheaf of relative differentials (of degree 1) of X over S* is the quasi-coherent sheaf

$$\Omega^1_{X/Y} \coloneqq \mathscr{C}_{X/X\times_S X}.$$

Remark 2.8.36. The sheaf of relative differentials also has a universal property like in the affine case [EGAIV$_4$, Proposition 16.5.3(ii)]. They also have analogues of the fundamental exact sequences in Theorems 2.8.30 and 2.8.32 since the formation of $\Omega^1_{B/A}$ is compatible with localization by Theorem 2.8.30 and Proposition 2.8.24.

Here is one fundamental computation.

Theorem 2.8.37 (Euler exact sequence). *Let A be a ring, let $Y = \operatorname{Spec}(A)$, and let $X = \mathbf{P}^n_A$. Then, there is a short exact sequence* [Har77, Thm. II.8.13]

$$0 \longrightarrow \Omega^1_{X/Y} \longrightarrow \mathcal{O}_X(-1)^{\oplus(n+1)} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

of $\mathcal{O}_X$-modules.

Proof. Let $S = A[x_0, x_1, \ldots, x_n]$ be the homogeneous coordinate ring of X. Let $E = S(-1)^{\oplus(n+1)}$, which is a graded S-module with basis $e_0, e_1, \ldots, e_n$ in degree 1. We then consider the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & E \arrow[r] & S \\ & & e_i \arrow[r, mapsto] & x_i \end{tikzcd}$$

of graded S-modules. This gives the short exact sequence

$$0 \longrightarrow \tilde{M} \longrightarrow \mathcal{O}_X(-1)^{\oplus(n+1)} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

of quasi-coherent $\mathcal{O}_X$-modules, where surjectivity on the right holds since while $E \to S$ is not surjective, it is surjective in all degrees ≥ 1.

It remains to show that $\tilde{M} \cong \Omega^1_{X/Y}$. If we localize at x_i, then $E_{x_i} \to S_{x_i}$ is a surjective map of free S_{x_i}-modules, and hence M_{x_i} is free of rank n with generators

$$\bigl\{e_j - (x_j/x_i)e_i \bigm| j \neq i\bigr\}.$$

Thus, if $U_i = D_+(x_i)$, then $\tilde{M}|_{U_i}$ is the free $\mathcal{O}_{U_i}$-module generated by the sections

$$\bigl\{(1/x_i)e_j - (x_j/x_i^2)e_i \bigm| j \neq i\bigr\}.$$

We define a map

$$\varphi_i \colon \Omega^1_{X/Y}|_{U_i} \longrightarrow \tilde{M}|_{U_i}$$

as follows. On each U_i, the domain is the free $\mathcal{O}_{U_i}$-module generated by

$$d(x_0/x_i), \ldots, \widehat{d(x_i/x_i)}, \ldots, d(x_n/x_i).$$

We may therefore define φ_i by

$$\varphi_i\bigl(d(x_j/x_i)\bigr) = (1/x_i^2)(x_i e_j - x_j e_i).$$

Thus, φ_i is an isomorphism. The φ_i glue together to give an isomorphism $\varphi \colon \Omega^1_{X/Y} \to \tilde{M}$ on all of X since in $\Omega^1_{X/Y}|_{U_i \cap U_j}$, we have $x_k/x_i = (x_k/x_j) \cdot (x_j/x_i)$ for all k, and hence

$$d\biggl(\frac{x_k}{x_i}\biggr) - \frac{x_k}{x_j} d\biggl(\frac{x_j}{x_i}\biggr) = \frac{x_j}{x_i} d\biggl(\frac{x_k}{x_j}\biggr).$$

Applying φ_i to the left-hand side and φ_j on the right-hand side, we get the same result $(1/x_i x_j)(x_j e_k - x_k e_j)$. □

2.8.8. (Formally) smooth, (formally) unramified, and (formally) étale morphisms. The corresponding notions of formally smooth, formally unramified, and formally étale morphisms are defined as follows.

[Har77, Exer. II.8.6] Definition 2.8.38 [EGAIV$_4$, Définition 17.1.1 and Remarque 17.1.2(*ii*)]. Let $f \colon X \to Y$ be a morphism of schemes. We say that f is *formally smooth* (resp. *formally unramified*, *formally étale*) if, for every affine scheme Y', every closed subscheme Y_0' defined by an ideal $\mathscr{I} \subseteq \mathcal{O}_{Y'}$ satisfying $\mathscr{I}^2 = 0$, and every morphism $Y' \to Y$ fitting into the commutative diagram

$$\begin{tikzcd}
X \arrow[d, "f"'] & Y_0' \arrow[l] \arrow[d, hook, "i"] \\
Y & Y' \arrow[l]
\end{tikzcd}$$

there exists a (resp. there exists at most one, there exists a unique) dashed morphism $T' \dashrightarrow X$ making the diagram

$$\begin{tikzcd}
X \arrow[d, "f"'] & Y_0' \arrow[l] \arrow[d, hook, "i"] \\
Y & Y' \arrow[l] \arrow[ul, dashed]
\end{tikzcd}$$

commute.

We can now define smooth, (G-)unramified, and étale morphisms. This definition for smooth morphisms is referred to as the *infinitesimal lifting criterion* for smoothness in [Stacks, Tag 02H6].

Definition 2.8.39 (Infinitesimal lifting criterion [EGAIV$_4$, Définitions 17.3.1 and 17.3.7; Ray70, Chapitre I, Définition 4; Stacks, Tag 02G4]). Let $f\colon X \to Y$ be a morphism of schemes. We say that f is *smooth* (resp. *G-unramified*, *étale*) if it is formally smooth (resp. formally unramified, formally étale) and of finite presentation. We say that f is *unramified* if it is formally unramified and of finite type.

We say that f has the corresponding property *at a point* $x \in X$ if there exists an open neighborhood U of x such that $U \to Y$ has that property.

[Har77, p. 268, Exer. III.10.3]

We follow the definition of unramified in [Ray70]. Following [Stacks], we refer to the original notion from [EGAIV$_4$] as *G-unramified.*

We are now ready to state the relationship between smoothness and differentials.

Proposition 2.8.40 [EGAIV$_4$, Proposition 17.2.3]. *Let $f\colon X \to Y$ be a formally smooth morphism.*

[Har77, Thm. II.8.17]

(i) *The $\mathcal{O}_X$-module $\Omega^1_{X/Y}$ is locally projective. If f is locally of finite type, then $\Omega^1_{X/Y}$ is locally free of finite type.*

(ii) *For every morphism $g\colon Y \to Z$, the sequence*

$$0 \longrightarrow f^*\Omega^1_{Y/Z} \longrightarrow \Omega^1_{X/Z} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

is exact, and for every $x \in X$, there exists an open neighborhood U of x on which the restriction of this exact sequence splits.

In this situation, we can define the canonical bundle.

Definition 2.8.41. Suppose that $f\colon X \to Y$ has $\Omega^1_{X/Y}$ locally free of finite rank n. The *relative canonical bundle* is

[Har77, p. 180]

$$\omega_{X/Y} \coloneqq \bigwedge^n \Omega^1_{X/Y}.$$

If $Y = \operatorname{Spec}(k)$ for a field k, we call $\omega_X \coloneqq \omega_{X/k}$ the *canonical bundle.*

Example 2.8.42. If $X = \mathbf{P}^n_A$, then

$$\omega_{X/\operatorname{Spec}(A)} = \mathcal{O}_X\bigl(-(n+1)\bigr)$$

by taking determinants in Theorem 2.8.37.

Under extra assumptions on $X \to Z$ and $Y \to Z$, the local freeness of $\Omega^1_{X/Y}$ can be turned into an if and only if.

[Har77, Thm. II.8.17]

Theorem 2.8.43 [EGAIV$_4$, Théorème 17.11.1]. *Let $f\colon Y \to S$ and $h\colon X \to S$ be morphisms locally of finite presentation. Let $g\colon X \to Y$ be an S-morphism. Let $x \in X$ with images $y \in Y$ and $s \in S$. The following are equivalent.*

(*a*) *f is smooth at y and g is smooth at x.*
(*b*) *g and h are smooth at x.*
(*c*) *h is smooth at x and the canonical map*

$$\left(g^*\Omega^1_{Y/S}\right)_x \longrightarrow \left(\Omega^1_{X/S}\right)_x$$

has a retraction, i.e., a left inverse.
(*c′*) *h is smooth at x and the canonical map*

$$\left(g^*\Omega^1_{Y/S}\right)_y \otimes_{\mathcal{O}_{Y,y}} k(y) \otimes_{k(y)} k(x) \longrightarrow \left(\Omega^1_{X/S}\right)_x \otimes_{\mathcal{O}_{X,x}} k(x)$$

is injective.

Suppose, moreover, that the map $k(y) \to k(x)$ is bijective. Then, the conditions above are equivalent to the following:

(*d*) *h is smooth at x and the canonical map*

$$T_{X/S}(x) \longrightarrow T_{Y/S}(y)$$

from the tangent space of X at x to the tangent space of Y at y is surjective.

See [EGAIV$_4$, (16.5.12)] for the definition of the *(geometric) tangent bundle* as

$$T_{X/S} \coloneqq \mathbf{V}_X(\Omega^1_{X/S}) \coloneqq \mathbf{Spec}_X\left(\mathrm{Sym}^\bullet_{\mathcal{O}_X}(\Omega^1_{X/S})\right).$$

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2.8.9. The Jacobian criterion. For completeness and to contextualize Bertini's theorem, we state the following result. This can be thought of as a relative, scheme-theoretic version of Zariski's Jacobian criterion [Zar47, Theorem 7] (see [Mur24ag, Theorem 1.5.4]). See [EGAIV$_1$, Chapitre 0, Théorème 22.6.1] for the algebraic version of the statement below.

[BLR90, Prop. 2.2/7]
[Stacks, Tag 01V9]

Theorem 2.8.44 (The Jacobian criterion [EGAIV$_4$, Théorème 17.5.1 and its proof]). *Let $f\colon X \to Y$ be a morphism locally of finite presentation, let $x \in X$ be a point, and let $y = f(x)$. The following conditions are equivalent:*

(*a*) *f is smooth at x.*
(*b*) *f is* regular *at x, i.e., f is flat at x and $\mathcal{O}_{f^{-1}(x),x}$ is geometrically regular over $k(y)$.*
(*c*) *There exist affine open neighborhoods $U = \mathrm{Spec}(A) \subseteq X$ of x and $V = \mathrm{Spec}(B) \subseteq Y$ of y such that $f(U) \subseteq V$, and such that there exists a presentation*

$$A = \frac{B[x_1, x_2, \ldots, x_n]}{(f_1, f_2, \ldots, f_c)}$$

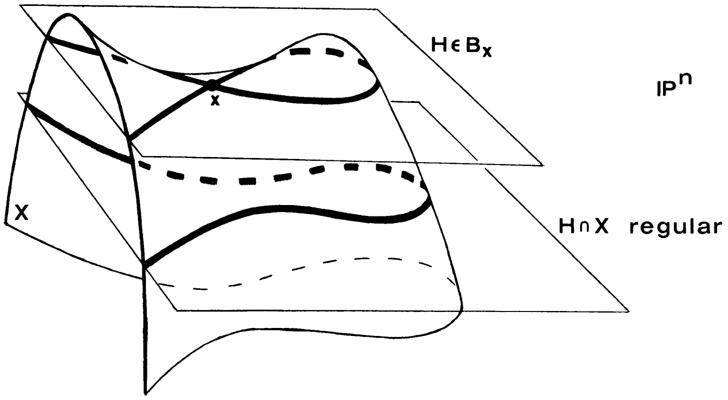

Figure 2.13. Hyperplane sections of a regular variety. From [Har77, Figure 10].

for which the element

$$\det \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_2}{\partial x_1} & \cdots & \dfrac{\partial f_c}{\partial x_1} \\ \dfrac{\partial f_1}{\partial x_2} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_c}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_1}{\partial x_n} & \dfrac{\partial f_2}{\partial x_n} & \cdots & \dfrac{\partial f_c}{\partial x_n} \end{pmatrix}$$

in $B[x_1, x_2, \ldots, x_n]$ maps to an element of A not in $\mathfrak{p}_x$.

2.8.10. Bertini's theorem for regularity. We now state and prove Bertini's theorem for regularity. See [Ber1882, p. 26; Ber1907, p. 227] for Bertini's original results and see [Kle98] for the history of Bertini's results.

Theorem 2.8.45 (Bertini's theorem for regularity). *Let k be an algebraically closed field and let $X \subseteq \mathbf{P}^n_k$ be a regular projective variety over k. Then, there exists a hyperplane $H \subseteq \mathbf{P}^n_k$, not containing X, such that $H \cap X$ is regular. Furthermore, the set of hyperplanes with this property forms an open dense subset of the complete linear system $|H|$ considered as*

[Har77, Thm. II.8.18]

$$\frac{H^0\big(\mathbf{P}^n_k, \mathcal{O}(1)\big) - \{\mathbf{0}\}}{k^*}$$

with the Zariski topology.

Proof. Let $x \in X$ be a closed point. We then consider the set

$$B_x = \left\{ \begin{matrix} \text{hyperplanes} \\ H \subseteq \mathbf{P}^n_k \end{matrix} \;\middle|\; \begin{array}{l} \bullet\ H \supseteq X; \text{ or} \\ \bullet\ H \not\supseteq X \text{ but } x \in H \cap X \text{ and } x \text{ is} \\ \quad \text{not a regular point of } H \cap X. \end{array} \right\}.$$

These are the "bad" hyperplanes with respect to the point x. See Figure 2.13. A hyperplane H is determined by a nonzero global section $f \in V = H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$.

Fix an element $f_0 \in V$ such that $x \notin H_0 = V_+(f_0)$. Then, we can define a map of k-vector spaces

$$\begin{aligned} \varphi_x \colon V &\longrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_x^2 \\ f &\longmapsto f/f_0 \pmod{\mathfrak{m}_x^2}. \end{aligned}$$

The scheme $H \cap X$ is defined at x by the ideal generated by f/f_0 in $\mathcal{O}_{X,x}$. So, $x \in H \cap X$ if and only if $\varphi_x(f) \in \mathfrak{m}_x$. Moreover, assuming that $H \not\supseteq X$, we see that x is not a regular point on $H \cap X$ if and only if $\varphi_x(f) \in \mathfrak{m}_x^2$ by definition of a regular local ring (see [EGAIV$_1$, Chapitre 0, Corollaire 17.1.8]). We therefore see that the hyperplanes $H \in B_x$ correspond exactly to elements $f \in \ker(\varphi_x)$ (note that if $H \supseteq X$, then $\varphi_x(f) = 0$).

Since x is a closed point and k is algebraically closed, $\mathfrak{m}_x$ is generated by linear forms in the coordinates on $\mathbf{P}^n_k$. Thus, the map φ_x is surjective. If $\dim(X) = r$, then the regularity of $\mathcal{O}_{X,x}$ implies $\dim_k(\mathcal{O}_{X,x}/\mathfrak{m}_x^2) = r + 1$. We have $\dim(V) = n + 1$, and hence

$$\dim\bigl(\ker(\varphi_x)\bigr) = n - r$$

by the rank–nullity theorem. This shows that B_x is a linear system of hyperplanes of dimension $n - r - 1$, i.e., B_x has dimension $n - r - 1$ thought of as a projective space over k.

We now consider the complete linear system $|H|$ as a projective space, and consider the incidence correspondence

$$B \coloneqq \bigl\{(x, H) \bigm| x \in X \text{ closed and } H \in B_x\bigr\} \subseteq X \times_k |H|.$$

Then, B is the set of closed points of a closed subset of $X \times_k |H|$ since vanishing at a point x corresponds to a set of polynomial conditions on the coefficients appearing in a linear form defining H. We also call this closed subset B, and consider it with the reduced closed subscheme structure. We then consider the two projection maps

$$\begin{tikzcd} B \arrow[r, hook] & X \times_k |H| \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & |H| \end{tikzcd}$$

from B. The first projection $p_1 \colon B \to X$ is surjective and the closed fibers of p_1 are projective spaces of dimension $n - r - 1$. Thus, B is irreducible (by working with the definition of irreducibility; see [EGAI$_{\text{new}}$, Chapitre 0, Proposition 2.1.13]) and has dimension $(n - r - 1) + r = n - 1$ (by the theorem on the dimension of fibers [Mur24ag, Theorem 1.8.5]; see [Har77, Exercise II.3.22] for the scheme-theoretic version). Considering the second projection $p_2 \colon B \to |H|$, we then see that $\dim(p_2(B)) \le n - 1$. Since $\dim|H| = n$, we conclude that $p_2(B) < |H|$. If $H \in |H| - p_2(B)$, then $H \not\supseteq X$ and every point of $H \cap X$ is regular, and hence such a hyperplane H satisfies the requirements of the theorem.

Finally, since X is projective, $p_2 \colon X \times_k |H| \to |H|$ is closed by Theorem 2.6.27(*i*). Since B is closed in $X \times_k |H|$, we see that $p_2(B)$ is closed in $|H|$, and hence $|H| - p_2(B)$ is an open dense subset of $|H|$. $\square$

2.8.11. Serre duality. Now that we have the language of sheaves of differentials and canonical bundles, we can prove the Serre duality theorem. We start with duality for $\mathbf{P}^n_k$.

Theorem 2.8.46 (Serre duality for $\mathbf{P}^n_k$). *Let $X = \mathbf{P}^n_k$ over a field k. We then have the following.* [Har77, Thm. III.7.1]

(*i*) *$H^n(X, \omega_X) \cong k$. Fix such an isomorphism.*

(*ii*) *For every coherent $\mathcal{O}_X$-module, the natural pairing*

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathscr{F}, \omega_X) \times H^n(X, \mathscr{F}) \longrightarrow H^n(X, \omega) \cong k$$

is a perfect pairing of finite-dimensional k-vector spaces.

(*iii*) *For every $i \geq 0$, there is a natural functorial isomorphism*

$$\operatorname{Ext}^i_{\mathcal{O}_X}(\mathscr{F}, \omega_X) \xrightarrow{\sim} H^{n-i}(X, \mathscr{F})^\vee,$$

where $(\,\cdot\,)^\vee$ denotes the dual vector space, which for $i = 0$ is the one induced by the pairing of (ii).

Proof. (*i*). By Example 2.8.42, we have $\omega_X \cong \mathcal{O}_X(-n-1)$. Thus, (*i*) follows from the computation of cohomology on $\mathbf{P}^n_k$ in Theorem 2.5.34(*c*).

(*ii*). The natural pairing is defined as follows: A morphism $\mathscr{F} \to \omega_X$ induces a map $H^n(X, \mathscr{F}) \to H^n(X, \omega_X)$ on cohomology. Now if $\mathscr{F} \cong \mathcal{O}(q)$ for some $q \in \mathbf{Z}$, then

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathscr{F}, \omega) \cong \operatorname{Hom}_{\mathcal{O}_X}\bigl(\mathcal{O}_X, \omega(-q)\bigr) \cong H^0\bigl(X, \omega_X(-q)\bigr).$$

Thus, the result follows from Theorem 2.5.34(*d*). Now an arbitrary coherent $\mathcal{O}_X$-module $\mathscr{F}$ fits into an exact sequence of the form

$$\mathscr{E}_1 \longrightarrow \mathscr{E}_0 \longrightarrow \mathscr{F} \longrightarrow 0$$

where the $\mathscr{E}_i$ are direct sums of sheaves of the form $\mathcal{O}(q_j)$. Now $\operatorname{Hom}_{\mathcal{O}_X}(\,\cdot\,, \omega_X)$ and $H^n(X, \,\cdot\,)^\vee$ are both left exact contravariant functors, and hence the five lemma impliex we have an isomorphism

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathscr{F}, \omega_X) \xrightarrow{\sim} H^n(X, \mathscr{F})^\vee.$$

(*iii*). Both sides are contravariant δ-functors indexed by $i \geq 0$ on $\mathsf{Coh}(\mathcal{O}_X)$. For $i = 0$, we have an isomorphism by (*ii*). Thus, to show they are isomorphic, it suffices to show that both sides are coeffaceable for $i > 0$ by Proposition 1.4.17. Given a coherent $\mathcal{O}_X$-module $\mathscr{F}$, by Serre's global generation theorem for Proj (Theorem 2.4.39(*iii*)), we can write $\mathscr{F}$ as a quotient of a sheaf

$$\mathscr{E} = \mathcal{O}(-q)^{\oplus N}$$

for $q \gg 0$. Then,

$$\operatorname{Ext}^i_{\mathcal{O}_X}(\mathscr{E}, \omega_X) \cong H^i\bigl(X, \omega_X(q)\bigr)^{\oplus N} = 0$$

for $i > 0$ by Theorem 2.5.34, where the first isomorphism holds by [Har77, Proposition III.6.3(c)]. On the other hand,

$$H^{n-i}(X, \mathscr{E})^\vee = \Bigl(H^{n-i}\bigl(X, \mathcal{O}(-q)\bigr)^\vee\Bigr)^{\oplus N}$$

which is 0 for $i > 0, q > 0$, again from Theorem 2.5.34. Thus, both sides are coeffaceable for $i > 0$. This shows that the δ-functors are universal and match for $i = 0$, and hence the δ-functors are isomorphic. □

To generalize Theorem 2.8.46 to other schemes, we take the properties in (*i*) and (*ii*) as our guide and define the following.

Definition 2.8.47. Let X be a complete scheme of dimension n over a field k. A *dualizing sheaf for X* is a coherent $\mathcal{O}_X$-module ω_X° together with a *trace morphism* [Har77, p. 241]

$$t\colon H^n(X,\omega_X^\circ)\longrightarrow k$$

such that for all coherent $\mathcal{O}_X$-modules $\mathscr{F}$ on X, the natural pairing

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathscr{F},\omega_X^\circ)\times H^n(X,\mathscr{F})\longrightarrow H^n(X,\omega_X^\circ)$$

followed by t induces an isomorphism

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathscr{F},\omega_X^\circ)\xrightarrow{\sim} H^n(X,\mathscr{F})^\vee$$

under tensor–Hom adjunction.

[Har77, Prop. III.7.2] Proposition 2.8.48. *Let X be a complete scheme over a field k. Then, a dualizing sheaf for X, if it exists, is unique. More precisely, if (ω°,t) and (ω',t') are two pairs consisting of a dualizing sheaf for X and a trace morphism, then there is a unique isomorphism*

$$\varphi\colon \omega^\circ\xrightarrow{\sim}\omega'$$

such that $t=t'\circ H^n(X,\varphi)$.

Proof. By the Yoneda lemma [Mur24ca, Lemma 4.6.1], it suffices to note that the pair (ω°,t) represents the functor

$$\begin{tikzcd}
\mathsf{Coh}(\mathcal{O}_X) \arrow[r] & \mathsf{Mod}(k)\\
\mathscr{F} \arrow[r, mapsto] & H^n(X,\mathscr{F})^\vee
\end{tikzcd}$$

and is therefore unique up to unique isomorphism. □

The existence of dualizing sheaves is more difficult. They in fact exist for any complete scheme X, but we will only prove existence for projective schemes over k. We need some preliminary results. Below, $\mathscr{E}xt^i_{\mathcal{O}_X}(\mathscr{F},\,\cdot\,)$ denotes the i-th right derived functor of $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\,\cdot\,)$.

[Har77, Lem. III.7.3] Lemma 2.8.49. *Let X be a closed subscheme of codimension r in $P=\mathbf{P}^N_k$, where k is a field. Then, we have*

$$\mathscr{E}xt^i_{\mathcal{O}_P}(\mathcal{O}_X,\omega_P)=0$$

for all $i<r$.

Proof. For any i, the sheaf $\mathscr{F}^i=\mathscr{E}xt^i_{\mathcal{O}_P}(\mathcal{O}_X,\omega_P)$ is coherent by [Har77, Exercise III.6.3]. Thus, after twisting by a suitably large integer q, it will be generated by global sections by Theorem 2.4.39(*iii*). To show that $\mathscr{F}^i$ is zero, it therefore suffices to show that $H^0(P,\mathscr{F}^i(q))=0$ for all $q\gg 0$. By [Har77, Propositions III.6.7 and III.6.9] (which boil down to coeffaceability and dimension shifting), we have

$$H^0\bigl(P,\mathscr{F}^i(q)\bigr)\cong \operatorname{Ext}^i_P\bigl(\mathcal{O}_X,\omega_P(q)\bigr)$$

for $q\ gg0$. By Theorem 2.8.46, the right-hand side is dual to $H^{N-i}(P,\mathcal{O}_X(-q))$. For $i<r$, we have $N-i>\dim(X)$, and hence this group is 0 by the Grothendieck vanishing Theorem 1.4.27. □

[Har77, Lem. III.7.4] Lemma 2.8.50. *Let X be a closed subscheme of codimension r in $P=\mathbf{P}^N_k$, where k is a field. Let $\omega_X^\circ=\mathscr{E}xt^r_{\mathcal{O}_P}(\mathcal{O}_X,\omega_P)$. Then, for any $\mathcal{O}_X$-module $\mathscr{F}$, there is a functorial isomorphism*

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathscr{F},\omega_X^\circ)\cong \operatorname{Ext}^r_{\mathcal{O}_P}(\mathscr{F},\omega_P).$$

Proof. Let $0 \to \omega_P \to \mathscr{I}^\bullet$ be an injective resolution of ω_P in $\mathsf{Mod}(\mathcal{O}_P)$. Then, we can calculate $\operatorname{Ext}^i_{\mathcal{O}_P}(\mathscr{F}, \omega_P)$ as the cohomology modules of the complex $\operatorname{Hom}_{\mathcal{O}_P}(\mathscr{F}, \mathscr{I}^\bullet)$. Since $\mathscr{F}$ is an $\mathcal{O}_X$-module, any morphism $\mathscr{F} \to \mathscr{I}^i$ factors though

$$\mathscr{J}^i = \mathscr{H}om_{\mathcal{O}_P}(\mathcal{O}_X, \mathscr{I}^i).$$

We therefore have

$$\operatorname{Ext}^i_{\mathcal{O}_P}(\mathscr{F}, \omega_P) = \mathbf{h}^i\big(\operatorname{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{J}^\bullet)\big).$$

Now each $\mathscr{J}^i$ is an injective $\mathcal{O}_X$-module since $\operatorname{Hom}_{\mathcal{O}_X}(\,\cdot\,, \mathscr{J}^i) = \operatorname{Hom}_{\mathcal{O}_P}(\,\cdot\,, \mathscr{I}^i)$ is an exact functor. Moreover, by Lemma 2.8.49, we have $\mathbf{h}^i(\mathscr{J}^\bullet) = 0$ for $i < r$, and hence $\mathscr{J}^\bullet$ is exact up to the r-th step. Since the $\mathscr{J}^i$ are injective, it is actually *split* exact up to the r-th step. This implies we can write the complex as a direct sum of two injective complexes

$$\mathscr{J}^\bullet = \mathscr{J}_1^\bullet \oplus \mathscr{J}_2^\bullet$$

where $\mathscr{J}_1^\bullet$ is concentrated in degrees $[0, r]$ and $\mathscr{J}_2^\bullet$ is concentrated in degrees $[r, \infty)$. It follows that

$$\omega_X^\bullet = \ker\big(d_r \colon \mathscr{J}_2^r \longrightarrow \mathscr{J}_2^{r+1}\big)$$

and that for any $\mathcal{O}_X$-module $\mathscr{F}$, we have

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathscr{F}, \omega_X^\bullet) \cong \operatorname{Ext}^r_{\mathcal{O}_P}(\mathscr{F}, \omega_P). \qquad \square$$

PROPOSITION 2.8.51. *Let X be a projective scheme over a field k. Then, X has a dualizing sheaf.* [Har77, Prop. III.7.5]

Proof. Embed X as a closed subscheme of $P = \mathbf{P}^N_k$ for some N, let r be its codimension, and let

$$\omega_X^\circ = \mathscr{E}xt^r_{\mathcal{O}_P}(\mathcal{O}_X, \omega_P).$$

Then, by Lemma 2.8.50, we have an isomorphism

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathscr{F}, \omega_X^\bullet) \cong \operatorname{Ext}^r_{\mathcal{O}_P}(\mathscr{F}, \omega_P)$$

for any $\mathcal{O}_X$-module $\mathscr{F}$. On the other hand, when $\mathscr{F}$ is coherent, the duality theorem for $P = \mathbf{P}^N_k$ (Theorem 2.8.46) gives an isomorphism

$$\mathscr{E}xt^r_{\mathcal{O}_P}(\mathcal{O}_X, \omega_P) \cong H^{N-r}(P, \mathscr{F})^\vee.$$

But $N - r = n = \dim(X)$ and $\mathscr{F}$ is a coherent sheaf on X, and hence we obtain an isomorphism

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathscr{F}, \omega_X^\circ) \cong H^n(X, \mathscr{F})^\vee$$

functorial in the coherent $\mathcal{O}_X$-module $\mathscr{F}$. In particular, taking $\mathscr{F} = \omega_X^\circ$, the element $\mathrm{id} \in \operatorname{Hom}_{\mathcal{O}_X}(\omega_X^\circ, \omega_X^\circ)$ gives a morphism

$$t \colon H^n(X, \omega_X^\circ) \longrightarrow k,$$

which we take as our trace map. By functoriality, we see that (ω_X°, t) is a dualizing sheaf for X. $\square$

We can now prove Serre duality for Cohen–Macaulay projective schemes X. A scheme is *Cohen–Macaulay* if all its local rings are Cohen–Macaulay. See [Mur24ca, Definition 8.16.4] for the definition of a Cohen–Macaulay ring.

Theorem 2.8.52 (Serre duality for projective schemes). *Let X be a projective scheme of dimension n over an algebraically closed field k. Let $\omega_X^\bullet$ be a dualizing sheaf for X and let $\mathcal{O}(1)$ be a very ample invertible $\mathcal{O}_X$-module. We have the following.* [Har77, Thm. III.7.6]

(*a*) *For all $i \geq 0$ and all coherent $\mathcal{O}_X$-modules $\mathscr{F}$, there are natural functorial maps*
$$\theta^i\colon \operatorname{Ext}^i_{\mathcal{O}_X}(\mathscr{F}, \omega_X^\circ) \longrightarrow H^{n-i}(X, \mathscr{F})^\vee$$
such that θ^0 is the map given in the definition for a dualizing sheaf above.

(*b*) *The following conditions are equivalent.*

  (*i*) *X is Cohen–Macaulay and equidimensional, i.e., all irreducible components of X have the same dimension.*

  (*ii*) *For any locally free $\mathcal{O}_X$-module $\mathscr{F}$, we have*
$$H^i\bigl(X, \mathscr{F}(-q)\bigr) = 0$$
for $i < n$ and $q \gg 0$.

  (*iii*) *The maps θ^i of (a) are isomorphisms for all $i \geq 0$ and all coherent $\mathcal{O}_X$-modules $\mathscr{F}$.*

Proof. (*a*). As in the proof of Theorem 2.8.46(*iii*), we can write any coherent $\mathcal{O}_X$-module $\mathscr{F}$ as a quotient of a sheaf
$$\mathscr{E} = \mathcal{O}_X(-q)^{\oplus N}$$
for $q \gg 0$. Then,
$$\operatorname{Ext}^i_{\mathcal{O}_X}(\mathscr{E}, \omega_X^\circ) \cong \Bigl(H^i\bigl(X, \omega_X^\circ(q)\bigr)\Bigr)^{\oplus N}$$
which is 0 for $i > 0$ and $q \gg 0$ by Serre vanishing (Proposition 2.7.2(*ii*)). Thus, the functor $\operatorname{Ext}^i(\,\cdot\,, \omega_X^\circ)$ is coeffaceable for $i > 0$, and hence the $\operatorname{Ext}^i(\,\cdot\,, \omega_X^\circ)$ form a universal contravariant δ-functor by Proposition 1.4.17. On the right-hand side, we have a contravariant δ-functor indexed by $i \geq 0$, and hence there is a unique morphism of δ-functors θ^i reducing to the given θ^0 for $i = 0$.

(*b*) (*i*) $\Rightarrow$ (*ii*). We embed X as a closed subscheme of $P = \mathbf{P}^N_k$. Then, for any locally free $\mathcal{O}_X$-module $\mathscr{F}$ and any closed point $x \in X$, we have
$$\operatorname{depth}_{\mathcal{O}_{X,x}}(\mathscr{F}_x) = n$$
since X is Cohen–Macaulay and equidimensional of dimension n. Let $A = \mathcal{O}_{P,x}$ be the local ring of P at x. Then, A is a regular local ring of dimension N. We then have
$$\operatorname{depth}_{\mathcal{O}_{X,x}}(\mathscr{F}_x) = \operatorname{depth}_A(\mathscr{F}_x)$$
by [EGAIV$_1$, Chapitre 0, Proposition 16.4.8]. By the Auslander–Buchsbaum formula [Mat89, Theorem 19.1], we see that
$$\operatorname{pd}_A(\mathscr{F}_x) = N - n.$$
Thus, by [Har77, Proposition III.6.8] and the fact that Ext for modules can be computed using projective resolutions for the first argument [Mat89, p. 279], we have
$$\mathscr{E}xt^i_{\mathcal{O}_P}(\mathscr{F}, \,\cdot\,) = 0$$
for all $i > N-n$. On the other hand, by Theorem 2.8.46, we know that $H^i(X, \mathscr{F}(-q))$ is dual to $\operatorname{Ext}^{N-i}_{\mathcal{O}_P}(\mathscr{F}, \omega_P(q))$. For $q \gg 0$, this Ext is isomorphic to
$$H^0\bigl(P, \mathscr{E}xt^{N-i}_{\mathcal{O}_P}\bigl(\mathscr{F}, \omega_P(q)\bigr)\bigr)$$

by [Har77, Proposition III.6.9]. This is 0 for $N - i > N - n$ as we saw above. In other words, $H^i(X, \mathscr{F}(-q)) = 0$ for $i < n$ and $q \gg 0$.

(ii) $\Rightarrow$ (i). Running the argument for (i) $\Rightarrow$ (ii) backwards and using condition (ii) for $\mathscr{F} = \mathcal{O}_X$, we have

$$\mathscr{E}xt^i_{\mathcal{O}_P}(\mathcal{O}_X, \omega_P) = 0$$

for $i > N - n$. This implies that over the local ring $A = \mathcal{O}_{P,x}$ as above, we have

$$\operatorname{Ext}^i_A(\mathcal{O}_{X,x}, A) = 0$$

for all $i > N - n$. Thus, by [Har77, Exercise III.6.6], we have $\operatorname{pd}_A(\mathcal{O}_{X,x}) \leq N - n$, and hence by the Auslander–Buchsbaum formula [Mat89, Theorem 19.1], we have $\operatorname{depth}_A(\mathcal{O}_{X,x}) \geq n$. But since $\dim(X) = n$, we must have equality for every closed point of X. This shows that X is Cohen–Macaulay and equidimensional.

(ii) $\Rightarrow$ (iii). Since we already know that $\operatorname{Ext}^i_{\mathcal{O}_X}(\,\cdot\,, \omega_X^\bullet)$ are universal contravariant δ-functors, to show that the θ^i are isomorphisms, it suffices to show that the δ-functor $(H^{n-i}(X, \,\cdot\,)^\vee)$ is universal also. By Proposition 1.4.17, it suffices to show that $H^{n-i}(X, \,\cdot\,)^\vee$ is coeffaceable for $i > 0$. Given a coherent $\mathcal{O}_X$-module $\mathscr{F}$, write $\mathscr{F}$ as a quotient of $\mathscr{E} = \mathcal{O}(-q)^{\oplus N}$ with $q \gg 0$ as before. Then, $H^{n-i}(X, \mathscr{E})^\vee = 0$ for $i > 0$ by (ii), and hence the functor is coeffaceable.

(iii) $\Rightarrow$ (ii). If θ^i are isomorphisms, then for all $\mathscr{F}$ locally free on X, we have

$$H^i\bigl(X, \mathscr{F}(-q)\bigr) \cong \operatorname{Ext}^{n-i}_{\mathcal{O}_X}\bigl(\mathscr{F}(-q), \omega_X^\bullet\bigr)^\vee.$$

But this Ext is isomorphic to $H^{n-i}(X, \mathscr{F}^\vee \otimes_{\mathcal{O}_X} \omega_X^\circ(q))$ by [Har77, Propositions III.6.3 and III.6.7], and is therefore 0 for $n - i > 0$ and $q \gg 0$ by Serre vanishing (Proposition 2.7.2(ii)). $\square$

In some cases, we get an explicit description of ω_X°. See [Har77] for proofs.

Theorem 2.8.53. *Let X be a closed subscheme of $P = \mathbf{P}^N_k$ which is a local complete intersection of codimension r. Let $\mathscr{I}$ be the ideal sheaf of X. Then,* [Har77, Thm. III.7.11]

$$\omega_X^\circ \cong \left(\omega_P \otimes_{\mathcal{O}_P} \bigwedge^r (\mathscr{I}/\mathscr{I}^2)^\vee\right)\bigg|_X.$$

In particular, ω_X° is an invertible $\mathcal{O}_X$-module.

Corollary 2.8.54. *If X is a projective regular variety over an algebraically closed field k, then the dualizing sheaf ω_X° is isomorphic to the canonical sheaf ω_X.* [Har77, Cor. III.7.12]

2.8.12. Riemann–Roch for curves. We now state and prove the Riemann–Roch theorem. For the statement below, when X is a smooth variety over a field k, we call any divisor K_X such that $\mathcal{O}_X(K_X) \cong \omega_X$ a *canonical divisor* on X. In this class, we have only defined the arithmetic genus $p_a(X)$. The genus below can be taken to be the arithmetic genus or the *geometric genus* $h^0(X, \omega_X)$, where here and below we use lowercase h^i to denote the k-vector space dimension of H^i.

Theorem 2.8.55 (Riemann–Roch for curves). *Let X be an integral nonsingular complete curve over an algebraically closed field k. Let D be a divisor on X of genus g. Then,* [Har77, Thm. IV.1.3]

$$h^0\bigl(X, \mathcal{O}_X(D)\bigr) - h^0\bigl(X, \mathcal{O}_X(K_X - D)\bigr) = \deg(D) + 1 - g.$$

Proof. By Serre duality (Theorem 2.8.52), we know that

$$H^0\big(X, \mathcal{O}_X(K_X - D)\big) \cong H^1\big(X, \mathcal{O}_X(D)\big)^\vee.$$

We therefore have to show that

$$\chi\big(\mathcal{O}_X(D)\big) = \deg(D) + 1 - g. \tag{2.8.56}$$

We first consider the case $D = 0$. Then, our formula says that

$$h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) = 0 + 1 - g.$$

This holds by definition of the arithmetic genus $p_a(X)$.

Now let D be any divisor and let P be any point. We will show that (2.8.56) is true for D if and only if (2.8.56) is true for $D + P$. Since any divisor can be reached from 0 in finitely many steps by adding or subtracting a point each time, this will show the result holds for all D.

Consider P as a closed subscheme of X. Its structure sheaf is a skyscraper sheaf $k(P)$ concentrated at P, and its ideal sheaf is $\mathcal{O}_X(-P)$. We therefore have the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-P) \longrightarrow \mathcal{O}_X \longrightarrow k(P) \longrightarrow 0.$$

Tensoring with $\mathcal{O}_X(D + P)$, we get

$$0 \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D + P) \longrightarrow k(P) \longrightarrow 0.$$

Now the Euler characteristic is additive on short exact sequences (take the associated long exact sequence on sheaf cohomology and use the rank–nullity theorem) and $\chi(k(P)) = 1$, which implies

$$\chi\big(\mathcal{O}_X(D + P)\big) = \chi\big(\mathcal{O}_X(D)\big) + 1.$$

On the other hand, $\deg(D + P) = \deg(D) + 1$, and hence (2.8.56) is true for D if and only if (2.8.56) is true for $D + P$. □

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Bibliography

[Abh70] Shreeram S. Abhyankar. “Polynomials and power series.” Written in August 1970 at the International Conference in Nice, France. Originally published in the *Mathematical Intelligencer*, no. 3 (Sept. 1972). *Algebra, arithmetic and geometry with applications (West Lafayette, IN, 2000)*. Berlin: Springer, 2004, pp. 783–784. DOI: `10.1007/978-3-642-18487-1_49`. MR: `2037080`. 149

[ACGH85] Enrico Arbarello, Maurizio Cornalba, Phillip A. Griffiths, and Joe Harris. *Geometry of algebraic curves. Vol. I.* Grundlehren Math. Wiss., Vol. 267. New York: Springer-Verlag, 1985. DOI: `10.1007/978-1-4757-5323-3`. MR: `770932`. 155

[AF94] Luchezar L. Avramov and Hans-Bjørn Foxby. “Grothendieck's localization problem.” *Commutative algebra: syzygies, multiplicities, and birational algebra (South Hadley, MA, 1992)*. Contemp. Math., Vol. 159. Providence, RI: Amer. Math. Soc., 1994, pp. 1–13. DOI: `10.1090/conm/159/01498`. MR: `1266174`. 188

[AK21] Allen B. Altman and Steven L. Kleiman. *A term of commutative algebra*. Version on ResearchGate. Cambridge, MA: Worldwide Center of Mathematics, 2021. DOI: `10.13140/RG.2.2.31866.62400`. 1, 2, 19, 57, 131–133

[And74a] Michel André. *Homologie des algèbres commutatives*. Grundlehren Math. Wiss., Vol. 206. Berlin-New York: Springer-Verlag, 1974. DOI: `10.1007/978-3-642-51449-4_18`. MR: `352220`. 186

[And74b] ———. “Localisation de la lissité formelle.” *Manuscripta Math.* 13 (1974), pp. 297–307. DOI: `10.1007/BF01168230`. MR: `357403`. 188

[Art62] Michael Artin. *Grothendieck topologies*. Cambridge, Mass.: Harvard University, 1962. URL: `https://hdl.handle.net/2027/mdp.39015056607966`. ZBL: `0208.48701`. 7

[ATJLL97] Leovigildo Alonso Tarrío, Ana Jeremías López, and Joseph Lipman. “Local homology and cohomology on schemes.” *Ann. Sci. École Norm. Sup. (4)* 30.1 (1997), pp. 1–39. NUMDAM: `10.1016/s0012-9593(97)89914-4`. DOI: `10.1016/S0012-9593(97)89914-4`. MR: `1422312`. 77

[Bae40] Reinhold Baer. “Abelian groups that are direct summands of every containing abelian group.” *Bull. Amer. Math. Soc.* 46 (1940), pp. 800–806. DOI: `10.1090/S0002-9904-1940-07306-9`. MR: `2886`. 23–25

[Ber1882] Eugenio Bertini. “Sui sistemi lineari.” *Ist. Lombardo, Rend., II. Ser.* 15 (1882), pp. 24–29. `ark:/13960/t7fr00h6z`. ZBL: `14.0433.02`. 195

[Ber1907] ———. *Introduzione alla geometria proiettiva degli iperspazi, con appendice sulle curve algebriche e loro singolarità*. Pisa: Enrico Spoerri, 1907. `ark:/13960/t47p9943b`. ZBL: `37.0580.02`. 195

[BH98] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*. Revised edition. Cambridge Stud. Adv. Math., Vol. 39. Cambridge: Cambridge Univ. Press, 1998. DOI: `10.1017/CBO9780511608681`. MR: `1251956`. 85, 86, 93

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*. Ergeb. Math. Grenzgeb. (3), Vol. 21. Berlin: Springer-Verlag, 1990. DOI: `10.1007/978-3-642-51438-8`. MR: `1045822`. 183, 184, 194

[Bor94a] Francis Borceux. *Handbook of categorical algebra. 1. Basic category theory*. Encyclopedia Math. Appl., Vol. 50. Cambridge: Cambridge Univ. Press, 1994. DOI: `10.1017/CBO9780511525858`. MR: `1291599`. 1–3, 9

[Bor94b] ———. *Handbook of categorical algebra. 2. Categories and structures*. Encyclopedia Math. Appl., Vol. 51. Cambridge: Cambridge Univ. Press, 1994. DOI: `10.1017/CBO9780511525865`. MR: `1313497`. 2, 9, 10, 21

[BouAlgI] Nicolas Bourbaki. *Algebra I. Chapters 1–3*. Translated from the French, Reprint of the 1989 English translation. Elem. Math. (Berlin). Berlin: Springer-Verlag, 1998. `ark:/13960/t2k727t4c`. MR: `1727844`. 28

[BouCA] ———. *Elements of mathematics. Commutative algebra*. Translated from the French. Paris: Hermann; Reading, MA: Addison-Wesley Publishing Co., 1972. `ark:/13960/t56f3ng94`. MR: `360549`. 63, 112, 131–133, 174

[BouGT] ———. *General topology. Chapters 1–4*. Translated from the French, Reprint of the 1989 English translation. Elem. Math. (Berlin). Berlin: Springer-Verlag, 1998. DOI: `10.1007/978-3-642-61701-0`. MR: `1726779`. 114, 131, 134, 135

[BS13] Markus P. Brodmann and Rodney Y. Sharp. *Local cohomology. An algebraic introduction with geometric applications*. Second edition. Cambridge Stud. Adv. Math., Vol. 136. Cambridge: Cambridge Univ. Press, 2013. Available through the Purdue library at `https://ebookcentral.proquest.com/lib/purdue/detail.action?docID=1042409`. DOI: `10.1017/CBO9781139044059`. MR: `3014449`. 93, 94, 99–101

[BS93] Mauro C. Beltrametti and Andrew J. Sommese. “On k-jet ampleness.” *Complex analysis and geometry*. Univ. Ser. Math. New York: Plenum, 1993, pp. 355–376. DOI: `10.1007/978-1-4757-9771-8_15`. MR: `1211891`. 153

[Buc55] David A. Buchsbaum. “Exact categories and duality.” *Trans. Amer. Math. Soc.* 80 (1955), pp. 1–34. DOI: `10.2307/1993003`. MR: `74407`. 10

[CE56] Henri Cartan and Samuel Eilenberg. *Homological algebra*. Princeton, NJ: Princeton Univ. Press, 1956. 1999 reprint available at `https://www.jstor.org/stable/j.ctt1bpmb49`. DOI: `10.1515/9781400883844`. MR: `77480`. 22, 23, 35

[CGM87] Caterina Cumino, Silvio Greco, and Mirella Manaresi. “Varietà algebriche debolmente normali e teoremi di Bertini.” *Rend. Sem. Mat. Fis. Milano* 57 (1987): *Proceedings of the Geometry Conference (Milan* *and Gargnano, 1987)*, pp. 135–148. doi: `10.1007/BF02925047`. mr: `1017925`. 155

[Cho57] Wei-Liang Chow. “On the projective embedding of homogeneous varieties.” *Algebraic geometry and topology. A symposium in honor of S. Lefschetz.* Princeton, NJ: Princeton Univ. Press, 1957, pp. 122–128. doi: `10.1515/9781400879915-009`. mr: `84851`. 174

[CM81] Caterina Cumino and Mirella Manaresi. “On the singularities of weakly normal varieties.” *Manuscripta Math.* 33.3-4 (1981), pp. 283–313. doi: `10.1007/BF01798229`. mr: `612614`. 155

[Con00] Brian Conrad. *Grothendieck duality and base change.* Lecture Notes in Math., Vol. 1750. Berlin: Springer-Verlag, 2000. “Clarifications and corrections” available at `https://math.stanford.edu/~conrad/papers/dualitycorrections.pdf`. “More examples” available at `https://math.stanford.edu/~conrad/papers/moreexample.pdf`. doi: `10.1007/b75857`. mr: `1804902`. 77

[Con06] ———. “Relative ampleness in rigid geometry.” *Ann. Inst. Fourier (Grenoble)* 56.4 (2006), pp. 1049–1126. doi: `10.5802/aif.2207`. mr: `2266885`. 49, 81

[CS53] Henri Cartan and Jean-Pierre Serre. “Un théorème de finitude concernant les variétés analytiques compactes.” *C. R. Acad. Sci. Paris* 237 (1953), pp. 128–130. `ark:/12148/bpt6k3189t/f128.item`. mr: `66010`. 180

[CT20] Paolo Cascini and Hiromu Tanaka. “Relative semi-ampleness in positive characteristic.” *Proc. Lond. Math. Soc. (3)* 121.3 (2020), pp. 617–655. doi: `10.1112/plms.12323`. mr: `4100119`. 144

[Dem12] Michel Demazure. “Résultant, discriminant.” *Enseign. Math. (2)* 58.3-4 (2012), pp. 333–373. doi: `10.4171/LEM/58-3-5`. mr: `3058604`. 148

[DM24] Rankeya Datta and Takumi Murayama. “Permanence properties of F-injectivity.” *Math. Res. Lett.* 31.4 (2024), pp. 985–1028. doi: `10.4310/MRL.241118233550`. mr: `4831046`. 155

[Doh08] Davis C. Doherty. “Singularities of generic projection hypersurfaces.” *Proc. Amer. Math. Soc.* 136.7 (2008), pp. 2407–2415. doi: `10.1090/S0002-9939-08-09286-1`. mr: `2390507`. 155

[Dol82] Igor Dolgachev. “Weighted projective varieties.” *Group actions and vector fields (Vancouver, B.C., 1981).* Lecture Notes in Math., Vol. 956. Berlin: Springer, 1982, pp. 34–71. doi: `10.1007/BFb0101508`. mr: `704986`. 89, 95

[EGAI] Alexander Grothendieck and Jean A. Dieudonné. “Éléments de géométrie algébrique. I. Le langage des schémas.” *Inst. Hautes Études Sci. Publ. Math.* 4 (1960), 228 pp. numdam: `PMIHES_1960__4__5_0`. doi: `10.1007/BF02684778`. mr: `217083`. xv, 64, 105–108, 111, 115, 118

[EGAI$_{\text{new}}$] ———. *Eléments de géométrie algébrique. I.* Grundlehren Math. Wiss., Vol. 166. Berlin: Springer-Verlag, 1971. `ark:/13960/t42s6kw4b`. mr: `3075000`. xv, 4, 47–53, 55–57, 59–63, 65, 67–77, 105, 108, 110, 111, 113–118, 120, 131–133, 135, 136, 141, 151, 168, 170, 173, 178, 187, 196

[EGAII] Alexander Grothendieck and Jean A. Dieudonné. "Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes." *Inst. Hautes Études Sci. Publ. Math.* 8 (1961), 222 pp. NUMDAM: `PMIHES_1961__8__5_0`. DOI: `10.1007/BF02699291`. MR: `217084`. xv, 76, 81–83, 85–88, 90, 91, 94–98, 102–104, 110, 111, 135, 137–150, 152, 170, 172, 174, 175

[EGAIII$_1$] ______. "Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I." *Inst. Hautes Études Sci. Publ. Math.* 11 (1961), 167 pp. NUMDAM: `PMIHES_1961__11__5_0`. DOI: `10.1007/BF02684273`, `10.1007/BF02684274`. MR: `217085`. xv, 65, 92, 99, 102, 103, 121, 122, 125, 128, 165, 171, 178–180

[EGAIII$_2$] ______. "Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. II." *Inst. Hautes Études Sci. Publ. Math.* 17 (1963), 91 pp. NUMDAM: `PMIHES_1963__17__5_0`. DOI: `10.1007/BF02684890`. MR: `163911`. xv

[EGAIV$_1$] ______. "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I." *Inst. Hautes Études Sci. Publ. Math.* 20 (1964), 259 pp. NUMDAM: `PMIHES_1964__20__5_0`. DOI: `10.1007/BF02684747`. MR: `173675`. xv, 76, 146, 170, 183–190, 194, 196, 200

[EGAIV$_2$] ______. "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II." *Inst. Hautes Études Sci. Publ. Math.* 24 (1965), 231 pp. NUMDAM: `PMIHES_1965__24__5_0`. DOI: `10.1007/BF02684322`. MR: `199181`. xv, 150, 180, 181, 188

[EGAIV$_3$] ______. "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III." *Inst. Hautes Études Sci. Publ. Math.* 28 (1966), 255 pp. NUMDAM: `PMIHES_1966__28__5_0`. DOI: `10.1007/BF02684343`. MR: `217086`. xv

[EGAIV$_4$] ______. "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV." *Inst. Hautes Études Sci. Publ. Math.* 32 (1967), 361 pp. NUMDAM: `PMIHES_1967__32__361_0`. DOI: `10.1007/BF02732123`. MR: `238860`. xv, 155–159, 168, 169, 187, 188, 191–194

[Eis05] David Eisenbud. *The geometry of syzygies. A second course in commutative algebra and algebraic geometry.* Grad. Texts in Math., Vol. 229. New York: Springer-Verlag, 2005. DOI: `10.1007/b137572`. MR: `2103875`. 92

[FAC] Jean-Pierre Serre. "Faisceaux algébriques cohérents." *Ann. of Math. (2)* 61 (1955), pp. 197–278. English translations available at `https://achinger.impan.pl/fac/fac.pdf` (by Piotr Achinger and Łukasz Krupa) and `https://www.jmilne.org/math/Documents/fac.pdf` (by John Milne). DOI: `10.2307/1969915`. MR: `68874`. 1, 41, 49, 50, 52, 53, 55, 62, 65, 85, 86, 91, 92, 96–99, 102, 103, 125, 128, 148

[Ful98] William Fulton. *Intersection theory.* Second edition. Ergeb. Math. Grenzgeb. (3), Vol. 2. Berlin: Springer-Verlag, 1998. DOI: `10.1007/978-1-4612-1700-8`. MR: `1644323`. 140, 142, 148, 156, 158

[Gab62] Pierre Gabriel. "Des catégories abéliennes." *Bull. Soc. Math. France* 90 (1962), pp. 323–448. DOI: `10.24033/bsmf.1583`. MR: `232821`. 103

[GH78] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry.* Pure Appl. Math. New York: Wiley-Interscience [John Wiley & Sons], 1978. DOI: `10.1002/9781118032527`. MR: `507725`. 155

[Gil11] William D. Gillam. “Localization of ringed spaces.” *Adv. Pure Math.* 1.5 (2011), pp. 250–263. DOI: `10.4236/apm.2011.15045`. ZBL: `1247.14001`. 105

[GIT] David Mumford, John Fogarty, and Frances Kirwan. *Geometric invariant theory.* Third edition. Ergeb. Math. Grenzgeb. (3), Vol. 34. Berlin: Springer-Verlag, 1994. Second edition available at `ark:/13960/s2xgpf0bmcg`. MR: `1304906`. 169

[God73] Roger Godement. *Topologie algébrique et théorie des faisceaux.* Troisième édition revue et corrigée. Actualités Sci. Indust., Vol. 1252. Publications de l’Institut de Mathématique de l’Université de Strasbourg, Vol. XIII. Paris: Hermann, 1973. MR: `345092`. 3–8, 10, 21, 28, 29, 38–41, 43, 121, 122, 178, 179

[GS01] Alexander Grothendieck and Jean-Pierre Serre. *Correspondance Grothendieck-Serre.* Ed. by Pierre Colmez and Jean-Pierre Serre. Bilingual edition. Translated by Catriona Maclean with the assistance of Leila Schneps and Jean-Pierre Serre. Doc. Math. (Paris), Vol. 2. Paris: Soc. Math. France, 2001. `ark:/13960/s23zs0g9zxd`. MR: `1942134`. 137

[GT80] Silvio Greco and Carlo Traverso. “On seminormal schemes.” *Compositio Math.* 40.3 (1980), pp. 325–365. NUMDAM: `CM_1980__40_3_325_0`. MR: `571055`. 155

[Har62] Robin Hartshorne. “Complete intersections and connectedness.” *Amer. J. Math.* 84.3 (1962), pp. 497–508. DOI: `10.2307/2372986`. MR: `142547`. 129

[Har68] ———. “Cohomological dimension of algebraic varieties.” *Ann. of Math. (2)* 88 (1968), pp. 403–450. DOI: `10.2307/1970720`. MR: `232780`. 128, 129

[Har77] ———. *Algebraic geometry.* Grad. Texts in Math., Vol. 52. New York-Heidelberg: Springer-Verlag, 1977. DOI: `10.1007/978-1-4757-3849-0`. MR: `463157`. xv, 3–8, 10, 11, 13–17, 21–24, 28–32, 34–36, 38–41, 43, 45, 47–53, 55–57, 59–65, 67–78, 81–86, 90–92, 95–99, 101, 102, 104, 105, 108–111, 113–117, 120–123, 125, 127–133, 135–150, 152–184, 186, 188–201

[Har94] ———. “Generalized divisors on Gorenstein schemes.” *K-Theory* 8.3 (1994): *Proceedings of Conference on Algebraic Geometry and Ring Theory in honor of Michael Artin, Part III (Antwerp, 1992)*, pp. 287–339. DOI: `10.1007/BF00960866`. MR: `1291023`. 173, 174

[Hoc14] Melvin Hochster. *Faithful flatness.* Supplementary material from a course taught at the University of Michigan, Winter 2014. URL: `https://dept.math.lsa.umich.edu/~hochster/615W14/fthflat.pdf`. 180

[Hoc17] ———. *Math 614 lecture notes.* Lecture notes from a course taught at the University of Michigan, Fall 2017. URL: `https://dept.math.lsa.umich.edu/~hochster/614F17/614.pdf`. 1–3, 18, 19

[Hör90] Lars Hörmander. *An introduction to complex analysis in several variables*. Third edition. North-Holland Math. Library, Vol. 7. Amsterdam: North-Holland Publishing Co., 1990. Second edition available at `https://www.sciencedirect.com/science/book/9780444105233`. MR: `1045639`. 56

[Jec03] Thomas Jech. *Set theory*. The third millennium edition, revised and expanded. Springer Monogr. Math. Berlin: Springer-Verlag, 2003. DOI: `10.1007/3-540-44761-X`. MR: `1940513`. 24, 26, 27

[Jim13] `Jim`. Answer to "Finite free graded modules and the grading of their duals". Mathematics Stack Exchange. Version of Jan. 17, 2014. URL: `https://math.stackexchange.com/a/602028`. 86

[Kem80] George R. Kempf. "Some elementary proofs of basic theorems in the cohomology of quasicoherent sheaves." *Rocky Mountain J. Math.* 10.3 (1980): *Surfaces (Stanford, Calif., 1982)*, pp. 637–645. DOI: `10.1216/RMJ-1980-10-3-637`. MR: `590225`. 66, 67, 77, 122, 179

[Kie67] Reinhardt Kiehl. "Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie." *Invent. Math.* 2 (1967), pp. 256–273. DOI: `10.1007/BF01425404`. MR: `210949`. 65

[Kle00] Steven L. Kleiman. "Cartier divisors versus invertible sheaves." *Comm. Algebra* 28.12 (2000): *Special issue in honor of Robin Hartshorne*, pp. 5677–5678. DOI: `10.1080/00927870008827181`. MR: `1808595`. 169

[Kle79] ———. "Misconceptions about K_X." *Enseign. Math. (2)* 25.3-4 (1979), 203–206 (1980). DOI: `10.5169/seals-50379`. MR: `570309`. 155, 156

[Kle98] ———. "Bertini and his two fundamental theorems." *Rend. Circ. Mat. Palermo (2)* Suppl. 55 (1998): *Studies in the history of modern mathematics, III*, pp. 9–37. MR: `1661859`. 195

[KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*. With the collaboration of C. Herbert Clemens and Alessio Corti, Translated from the 1998 Japanese original. Cambridge Tracts in Math., Vol. 134. Cambridge: Cambridge Univ. Press, 1998. DOI: `10.1017/CBO9780511662560`. MR: `1658959`. 150

[KMM87] Yujiro Kawamata, Katsumi Matsuda, and Kenji Matsuki. "Introduction to the minimal model problem." *Algebraic geometry, Sendai, 1985*. Adv. Stud. Pure Math., Vol. 10. Amsterdam: North-Holland, 1987, pp. 283–360. DOI: `10.2969/aspm/01010283`. MR: `946243`. 144

[KS06] Masaki Kashiwara and Pierre Schapira. *Categories and sheaves*. Grundlehren Math. Wiss., Vol. 332. Berlin: Springer-Verlag, 2006. DOI: `10.1007/3-540-27950-4`. MR: `2182076`. 31, 53, 55, 68, 102, 153

[Kun86] Ernst Kunz. *Kähler differentials*. Adv. Lectures Math. Braunschweig: Friedr. Vieweg & Sohn, 1986. DOI: `10.1007/978-3-663-14074-0`. MR: `864975`. 183–185

[Kür13] Alex Küronya. "Positivity on subvarieties and vanishing of higher cohomology." *Ann. Inst. Fourier (Grenoble)* 63.5 (2013), pp. 1717–1737. DOI: `10.5802/aif.2812`. MR: `3186506`. 175

[Kur1922] Casimir Kuratowski. "Une méthode d'élimination des nombres transfinis des raisonnements mathématiques." *Fundamenta Mathematicae* 3

(1922), pp. 76–108. DOI: 10.4064/fm-3-1-76-108. ZBL: 48.0205.04. 25, 79

[Lan02] Serge Lang. *Algebra*. Third edition. Grad. Texts in Math., Vol. 211. New York: Springer-Verlag, 2002. DOI: 10.1007/978-1-4613-0041-0. MR: 1878556. 18

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series*. Ergeb. Math. Grenzgeb. (3), Vol. 48. Berlin: Springer-Verlag, 2004. DOI: 10.1007/978-3-642-18808-4. MR: 2095471. 168, 171

[Lee13] John M. Lee. *Introduction to smooth manifolds*. Second edition. Grad. Texts in Math., Vol. 218. New York: Springer, 2013. DOI: 10.1007/978-1-4419-9982-5. MR: 2954043. 183

[Lip09] Joseph Lipman. “Notes on derived functors and Grothendieck duality.” *Foundations of Grothendieck duality for diagrams of schemes*. Lecture Notes in Math., Vol. 1960. Berlin: Springer, 2009, pp. 1–259. DOI: 10.1007/978-3-540-85420-3. MR: 2490557. 179

[Liu88] Qing Liu. “Un contre-exemple au “critère cohomologique d’affinoïdicité”.” *C. R. Acad. Sci. Paris Sér. I Math.* 307.2 (1988), pp. 83–86. ark:/12148/bpt6k54938197/f31.image. MR: 954265. 81

[Mac94] Francis S. Macaulay. *The algebraic theory of modular systems*. Revised reprint of the 1916 original, With an introduction by Paul Roberts. Cambridge Math. Lib. Cambridge: Cambridge Univ. Press, 1994. Original 1916 edition available at ark:/13960/t2d79897m. MR: 1281612. 143

[Mat89] Hideyuki Matsumura. *Commutative ring theory*. Translated from the Japanese by M. Reid. Second edition. Cambridge Stud. Adv. Math., Vol. 8. Cambridge: Cambridge Univ. Press, 1989. DOI: 10.1017/CBO9781139171762. MR: 1011461. 183–190, 200, 201

[McL10] Colin McLarty. “What does it take to prove Fermat’s last theorem? Grothendieck and the logic of number theory.” *Bull. Symbolic Logic* 16.3 (2010), pp. 359–377. DOI: 10.2178/bsl/1286284558. MR: 2731249. 36

[MM16] MooS and Takumi Murayama. Answer to “Extension by zero not Quasi-coherent”. Mathematics Stack Exchange. Version of Mar. 4, 2016. URL: https://math.stackexchange.com/a/1682649. 49

[Mor79] Shigefumi Mori. “Projective manifolds with ample tangent bundles.” *Ann. of Math. (2)* 110.3 (1979), pp. 593–606. DOI: 10.2307/1971241. MR: 554387. 150

[mst16] msteve. “Detecting k-affinoid spaces by vanishing cohomology.” MathOverflow. Version of Jan. 14, 2016. URL: https://mathoverflow.net/q/228366. 81

[Mum66] David Mumford. *Lectures on curves on an algebraic surface*. With a section by George M. Bergman. Ann. of Math. Stud., Vol. 59. Princeton, NJ: Princeton Univ Press, 1966. Available at https://www.jstor.org/stable/j.ctt1b9x2g3. DOI: 10.1515/9781400882069. MR: 209285. 157, 174

[Mum67] David Mumford. *Introduction to algebraic geometry. Preliminary version of the first 3 chapters.* Cambridge, Mass.: Department of Mathematics, Harvard University, 1967. Republished as [Mum88]. 56–58

[Mum88] ———. *The red book of varieties and schemes.* Lecture Notes in Math., Vol. 1358. Berlin: Springer-Verlag, 1988. DOI: 10.1007/978-3-662-21581-4. MR: 971985. 216

[Mun00] James R. Munkres. *Topology.* Second edition. Upper Saddle River, NJ: Prentice Hall, Inc., 2000. MR: 3728284. 131, 134

[Mur18a] Takumi Murayama. Answer to "Closed immersions are stable under base change". Mathematics Stack Exchange. Version of Feb. 6, 2018. URL: https://math.stackexchange.com/a/1081279. 114

[Mur18b] ———. Answer to "Integral Algebraic Scheme has Cartier Divisors". Mathematics Stack Exchange. Version of Nov. 29, 2018. URL: https://math.stackexchange.com/a/3019060. 168

[Mur22] Takumi Murayama. "A uniform treatment of Grothendieck's localization problem." *Compos. Math.* 158.1 (2022), pp. 57–88. DOI: 10.1112/s0010437x21007715. MR: 4371042. 188

[Mur24ag] ———. *MA595AG: Algebraic Geometry I.* Lecture notes from a course taught at Purdue University, Fall 2024. 1–5, 8, 12, 13, 17–19, 21, 28, 43, 49, 59–61, 69, 71, 74, 77, 79, 81, 100, 101, 105–107, 132, 136, 140, 143, 144, 148, 162–164, 166, 194, 196

[Mur24ca] ———. *MA557: Commutative algebra.* Lecture notes from a course taught at Purdue University, Fall 2024. 1, 23, 49–51, 56, 60, 62–65, 72, 73, 75, 76, 99, 106, 109, 110, 112, 113, 128, 143, 153, 156, 158, 171, 181, 182, 187, 188, 198, 199

[Nag56] Masayoshi Nagata. "The theory of multiplicity in general local rings." *Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955.* Tokyo: Science Council of Japan, 1956, pp. 191–226. MR: 82966. Available at https://www.jmilne.org/math/Documents/TokyoNikko1955.pdf. 184

[Nag75] ———. *Local rings.* Corrected reprint. Huntington, NY: Robert E. Krieger Publishing Co., 1975. ark:/13960/s2r2641wnj6. MR: 460307. 184

[Nak63] Yoshikazu Nakai. "Some fundamental lemmas on projective schemes." *Trans. Amer. Math. Soc.* 109 (1963), pp. 296–302. DOI: 10.2307/1993908. MR: 155830. 169

[Oja08] Manuel Ojanguren. "Un ouvert bizarre." Version of Apr. 20, 2008. URL: https://web.archive.org/web/20240917152728/https://sma.epfl.ch/~ojangure/nichtnoethersch.pdf. 72, 84

[Oka50] Kiyoshi Oka. "Sur les fonctions analytiques de plusieurs variables. VII. Sur quelques notions arithmétiques." *Bull. Soc. Math. France* 78 (1950), pp. 1–27. DOI: 10.24033/bsmf.1408. MR: 35831. 52, 56

[Per41/43] Oskar Perron. "Über die Bedingungen, dass eine binäre Form n-ten Grades eine n-te Potenz ist, und über die rationale Kurve n-ter Ordnung im R_m." *Math. Ann.* 118 (1941/1943), pp. 305–309. DOI: 10.1007/BF01487372. MR: 8910. 129

[Poo10] Bjorn Poonen. Answer to "Is a universally closed morphism of schemes quasi-compact?" MathOverflow. Version of May 5, 2010. URL: https://mathoverflow.net/q/23528. 137

[Ray70] Michel Raynaud. *Anneaux locaux henséliens*. Lecture Notes in Math., Vol. 169. Berlin-New York: Springer-Verlag, 1970. DOI: 10.1007/BFb0069571. MR: 277519. 193

[Rei80] Miles Reid. "Canonical 3-folds." *Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*. Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980, pp. 273–310. MR: 605348. Available at https://www.maths.ed.ac.uk/cheltsov/quotient/pdf/reid1.pdf. 168, 174

[Rei95] ———. *Undergraduate commutative algebra*. London Math. Soc. Stud. Texts, Vol. 29. Cambridge: Cambridge Univ. Press, 1995. DOI: 10.1017/CBO9781139172721. MR: 1458066. 57

[Rob71] Joel Roberts. "Generic projections of algebraic varieties." *Amer. J. Math.* 93 (1971), pp. 191–214. DOI: 10.2307/2373457. MR: 277530. 155

[Rob75] ———. "Singularity subschemes and generic projections." *Trans. Amer. Math. Soc.* 212 (1975), pp. 229–268. DOI: 10.2307/1998623. MR: 422274. 155

[RZN84] Joel Roberts and Rahim Zaare-Nahandi. "Transversality of generic projections and seminormality of the image hypersurfaces." *Compositio Math.* 52.2 (1984), pp. 211–220. NUMDAM: CM_1984__52_2_211_0. MR: 750357. 155

[Sam66] Pierre Samuel. *Lectures on old and new results on algebraic curves*. Notes by S. Anantharaman. Tata Inst. Fundam. Res. Lect. Math., Vol. 36. Bombay: Tata Institute of Fundamental Research, 1966. Retypeset version available at https://mathweb.tifr.res.in/sites/default/files/publications/ln/tifr36.pdf. ark:/13960/s2smq9qdmq7. MR: 222088. 155

[SCC55/56] *Séminaire H. Cartan et C. Chevalley, 8e année: 1955/1956. Géométrie algébrique*. Paris: Secrétariat mathématique, 11 rue Pierre Curie, 1956. NUMDAM: SHC_1955-1956__8_. MR: 97402. 131, 137

[Sch00] Stefan Schröer. "Remarks on the existence of Cartier divisors." *Arch. Math. (Basel)* 75.1 (2000), pp. 35–38. DOI: 10.1007/s000130050470. MR: 1764889. 169

[Sch03] Peter Schenzel. "Proregular sequences, local cohomology, and completion." *Math. Scand.* 92.2 (2003), pp. 161–180. DOI: 10.7146/math.scand.a-14399. MR: 1973941. 95

[Sch05] Karl Schwede. "Gluing schemes and a scheme without closed points." *Recent progress in arithmetic and algebraic geometry*. Contemp. Math., Vol. 386. Providence, RI: Amer. Math. Soc., 2005, pp. 157–172. DOI: 10.1090/conm/386/07222. MR: 2182775. 79

[Ser53] Jean-Pierre Serre. "Groupes d'homotopie et classes de groupes abéliens." *Ann. of Math. (2)* 58 (1953), pp. 258–294. DOI: 10.2307/1969789. MR: 59548. 103

[Ser57] ———. "Sur la cohomologie des variétés algébriques." *J. Math. Pures Appl. (9)* 36 (1957), pp. 1–16. MR: 83813. 41, 76

[SGA2_{new}] Alexander Grothendieck. *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux*. Séminaire de Géométrie Algébrique du Bois-Marie 1962 (SGA 2). Augmenté d'un exposé de Michèle Raynaud, Revised reprint of the 1968 French original, With a preface and edited by Yves Laszlo. Doc. Math. (Paris), Vol. 4. Paris: Soc. Math. France, 2005. Available at `https://www.cmls.polytechnique.fr/perso/laszlo/sga2/sga2-smf.pdf`. MR: `2171939`. 95, 127, 128

[SGA4_2] *Théorie des topos et cohomologie étale des schémas. Tome 2.* Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par Michael Artin, Alexander Grothendieck et Jean-Louis Verdier. Avec la collaboration de Nicolas Bourbaki, Pierre Deligne et Bernard Saint-Donat. Lecture Notes in Math., Vol. 270. Berlin-New York: Springer-Verlag, 1972. Revised version of Jul. 30, 2024 available at `http://fabrice.orgogozo.perso.math.cnrs.fr/SGA4`. DOI: `10.1007/BFb0061319`. MR: `354653`. 77

[SGA6] *Théorie des intersections et théorème de Riemann-Roch*. Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Dirigé par Pierre Berthelot, Alexander Grothendieck et Luc Illusie. Avec la collaboration de Daniel Ferrand, Jean-Pierre Jouanolou, Olli Jussila, Steven Kleiman, Michel Raynaud et Jean-Pierre Serre. Lecture Notes in Math., Vol. 225. Berlin-New York: Springer-Verlag, 1971. DOI: `10.1007/BFb0066283`. MR: `354655`. 77

[SHC50/51] *Séminaire Henri Cartan de l'Ecole Normale Supérieure, 1950/1951. Cohomologie des groupes, suite spectrale, faisceaux.* 2e édition. Paris: Secrétariat mathématique, 11 rue Pierre Curie, 1955. NUMDAM: `SHC_1950-1951__3_`. MR: `77921`. 7

[SHC51/52] *Séminaire Henri Cartan de l'Ecole Normale Supérieure, 1951/1952. Fonctions analytiques de plusieurs variables complexes.* NUMDAM: `SHC_1951-1952__4_`. 52, 65, 76

[Sin11] Balwant Singh. *Basic commutative algebra*. Hackensack, NJ: World Scientific Publishing Co. Pte. Ltd., 2011. DOI: `10.1142/7811`. MR: `2752311`. 189, 190

[Smi04] S. Paul Smith. "Maps between non-commutative spaces." *Trans. Amer. Math. Soc.* 356.7 (2004), pp. 2927–2944. See also [Smi16]. DOI: `10.1090/S0002-9947-03-03411-1`. MR: `2052602`. 103

[Smi16] ______. "Corrigendum to "Maps between non-commutative spaces"." *Trans. Amer. Math. Soc.* 368.11 (2016), pp. 8295–8302. DOI: `10.1090/tran/6908`. MR: `3546801`. 218

[Stacks] The Stacks project authors. *The Stacks project*. `https://stacks.math.columbia.edu`. 2025. 6, 8, 24, 74–79, 82, 105, 133, 137, 147, 149, 150, 171, 176, 191, 193, 194

[Ste56] Karl Stein. "Analytische Zerlegungen komplexer Räume." *Math. Ann.* 132 (1956), pp. 63–93. DOI: `10.1007/BF01343331`. MR: `83045`. 165

[Sup72] Patrick Suppes. *Axiomatic set theory.* Unabridged and corrected republication of the 1960 original with a new preface and a new section (8.4). New York: Dover, 1972. `ark:/13960/t8nc9z55b`. MR: `349389`. 26, 27

[TohokuI] Alexander Grothendieck. “Sur quelques points d’algèbre homologique. I.” *Tohoku Math. J. (2)* 9.2 (1957), pp. 119–183. English translation by Marcia L. Barr and Michael Barr available at `https://math.mcgill.ca/barr/papers/gk.pdf`. DOI: `10.2748/tmj/1178244839`. MR: `102537`. 3, 7, 9–11, 17, 18, 20–25, 28–30, 33–41, 43, 44, 103, 178

[TohokuII] ______. “Sur quelques points d’algèbre homologique. II.” *Tohoku Math. J. (2)* 9.3 (1957), pp. 185–221. English translation by Marcia L. Barr and Michael Barr available at `https://math.mcgill.ca/barr/papers/gk.pdf`. DOI: `10.2748/tmj/1178244774`. MR: `102537`. 34, 49, 51

[use19] `user26857`. Comment to answer on “Is the global section ring of a Noetherian Scheme Noetherian as well?” Mathematics Stack Exchange. Version of Sep. 9, 2019. URL: `https://math.stackexchange.com/questions/818578/is-the-global-section-ring-of-a-noetherian-scheme-noetherian-as-well/818591#comment6895756_818591`. 85

[Vak] Ravi Vakil. *The rising sea. Foundations of algebraic geometry.* To be published (2025) by Princeton Univ. Press, Princeton, NJ. Draft version of Sep. 8, 2024 available at `https://math.stanford.edu/~vakil/216blog/FOAGsep0824public.pdf`. 77, 78, 110, 115, 132, 150

[Wei48] André Weil. *Sur les courbes algébriques et les variétés qui s’en déduisent.* Publ. Inst. Math. Univ. Strasbourg, Vol. 7 (1945); Actualités Sci. Indust., No. 1041. Paris: Hermann & Cie, 1948. `ark:/12148/bpt6k3372566b`. MR: `27151`. 148

[Wei62] ______. *Foundations of algebraic geometry.* Revised edition. Amer. Math. Soc. Colloq. Publ., Vol. 29. Providence, RI: Amer. Math. Soc., 1962. DOI: `10.1090/coll/029`. MR: `144898`. 148, 149

[Wei94] Charles A. Weibel. *An introduction to homological algebra.* Cambridge Stud. Adv. Math., Vol. 38. Cambridge: Cambridge Univ. Press, 1994. Corrections available at `https://sites.math.rutgers.edu/~weibel/Hbook-corrections.html`. DOI: `10.1017/CBO9781139644136`. MR: `1269324`. 1–3, 9, 10, 20, 21, 23, 28–35, 38, 39, 122

[Yon54] Nobuo Yoneda. “On the homology theory of modules.” *J. Fac. Sci. Univ. Tokyo Sect. I* 7 (1954), pp. 193–227. MR: `68832`. 65

[Zar47] Oscar Zariski. “The concept of a simple point of an abstract algebraic variety.” *Trans. Amer. Math. Soc.* 62 (1947), pp. 1–52. DOI: `10.2307/1990628`. MR: `21694`. 194

[ZN84] Rahim Zaare-Nahandi. “Seminormality of certain generic projections.” *Compositio Math.* 52.2 (1984), pp. 245–274. NUMDAM: `CM_1984__52_2_245_0`. MR: `750360`. 155

[Zor35] Max Zorn. “A remark on method in transfinite algebra.” *Bull. Amer. Math. Soc.* 41.10 (1935), pp. 667–670. DOI: `10.1090/S0002-9904-1935-06166-X`. MR: `1563165`. 25, 79