MA665: Algebraic Geometry II

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Contents

List of Symbols

Symbol	Description
0	zero element, zero ring, or zero object, 34
1	identity element in a ring R, ix
$\sim$	homotopy equivalence of morphisms of chain complexes, 70
$\twoheadrightarrow$	epimorphism, 58
$\rightsquigarrow$	specialization, 92
$\hookrightarrow$	monomorphism, 58
$\langle U, f\rangle$	representative of a germ of a regular or rational function f, or a section f of a sheaf, 17
$\coprod_i X_i$	coproduct of objects X_i, 39
$\bigoplus_i X_i$	direct sum of objects X_i in an Abelian category, 50
$\prod_i X_i$	product of objects X_i, 39
$\Gamma(U, \mathscr{F})$	sections of a sheaf $\mathscr{F}$ on an open set U, 16
$\Gamma_*(\mathscr{F})$	graded module associated to a sheaf $\mathscr{F}$, 155
$\Gamma_Z(U, \mathscr{F})$	sections of a sheaf $\mathscr{F}$ on an open set U with support in Z, 74
Ω_X^p	sheaf of differential p-forms, 35
$\mathbf{C}$	complex numbers
$\mathbf{N}$	natural numbers $\{0, 1, 2, \ldots\}$
$\mathbf{P}_A^n$	projective n-space over A, 146
$\mathbf{Q}$	rational numbers
$\mathbf{R}$	real numbers
Ab	category of Abelian groups, 8, 42
$\mathsf{Ab}(X)$	category of Abelian sheaves on a topological space X, 18
$\mathsf{Ch}(\mathscr{A})$	category of (cochain) complexes in $\mathscr{A}$, 67
$\mathsf{Coh}(\mathscr{O}_X)$	category of coherent sheaves of $\mathscr{O}_X$-modules, 113
$\mathsf{ind}_I(\mathscr{C})$	category of direct systems indexed by I in $\mathscr{C}$, 85
LRS	category of locally ringed spaces, 100
$\mathsf{Mod}(\mathscr{O}_X)$	category of sheaves of $\mathscr{O}_X$-modules, 33
$\mathsf{Mod}(R)$	category of R-modules, 8, 42
$^*\mathsf{Mod}(S)$	category of graded S-modules, 148
$\mathsf{PAb}(X)$	category of Abelian presheaves on a topological space X, 18

Symbol	Description
$\mathsf{PSh}(X)$	category of presheaves of sets on a topological space X, 18
$\mathsf{QCoh}(\mathscr{O}_X)$	category of quasi-coherent sheaves of $\mathscr{O}_X$-modules, 108
Rings	category of rings, 8
RS	category of ringed spaces, 33
Sch	category of schemes, 105
Sets	category of sets, 8
$\mathsf{Sh}(X)$	category of sheaves of sets on a topological space X, 18
$\mathsf{Top}(X)$	category of open subsets of a topological space X, 15
Var_k	category of varieties over k, 172
$\mathfrak{m}_x$	maximal ideal of the local ring of a locally ringed space at a point x, 100
$\mathfrak{N}$	nilradical of a ring, 133
$\mathfrak{p}_x$	prime ideal corresponding to a point $x \in \operatorname{Spec}(A)$, 91
$\mathscr{C}^{\mathrm{op}}$	opposite category, 8
$\mathscr{F}(n)$	n-th twist of an $\mathscr{O}_X$-module $\mathscr{F}$, 149
$\mathscr{F}/\mathscr{F}'$	quotient of a sheaf of Abelian groups or $\mathscr{O}_X$-modules, 35
$\mathscr{F}/R$	quotient of a sheaf of sets by an equivalence relation, 23
$\mathscr{F}^{\#}$	sheafification of a presheaf $\mathscr{F}$, 20
$\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$	tensor product of $\mathscr{O}_X$-modules, 38
$\mathscr{F}_P$	stalk of a sheaf $\mathscr{F}$ at P, 17
$\mathscr{F}_U$	$j_! j^* \mathscr{F}$ where $j \colon U \hookrightarrow X$ is an open inclusion, 87
${}_U\mathscr{F}$	$j_* j^* \mathscr{F}$ where $j \colon U \hookrightarrow X$ is an open inclusion, 124
$\mathscr{F}_Y$	$i_* i^* \mathscr{F}$ where $i \colon Y \hookrightarrow X$ is a closed inclusion, 87
$\mathscr{H}om$(F,G)	sheaf hom, 32
$\mathscr{I}_Y$	sheaf of ideals of a subset Y, 36
$\mathscr{K}_X$	sheaf of rational functions, 37
$\mathscr{N}_X$	nilradical of a scheme, 133
$\mathscr{O}_U$	structure sheaf of an open subspace $U \subseteq X$, 104
$\mathscr{O}_X$	structure sheaf of a ringed space, e.g., sheaf of regular functions on a variety X, 17
$\mathscr{O}_X(n)$	n-th twisting sheaf of Serre, 149
$\mathscr{O}_{X,x}$	stalk of the structure sheaf of a ringed space X at a point x, e.g., local ring of a variety X at a point x, 100
$\mathscr{P}(X)$	power set, 59
$A(X)$	ring of an affine scheme, or affine coordinate ring of an algebraic set $X \subseteq \mathbf{A}^n_k$, 101
$A - B$	set difference, ix
$A^\bullet$	complex in an Abelian category, 67
$\underline{A}_X$	constant sheaf determined by A, 17
$B^i(A^\bullet)$	i-th coboundaries of a complex $A^\bullet$, 68
$C(X)$	affine cone over $X = \operatorname{Proj}(S)$, 159
$\operatorname{coim}(f)$	coimage of a morphism, 41
$\operatorname{coker}(f)$	cokernel of a morphism, 35, 41
$D(f)$	distinguished open set in $\operatorname{Spec}(A)$, 94

Symbol	Description
$D_+(f)$	distinguished open set in $\mathrm{Proj}(S)$, 144
$f_*\mathscr{F}$	direct image of a sheaf $\mathscr{F}$, 23
$f^*\mathscr{G}$	inverse image of an $\mathcal{O}_X$-module $\mathscr{G}$, 38
$f^{-1}\mathscr{G}$	inverse image of a sheaf $\mathscr{G}$, 23
$f_{\mathcal{P}}^{-1}\mathscr{G}$	inverse image presheaf of a presheaf $\mathscr{G}$, 23
$f^\#$	map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ on sheaves associated to a morphism of ringed spaces, 33
$\mathbf{h}^i(A^\bullet)$	i-th cohomology object of a complex, 68
$H^i(X, \mathscr{F})$	i-th sheaf cohomology module, 74
$H^i_{\mathrm{dR}}(X)$	i-th de Rham cohomology group of a smooth manifold X, 36
$H^i_Z(X, \mathscr{F})$	i-th sheaf cohomology module with support in Z, 74
$\mathrm{im}(f)$	image of a morphism, 22, 41
$i_P(A)$	skyscraper sheaf at P with value A, 24
$K(f)$	link of a polynomial f, 1
$K(R)$	Grothendieck group of a ring R, 11
$k(x)$	residue field at x, 175
$\ker(f)$	kernel of a morphism, 34, 41
$M(n)$	n-th twist of a graded module M, 148
$\tilde{M}$	sheaf on $\mathrm{Spec}(A)$ (resp. $\mathrm{Proj}(S)$) associated to a module (resp. a graded module), 95, 145
$\mathrm{Proj}(S)$	homogeneous spectrum of a graded ring S, 143
s_P	germ of a section of a sheaf at P, 17
$\mathrm{sp}(X)$	underlying topological space of a ringed space or scheme, 105
$\mathrm{Spé}(\mathscr{F})$	the espace étalé of a presheaf $\mathscr{F}$, 20
$\mathrm{Spec}(A)$	spectrum of a ring A, 91
$T \star S$	composition of two natural transformations T and S, 10
$V(I)$	closed set in $\mathrm{Spec}(A)$ defined by an ideal $I \subseteq A$, 91
$V_+(E)$	vanishing set in $\mathrm{Proj}(S)$, 143
X_f	subset of a locally ringed space where f_x is iinveritble, 100
X_{red}	reduced scheme associated to X, 134
$X \times_S Y$	fiber product of X and Y over S, 171
X_y	fiber of x over $y \in Y$, 175
$Z^i(A^\bullet)$	i-th cocycles of a complex $A^\bullet$, 68

Conventions

(1) Let A and B be subsets of a set X. The *set difference* is denoted

$$A - B \coloneqq \left\{ x \in X \;\middle|\; x \in A \text{ and } x \notin B \right\}.$$

(2) All rings R will be assumed to be commutative with an identity element 1, unless stated otherwise. We may sometimes denote 1 by 1_R for clarity.

(3) All ring maps $\varphi\colon R \to S$ will be assumed to respect the identity element, i.e., $\varphi(1_R) = 1_S$.

Preface

These are notes for the second semester of an introductory graduate sequence on algebraic geometry (MA665) taught at Purdue University in Spring 2025 and Spring 2026. The official course text is [Har1977]. We also suggest [EGAI; EGAInew; EGAII; EGAIII$_1$; EGAIII$_2$; EGAIV$_1$; EGAIV$_2$; EGAIV$_3$; EGAIV$_4$] as additional references. The notes in the margins point to where in these texts (and sometimes others) one can find the material written down in these notes.

These notes will be continually updated throughout the semester.

I would like to thank Farrah Yhee for innumerable helpful conversations.

Chapter 0

Introduction

0.1 Motivation

Algebraic geometry is the geometric study of solutions to systems of polynomial equations. To motivate this subfield (or other fields) of mathematics, we can ask ourselves the following: $\boxed{1/13}$ [Har1977, §I.8]

Question 0.1.1.

(1) What are the basic objects we are interested in?

(2) Why are they interesting?

(3) How can we study these objects? What theory do we need to develop to study and classify them?

For (1), so far, we have studied quasi-projective varieties over algebraically closed fields in MA595AGI [MurAGI].

For (2), as we discussed at the beginning in [MurAGI, §1.0], the interest in quasi-projective varieties goes back at least to the classical problems of antiquity [BK1986, (1.2)]. We also stated various questions about quasi-projective varieties that are interesting and still open.

We give another example which shows why varieties are important for other fields of mathematics.

Example 0.1.2. Let $Z(f) \subseteq \mathbf{C}^{n+1}$ be an affine hypersurface that has an isolated singularity at the origin 0, i.e., there exists an open neighborhood U of 0 such that $U - \{0\}$ is nonsingular. We can intersect $Z(f)$ with a small $(2n+1)$-sphere S_ε^{2n+1} of radius ε around the origin 0 to define the *link of f* [Mil1968, §1] [Mur2017]

$$K(f) \coloneqq Z(f) \cap S_\varepsilon^{2n+1}.$$

By Sard's theorem and Ehresmann's theorem, K is a smooth $(2n-1)$-fold whose diffeomorphism class does not depend on ε [Mil1968, Corollary 2.9].

We write down some examples of links associated to complex isolated hypersurface singularities.

(1) (The case $n = 1$) Let
$$f = z_1^p + z_2^q$$
for positive integers p, q that are coprime. Then, $K(f)$ is a (p, q)-torus knot. See [Mil1968, p. 4] or [BK1986, Proposition 1 on p. 224].

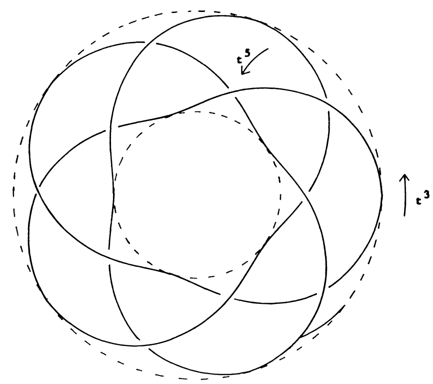

Figure 0.1: The $(3, 5)$-torus knot. From [BK1986, p. 432].

(2) ($n > 1$) Brieskorn [Bri1966a; Bri1966b] considered links associated to the polynomials
$$f = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}$$
where $a_i \geq 2$ for all i. The resulting manifolds are called *Brieskorn–Pham manifolds* (Pham studied related manifolds earlier in [Pha1965]). The link $K(x^3 + y^5 + z^2)$ is a Poincaré homology sphere, i.e., a manifold with the same homology as S^3 that is not homeomorphic to S^3 [Mum1961, §IV]. The links
$$K\Big(z_1^2 + z_2^2 + \cdots + z_{n-1}^2 + z_n^3 + z_{n+1}^m\Big)$$
for various $n \geq 3$ and $m \geq 2$ give *exotic spheres*, i.e., smooth manifolds that are homeomorphic to spheres but with a different differential structure. Setting $m = 6k - 1$ for $k \in \{1, 2, \dots, 28\}$ yields the 28 different differential structures on the 7-sphere [Bri1966b, p. 2]. Note that Milnor found the first exotic 7-sphere [Mil1956]. The $n = 5, m = 2$ case yields Kervaire's exotic 9-sphere [Ker1960] (see [Mil1968, p. 72]).

Example 0.1.2 shows how even in algebraic geometry, the answer to Question 0.1.1(3) involves many subjects. In [MurAGI], we used algebraic methods to study quasi-projective varieties. Example 0.1.2 shows how to study those varieties, we often want to have available tools from differential topology. We will not be able to cover this differential topological aspect of algebraic geometry in this course, but it shows how interwoven algebraic geometry is with other subfields of mathematics.

For the rest of this section, we want to explain how different answers to Question 0.1.1(2),(3) require us to change our answer to Question 0.1.1(1): Even if one's primary motivation is to study quasi-projective varieties (or even nonsingular ones!), there are reasons why we need foundations for algebraic geometry that encompass more general objects. We list some of these reasons.

(1) (Reducible and/or non-reduced) In [MurAGI, §1.7], we studied intersections in projective space. We saw that sometimes, intersecting nonsingular varieties results in (a) reducible algebraic sets, or (b) intersections with intersection multipicities greater than 1 along a component.

For example, consider the intersection of a conic and a line:

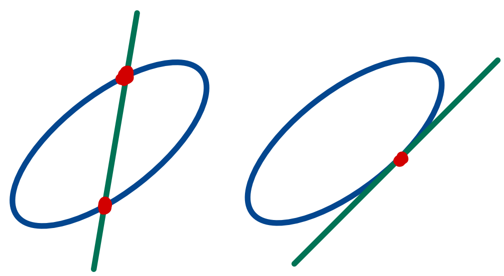

Figure 0.2: A conic and a line in $\mathbf{P}^2$ can intersect in two ways over an algebraically closed field.

In the left situation (a), the algebraic set consisting of two points is not a variety. In the right situation (b), while the algebraic set consisting of one point is a variety, thinking of the intersection point as just a point loses crucial algebraic information. Affine-locally, the corresponding ring is

$$\frac{k[x,y]}{(y,y-x^2)} \cong \frac{k[x]}{(x^2)},$$

which is a non-reduced ring with dimension 2 as a k-vector space.

Schemes will allow us to think of both situations on the same footing as varieties. In situation (a), we will allow the underlying topological spaces to be reducible. In situation (b), we will keep track of the regular functions on the topological space by attaching a *structure sheaf* $\mathcal{O}_X$ to the topological space consisting of a point. This structure sheaf will encode the ring of regular functions for every open subset $U \subseteq X$. For the example on the right of Figure 0.2, this structure sheaf will return the ring $\mathcal{O}_X(X) \cong k[x]/(x^2)$.

You might wonder why we need to keep track of an entire *sheaf* of regular functions, instead of just keeping track of the intersection multiplicity as defined in [MurAGI, Definition 1.7.17], which equals 2 at the point. This is because just keeping track of the multiplicity cannot distinguish between different rings that may arise from different geometric situations. For example, there are multiple 0-dimensional finite type k-algebras of multiplicity 3 that have a unique prime ideal:

$$\frac{k[x]}{(x^3)} \ncong \frac{k[x,y]}{(x^2,xy,y^2)}.$$

(2) (Non-algebraically closed fields) For applications to number theory, it becomes necessary to work over non-algebraically closed fields. For example, Fermat's "last theorem" [Dio1670, Arithmeticorum Liber II, Quæstio VIII, Observatio Domini Petri de Fermat] (now a theorem due to Wiles [Wil1995] and Taylor–Wiles [TW1995]) asks whether the algebraic curve

$$\{x^n + y^n = z^n\} \subseteq \mathbf{P}^2_{\mathbf{Q}}$$

has any points over $\mathbf{Q}$ when $n > 2$. Over finite fields, the Weil conjectures [Wei1949] (now theorems due to Dwork [Dwo1960], Grothendieck [Gro1968], and Deligne [Del1974]) were one of the motivations for Weil to develop his foundations for algebraic geometry [Wei1946; Wei1962].

(3) (Non-quasi-projective varieties via gluing) Weil [Wei1946; Wei1962] and later Serre [FAC] developed ways to discuss abstract varieties that are not necessarily quasi-projective varieties. This becomes necessary because sometimes it is easier to construct objects by gluing together affine varieties – for example Abelian varieties – in which case one must show that the resulting object is a quasi-projective variety.

(4) (Other ground rings) In commutative algebra, many theorems can be stated (and are easier to prove) in the general framework of Noetherian rings. In algebraic geometry, we want to be able to do something similar, i.e., to put all Noetherian rings (and possibly more) on the same footing. This allows one to take localizations and completions freely without leaving the category in question, which is essential (for example) in Hironaka's proof of resolutions of singularities [Hir1964a; Hir1964b] for varieties over fields of characteristic zero. (Hironaka's inductive proof in fact works with schemes of finite type over quasi-excellent local $\mathbf{Q}$-algebras.) The case of Dedekind domains of possibly mixed characteristic and objects of finite type over them (considered by Nagata [Nag1956] before Grothendieck) is one important example of this, since these schemes can interpolate between positive characteristic and characteristic zero.

(5) (Other sheaves) We mentioned one example of a sheaf above in (1): the structure sheaf $\mathcal{O}_X$. However, just like in commutative algebra, it is useful to consider sheaves of *modules* over $\mathcal{O}_X$. Examples of such objects include ideal sheaves and (sheaves associated to) vector bundles. We have already seen examples where considering modules was indispensable, namely the definition of intersection multiplicity and the proof of Bézout's theorem [MurAGI, Theorem 1.7.18].

Schemes will give us a common language that encompass all these features! Schemes were developed by Grothendieck and Dieudonné in [EGAI; EGAII; EGAIII$_1$; EGAIII$_2$; EGAIV$_1$; EGAIV$_2$; EGAIV$_3$; EGAIV$_4$; EGAInew].

Remark 0.1.3. We note that there are some geometric objects from last semester that we will not discuss. In [MurAGI, §1.6], we discussed *abstract nonsingular curves*. These are examples of *Zariski–Riemann spaces* [Zar1940; Zar1944; Nag1963]. Zariski–Riemann spaces were used in Zariski's algebraic proof of resolution of singularities for surfaces over fields of characteristic zero [Zar1939] (this was the first algebraic proof for surfaces) and his proof of resolution of singularities for threefolds over fields of characteristic zero [Zar1944] before Hironaka's work mentioned in (4). They are also critical for Nagata's proof of his compactification theorem [Nag1962; Nag1963]. So, in your algebraic geometry journey, you may need to go even further.

0.2 Structure of the course

The goal of this semester is to build enough of a firm foundation for schemes so that you are able to learn more algebraic geometry on your own. After all, there is only so much we can cover in one semester (or a year, if we count last semester).

In last semester's course [MurAGI], we did almost all of the exercises in [Har1977, Chapter I] either in class, problem sessions, or homework. A goal many algebraic geometers set is to be able to do all the exercises in [Har1977]. This course will cover much of the material [Har1977, Chapters II and III], although we will proceed in a different order and more closely follow the conventions from [EGAI; EGAII; EGAIII$_1$; EGAIII$_2$; EGAIV$_1$; EGAIV$_2$; EGAIV$_3$; EGAIV$_4$; EGAInew].

There will be weekly homework assignments and oral exams where you will present some of the homework problems. We will not be able to do all the exercises from [Har1977, Chapters II–V], but I encourage you to try solving the ones we do not cover on your own (probably after the semester is over).

The broad structure of this course is as follows.

(I) **Sheaves and sheaf cohomology.** Sheaves will allow us to keep track of regular functions on open subsets in a way that gluing becomes easier. From a commutative-algebraic point of view, sheaves (more precisely, quasi-coherent sheaves) are what correspond to modules.

We will then construct cohomology groups from sheaves. This will provide a good source of invariants we can use to differentiate different varieties or schemes. For example, sheaf cohomology can be used to distinguish affine varieties from non-affine ones, and similarly for schemes.

(II) **Schemes.** After defining sheaves, we can define schemes. A scheme is a topological space equipped with a structure sheaf $\mathcal{O}_X$ that is locally isomorphic to an affine scheme.

To make sense of this, we first define affine schemes. Topologically, these are just spectra of rings as you have seen in commutative algebra [MurCA], so we will need to define structure sheaves on them.

We will then define the category of *locally ringed spaces*, and define schemes as locally ringed spaces that are locally isomorphic (as locally ringed spaces) to affine schemes.

After defining schemes, we will spend the rest of the semester studying them.

Chapter 1

Sheaves and sheaf cohomology

For the first part of the course, we will develop the theory of sheaf cohomology that will be used throughout the rest of the course. Sheaf cohomology was introduced to algebraic geometry by Serre in [FAC]. The theory works surprisingly well given the coarseness of the Zariski topology! However, since the theory works for basically any topological space, we will start the course by developing the theory of sheaf cohomology for arbitrary topological spaces.

1.1 Categories and functors

We begin with some background material on categories and homological algebra. See also [MurCA, §2.1] or [MurAGI, Appendix A].

1.1.1 Categories

Definition 1.1.1. A *category* $\mathscr{C}$ consists of the following data:

[Wei1994, Def. A.1.1]
[Bor1994a, Def. 1.2.1]
[AK2021, (6.1)]
[Hoc2017, p. 8]

(1) A class[1] of *objects*.

(2) For every pair of objects A and B in $\mathscr{C}$, a set $\operatorname{Hom}_{\mathscr{C}}(A,B)$ of *maps* or *morphisms*, such that $\operatorname{Hom}_{\mathscr{C}}(A,B)$ and $\operatorname{Hom}_{\mathscr{C}}(A',B')$ are disjoint unless $A=A'$ and $B=B'$. We write $f\colon A\to B$ to mean that $f\in\operatorname{Hom}_{\mathscr{C}}(A,B)$.

[1]This means that the objects of a category do not necessarily form a set. However, the condition that some mathematical object is an object in the category should be set-theoretic. We use the word "class" here because it is possible to add classes to the usual foundations of mathematics (ZFC) without introducing new theorems. See [McL2020, §4.1].

(3) For every triple of objects A, B, and C in $\mathscr{C}$, a *composition law*

$$\begin{aligned} \operatorname{Hom}_{\mathscr{C}}(A,B) \times \operatorname{Hom}_{\mathscr{C}}(B,C) &\longrightarrow \operatorname{Hom}_{\mathscr{C}}(A,C) \\ (f,g) &\longmapsto g \circ f \end{aligned}$$

satisfying the following axioms:

(a) For every object B, there is a distinguished *identity* morphism $\mathrm{id}_B\colon B \to B$ such that for every morphism $f\colon A \to B$, we have $\mathrm{id}_A \circ f = f$, and for every morphism $g\colon B \to C$, we have $g \circ \mathrm{id}_B = g$.

(b) Composition is associative: if $f\colon A \to B$, $g\colon B \to C$, and $g\colon C \to D$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

We say that $f\colon A \to B$ is a *isomorphism* with *inverse* $g\colon B \to A$ if $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. If such an inverse exists, it is unique, and is also an isomorphism $g\colon B \to A$. If there is an isomorphism between a pair of objects A and B, we say that A and B are *isomorphic*.

[Wei1994, A.1.7]
[Hoc2017, p. 11]

Given a category $\mathscr{C}$, we can construct the *opposite category* $\mathscr{C}^{\mathrm{op}}$. It has the same objects as $\mathscr{C}$, and the morphisms are given by

$$\operatorname{Hom}_{\mathscr{C}^{\mathrm{op}}}(A,B) = \operatorname{Hom}_{\mathscr{C}}(B,A).$$

If $f \in \operatorname{Hom}_{\mathscr{C}^{\mathrm{op}}}(A,B)$ and $g \in \operatorname{Hom}_{\mathscr{C}^{\mathrm{op}}}(B,C)$, then composition is given by

$$g \circ_{\mathscr{C}^{\mathrm{op}}} f = f \circ_{\mathscr{C}} g.$$

[Hoc2017, pp. 9–10]

Example 1.1.2. The category Sets of sets, with functions as morphisms. The category Rings of rings, with ring homomorphisms as morphisms.

[Wei1994, Ex. A.1.3]
[Bor1994b, Ex. 1.4.6.a]

Example 1.1.3. Let R be a ring. The class of R-modules together with R-module homomorphisms forms a category $\mathsf{Mod}(R)$. When $R = \mathbf{Z}$, the category $\mathsf{Mod}(\mathbf{Z})$ coincides with the category Ab of Abelian groups with group homomorphisms.

1.1.2 Functors

The utility of categories is really in relationships between them, given by functors.

[Wei1994, §A.2]
[Bor1994a, Def. 1.2.2]
[AK2021, (6.2)]
[Hoc2017, p. 11]

Definition 1.1.4. Given two categories $\mathscr{C}$ and $\mathscr{D}$, a *(covariant) functor*

$$F\colon \mathscr{C} \longrightarrow \mathscr{D}$$

is a rule that assigns to each object A of $\mathscr{C}$ an object $F(A)$ of $\mathscr{D}$, and assigns to each morphism $f\colon A \to B$ in $\mathscr{C}$ a morphism $F(f)\colon F(A) \to F(B)$ in $\mathscr{D}$, such that

(1) For all objects A in $\mathscr{C}$, we have $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

(2) For all morphisms $f\colon A \to B$ and $g\colon B \to C$ in $\mathscr{C}$, we have $F(g \circ f) = F(g) \circ F(f)$.

Note that a functor preserves isomorphisms.

A *contravariant functor* from $\mathscr{C}$ to $\mathscr{D}$ is a covariant functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}$. We will say "let $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}$ be a contravariant functor." [Wei1994, A.2.5] [Bor1994a, Def. 1.4.1]

Example 1.1.5. Here are some examples of functors. [Hoc2017, pp. 11–12]

(1) Given any category $\mathscr{C}$, there is an identity functor $\mathrm{id}_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{C}$ that sends objects A to A itself and morphisms f to f itself.

(2) Given a category $\mathscr{C}$ whose objects have underlying sets and where composition coincides with composition of functions, there is a *forgetful functor* $\mathsf{Forget}\colon \mathscr{C} \to \mathsf{Sets}$ sending objects to their underlying sets, and morphisms to their underlying functions. [Wei1994, A.2.2] [Bor1994a, Ex. 1.2.8.a]

(3) A category $\mathscr{C}$ is a *full subcategory* of another category $\mathscr{D}$ if the objects of $\mathscr{C}$ form a subclass of objects in $\mathscr{D}$, and if $\mathrm{Hom}_{\mathscr{C}}(A, B) = \mathrm{Hom}_{\mathscr{D}}(A, B)$ for every pair of objects A and B in $\mathscr{C}$. For example, finite sets form a full subcategory of Sets, Ab is a full subcategory of Grp. [Wei1994, A.2.3] [Bor1994a, Def. 1.5.4]

(4) The spectrum $\mathrm{Spec}(R)$ defines a contravariant functor [MurAGI, Cor. 2.2.19]

$$\mathrm{Spec}\colon \mathsf{Rings}^{\mathrm{op}} \longrightarrow \mathsf{Top}$$

to the category of topological spaces.

(5) Here is a non-example: Mimicking the definition of $\mathrm{Spec}(\varphi)$ for the maximal spectrum MaxSpec does not define a functor $\mathsf{Rings}^{\mathrm{op}} \to \mathsf{Top}$ (where $\mathrm{MaxSpec}(R)$ is given the subspace topology), or even to Sets, since the inverse image of a maximal ideal is not always maximal. One example of this is $\mathbf{Z} \to \mathbf{Q}$, where the maximal ideal $(0) \subseteq \mathbf{Q}$ has inverse image $(0) \subseteq \mathbf{Z}$, which is not maximal. The same thing occurs for $k[x] \to k(x)$.

1.1.3 Natural transformations and equivalences of categories

We also define natural transformations and isomorphisms of functors.

Definition 1.1.6. Let $F, G\colon \mathscr{C} \to \mathscr{D}$ be two functors. A *natural transformation* $T\colon F \Rightarrow G$ assigns to every object X in $\mathscr{C}$ a morphism $T_X\colon F(X) \to G(X)$ such that for all morphisms $f\colon X \to Y$ in $\mathscr{C}$, there is a commutative diagram [Wei1994, §A.3] [Bor1994a, Def. 1.3.1] [Hoc2017, pp. 13–15]

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "T_X"'] & F(Y) \arrow[d, "T_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y). \end{tikzcd}$$

We can visualize a natural transformation as

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow T"] & \mathscr{D}. \end{tikzcd}$$

Natural transformations $S\colon F \Rightarrow G$ and $T\colon G \Rightarrow H$ can be composed to form the natural transformation $T \star S\colon F \Rightarrow H$ given by the rule

$$(T \star S)_X \coloneqq T_X \circ S_X.$$

There is an identity natural transformation id_F from $F\colon \mathscr{C} \to \mathscr{D}$ to itself. Two functors $F, G\colon \mathscr{C} \to \mathscr{D}$ are *isomorphic* if there are natural transformations

$$T\colon F \Longrightarrow G \qquad \text{and} \qquad T'\colon G \Longrightarrow F$$

such that $T' \star T = \mathrm{id}_F$ and $T \star T' = \mathrm{id}_G$. In fact, T is an isomorphism if and only if the morphisms T_X are isomorphisms for all objects X, in which case

$$(T_X^{-1}) = (T_X)^{-1}.$$

Using isomorphisms of functors, we can define equivalences of categories.

[Wei1994, A.3.2] [Bor1994a, Def. 3.4.4] [Hoc2017, p. 15]

Definition 1.1.7. Two categories $\mathscr{C}$ and $\mathscr{D}$ are *equivalent* if there exist functors $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ such that $G \circ F$ is isomorphic to the identity functor on $\mathscr{C}$ and $F \circ G$ is isomorphic to the identity functor on $\mathscr{D}$. Two categories $\mathscr{C}$ and $\mathscr{D}$ are *antiequivalent* if $\mathscr{C}^{\mathrm{op}}$ is equivalent to $\mathscr{D}$.

$\boxed{1/15}$

Example 1.1.8. Here is an example of a category $\mathscr{D}$ that has a full subcategory $\mathscr{C}$ such that the inclusion functor $\mathscr{C} \to \mathscr{D}$ is an equivalence.

Let $\mathscr{D} = \mathsf{Mod}_{\mathrm{fp}}(R)$ be the category of finitely presented modules over a ring R. We claim that the objects of $\mathscr{D}$ do not form a set. We argue in a set theory with proper classes, so it makes sense to talk about subclasses of classes. Suppose that $\mathscr{D}$ did form a set. We can then consider the subset Σ of all objects of the form

$$R^S = \big\{ f\colon S \to R \big\}$$

where S is a finite set. For every set T, the module $R^{\{T\}}$ lies in Σ. This implies that the class of all sets (which is not a set!) is in bijection with a subset of Σ, a contradiction.

On the other hand, there is a full subcategory $\mathscr{C}$ whose objects form a set of objects of $\mathscr{D}$ such that every object of $\mathscr{D}$ is isomorphic to an object of $\mathscr{C}$. A category whose objects and morphisms form sets is called *small* and a category that is equivalent to a small category is called *essentially small*. Consider the set

$$\bigcup_{m,n \in \mathbf{N}} \left\{ \operatorname{coker}\Big(R^{\oplus n} \xrightarrow{f} R^{\oplus m} \Big) \;\middle|\; f \in \operatorname{Hom}_R\Big(R^{\oplus n}, R^{\oplus m} \Big) \right\}.$$

The full subcategory $\mathscr{C}$ of $\mathscr{D}$ whose objects lie in this set is a small category. The inclusion functor $\mathscr{C} \hookrightarrow \mathscr{D}$ is an equivalence: the inverse is defined by assigning to each object M an object $M_{m,n} = \operatorname{coker}(R^{\oplus n} \to R^{\oplus m})$ of $\mathscr{C}$, and then figuring out how to map morphisms using the fully faithfulness of the inclusion functor. See [Bor1994a, Proposition 3.4.3] for more details.

The reason we might want to do this is that if the objects do not form a set, there is no way to make sense of constructions that quantify over all objects in the category. For example, this is essential when discussing Grothendieck groups, which were introduced by Grothendieck around 1957 [SGA6, Exposé 0, Appendice; BS1958, p. 105]. The *Grothendieck group* $K(R)$ of a ring is the quotient of the free Abelian group generated by all isomorphism classes of $\mathsf{Mod}_{\mathrm{fp}}(R)$ quotiented out by the subgroup generated by $[M] - [M'] - [M'']$ for short exact sequences $0 \to M' \to M \to M'' \to 0$.

A fun exercise is to use the classification of finitely generated modules over a PID to prove that the Grothendieck group of a PID is always $\mathbf{Z}$ [AM1969, Chapter 7, Exercise 26; Har1977, Chapter II, Exercise 6.10(*a*)].

1.1.4 Direct limits and inverse limits (not covered in class)

Definition 1.1.9. (Filtered partially ordered set) A partially ordered set $(I, \leq)$ is *filtered* or *directed* if, for any two elements $i, j \in I$, there exists $k \in I$ with $i \leq k$ and $j \leq k$. [Hoc2017, p. 153]

In other words, any two elements of I have an upper bound.

Examples 1.1.10. Here are examples of filtered partially ordered sets. [Hoc2017, p. 153]

(1) Totally ordered sets, for example, the natural numbers $\mathbf{N}$ and the positive integers.

(2) Finite subsets of a given set under $\subseteq$.

(3) Finitely generated R-submodules of an R-module under $\subseteq$.

(4) Finitely generated R-subalgebras of an R-algebra under $\subseteq$.

(5) Open neighborhoods of a point $x \in X$, where X is a topological space, under $\supseteq$.

We now define direct systems and direct limits.

Definition 1.1.11. (Direct system) Let $(I, \leq)$ be a partially ordered set. A *direct system* or *direct limit system* indexed by I in a category $\mathscr{C}$ is a functor $I \to \mathscr{C}$ where I is considered as a category as in [MurAGI, Example A.1.2(2)]: The objects of the category are the elements in I, and [Hoc2017, p. 153]

$$\operatorname{Hom}(x, y) = \begin{cases} \{*\} & \text{if } x \leq y, \\ \varnothing & \text{otherwise.} \end{cases}$$

Explicitly, the data of a direct system consist of an object X_i for every $i \in I$ and morphisms

$$f_{ij} \colon X_i \longrightarrow X_j$$

for all pairs $i, j \in I$ such that such that

(1) $f_{ii} = \mathrm{id}_{X_i}$ for every $i \in I$; and

(2) Whenever $i \leq j \leq k$, we have $f_{ik} = f_{jk} \circ f_{ij}$.

If I is filtered, we say that the direct system is a *filtered direct system*.

[Hoc2017, p. 153]
[Lan2002, Thm. III.10.1]
[TohokuI, §1.8]

Definition 1.1.12. (Direct limit) Let $\{X_i, f_{ij}\}$ be a direct system of objects and morphisms in a category $\mathscr{C}$. The *direct limit* of the system $\{X_i, f_{ij}\}$ is an object of $\mathscr{C}$ denoted by $\lim X_i$ with morphisms $\{\varphi_i \colon X_i \to \lim X_i\}$ such that $\varphi_j \circ f_{ij} = \varphi_i$ for all $i \leq j$. Moreover, if $Y \in \mathscr{C}$ is another object with morphisms $\{\psi_i \colon X_i \to Y\}$ such that $\psi_j \circ f_{ij} = \psi_i$ for all $i \leq j$, then there is a unique morphism $u\colon \lim X_i \to Y$ making the following diagram commute:

$$\begin{tikzcd}
X_i \arrow[rr, "f_{ij}"] \arrow[dr, "\varphi_i"] \arrow[ddr, bend right, "\psi_i"'] & & X_j \arrow[dl, "\varphi_j"'] \arrow[ddl, bend left, "\psi_j"] \\
& \lim\limits_i X_i \arrow[d, dashed, "u"] & \\
& Y &
\end{tikzcd}$$

If the direct system is filtered, we call the direct limit a *filtered direct limit*.

Examples 1.1.13. Let $\mathscr{C}$ be the category of sets, groups, Abelian groups, rings, R-modules, or R-algebras. (The examples below will work for any category where the objects have an underlying set and a morphism is a function possibly satisfying additional conditions.)

(1) Let Z be a fixed object in $\mathscr{C}$ and let $\{X_i\}$ be a filtered set of subobjects of Z, partially ordered by inclusion. Then, the direct limit of the X_i is the union of these subobjects, and is called the *filtered union*.

[AK2021, (7.4)]

(2) Let $\{X_i, f_{ij}\}$ be a filtered direct system of objects and morphisms in $\mathscr{C}$. Then, the filtered direct limit exists and can be constructed as

$$\lim_i X_i \coloneqq \bigsqcup_i X_i \Big/ f_{ij}(x_i) \sim x_j \text{ for all } i \leq j.$$

If $\mathscr{C}$ is the category of Abelian groups or R-modules, then the direct limit exists and can be constructed as

$$\lim_{i} X_i \coloneqq \bigoplus_i X_i \Bigg/ \big(f_{ij}(x_i) - x_j\big)_{i \leq j}$$

where $(f_{ij}(x_i) - x_j)_{i \leq j}$ is the subobject generated by the elements $f_{ij}(x_i) - x_j$.

Next, we define inverse systems and inverse limits.

Definition 1.1.14. (Inverse system) Let $(I, \leq)$ be a partially ordered set. [Hoc2017, p. 155] An *inverse system* or *inverse limit system* indexed by I in a category $\mathscr{C}$ is a functor $I^{\mathrm{op}} \to \mathscr{C}$ where I is considered as a category as in [MurAGI, Example A.1.2(2)]. Explicitly, for every element $i \in I$, we have an object X_i in $\mathscr{C}$, and for all pairs $i, j \in I$, we have a morphism

$$f_{ij} \colon X_j \longrightarrow X_i$$

such that

(1) $f_{ii} = \mathrm{id}_{X_i}$ for every $i \in I$.

(2) Whenever $i \leq j \leq k$, we have $f_{ik} = f_{ij} \circ f_{jk}$.

If I is filtered, we say the inverse system is a *cofiltered inverse system*.

Definition 1.1.15. (Inverse limit) Let $\{X_i, f_{ij}\}$ be a inverse system of objects and morphisms in a category $\mathscr{C}$. [Hoc2017, p. 155] The *inverse limit* of the system $\{X_i, f_{ij}\}$ is an object of $\mathscr{C}$ denoted by $\lim X_i$ with morphisms $\{\pi_i \colon \lim X_i \to X_i\}$ such that $f_{ij} \circ \pi_j = \pi_i$ for all $i \leq j$. Moreover, if Y is another object with morphisms $\{\psi_i \colon Y \to X_i\}$ such that $f_{ij} \circ \psi_j = \psi_i$ for all $i \leq j$, then there is a unique morphism $u \colon Y \to \lim X_i$ making the following diagram commute:

$$\begin{tikzcd}
& Y \arrow[d, dashed, "u"] \arrow[ddl, bend right, "\psi_j"'] \arrow[ddr, bend left, "\psi_i"] & \\
& \lim\limits_i X_i \arrow[dl, "\pi_j"] \arrow[dr, "\pi_i"'] & \\
X_j \arrow[rr, "f_{ij}"'] & & X_i
\end{tikzcd}$$

If I is filtered, we say the inverse limit is a *cofiltered inverse limit*.

Example 1.1.16. Let $\mathscr{C}$ be the category of sets, Abelian groups, rings, R-modules, or R-algebras. [Hoc2017, pp. 155–156] The inverse limit exists and can be constructed as

$$\lim_{i} X_i \coloneqq \left\{ (x_1, x_2, \ldots) \in \prod_i X_i \;\middle|\; f_{ij}(x_j) = x_i \text{ for all } i \geq j \right\}.$$

1.1.5 Filtered direct limits are exact (not covered in class)

An important result when working with sheaves is the following.

[AK2021, (7.9)]

Theorem 1.1.17. (Filtered direct limits are exact) *Let R be a ring and let I be a filtered partially ordered set. Let $\mathscr{C}$ be the category of 3-term exact sequences of R-modules: its objects are 3-term exact sequences, and its morphisms are commutative diagrams of the form*

$$\begin{array}{ccccc} L & \longrightarrow & M & \longrightarrow & N \\ \downarrow & & \downarrow & & \downarrow \\ L' & \longrightarrow & M' & \longrightarrow & N'. \end{array}$$

Then, for any filtered direct system

$$\left\{L_i \xrightarrow{\beta_i} M_i \xrightarrow{\gamma_i} N_i\right\}_{i\in I}$$

the induced sequence

$$\lim_i L_i \xrightarrow{\beta} \lim_i M_i \xrightarrow{\gamma} \lim_i N_i$$

is exact.

Proof. Abusing notation, we denote by f_{ij} the transition maps connecting the L_i, M_i, or N_i, and by φ_i the canonical morphisms mapping the L_i, M_i, or N_i to their respective direct limits.

We first show that $\operatorname{im}(\beta) \subseteq \ker(\gamma)$. Suppose $\ell \in \operatorname{im}(\beta)$. By definition of direct limits (Example 1.1.13(2)), there exist $i \in I$ and $\ell_i \in L_i$ such that $\varphi_i(\ell_i) = \ell$. We then consider the commutative diagram

$$\begin{array}{ccccc} L_i & \xrightarrow{\beta_i} & M_i & \xrightarrow{\gamma_i} & N_i \\ {\scriptstyle\varphi_i}\downarrow & & \downarrow{\scriptstyle\varphi_i} & & \downarrow{\scriptstyle\varphi_i} \\ \lim_i L_i & \xrightarrow{\beta} & \lim_i M_i & \xrightarrow{\gamma} & \lim_i N_i. \end{array} \tag{1.1.18}$$

Since $\operatorname{im}(\beta_i) \subseteq \ker(\gamma_i)$ by assumption, we see that

$$(\gamma \circ \beta)(\ell) = (\gamma \circ \beta)\big(\varphi_i(\ell_i)\big) = \varphi_i\big((\gamma_i \circ \beta_i)(\ell_i)\big) = 0.$$

We now show that $\operatorname{im}(\beta) \supseteq \ker(\gamma)$. Suppose $m \in \ker(\gamma)$. By construction of filtered direct limits (Example 1.1.13(2)), there exist $i \in I$ and $m_i \in M_i$ such that $\varphi_i(m_i) = m$ and hence $\varphi_i(\gamma_i(m_i)) = 0$ by the commutativity of (1.1.18). By construction of filtered direct limits (Example 1.1.13(2)), we know there exists $j \geq i$ such that $f_{ij}(m_i) = 0$. We now consider the commutative diagram

$$\begin{tikzcd}
L_i \arrow[r, "\beta_i"] \arrow[d, "f_{ij}"'] & M_i \arrow[r, "\gamma_i"] \arrow[d, "f_{ij}"] & N_i \arrow[d, "f_{ij}"] \\
L_j \arrow[r, "\beta_j"] \arrow[d, "\varphi_j"'] & M_j \arrow[r, "\gamma_j"] \arrow[d, "\varphi_j"] & N_j \arrow[d, "\varphi_j"] \\
\lim\limits_i L_i \arrow[r, "\beta"] & \lim\limits_i M_i \arrow[r, "\gamma"] & \lim\limits_i N_i.
\end{tikzcd}$$

By exactness in the j-th row, we know there exists $\ell_j \in L_j$ such that $\beta_j(\ell_j) = f_{ij}(m_i)$. Let $\ell = \varphi_j(\ell_j)$. By the commutativity of the diagram in the bottom two rows, we see that

$$\beta(\ell) = \beta\bigl(\varphi_j(\ell_j)\bigr) = \varphi_j\bigl(\beta_j(\ell_j)\bigr) = \varphi_j(m_j) = m. \quad \square$$

1.2 Sheaves

We review the notion of a sheaf. These provide a systematic way to keep track of local algebraic data on a topological space. For example, for a quasi-projective variety X, the structure sheaf $\mathcal{O}_X$ keeps track of the regular functions that are defined on an open subset U. $\boxed{1/13}$

1.2.1 Presheaves

We start with presheaves.

Definition 1.2.1. Let X be a topological space. We can consider the category $\mathsf{Top}(X)$ whose objects are the open subsets of X, and whose morphisms are the inclusion maps. Let $\mathscr{C}$ be a category. A *presheaf $\mathscr{F}$ on X with values in $\mathscr{C}$* is a contravariant functor [MurAGI, Def. 2.1.1] [Har1977, p. 61] [God1973, I.1.9]

$$\mathscr{F} \colon \mathsf{Top}(X)^{\mathrm{op}} \longrightarrow \mathscr{C}.$$

If $\mathscr{C}$ is the category of Abelian groups, rings, or sets, then we say that $\mathscr{F}$ is a presheaf of Abelian groups, rings, or sets. For Abelian groups, we sometimes say *Abelian presheaf*. [TohokuI, p. 154]

We spell out what a presheaf $\mathscr{F}$ of sets (resp. Abelian groups) is. A presheaf $\mathscr{F}$ of sets (resp. Abelian groups) consists of the following data:

(a) For every open subset $U \subseteq X$, a set (resp. an Abelian group) $\mathscr{F}(U)$.

(b) For every inclusion $V \subseteq U$ of open subsets of X, a function (resp. a group homomorphism) $\rho^U_V \colon \mathscr{F}(U) \to \mathscr{F}(V)$. We call these *restriction maps*.

[Har1977, p. 61] has the extra condition that $\mathscr{F}(\varnothing) = 0$. This is not standard.

These data are subject to the following conditions:

(1) ρ_U^U is the identity map $\mathscr{F}(U) \to \mathscr{F}(U)$.

(2) If $W \subseteq V \subseteq U$ are three open subsets, then $\rho_W^U = \rho_W^V \circ \rho_V^U$.

[MurAGI, Def. 2.1.3]
[Har1977, p. 61]
[God1973, I.1.9]

Definition 1.2.2. If $\mathscr{F}$ is a presheaf on X, we refer to $\mathscr{F}(U)$ as the *sections* over the open set U. We sometimes use the notation

$$\Gamma(U, \mathscr{F}) := \mathscr{F}(U).$$

If $V \subseteq U$ is an inclusion of open sets, we sometimes write $s|_V$ instead of $\rho_V^U(s)$ for $s \in \mathscr{F}(U)$.

1.2.2 Sheaves

We can now define sheaves as a kind of presheaf that is determined by local data.

[MurAGI, Def. 2.1.4]
[Har1977, p. 61]
[God1973, II.1.1]

Definition 1.2.3. Let X be a topological space and let $\mathscr{F}$ be a presheaf on X with values in sets or a category $\mathscr{C}$ like Abelian groups or rings. We say that $\mathscr{F}$ is a *sheaf with values in* $\mathscr{C}$ if it satisfies the following conditions:

(3) Let $\{U_i\}_{i\in I}$ be a family of open subsets of X with union U. Let $s', s'' \in \mathscr{F}(U)$. If $s'|_{U_i} = s''|_{U_i}$ for every i, then $s' = s''$.

(4) Let $\{V_i\}_{i\in I}$ be a family of open subsets of X with union V. Let $s_i \in \mathscr{F}(V_i)$ be elements such that for all $i, j \in I$, we have

$$s_i|_{V_i\cap V_j} = s_j|_{V_i \cap V_j}.$$

Then, there exists $s \in \mathscr{F}(V)$ such that $s|_{V_i} = s_i$ for all $i \in I$. (Note (3) implies that s is unique.)

[MurAGI, Rem. 2.1.5]
[EGAInew, (**0**, 3.1.1)]

Remark 1.2.4. (The sheaf conditions in terms of an equalizer) A convenient way (that you do not need to know!) to package the definition of a sheaf is that a presheaf $\mathscr{F}$ of sets is a sheaf if the diagram

$$\mathscr{F}(U) \longrightarrow \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \cap U_j)$$

is an equalizer diagram for every family of open sets $\{U_i\}_{i\in I}$. See [MurAGI, Definition A.4.1] for the definition of an equalizer. This means that the elements of $\prod_i \mathscr{F}(U_i)$ that map to the same element in $\prod_{i,j} \mathscr{F}(U_i \cap U_j)$ are *exactly* the elements in the image of $\mathscr{F}(U)$. An Abelian presheaf is a sheaf if the sequence

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\prod_i \rho_{U_i}^U} \prod_i \mathscr{F}(U_i) \xrightarrow{\prod_{i,j}(\rho_{U_i\cap U_j}^{U_i} - \rho_{U_i\cap U_j}^{U_j})} \prod_{i,j} \mathscr{F}(U_i \cap U_j)$$

is exact for every family of open sets $\{U_i\}_{i\in I}$.

Example 1.2.5. Let X be a quasi-projective variety over an algebraically closed field k. For each open subset $U \subseteq X$, let $\mathcal{O}_X(U)$ be the ring of regular functions from U to k, and for each $V \subseteq U$, let [MurAGI, Ex. 2.1.7] [Har1977, Ex. II.1.0.1]

$$\rho^U_V \colon \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V)$$

be the restriction map (in the usual sense). Then, $\mathcal{O}_X$ is a presheaf of rings on X by definition of a regular function, and is a sheaf on X since a function that is locally 0 is 0, and a function which is regular locally is regular. We call $\mathcal{O}_X$ the *sheaf of regular functions* on X.

Example 1.2.6. Similarly, one can define the sheaf of continuous real-valued functions on any topological space, the sheaf of differentiable functions on a differentiable manifold, the sheaf of holomorphic functions on a complex manifold, etc. [MurAGI, Ex. 2.1.8] [Har1977, Ex. II.1.0.2]

Example 1.2.7. Let X be a topological space and let A be an object in $\mathscr{C}$. The *constant sheaf* $\underline{A}_X$ on X determined by A is defined as follows. Consider A with the discrete topology. For any open set $U \subseteq X$, we set [MurAGI, Ex. 2.1.9] [Har1977, Ex. II.1.0.3] [God1973, II.1.4]

We use this notation instead of $\mathscr{A}$ since $\underline{\mathbf{Z}}$ is a useful example.

$$\underline{A}_X(U) \coloneqq \left\{\begin{matrix}\text{continuous maps}\\ U \to A\end{matrix}\right\}.$$

Together with the usual restriction maps, we obtain a sheaf $\underline{A}_X$. We sometimes drop the subscript X.

If U is an open set whose connected components are open (which is always true if X is locally connected), then $\underline{A}_X(U)$ is a direct product of copies of A, one for each connected component of U.

1.2.3 Stalks

The following definition gives the analogue of a germ of a function and the local rings of a quasi-projective variety for arbitrary sheaves.

Definition 1.2.8. Let X be a topological space and let $\mathscr{F}$ be a presheaf on X with values in a cocomplete category (i.e., a category with arbitrary small colimits, in particular direct limits). Let $P \in X$ be a point. The *stalk* of $\mathscr{F}$ at P is [MurAGI, Def. 2.1.10] [Har1977, p. 62] [God1973, II.1.2]

$$\mathscr{F}_P \coloneqq \varinjlim_{U \ni P} \mathscr{F}(U)$$

where the transition maps in the direct system are the restriction maps. The elements of $\mathscr{F}_P$ are called *germs* of sections of $\mathscr{F}$ at the point P. If $s \in \mathscr{F}(U)$ is a section, then the germ of s at $P \in U$ is s_P.

Spelling out the definition of a direct limit, the elements of $\mathscr{F}_P$ are represented by pairs $\langle U, s\rangle$ where U is an open neighborhood of P and $s \in \mathscr{F}(U)$, subject to the equivalence relation that

$$\langle U, s\rangle \sim \langle V, t\rangle \iff \begin{matrix}\text{there exists an open neighborhood}\\ W \subseteq U \cap V \text{ of } P \text{ such that } s|_W = t|_W.\end{matrix}$$

1/15
[Har1977, p. 62]

Example 1.2.9. Let X be a quasi-projective variety over an algebraically closed field k. Then, the stalk $\mathcal{O}_{X,P}$ of the structure sheaf is the same thing as the local ring of P on X.

1.2.4 Morphisms

We now define morphisms of (pre)sheaves.

[MurAGI, Def. 2.1.12]
[Har1977, p. 62]
[God1973, I.1.9]

Definition 1.2.10. Let X be a topological space. Let $\mathscr{F}, \mathscr{G}$ be two presheaves on X with values in $\mathscr{C}$. A *morphism* $\varphi\colon \mathscr{F} \to \mathscr{G}$ is a natural transformation $\mathscr{F} \Rightarrow \mathscr{G}$. More explicitly, φ consists of the data of a morphism $\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$ in $\mathscr{C}$ such that whenever $V \subseteq U$ is an inclusion, the diagram

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho_V^U"'] & \mathscr{G}(U) \arrow[d, "\rho_V'^U"] \\ \mathscr{F}(V) \arrow[r, "\varphi(V)"] & \mathscr{G}(V) \end{tikzcd}$$

commutes, where the vertical maps are the restriction maps for $\mathscr{F}$ and $\mathscr{G}$, respectively. We use the same definition for sheaves. An *isomorphism* is a morphism which has a two-sided inverse.

We therefore obtain the categories

$$\mathsf{Sh}(X) \hookrightarrow \mathsf{PSh}(X)$$

of sheaves and presheaves of sets on X, respectively, where the arrow denotes that $\mathsf{Sh}(X)$ forms a full subcategory of $\mathsf{PSh}(X)$. The analogues for Abelian (pre)sheaves are

$$\mathsf{Ab}(X) \hookrightarrow \mathsf{PAb}(X)$$

which are connected to the categories for sheaves via forgetful functors. Since these forgetful functors reflect isomorphisms, we can apply statements about sheaves of sets to Abelian sheaves.

To prove Proposition 1.2.12 below, we define the following:

[Stacks, Tag 006X]
[God1973, II.3.1]
[Har1977, Exer. II.1.16(e)]

Example 1.2.11. Let X be a topological space and let $\{A_P\}_{P \in X}$ be a family of sets indexed by X. Then, the assignment

$$U \longmapsto \prod_{P \in U} A_P$$

defines a sheaf Π.

By the definition of direct limits, a morphism $\varphi\colon \mathscr{F} \to \mathscr{G}$ of presheaves on X induces a morphism

$$\varphi_P\colon \mathscr{F}_P \longrightarrow \mathscr{G}_P$$

on stalks for any point $P \in X$. The following result illustrates the local nature of sheaves.

Proposition 1.2.12. *Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves of sets on a topological space X. If φ is an isomorphism, then the induced maps $\varphi_P\colon \mathscr{F}_P \to \mathscr{G}_P$ on stalks are isomorphisms for every $P \in X$. Conversely:* [Har1977, Prop. II.1.1]

(a) *If φ_P is injective for every $P \in X$ and if $\mathscr{F}$ satisfies the sheaf property (3), then $\varphi(U)$ is injective for every open set $U \subseteq X$.*

(b) *If φ_P is an isomorphism for every $P \in X$, the presheaf $\mathscr{F}$ is a sheaf, and $\mathscr{G}$ satisfies the sheaf property (3), then $\varphi(U)$ is an isomorphism for every open set $U \subseteq X$.*

As a consequence, if $\mathscr{F}, \mathscr{G}$ are sheaves and the maps φ_P are isomorphisms for every P, then φ is an isomorphism.

Proof. The direction $\Rightarrow$ follows from the construction of direct limits.

For the converse, denote by $\Pi(\mathscr{F})$ the sheaf in Example 1.2.11 obtained from the family $\{\mathscr{F}_P\}_{P\in X}$. We first show:

Claim 1.2.13. *If $\mathscr{F}$ satisfies the sheaf property (3), then the map*

$$\mathscr{F}(U) \longrightarrow \Pi(\mathscr{F})(U)$$

is injective.

If $s, t \in \mathscr{F}(U)$ map to the same elements, then there is an open covering W_P of U for which $s|_{W_P} = t|_{W_P}$ for every P by the definition of germs. The sheaf property (3) then shows that $s = t$.

We now show (a). If φ_P is injective for every $\varphi(U)$, then the commutativity of the diagram

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, hook] & \mathscr{G}(U) \arrow[d] \\ \prod\limits_{P\in U} \mathscr{F}_P \arrow[r, "\prod_P \varphi_P"] & \prod\limits_{P\in U} \mathscr{G}_P. \end{tikzcd} \tag{1.2.14}$$

shows that the top horizontal map $\varphi(U)$ is injective. Here we use Claim 1.2.13 to show that the left vertical map is injective.

We now show (b). Suppose we have a section $t \in \mathscr{G}(U)$. Then, for every $P \in U$, there are an open neighborhood $V_P \ni P$ and a section $s(P) \in \mathscr{F}(V_P)$ such that

$$\varphi_P\Big(s(P)_P\Big) = t_P$$

for every $P \in U$. After possibly shrinking the V_P, we may assume that

$$\varphi\Big(s(P)\Big)|_{V_P} = t|_{V_P}$$

for every P. Now if P, Q are two points, then $s(P)|_{V_P\cap V_Q}$ and $s(Q)|_{V_P\cap V_Q}$ are two sections of $\mathscr{F}(V_P \cap V_Q)$, which are both sent by φ to $t|_{V_P\cap V_Q}$. By the injectivity shown in (a), this implies that these two sections $s(P)|_{V_P \cap V_Q}$ and $s(Q)|_{V_P \cap V_Q}$ are equal. By the sheaf property (4), we then see there is a section $s \in \mathscr{F}(U)$ such that $s|_{V_P} = s(P)$ for every P. Finally, we have $\varphi(U)(s) = t$ by the commutativity of (1.2.14) and Claim 1.2.13 applied to $\mathscr{G}$.

We can therefore define an inverse morphism $\varphi^{-1} \colon \mathscr{G} \to \mathscr{F}$, where compatibility with restriction maps holds by the fact that $\varphi^{-1}(U)$ is the inverse for $\varphi(U)$ on every open subset $U \subseteq X$. □

1.2.5 Espace étalé and sheafification

We now recall the *sheafification* construction, which turns a presheaf into a sheaf.

[Har1977, Exer. II.1.13, Prop.-Def. II.1.2] [God1973, II.1.2]

Definition 1.2.15. (Espace étalé and sheafification) Let $\mathscr{F}$ be a presheaf of sets on X. We define a topological space $\operatorname{Spé}(\mathscr{F})$ called the *espace étalé* of $\mathscr{F}$ as follows.

(i) As a set,
$$\operatorname{Spé}(\mathscr{F}) := \bigsqcup_{P \in X} \mathscr{F}_P.$$
We define a projection map
$$\begin{aligned} \pi \colon \operatorname{Spé}(\mathscr{F}) &\longrightarrow X \\ \mathscr{F}_P \ni s &\longmapsto P. \end{aligned}$$
For each open set $U \subseteq X$ and each section $s \in \mathscr{F}(U)$, we obtain a map
$$\begin{aligned} \bar{s} \colon U &\longrightarrow \operatorname{Spé}(\mathscr{F}) \\ P &\longmapsto s_P \end{aligned}$$
which is a section of π over U, i.e., $\pi \circ \bar{s} = \mathrm{id}_U$. Note that for every inclusion $V \subseteq U$, we have
$$\bar{s}|_V = \overline{s|_V}.$$

[TohokuI, p. 154]

(ii) The topology on $\operatorname{Spé}(\mathscr{F})$ is the topology generated by all subsets of $\operatorname{Spé}(\mathscr{F})$ of the form $\bar{s}(U)$ for all $U \subseteq X$ open and all $s \in \mathscr{F}(U)$. Note that π is a continuous map.

The espace étalé description of a sheaf is from [SHC50/51, Exp. XIV] and predates the one given above. The notation for the sheafification in [Har1977] is $\mathscr{F}^+$. We use the notation $\mathscr{F}^\#$ to match other sources, for example [Art1962, §2.1].

The espace étalé defines a *sheaf* $\mathscr{F}^\#$, where
$$\mathscr{F}^\#(U) := \bigl\{\text{continuous sections } U \to \operatorname{Spé}(\mathscr{F})\bigr\}.$$
We obtain a morphism
$$\theta \colon \mathscr{F} \longrightarrow \mathscr{F}^\#$$
which we call the *sheafification* of $\mathscr{F}$.

We now have the following:

Proposition 1.2.16. *Let X be a topological space and let $\mathscr{F}$ be a presheaf of sets on X.* [Har1977, Exer. II.1.13, Prop.-Def. II.1.2] [God1973, II.1.2] [SGA4$_1$, Exp. II, §3]

(*i*) *θ_P is bijective for every $P \in X$.*

(*ii*) *If $\mathscr{F}$ satisfies the sheaf property (3), then $\theta(U)$ is injective for every open set $U \subseteq X$.* [Stacks, Tag 007Z]

(*iii*) *If $\mathscr{F}$ is a sheaf, then θ is an isomorphism.*

(*iv*) *The sheafification is the left adjoint of the inclusion functor $\mathsf{Sh}(X) \hookrightarrow \mathsf{PSh}(X)$ or $\mathsf{Ab}(X) \hookrightarrow \mathsf{PAb}(X)$, i.e., for every sheaf $\mathscr{G}$ on X, there is a bijection* [Stacks, Tag 0080] [SGA4$_1$, Exp. II, Thm. 3.4]

$$\theta^* \colon \operatorname{Hom}(\mathscr{F}^\#, \mathscr{G}) \xrightarrow{\sim} \operatorname{Hom}(\mathscr{F}, \mathscr{G})$$

induced by precomposition by θ that is functorial in both $\mathscr{F}$ and $\mathscr{G}$.

Proof. For the statement (*i*), we first note that the map [Stacks, Tag 007D]

$$\mathscr{F}_P \longrightarrow \Pi(\mathscr{F})_P$$

has a retraction mapping a section $s \in \Pi(\mathscr{F})(U) = \prod_{P \in U} \mathscr{F}_P$ to $s_P \in \mathscr{F}_P$. Since $\mathscr{F} \to \Pi(\mathscr{F})$ factors as

$$\mathscr{F} \longrightarrow \mathscr{F}^\# \longrightarrow \Pi(\mathscr{F}),$$

we see that $\mathscr{F}_P \to \mathscr{F}_P^\#$ is injective. For surjectivity, suppose we have a germ $\langle U, s \rangle \in \mathscr{F}_P^\#$, where s is a continuous section of the map $\operatorname{Spé}(\mathscr{F})|_U \to U$. Now look at the image $s(P) = \langle V, \sigma \rangle \in \mathscr{F}_P$. By definition of $\operatorname{Spé}(\mathscr{F})$, this section $\sigma \in \mathscr{F}(V)$ yields a continuous section $\bar{\sigma} \in \mathscr{F}^\#(V)$ such that $s(P) = \bar{\sigma}(P) = \sigma_P$. Since s is continuous and by the definition of the topology on $\operatorname{Spé}(\mathscr{F})$, the set

$$W = s^{-1}\big(\bar{\sigma}(V)\big) = \big\{ Q \in U \bigm| s(Q) \in \bar{\sigma}(V) \big\} = \big\{ Q \in U \bigm| s(Q) = \bar{\sigma}(Q) \big\}$$

is an open subset on which $s(Q) = \bar{\sigma}(Q)$ for all $Q \in W$. This shows that $\sigma_Q \in \mathscr{F}_P$ maps to $s \in \mathscr{F}_P^\#$.

Next, statements (*ii*) and (*iii*) follow from (*i*) and Proposition 1.2.12.

For (*iv*), naturality in $\mathscr{G}$ holds by definition as precomposition by θ, and naturaliy in $\mathscr{F}$ holds by the fact that if $\mathscr{F}_1 \to \mathscr{F}_2$ is a morphism of presheaves, we have a corresponding morphism $\mathscr{F}_1^\# \to \mathscr{F}_2^\#$ of sheaves.

We want to show that θ^* is bijective. Consider the commutative diagram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\theta_{\mathscr{F}}(U)"] \arrow[d] & \mathscr{F}^\#(U) \arrow[r, hook] \arrow[d] \arrow[dl, dashed, "\exists"'] & \prod_{P \in U} \mathscr{F}_P \arrow[d] \\
\mathscr{G}(U) \arrow[r, "\theta_{\mathscr{G}}(U)", "\sim"'] & \mathscr{G}^\#(U) \arrow[r, hook] & \prod_{P \in U} \mathscr{G}_P
\end{tikzcd}$$

where $\theta_{\mathscr{G}}(U)$ is an isomorphism by (*iii*). By composition, we obtain the dashed arrow in the diagram above, which is compatible with restriction by the fact that we can draw the same commutative diagram above for $V \subseteq U$, and connect the two with the restriction maps ρ_V^U. This shows that θ^* is surjective. Finally, θ^* is injective because the right vertical map $\prod_{P\in U} \mathscr{F}_P \to \prod_{P\in U} \mathscr{G}_P$ is uniquely determined, and the horizontal maps on the right are injective by Proposition 1.2.12. □

1.2.6 Properties of morphisms of sheaves of sets

We now define some properties of morphisms of sheaves.

[God1973, II.1.6] [Har1977, p. 64]

Definition 1.2.17. Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of sets on a topological space X. We say that φ is *injective* (resp. *surjective*) if φ_P is injective (resp. surjective) for every $P \in X$.

[God1973, II.1.6]

Example 1.2.18. Note that $\varphi\colon \mathscr{F} \to \mathscr{G}$ injective implies $\varphi(U)$ is injective for every open set $U \subseteq X$ by Proposition 1.2.12(a).

1/20

The analogue for surjectivity does not hold. For example, let $\mathscr{O}_{\mathbf{C}}$ denote the sheaf consisting of holomorphic functions $U \mapsto \mathbf{C}$ for all $U \subseteq \mathbf{C}$ under addition, and let $\mathscr{O}_{\mathbf{C}}^\times$ denote the sheaf consisting of holomorphic functions $U \mapsto \mathbf{C}^\times$ for all $U \subseteq \mathbf{C}$ under multiplication. These are sheaves since a function on U is holomorphic if and only if it is holomorphic on each set of any open cover of U. Now consider the exponential map

$$\begin{aligned} \exp\colon \mathscr{O}_{\mathbf{C}} &\longrightarrow \mathscr{O}_{\mathbf{C}}^\times \\ f &\longmapsto \exp(f). \end{aligned}$$

This is surjective: For every $P \in \mathbf{C}$, we can choose a simply connected open neighborhood $U \ni P$, and $\exp(U)$ is surjective by, e.g., [Rud1987, Theorem 13.11] or [SS2003, Chapter 3, Theorem 6.2] (an application of Cauchy's integral formula). On the other hand, if $U = \mathbf{C}^\times$, then $\exp(U)$ is not surjective since the function z does not have a logarithm on $\mathbf{C}^\times$ [SS2003, pp. 97–98].

[God1973, II.1.8] [Har1977, p. 64]

Definition 1.2.19. Let $\mathscr{F}$ be a sheaf of sets on a topological space X. A sheaf $\mathscr{G}$ is a *subsheaf* of $\mathscr{F}$ if, for every open subset $U \subseteq X$, the sets $\mathscr{G}(U)$ are subsets of $\mathscr{F}(U)$, and the restriction maps are the restriction maps for $\mathscr{F}$.

Now let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of sets on a topological space X. The *image* $\operatorname{im}(f)$ of φ is the sheafification

$$\operatorname{im}(\varphi) \coloneqq \Bigl(U \longmapsto \operatorname{im}\bigl(\varphi(U)\bigr)\Bigr)^{\#}.$$

Equivalently (up to isomorphism), the image can be described as the sheaf of sections of the image of $\operatorname{Sp\acute{e}}(\mathscr{F})$ in $\operatorname{Sp\acute{e}}(\mathscr{G})$. The description in terms of the espace étalé shows that the image $\operatorname{im}(\varphi)$ can be identified with a subsheaf of $\mathscr{G}$.

[God1973, II.1.9] [Har1977, p. 65]

Definition 1.2.20. Let $\mathscr{F}$ be a sheaf of sets on a topological space X. Suppose that for every open $U \subseteq X$, we have an equivalence relation $R(U)$ on $\mathscr{F}(U)$. We say that the collection $\{R(U)\}_{U\subseteq X}$ is an equivalence relation on $\mathscr{F}$ if it satisfies the following property: For $s, t \in \mathscr{F}(U)$, we have $s \equiv t \bmod R(U)$ if and only if there exists an open cover $U = \bigcup_i U_i$ such that $s|_{U_i} \equiv t|_{U_i} \bmod R(U_i)$ for every i. We then define the *quotient sheaf* to be the sheafification

$$\mathscr{F}/R := \Big(U \longmapsto \mathscr{F}(U)/R(U)\Big)^{\#}.$$

This presheaf satisfies the sheaf property (3), and hence injects into $\mathscr{F}/R$. The canonical morphism $\mathscr{F} \to \mathscr{F}/R$ is surjective.

1.2.7 Functors associated to continuous maps

We now define some basic operations associated to continuous maps of topological spaces.

Definition 1.2.21. Let $f\colon X \to Y$ be a continuous map of topological spaces. [Har1977, p. 65]

(i) Let $\mathscr{F}$ be a sheaf of sets or Abelian groups on X. The *direct image* or *push forward sheaf* $f_*\mathscr{F}$ on Y is defined by [God1973, II.1.13]

$$(f_*\mathscr{F})(V) := \mathscr{F}\Big(f^{-1}(V)\Big)$$

for any open subset $V \subseteq Y$ (you should check for yourself that this defines a sheaf). For morphisms $\varphi\colon \mathscr{F} \to \mathscr{G}$ and $\psi\colon \mathscr{G} \to \mathscr{H}$, we have the commutative diagram

$$\begin{tikzcd}
\mathscr{F}\Big(f^{-1}(V)\Big) \arrow[r, "\varphi(f^{-1}(V))"] \arrow[d, "\rho^V_{V'}"'] & \mathscr{G}\Big(f^{-1}(V)\Big) \arrow[r, "\psi(f^{-1}(V))"] \arrow[d, "\rho'^V_{V'}"] & \mathscr{H}\Big(f^{-1}(V)\Big) \arrow[d, "\rho''^V_{V'}"] \\
\mathscr{F}\Big(f^{-1}(V')\Big) \arrow[r, "\varphi(f^{-1}(V'))"] & \mathscr{G}\Big(f^{-1}(V')\Big) \arrow[r, "\psi(f^{-1}(V'))"] & \mathscr{H}\Big(f^{-1}(V')\Big)
\end{tikzcd}$$

for all inclusions $V' \subseteq V$ of open sets in Y. Thus, the direct image is functorial in $\mathscr{F}$, and we obtain functors

$$\begin{aligned} f_*\colon \mathsf{Sh}(X) &\longrightarrow \mathsf{Sh}(Y), \\ f_*\colon \mathsf{Ab}(X) &\longrightarrow \mathsf{Ab}(Y). \end{aligned}$$

(ii) Let $\mathscr{G}$ be a sheaf of sets or Abelian groups on Y. The *inverse image* or *pullback presheaf* $f_{\mathcal{P}}^{-1}\mathscr{G}$ on X is defined by [God1973, II.1.12]

$$(f_{\mathcal{P}}^{-1}\mathscr{G})(U) := \varinjlim_{V \supseteq f(U)} \mathscr{G}(V)$$

for any open subset $U \subseteq X$, where the direct limit is taken over all open sets $V \subseteq Y$ containing $f(U)$. The *inverse image* or *pullback sheaf* $f^{-1}\mathscr{G}$ is the sheafification

$$f^{-1}\mathscr{G} := (f_{\mathcal{P}}^{-1}\mathscr{G})^{\#}.$$

We describe how $f_{\mathcal{P}}^{-1}$ acts on maps of presheaves. For morphisms $\varphi\colon \mathscr{F} \to \mathscr{G}$ and $\psi\colon \mathscr{G} \to \mathscr{H}$, we have the commutative diagram

$$\begin{tikzcd}
\mathscr{F}(V) \arrow[r, "\varphi(V)"] \arrow[d, "\rho^V_{V'}"'] & \mathscr{G}(V) \arrow[r, "\psi(V)"] \arrow[d, "\rho^V_{V'}"] & \mathscr{H}(V) \arrow[d, "\rho^V_{V'}"] \\
\mathscr{F}(V') \arrow[r, "\varphi(V')"] & \mathscr{G}(V') \arrow[r, "\psi(V')"] & \mathscr{H}(V')
\end{tikzcd}$$

where the vertical morphisms are the restriction morphisms $\rho^V_{V'}$ running over $V \supseteq V' \supseteq f(U)$. Taking direct limits in each column, we obtain the morphisms

$$f_{\mathcal{P}}^{-1}\mathscr{F}(U) \xrightarrow{f_{\mathcal{P}}^{-1}\varphi(U)} f_{\mathcal{P}}^{-1}\mathscr{G}(U) \xrightarrow{f_{\mathcal{P}}^{-1}\psi(U)} f_{\mathcal{P}}^{-1}\mathscr{H}(U).$$

by the universal property for direct limits (see Definition 1.1.12). Note by the universal property of direct limits applied to $\psi(V) \circ \varphi(V)$, we get that

$$f_{\mathcal{P}}^{-1}(\psi \circ \varphi)(U) = f_{\mathcal{P}}^{-1}(\psi)(U) \circ f_{\mathcal{P}}^{-1}(\varphi)(U).$$

Thus, $f_{\mathcal{P}}^{-1}$ defines a functor on presheaves. Taking sheafifications, we obtain the functors

$$f^{-1}\colon \mathsf{Sh}(Y) \longrightarrow \mathsf{Sh}(X),$$
$$f^{-1}\colon \mathsf{Ab}(Y) \longrightarrow \mathsf{Ab}(X).$$

As a special case, if f is an inclusion of a subset $X \subseteq Y$ with the induced topology, we call

$$\mathscr{G}|_X \coloneqq i^{-1}\mathscr{G}$$

[God1973, II.1.5] the *restriction* of $\mathscr{G}$ to X. Note the stalk of $\mathscr{G}|_X$ at $P \in X$ is just $\mathscr{F}_P$.

[Har1977, Exer. II.1.17] **Example 1.2.22. (Skyscraper sheaves)** Let X be a topological space and let $P \in X$ be a point. Let A be a set or an Abelian group. The *skyscraper sheaf at P with value A* is the sheaf $i_P(A)$ defined by

$$i_P(A)(U) = \begin{cases} A & \text{if } P \in U \\ \{*\} & \text{otherwise} \end{cases}$$

for sets and

$$i_P(A)(U) = \begin{cases} A & \text{if } P \in U \\ 0 & \text{otherwise} \end{cases}$$

for Abelian groups. If $i_P\colon \{P\} \hookrightarrow X$ is the inclusion, we can also describe the skyscraper sheaf as $i_{P*}\underline{A}_{\{P\}}$.

An important result connecting these functors is the following.

Proposition 1.2.23. *Let $f\colon X \to Y$ be a continuous map of topological spaces. For sheaves $\mathscr{F}$ and $\mathscr{G}$ of sets or Abelian groups, there is a bijection of sets* [Har1977, Exer. II.1.18] [EGAI, (**0**, 3.5.4)] [EGAInew, (**0**, 3.5.4)]

$$\operatorname{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) \xrightarrow{\sim} \operatorname{Hom}_Y(\mathscr{G}, f_*\mathscr{F}) \tag{1.2.24}$$

natural in both $\mathscr{F}$ and $\mathscr{G}$. Here we say that f^{-1} is the left adjoint *of f_*, and that f_* is a* right adjoint *of f^{-1}.*

Proof. We proceed in steps.

Step 1. *It suffices to show the statement after replacing f^{-1} by the functor $f_{\mathcal{P}}^{-1}$ on presheaves, where*

$$f_{\mathcal{P}}^{-1}\mathscr{G}(U) = \varinjlim_{V \supseteq f(U)} \mathscr{G}(U).$$

By Proposition 1.2.16(*iv*), the left diagonal map in the diagram

$$\begin{tikzcd} \operatorname{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) \arrow[rr] \arrow[dr, "\sim"'] & & \operatorname{Hom}_Y(\mathscr{G}, f_*\mathscr{F}) \\ & \operatorname{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, \mathscr{F}) \arrow[ur] & \end{tikzcd}$$

is natural in both $\mathscr{F}$ and $\mathscr{G}$. If we construct a natural right diagonal map that is a bijection, we can define the top horizontal map as the composition. This composition will be natural in $\mathscr{F}$ and $\mathscr{G}$ since the two diagonal maps are.

It therefore suffices to construct a natural bijection of sets

$$\operatorname{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, \mathscr{F}) \longrightarrow \operatorname{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$$

where $\mathscr{F}$ and $\mathscr{G}$ are presheaves on X and Y, respectively.

Step 2. *Constructing a natural transformation*

$$\varepsilon_{\mathcal{P}}\colon f_{\mathcal{P}}^{-1} f_* \Longrightarrow \mathrm{id}$$

of functors $\mathsf{Sh}(X) \to \mathsf{Sh}(X)$ or $\mathsf{Ab}(X) \to \mathsf{Ab}(X)$. In other words, we construct morphisms

$$\varepsilon_{\mathcal{P}}(\mathscr{F})\colon f_{\mathcal{P}}^{-1} f_*\mathscr{F} \longrightarrow \mathscr{F}$$

natural in $\mathscr{F}$.

First, note that

$$(f_{\mathcal{P}}^{-1} f_*\mathscr{F})(U) = \varinjlim_{V \supseteq f(U)} f_*\mathscr{F}(V) = \varinjlim_{V \supseteq f(U)} \mathscr{F}\big(f^{-1}(V)\big).$$

For this step, we fix the following notation. Let $U' \subseteq U$ be open subsets of X, and let V, W, V', W' be open subsets of Y satisfying the inclusions

$$\begin{array}{ccccc} V & \supseteq & W & \supseteq & f(U) \\ \cup\vert & & \cup\vert & & \cup\vert \\ V' & \supseteq & W' & \supseteq & f(U') \end{array}$$

which, after taking inverse images, implies

$$f^{-1}(V) \supseteq f^{-1}(W) \supseteq f^{-1}\big(f(U)\big) \supseteq U$$

and similarly for U', V', W'.

We now construct $\varepsilon_{\mathcal{P}}$ using the universal property of the direct limit:

$$\begin{tikzcd}
f_*\mathscr{F}(V) \arrow[rr, "\rho^{f^{-1}(V)}_{f^{-1}(W)}"] \arrow[dr, "\iota_V"] \arrow[ddr, bend right, "\rho^{f^{-1}(V)}_{U}"'] & & f_*\mathscr{F}(W) \arrow[dl, "\iota_W"'] \arrow[ddl, bend left, "\rho^{f^{-1}(W)}_{U}"] \\
& f_{\mathcal{P}}^{-1} f_*\mathscr{F}(U) \arrow[d, dashed, "\exists!"', "\varepsilon_{\mathcal{P}}(\mathscr{F})(U)"] & \\
& \mathscr{F}(U). &
\end{tikzcd} \tag{1.2.25}$$

To show that $\varepsilon_{\mathcal{P}}$ defines a morphism of presheaves, we consider the diagram

$$\begin{tikzcd}
f_*\mathscr{F}(V) \arrow[rrrrr, "\rho^{f^{-1}(V)}_{f^{-1}(V')}"] \arrow[drr, "\rho^{f^{-1}(V)}_{f^{-1}(W)}"] \arrow[ddr, "\iota_V"] \arrow[dddr, bend right, "\rho^{f^{-1}(V)}_{U}"'] & & & & & f_*\mathscr{F}(V') \arrow[dll, "\rho^{f^{-1}(V')}_{f^{-1}(W')}"'] \arrow[ddl, "\iota_{V'}"'] \arrow[dddl, bend left, "\rho^{f^{-1}(V')}_{U'}"] \\
& & f_*\mathscr{F}(W) \arrow[r, "\rho^{f^{-1}(W)}_{f^{-1}(W')}"] \arrow[dl, "\iota_W"'] \arrow[ddl, "\rho^{f^{-1}(W)}_{U}"] & f_*\mathscr{F}(W') \arrow[dr, "\iota_{W'}"] \arrow[ddr, "\rho^{f^{-1}(W')}_{U'}"'] & & \\
& f_{\mathcal{P}}^{-1} f_*\mathscr{F}(U) \arrow[rrr, red, "\rho^{U}_{U'}"] \arrow[d, red, dashed, "\varepsilon_{\mathcal{P}}(\mathscr{F})(U)"', "\exists!"] & & & f_{\mathcal{P}}^{-1} f_*\mathscr{F}(U') \arrow[d, red, dashed, "\exists!"', "\varepsilon_{\mathcal{P}}(\mathscr{F})(U')"] & \\
& \mathscr{F}(U) \arrow[rrr, red, "\rho^{U}_{U'}"] & & & \mathscr{F}(U') &
\end{tikzcd}$$

obtained by combining two copies of the commutative diagram (1.2.25). The horizontal map

$$\rho^{U}_{U'} \colon f_{\mathcal{P}}^{-1} f_*\mathscr{F}(U) \longrightarrow f_{\mathcal{P}}^{-1} f_*\mathscr{F}(U')$$

is induced by the universal property of the direct limit defining $f_{\mathcal{P}}^{-1} f_*\mathscr{F}(U)$. The composition $f_{\mathcal{P}}^{-1} f_*\mathscr{F}(U) \to \mathscr{F}(U')$ is unique by the universal property of $f_{\mathcal{P}}^{-1} f_*\mathscr{F}(U)$. This shows that the red square commutes, i.e., $\varepsilon_{\mathcal{P}}$ is a morphism of presheaves.

Next, we show that the construction of $\varepsilon_{\mathcal{P}}(\mathscr{F})$ is natural in $\mathscr{F}$, and hence we have a natural transformation $\varepsilon_{\mathcal{P}}\colon f_{\mathcal{P}}^{-1} f_* \Rightarrow \mathrm{id}$. Let $\xi\colon \mathscr{F} \to \mathscr{F}'$ be a morphism. We consider the diagram

$$\begin{tikzcd}
f_*\mathscr{F}(V) \arrow[rrrrr, "f_*\xi(V)"] \arrow[rr, "\rho^{f^{-1}(V)}_{f^{-1}(W)}"] \arrow[dr, "\iota_V"] \arrow[ddr, bend right, "\rho_U^{f^{-1}(V)}"'] & & & & & f_*\mathscr{F}'(V) \arrow[dll, "\rho^{f^{-1}(V)}_{f^{-1}(W)}"'] \arrow[ddl, "\iota_V"'] \arrow[dddl, bend left, "\rho_U^{f^{-1}(V)}"] \\
& & f_*\mathscr{F}(W) \arrow[r, "f_*\xi(W)"] \arrow[dl, "\iota_W"'] \arrow[ddl, bend left, "\rho_U^{f^{-1}(W)}"] & f_*\mathscr{F}'(W) \arrow[dr, "\iota_W"] \arrow[ddr, bend right, "\rho_U^{f^{-1}(W)}"'] & & \\
& f_{\mathcal{P}}^{-1} f_*\mathscr{F}(U) \arrow[rrr, "f_{\mathcal{P}}^{-1} f_*\xi(U)"] \arrow[d, dashed, "\varepsilon_{\mathcal{P}}(\mathscr{F})(U)"', "\exists!"] & & & f_{\mathcal{P}}^{-1} f_*\mathscr{F}'(U) \arrow[d, dashed, "\exists!"', "\varepsilon_{\mathcal{P}}(\mathscr{F}')(U)"] & \\
& \mathscr{F}(U) \arrow[rrr, "\xi(U)"] & & & \mathscr{F}'(U) &
\end{tikzcd}$$

obtained by combining two copies of the commutative diagram (1.2.25) using the fact that ξ is a morphism of presheaves. The horizontal map

$$f_{\mathcal{P}}^{-1} f_*\xi(U)\colon f_{\mathcal{P}}^{-1} f_*\mathscr{F}(U) \longrightarrow f_{\mathcal{P}}^{-1} f_*\mathscr{F}'(U)$$

is induced by the universal property of the direct limit defining $f_{\mathcal{P}}^{-1} f_*\mathscr{F}(U)$. This shows that the red square commutes. We can imagine connecting two copies of this diagram for the open sets U, V, W and for U', V', W'. Using the universal properties for $f_{\mathcal{P}}^{-1} f_*\mathscr{F}(U)$, $f_{\mathcal{P}}^{-1} f_*\mathscr{F}'(U)$, $f_{\mathcal{P}}^{-1} f_*\mathscr{F}(U')$, and $f_{\mathcal{P}}^{-1} f_*\mathscr{F}'(U')$, we see that we have a commutative diagram

$$\begin{tikzcd}
f_{\mathcal{P}}^{-1} f_*\mathscr{F} \arrow[r, "f_{\mathcal{P}}^{-1} f_*\xi"] \arrow[d, "\varepsilon_{\mathcal{P}}(\mathscr{F})"'] & f_{\mathcal{P}}^{-1} f_*\mathscr{F}' \arrow[d, "\varepsilon_{\mathcal{P}}(\mathscr{F}')"] \\
\mathscr{F} \arrow[r, "\xi"] & \mathscr{F}'
\end{tikzcd}$$

i.e., the morphisms $\varepsilon_{\mathcal{P}}(\mathscr{F})$ define a natural transformation $\varepsilon_{\mathcal{P}}\colon f_{\mathcal{P}}^{-1} f_* \Rightarrow \mathrm{id}$.

Step 3. *Constructing a natural transformation*

$$\eta_{\mathcal{P}}\colon \mathrm{id} \Longrightarrow f_* f_{\mathcal{P}}^{-1}$$

of functors $\mathsf{Sh}(Y) \to \mathsf{Sh}(Y)$ *or* $\mathsf{Ab}(Y) \to \mathsf{Ab}(Y)$. *In other words, we construct morphisms*

$$\eta_{\mathcal{P}}(\mathscr{G})\colon \mathscr{G} \longrightarrow f_* f_{\mathcal{P}}^{-1}\mathscr{G}$$

natural in $\mathscr{G}$.

First, note that

$$f_* f_{\mathcal{P}}^{-1}\mathscr{G}\big(U\big) = f_{\mathcal{P}}^{-1}\mathscr{G}\big(f^{-1}(U)\big) = \varinjlim_{V \supseteq f(f^{-1}(U))} \mathscr{G}(V). \tag{1.2.26}$$

For this step, we fix the following notation. Let $U' \subseteq U$ be open subsets of Y, and let V, W, V', W' be open subsets of Y satisfying the inclusions

$$\begin{array}{ccccccc} U & \supseteq & V & \supseteq & W & \supseteq & f\Big(f^{-1}(U)\Big) \\ \cup\!\shortmid & & \cup\!\shortmid & & \cup\!\shortmid & & \cup\!\shortmid \\ U' & \supseteq & V' & \supseteq & W' & \supseteq & f\Big(f^{-1}(U')\Big). \end{array}$$

We can define $\eta_{\mathcal{P}}(\mathscr{G})\colon \mathscr{G} \to f_* f_{\mathcal{P}}^{-1}\mathscr{G}$ by considering the commutative diagram

$$\begin{tikzcd}
 & \mathscr{G}(U) \arrow[rrr, red, "\rho^U_{U'}"] \arrow[dl, "\rho^U_V"'] \arrow[ddr, bend left, "\rho^U_W"] \arrow[ddd, red, "\eta_{\mathcal{P}}(\mathscr{G})(U)"] & & & \mathscr{G}(U') \arrow[dr, "\rho^{U'}_{V'}"] \arrow[ddl, bend right, "\rho^{U'}_{W'}"'] \arrow[ddd, red, "\eta_{\mathcal{P}}(\mathscr{G})(U')"'] & \\
\mathscr{G}(V) \arrow[rrrrr, "\rho^V_{V'}"] \arrow[drr, bend right, "\rho^V_W"'] \arrow[ddr, bend right, "\iota_V"'] & & & & & \mathscr{G}(V') \arrow[dll, bend left, "\rho^{V'}_{W'}"] \arrow[ddl, bend left, "\iota_{V'}"] \\
 & & \mathscr{G}(W) \arrow[r, "\rho^W_{W'}"] \arrow[dl, "\iota_W"] & \mathscr{G}(W') \arrow[dr, "\iota_{W'}"'] & & \\
 & f_* f_{\mathcal{P}}^{-1}\mathscr{G}(U) \arrow[rrr, red, dashed, "\rho^U_{U'}"] & & & f_* f_{\mathcal{P}}^{-1}\mathscr{G}(U') &
\end{tikzcd}$$

where the bottom horizontal map is induced by the universal property for the direct limit defining $f_* f_{\mathcal{P}}^{-1}\mathscr{G}(U)$ (see the definition of inverse image sheaves).

Next, we show that the construction of $\eta_{\mathcal{P}}(\mathscr{G})$ is natural in $\mathscr{G}$, and hence we have a natural transformation $\eta_{\mathcal{P}}\colon \mathrm{id} \Rightarrow f_* f_{\mathcal{P}}^{-1}$. Let $\xi\colon \mathscr{G} \to \mathscr{G}'$ be a morphism. We consider the commutative diagram

$$\begin{tikzcd}
 & \mathscr{G}(U) \arrow[rrr, red, "\xi(U)"] \arrow[dl, "\rho^U_V"'] \arrow[ddr, bend left, "\rho^U_W"] \arrow[ddd, red, "\eta_{\mathcal{P}}(\mathscr{G})(U)"] & & & \mathscr{G}'(U) \arrow[dr, "\rho^U_V"] \arrow[ddl, bend right, "\rho^U_W"'] \arrow[ddd, red, "\eta_{\mathcal{P}}(\mathscr{G}')(U)"'] & \\
\mathscr{G}(V) \arrow[rrrrr, "\xi(V)"] \arrow[drr, bend right, "\rho^V_W"'] \arrow[ddr, bend right, "\iota_V"'] & & & & & \mathscr{G}'(V) \arrow[dll, bend left, "\rho^V_W"] \arrow[ddl, bend left, "\iota_V"] \\
 & & \mathscr{G}(W) \arrow[r, "\xi(W)"] \arrow[dl, "\iota_W"] & \mathscr{G}'(W) \arrow[dr, "\iota_W"'] & & \\
 & f_* f_{\mathcal{P}}^{-1}\mathscr{G}(U) \arrow[rrr, red, dashed, "f_* f_{\mathcal{P}}^{-1}\xi(U)"] & & & f_* f_{\mathcal{P}}^{-1}\mathscr{G}'(U) &
\end{tikzcd}$$

where the bottom horizontal map is induced by the universal property of the direct limit defining $f_* f_{\mathcal{P}}^{-1}\mathscr{G}(U)$ (see the definition of the inverse image functor). We can then connect two copies of this diagram for the open sets U, V, W and for U', V, W'. The red squares in these two copies of the previous diagram yield the commutative cube

$$\begin{tikzcd}
\mathscr{G}(U) \arrow[rr, "\xi(U)"] \arrow[dr, "\rho^U_{U'}"'] \arrow[dd, "\eta_{\mathcal{P}}(\mathscr{G})(U)"'] & & \mathscr{G}'(U) \arrow[dr, "\rho^U_{U'}"] \arrow[dd, "\eta_{\mathcal{P}}(\mathscr{G}')(U)"' near start] & \\
& \mathscr{G}(U') \arrow[rr, "\xi(U')"' near end] \arrow[dd, "\eta_{\mathcal{P}}(\mathscr{G})(U')"' near start] & & \mathscr{G}'(U') \arrow[dd, "\eta_{\mathcal{P}}(\mathscr{G}')(U')"] \\
f_* f_{\mathcal{P}}^{-1}\mathscr{G}(U) \arrow[rr, "f_* f_{\mathcal{P}}^{-1}\xi(U)"' near end] \arrow[dr, "\rho^U_{U'}"'] & & f_* f_{\mathcal{P}}^{-1}\mathscr{G}'(U) \arrow[dr, "\rho^U_{U'}"] & \\
& f_* f_{\mathcal{P}}^{-1}\mathscr{G}(U') \arrow[rr, "f_* f_{\mathcal{P}}^{-1}\xi(U')"] & & f_* f_{\mathcal{P}}^{-1}\mathscr{G}'(U').
\end{tikzcd}$$

Using the universal property for $f_* f_{\mathcal{P}}^{-1}\mathscr{G}(U)$, the bottom face of this cube commutes. We therefore obtain the commutative diagram

$$\begin{tikzcd}
\mathscr{G} \arrow[r, "\xi"] \arrow[d, "\eta_{\mathcal{P}}(\mathscr{G})"'] & \mathscr{G}' \arrow[d, "\eta_{\mathcal{P}}(\mathscr{G})"] \\
f_* f_{\mathcal{P}}^{-1}\mathscr{G} \arrow[r, "f_* f_{\mathcal{P}}^{-1}\xi"] & f_* f_{\mathcal{P}}^{-1}\mathscr{G}'
\end{tikzcd}$$

i.e., the morphisms $\eta_{\mathcal{P}}(\mathscr{G})$ define a natural transformation $\eta_{\mathcal{P}}\colon \mathrm{id} \Rightarrow f_* f_{\mathcal{P}}^{-1}$.

Step 4. *The bijection* (1.2.24).

Now we want to define the maps defining the bijection in (1.2.24). The map from right to left is defined as

$$H_1\colon \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F}) \xrightarrow{f_{\mathcal{P}}^{-1}} \mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, f_{\mathcal{P}}^{-1} f_*\mathscr{F}) \xrightarrow{\varepsilon_{\mathcal{P}}(\mathscr{F})\circ -} \mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, \mathscr{F}).$$

The map from left to right is defined as

$$H_2\colon \mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, \mathscr{F}) \xrightarrow{f_*} \mathrm{Hom}_Y(f_* f_{\mathcal{P}}^{-1}\mathscr{G}, f_*\mathscr{F}) \xrightarrow{-\circ\eta_{\mathcal{P}}(\mathscr{G})} \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F}).$$

We need to show they are inverse to each other.

To show that $H_1 \circ H_2 = \mathrm{id}$, let $\varphi\colon f_{\mathcal{P}}^{-1}\mathscr{G} \to \mathscr{F}$. Let U be an open subset in

X and let $V \supseteq W \supseteq f(U)$ be open subsets in Y. We have the diagram

$$\begin{tikzcd}
f_* f_{\mathcal{P}}^{-1}\mathscr{G}(V) \arrow[rr, "\rho_W^V"] \arrow[dr] \arrow[dd, "f_*\varphi(V)"'] & & f_* f_{\mathcal{P}}^{-1}\mathscr{G}(W) \arrow[dl] \arrow[dd, "f_*\varphi(W)"] \\
& f_{\mathcal{P}}^{-1}\mathscr{G}(U) \arrow[dd, dashed, "f_{\mathcal{P}}^{-1}(f_*\varphi\circ\eta_{\mathcal{P}}(\mathscr{G}))(U)"] \arrow[ddd, bend right, "\varphi(U)"'] & \\
f_*\mathscr{F}(V) \arrow[rr] \arrow[dr] \arrow[ddr, bend right, "\rho_U^{f^{-1}(V)}"'] & & f_*\mathscr{F}(W) \arrow[dl] \arrow[ddl, bend left, "\rho_U^{f^{-1}(W)}"] \\
& f_{\mathcal{P}}^{-1} f_*\mathscr{F}(U) \arrow[d, "\varepsilon_{\mathcal{P}}(\mathscr{F})(U)"] & \\
& \mathscr{F}(U). &
\end{tikzcd}$$

The morphism $\varphi(U)$ fits into the commutative diagram without $f_{\mathcal{P}}^{-1} f_*\mathscr{F}(U)$, and is the unique morphism making this diagram without $f_{\mathcal{P}}^{-1} f_*\mathscr{F}(U)$ commute by the universal property defining $f_{\mathcal{P}}^{-1}\mathscr{G}(U)$. By the definition of the functor $f_{\mathcal{P}}^{-1}$, we can fill in the top half of the diagram with the dashed arrow equaling $f_{\mathcal{P}}^{-1}(f_*\varphi \circ \eta_{\mathcal{P}}(\mathscr{G}))(U)$. The commutativity of the diagram therefore implies

$$\varphi = \varepsilon_{\mathcal{P}}(\mathscr{F}) \circ f_{\mathcal{P}}^{-1}\Big(f_*\varphi \circ \eta_{\mathcal{P}}(\mathscr{G})\Big),$$

i.e., $H_1 \circ H_2 = \mathrm{id}$.

Next, we show that $H_2 \circ H_1 = \mathrm{id}$. Let $\psi\colon \mathscr{G} \to f_*\mathscr{F}$. Let U be an open subset in X and let $V \supseteq W \supseteq f(U)$ be open subsets in Y. We have the commutative diagram

$$\begin{tikzcd}
\mathscr{G}(V) \arrow[rr, "\rho_W^V"] \arrow[dr, "\iota_V"'] \arrow[dd, "\psi(V)"'] & & \mathscr{G}(W) \arrow[dl, "\iota_W"] \arrow[dd, "\psi(W)"] \\
& f_{\mathcal{P}}^{-1}\mathscr{G}(U) \arrow[dd, dashed, "f_{\mathcal{P}}^{-1}\psi(U)"] & \\
f_*\mathscr{F}(V) \arrow[rr] \arrow[dr] \arrow[ddr, bend right, "\rho_U^{f^{-1}(V)}"'] & & f_*\mathscr{F}(W) \arrow[dl] \arrow[ddl, bend left, "\rho_U^{f^{-1}(W)}"] \\
& f_{\mathcal{P}}^{-1} f_*\mathscr{F}(U) \arrow[d, "\varepsilon_{\mathcal{P}}(\mathscr{F})(U)"] & \\
& \mathscr{F}(U) &
\end{tikzcd}$$

where the dashed arrow is equal to $f_{\mathcal{P}}^{-1}\psi(U)$ by the definition of the functor $f_{\mathcal{P}}^{-1}$. We want to apply this diagram to an open subset of the form $U = f^{-1}(U')$

for an open subset $U' \subseteq Y$ and open subsets $V \supseteq W$ such that

$$U' \supseteq V \supseteq W \supseteq f\Big(f^{-1}(U')\Big).$$

After taking inverse images, we have

$$f^{-1}(U') \supseteq f^{-1}(V) \supseteq f^{-1}(W) \supseteq f^{-1}\Big(f\Big(f^{-1}(U')\Big)\Big) = f^{-1}(U'),$$

and hence equality holds throughout. We therefore have the commutative diagram

$$\begin{tikzcd}
& \mathscr{G}(U') \arrow[dl, "\rho_V^{U'}"'] \arrow[dr, "\rho_W^{U'}"] \arrow[dd, "\eta_{\mathcal{P}}(\mathscr{G})(U')"] & \\
\mathscr{G}(V) \arrow[rr, "\rho_W^V"] \arrow[dr, "\iota_V"'] \arrow[dd, "\psi(V)"'] & & \mathscr{G}(W) \arrow[dl, "\iota_W"] \arrow[dd, "\psi(W)"] \\
& f_* f_{\mathcal{P}}^{-1} \mathscr{G}(U') \arrow[dd, dashed, "f_* f_{\mathcal{P}}^{-1} \psi(U')"] & \\
f_* \mathscr{F}(V) \arrow[rr, equal] \arrow[dr] \arrow[ddr, equal, bend right] & & f_* \mathscr{F}(W) \arrow[dl] \arrow[ddl, equal, bend left] \\
& f_* f_{\mathcal{P}}^{-1} f_* \mathscr{F}(U') \arrow[d, "f_* \varepsilon_{\mathcal{P}}(\mathscr{F})(U')"] & \\
& f_* \mathscr{F}(U'). &
\end{tikzcd}$$

The composition $\mathscr{G}(U') \to f_* \mathscr{F}(U')$ is equal to $\psi(U')$ because of the commutative diagram

$$\begin{tikzcd}
\mathscr{G}(U') \arrow[r, "\psi(U')"] \arrow[d, "\rho_V^{U'}"'] & f_* \mathscr{F}(U') \arrow[d, equal] \\
\mathscr{G}(V) \arrow[r, "\psi(V)"] & f_* \mathscr{F}(V).
\end{tikzcd}$$

We therefore see that

$$\psi = f_* \varepsilon_{\mathcal{P}}(\mathscr{F}) \circ f_* f_{\mathcal{P}}^{-1} \psi \circ \eta_{\mathcal{P}}(\mathscr{G}) = f_* \Big(\varepsilon_{\mathcal{P}}(\mathscr{F}) \circ f_{\mathcal{P}}^{-1} \psi\Big) \circ \eta_{\mathcal{P}}(\mathscr{G})$$

where the second equality holds because f_* is a functor. This shows that $H_2 \circ H_1 = \mathrm{id}$.

Step 5. *Naturality of* (1.2.24) *in* $\mathscr{F}$ *and* $\mathscr{G}$.

Now, naturality in $\mathscr{F}$ follows since if $\varphi\colon \mathscr{F} \to \mathscr{F}'$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F}) \arrow[r, "\mathrm{Hom}_Y(\mathscr{G}, f_*\varphi)"] \arrow[d, "f_{\mathcal{P}}^{-1}"'] & \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F}') \arrow[d, "f_{\mathcal{P}}^{-1}"] \\
\mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, f_{\mathcal{P}}^{-1}f_*\mathscr{F}) \arrow[r, "\mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, f_{\mathcal{P}}^{-1}f_*\varphi)"] \arrow[d, "\mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, \varepsilon_{\mathcal{P}}(\mathscr{F}))"'] & \mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, f_{\mathcal{P}}^{-1}f_*\mathscr{F}') \arrow[d, "\mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, \varepsilon_{\mathcal{P}}(\mathscr{F}))"] \\
\mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, \mathscr{F}) \arrow[r, "\mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, \varphi)"] & \mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, \mathscr{F}')
\end{tikzcd}$$

commutes by functoriality of $f_{\mathcal{P}}^{-1}$ in the top square and naturality of $\varepsilon_{\mathcal{P}}$ in the bottom square, and then since the composition of the vertical maps gives H_1. Finally, naturality in $\mathscr{G}$ follows since if $\psi\colon \mathscr{G} \to \mathscr{G}'$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}, \mathscr{F}) \arrow[d, "f_*"'] & \mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\mathscr{G}', \mathscr{F}) \arrow[l, "\mathrm{Hom}_X(f_{\mathcal{P}}^{-1}\psi, \mathscr{F})"'] \arrow[d, "f_*"] \\
\mathrm{Hom}_Y(f_*f_{\mathcal{P}}^{-1}\mathscr{G}, f_*\mathscr{F}) \arrow[d, "\mathrm{Hom}_Y(\eta_{\mathcal{P}}(\mathscr{G}), f_*\mathscr{F})"'] & \mathrm{Hom}_Y(f_*f_{\mathcal{P}}^{-1}\mathscr{G}', f_*\mathscr{F}) \arrow[l, "\mathrm{Hom}_Y(f_*f_{\mathcal{P}}^{-1}\psi, f_*\mathscr{F})"'] \arrow[d, "\mathrm{Hom}_Y(\eta_{\mathcal{P}}(\mathscr{G}), f_*\mathscr{F})"] \\
\mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F}) & \mathrm{Hom}_Y(\mathscr{G}', f_*\mathscr{F}) \arrow[l, "\mathrm{Hom}_Y(\psi, f_*\mathscr{F})"']
\end{tikzcd}$$

commutes by functoriality of f_* in the top square and naturality of $\eta_{\mathcal{P}}(\mathscr{G})$ in the bottom square, and then since the composition of the vertical maps gives H_2. $\square$

1.2.8 Sheaf hom

We also define a sheafy version of Hom.

[Har1977, Exer. II.1.15]
[God1973, II.1.7]

Definition 1.2.27. (Sheaf $\mathscr{H}om$) Let X be a topological space. Let $\mathscr{F}$, $\mathscr{G}$ be sheaves of sets or Abelian groups on X. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves of Abelian groups, the set $\mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ of morphisms of the restricted shaves has a natural structure of Abelian group for every open set $U \subseteq X$. The presheaf

$$U \longmapsto \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$$

is a sheaf called the *sheaf of local morphisms* of $\mathscr{F}$ into $\mathscr{G}$, "sheaf hom" for short, and is denoted

$$\mathscr{H}om(\mathscr{F}, \mathscr{G})$$

Proof that $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ is a sheaf. To show that $U \mapsto \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is a sheaf, suppose $\{V_i\}$ is an open cover of U and $s, s' \in \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is such that

$$s|_{V_i} = s(V_i) = s'(V_i) = s'|_{V_i}$$

for all i. Let $f \in \mathscr{F}(V)$ be arbitrary for some $V \subset U$. Then,

$$s(V_i \cap V)(f|_{V_i \cap V}) = s'(V_i \cap V)(f|_{V_i \cap V}),$$

and so since $\mathscr{G}$ is a sheaf, $s(V)(f) = s'(V)(f)$. Thus, $s = s'$. Now suppose that we have elements $s_i \in \operatorname{Hom}(\mathscr{F}|_{V_i}, \mathscr{G}|_{V_i})$ such that for each i, j,

$$s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}.$$

If $f \in \mathscr{F}(V)$, then

$$s_i(V \cap V_i \cap V_j)(f|_{V \cap V_i \cap V_j}) = s_j(V \cap V_i \cap V_j)(f|_{V \cap V_i \cap V_j}) \in \mathscr{G}(V \cap V_i \cap V_j)$$

gives that there exists a well-defined image $s(V)(f) \in \mathscr{G}(V)$ since $\mathscr{G}$ is a sheaf. □

1.3 Sheaves of modules

We now discuss sheaves of modules. These include sheaves of Abelian groups as a special case, and will be the category of sheaves that we will work with for much of the first part of this course.

1.3.1 Ringed spaces and sheaves of modules

Definition 1.3.1. ([EGAI, (**0**, 4.1.1); EGAInew, (**0**, 4.1.1)]) A *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X called the *structure sheaf*. A *morphism* of ringed spaces [Har1977, p. 70]

$$(f, f^\#)\colon (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

consists of a continuous map $f\colon X \to Y$ and a map

$$f^\#\colon \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$$

of sheaves of rings on Y. This forms a category RS.

Definition 1.3.2. Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf $\mathscr{F}$ of $\mathcal{O}_X$-modules*, or an *$\mathcal{O}_X$-module* for short, is a sheaf of Abelian groups such that every $\mathscr{F}(U)$ is a $\mathcal{O}_X(U)$-module and the restriction maps $\rho_V^U\colon \mathscr{F}(U) \to \mathscr{F}(V)$ are maps of modules compatible with the ring maps $\rho_V^U\colon \mathcal{O}_X(U) \to \mathcal{O}_X(V)$. A *morphism* $\varphi\colon \mathscr{F} \to \mathscr{G}$ of $\mathcal{O}_X$-modules is a morphism of sheaves of sets such that each $\mathscr{F}(U) \to \mathscr{G}(U)$ is a morphism of $\mathcal{O}_X(U)$-modules. This forms the category $\mathsf{Mod}(\mathcal{O}_X)$. [MurAGI, Def. 2.1.29] [God1973, II.2.1, II.2.2] [Har1977, pp. 64–65, p. 109]

Example 1.3.3. The zero sheaf 0 is a sheaf of $\mathcal{O}_X$-modules. By properties of categories of modules, we know that there are always unique morphisms $0 \to \mathscr{F}$ and $\mathscr{F} \to 0$. In other words, the 0 sheaf is a *zero object* in the category $\mathsf{Mod}(\mathcal{O}_X)$. This means that the zero sheaf is both an initial object (there is a unique map from 0) and a final object (there is a unique map to 0) in the category. [MurAGI, Ex. 2.1.30]

We take this opportunity to define 0 objects.

Definition 1.3.4. Let $\mathscr{C}$ be a category.

[Bor1994b, Def. 2.3.1]

(*i*) An object $\varnothing$ of $\mathscr{C}$ is an *initial object* if, for every object T of $\mathscr{C}$, there exists a unique morphism $\varnothing \to T$.

[Bor1994b, Def. 2.3.1]

(*ii*) An object $\{*\}$ of $\mathscr{C}$ is a *final object* if, for every object S of $\mathscr{C}$, there exists a unique morphism $S \to \{*\}$.

[Bor1994b, Def. 1.1.1]

(*iii*) An object 0 of $\mathscr{C}$ is a *zero object* if it is both an initial and a final object of $\mathscr{C}$.

Note that since these notions are defined using universal properties, they are automatically unique.

[Bor1994b, Def. 1.1.2]
[Bor1994b, Prop. 1.1.3]

Definition 1.3.5. Let $\mathscr{C}$ be a category with a zero object 0. A morphism $X \to Y$ is a *zero morphism* if it factors through 0. Since the morphisms $X \to 0$ and $0 \to Y$ are unique, such a zero morphism always exists and is unique.

Example 1.3.6. If $\mathscr{O} = \underline{\mathbf{Z}}_X$, then $\mathsf{Mod}(\underline{\mathbf{Z}}_X) = \mathsf{Ab}(X)$.

Remark 1.3.7. It is often useful to think about sheaves of non-commutative rings, for example the sheaf of rings of differential operators $\mathscr{D}_X$ [EGAIV$_4$, Corollaire 16.8.10]. We will stick to commutative rings for simplicity.

One fact you will verify on the homework is that the category $\mathsf{Mod}(\mathscr{O}_X)$ of $\mathscr{O}_X$-modules – in particular, the category $\mathsf{Ab}(X)$ of Abelian sheaves – are Abelian categories. For now, we explain what exact sequences (and related notions) are in this category.

[Har1977, pp. 64–65]
[God1973, II.2.4]

Definition 1.3.8. Let $(X, \mathscr{O}_X)$ be a ringed space.

(*i*) Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of $\mathscr{O}_X$-modules. The *kernel* $\ker(f)$ of φ is

$$\ker(\varphi) \coloneqq \Bigl(U \longmapsto \ker\bigl(\varphi(U)\bigr)\Bigr).$$

This is already a sheaf (by the sheaf condition (3)) and is a subsheaf of $\mathscr{F}$. Note that φ is injective on an open set U if and only if $\ker(\varphi)(U) = 0$ because by definition of stalks using direct limits, we have

[Har1977, Exer. II.1.2(*a*)]

$$\ker(\varphi)_P \xrightarrow{\sim} \ker(\varphi_P) \tag{1.3.9}$$

for all points $P \in X$. To show the isomorphism Definition 1.3.9, consider the commutative diagram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho^U_V"'] & \mathscr{G}(U) \arrow[d, "\rho'^U_V"] \\
\mathscr{F}(V) \arrow[r, "\varphi(V)"] \arrow[d, "\pi_V"'] & \mathscr{G}(V) \arrow[d, "\pi'_V"] \\
\mathscr{F}_P \arrow[r, "\varphi_P"] & \mathscr{G}_P
\end{tikzcd}$$

for open subsets $V \subseteq U \subseteq X$ containing P. The commutativity of the diagram shows that $\ker(\varphi)_P \subseteq \ker(\varphi_P)$. Conversely, if $\langle U, f\rangle \in \ker(\varphi_P)$, then we know that $\langle U, \varphi(U)(f)\rangle = 0$. By the sheaf condition (3), this implies there exists an open subset $V \subseteq U$ such that

$$\varphi(U)(f)|_V = \varphi(V)\big(f|_V\big) = 0.$$

(*ii*) We say that a sequence of $\mathscr{O}_X$-modules [God1973, II.2.5]

$$\cdots \longrightarrow \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \longrightarrow \cdots$$

is *exact* if $\ker(\varphi^i) = \operatorname{im}(\varphi^{i-1})$ as subsheaves of $\mathscr{F}^i$ for every i.

(*iii*) Let $\mathscr{F}'$ be a sheaf of sub-$\mathscr{O}_X$-modules of $\mathscr{F}$, i.e., every $\mathscr{F}'(U)$ is a sub-$\mathscr{O}_X(U)$-module of $\mathscr{F}(U)$. The *quotient sheaf* $\mathscr{F}/\mathscr{F}'$ is the sheafification [God1973, II.2.3]

$$\mathscr{F}/\mathscr{F}' \coloneqq \big(U \longmapsto \mathscr{F}(U)/\mathscr{F}'(U)\big)^{\#}.$$

This is a special case of the construction for sets. The stalks are $(\mathscr{F}/\mathscr{F}')_P = \mathscr{F}_P/\mathscr{F}'_P$ by Proposition 1.2.16(i).

(*iv*) Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of $\mathscr{O}_X$-modules. The *cokernel* $\operatorname{coker}(f)$ of φ is the sheafification [God1973, II.2.4]

$$\operatorname{coker}(\varphi) \coloneqq \Big(U \longmapsto \operatorname{coker}\big(\varphi(U)\big)\Big)^{\#}.$$

This is a quotient sheaf of $\mathscr{F}$.

We give an example of an exact sequence from toplogy.

Example 1.3.10. Let X be a smooth manifold of dimension n. For every $p \geq 0$, consider the sheaf Ω^p_X of differential forms of degree p on X, which is defined by setting [God1973, Ex. II.2.5.1]

$$\Omega^p_X(U) \coloneqq \big\{\text{differential forms of degree } p \text{ on } U\big\}.$$

The exterior derivative

$$\Omega^p_X(U) \longrightarrow \Omega^{p+1}_X(U)$$

defines morphisms of sheaves

$$d^p\colon \Omega^p_X \longrightarrow \Omega^{p+1}_X$$

of sheaves of real vector spaces. We have an injection

$$j\colon \underline{\mathbf{R}}_X \hookrightarrow \Omega^0_X$$

of the constant sheaf with value $\mathbf{R}$ into Ω_X^0 by considering locally constant real-valued functions as smooth functions on X. The *Poincaré lemma* then implies that the sequence

$$0 \longrightarrow \underline{\mathbf{R}}_X \xrightarrow{j} \Omega_X^0 \xrightarrow{d^0} \Omega_X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \Omega_X^n \longrightarrow 0 \tag{1.3.11}$$

is exact: Since exactness is a local question, we can work locally to assume that $X = \mathbf{R}^n$, in which case this is the original Poincaré lemma. See [Lee2013, Corollary 17.15].

Note that if we take sections in Example 1.3.11, we do not get an exact sequence! This is where de Rham cohomology comes from: the i-th de Rham cohomology group is defined as

$$H^i_{\mathrm{dR}}(X) \coloneqq \mathbf{h}^i\Big(0 \longrightarrow \Omega_X^0(X) \longrightarrow \Omega_X^1(X) \longrightarrow \cdots \longrightarrow \Omega_X^n(X) \longrightarrow 0\Big).$$

For example, letting $X = S^1$ and taking global sections, we have

$$H^1_{\mathrm{dR}}(S^1) \cong \mathbf{R}.$$

See [BT1982, Example 2.6] or [Lee2013, Theorem 17.21].

$\boxed{1/22}$ [Har1977, Exer. II.1.21]

Example 1.3.12. Let X be a quasi-projective variety over an algebraically closed field k, which we consider as a ringed space by letting $\mathcal{O}_X$ be the sheaf of regular functions on X.

(*a*) (Sheaves of ideals) Let $Y \subseteq X$ be a closed subset. For each open set $U \subseteq X$, let

$$\mathscr{I}_Y(U) \coloneqq \big\{ f \in \mathcal{O}_X(U) \bigm| f(P) = 0 \text{ for all } P \in Y \cap U \big\}.$$

This forms a presheaf since the restriction maps on $\mathcal{O}_X$ restrict to restriction maps on $\mathscr{I}_Y$. We claim $\mathscr{I}_Y$ is in fact a sheaf. Since $\mathscr{I}_Y(U) \subseteq \mathcal{O}_X(U)$, we have that $\mathscr{I}_Y$ satisfies the sheaf property (3). If we have an open set $U = \bigcup V_i$ and we have a compatible family of regular functions $s_i \in \mathcal{O}_X(V_i)$ that vanish on $Y \cap V_i$ for all i, then they glue together to a regular function $s \in \mathcal{O}_X(U)$, that moreover vanishes on $Y \cap U$ since regular functions are defined locally. Thus, $\mathscr{I}_Y$ is a sheaf, called the *sheaf of ideals of* Y and is a subsheaf of $\mathcal{O}_X$.

(*b*) (The restriction short exact sequence) If Y is a subvariety, then the quotient sheaf $\mathcal{O}_X/\mathscr{I}_Y$ is isomorphic to $i_*(\mathcal{O}_Y)$, where $i\colon Y \hookrightarrow X$ is the inclusion, and $\mathcal{O}_Y$ is the sheaf of regular functions on Y.

To prove this, first note we have a sheaf morphism

$$\psi\colon \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y$$

given by restricting regular functions. The kernel of this morphism consists of exactly those regular functions that vanish on all points of $Y \cap U$, giving us the sequence

$$0 \longrightarrow \mathscr{I}_Y \longrightarrow \mathcal{O}_X \xrightarrow{\psi} i_*\mathcal{O}_Y \longrightarrow 0$$

which is exact at $\mathscr{I}_Y$ and $\mathscr{O}_X$ by (a), and at $i_*\mathscr{O}_Y$ by definition of a regular function locally on affines as a restriction of a polynomial function on $\mathbf{A}_k^n$. (Note: This requires some argument!)

We now look at the specific example of $X = \mathbf{P}_k^1$.

(c) (Taking global sections is not necessarily exact) Let $Y = \{P, Q\}$ for distinct points $P, Q \in \mathbf{P}_k^1$. We then have the exact sequence

$$0 \longrightarrow \mathscr{I}_Y \longrightarrow \mathscr{O}_X \longrightarrow i_{P*}\Big(\underline{k}_{\{P\}}\Big) \oplus i_{Q*}\Big(\underline{k}_{\{Q\}}\Big) \longrightarrow 0$$

by (b). However, the induced map of k-vector spaces

$$k = \Gamma(X, \mathscr{O}_X) \longrightarrow \Gamma\Big(X, i_{P*}\Big(\underline{k}_{\{P\}}\Big) \oplus i_{Q*}\Big(\underline{k}_{\{Q\}}\Big)\Big) \cong k^{\oplus 2}$$

of global sections is the map $a \mapsto (a, a)$, which is not surjective.

(d) (The sheaf of rational functions) Let $\mathscr{K}_X$ be the constant sheaf on X with value $K(X)$, the function field of X. We call $\mathscr{K}_X$ the *sheaf of rational functions on X*. We claim we have the sequence

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{K}_X \longrightarrow \bigoplus_{P \in X} i_P\left(\frac{K(X)}{\mathscr{O}_{X,P}}\right) \longrightarrow 0. \tag{1.3.13}$$

The second map is defined by

$$\begin{tikzcd} \mathscr{K}_X \arrow[r] \arrow[dr, dashed] & \displaystyle\prod_{P \in X} i_P\left(\frac{K(X)}{\mathscr{O}_{X,P}}\right) \\ & \displaystyle\bigoplus_{P \in X} i_P\left(\frac{K(X)}{\mathscr{O}_{X,P}}\right) \arrow[u, hook] \end{tikzcd}$$

where the top horizontal map is the evaluation map, and the diagonal dashed map exists since a rational map on $\mathbf{P}_k^1$ is not regular at only finitely many points on $\mathbf{P}_k^1$. The sequence (1.3.13) is exact since after taking stalks, we have

$$0 \longrightarrow \mathscr{O}_{X,P} \longrightarrow K(X) \longrightarrow \frac{K(X)}{\mathscr{O}_{X,P}} \longrightarrow 0$$

which is exact. We can therefore identify

$$\mathscr{K}_X / \mathscr{O}_X \cong \bigoplus_{P \in X} i_P\left(\frac{K(X)}{\mathscr{O}_{X,P}}\right).$$

(*e*) (The first Cousin problem) While taking global sections is not exact in general, the sequence

$$0 \longrightarrow \Gamma(X, \mathscr{O}_X) \longrightarrow \Gamma(X, \mathscr{K}_X) \longrightarrow \Gamma(X, \mathscr{K}_X/\mathscr{O}_X) \longrightarrow 0$$

is exact for $X = \mathbf{P}^1_K$. In this case, you can compute this directly (Homework 2, Problem 2). Algebraically, this follows from the fact that a certain sheaf cohomology group $H^1(X, \mathscr{O}_X)$ vanishes. Analytically, this is the first Cousin problem in several complex variables for the complex manifold $\mathbf{CP}^1$. For example, see [Hör1990, Theorem 5.5.1] for the first Cousin problem over Stein manifolds and see [GR1979, Chapter V, §2, Theorem 1] for the first Cousin problem over more general complex manifolds.

We also define some operations on sheaves of $\mathscr{O}_X$-modules.

[God1973, II.2.8]
[Har1977, p. 109]

Definition 1.3.14. Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}$ and $\mathscr{G}$ be $\mathscr{O}_X$-modules. The *tensor product of* $\mathscr{F}$ *and* $\mathscr{G}$ is the sheaf $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ associated to the presheaf

$$U \longmapsto \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U).$$

We will often drop the subscript $\mathscr{O}_X$ from our notation.

[Har1977, p. 110]

Definition 1.3.15. ([EGAI, (**0**, 4.3.1); EGAInew, (**0**, 4.3.1)]) Let $f\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces and let $\mathscr{G}$ be an $\mathscr{O}_Y$-module. Then, $f^{-1}\mathscr{G}$ is an $f^{-1}\mathscr{O}_Y$-module. Since f^{-1} is the left adjoint of f_* Proposition 1.2.23, we have the morphism $f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ of sheaves of rings on X. The *inverse image of* $\mathscr{G}$ *by the morphism* f is

$$f^*\mathscr{G} \coloneqq f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X.$$

This defines a functor $f^*\colon \mathsf{Mod}(\mathscr{O}_Y) \to \mathsf{Mod}(\mathscr{O}_X)$.

There is an analogue of Proposition 1.2.23 for f^* and f_*.

[Har1977, p. 110]
[EGAI, (**0**, 4.4.3)]
[EGAInew, (**0**, 4.4.3)]

Proposition 1.3.16. *Let* $f\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ *be a morphism of ringed spaces. For an* $\mathscr{O}_X$*-module* $\mathscr{F}$ *and* $\mathscr{O}_Y$*-module* $\mathscr{G}$*, there is a bijection of sets*

$$\mathrm{Hom}_{\mathscr{O}_X}(f^*\mathscr{G}, \mathscr{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{G}, f_*\mathscr{F})$$

natural in both $\mathscr{F}$ *and* $\mathscr{G}$*. Thus,* f^* *is the left adjoint of* f_* *and* f_* *is the right adjoint of* f^**.*

Proof. This will be on Homework 3. □

1.3.2 (Pre)additive categories

A very important fact about the category of sheaves of modules is that it "behaves similarly" to the category of modules over a ring. The precise version of this statement is that $\mathsf{Mod}(\mathscr{O}_X)$ is an *Abelian category*. For the next few subsections, we define the necessary terminology from category theory to make sense of this notion.

To define (pre)additive categories, we need to define (co)products.

Definition 1.3.17. Let $\mathscr{C}$ be a category. Let $\{X_i\}$ be a set of objects in $\mathscr{C}$. [Wei1994, A.1.9]

(1) The *product* of the $\{X_i\}$ is an object $\prod_i X_i$ together with projection maps $\pi_i\colon \prod_i X_i \to X_i$ such that for every object S with maps $f_i\colon S \to X_i$, there is a unique dashed morphism making the diagram [Bor1994a, Def. 2.1.1]

$$\begin{tikzcd}
& & X_i \\
S \arrow[r, dashed, "\exists!"] \arrow[urr, bend left, "f_i"] \arrow[drr, bend right, "f_j"'] & \prod\limits_i X_i \arrow[ur, "\pi_i"'] \arrow[dr, "\pi_j"] & \\
& & X_j
\end{tikzcd}$$

commute for all i, j. If $\{X_i\}$ is indexed over the natural numbers, we also write $X_1 \times X_2 \times \cdots$ for $\prod_i X_i$, especially for finite products.

(2) The *coproduct* of the $\{X_i\}$ is an object $\coprod_i X_i$ together with inclusion maps $\iota_i\colon X_i \to \coprod_i X_i$ such that for every object T with maps $g_i\colon X_i \to T$, there is a unique dashed morphism making the diagram [Bor1994a, Def. 2.2.1]

$$\begin{tikzcd}
X_i \arrow[dr, "\iota_i"'] \arrow[drr, bend left, "g_i"] & & \\
& \coprod\limits_i X_i \arrow[r, dashed, "\exists!"] & T \\
X_j \arrow[ur, "\iota_j"] \arrow[urr, bend right, "g_j"'] & &
\end{tikzcd}$$

commute for all i, j. If $\{X_i\}$ is indexed over the natural numbers, we also write $X_1 \sqcup X_2 \sqcup \cdots$ for $\coprod_i X_i$, especially for finite coproducts.

Definition 1.3.18. Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ is an *preadditive category* if the following condition holds. [Bor1994b, Defs. 1.2.1, 1.2.6]

(1) For all objects X, Y, Z in $\mathscr{C}$, the hom sets $\operatorname{Hom}_{\mathscr{C}}(X, Y)$ are equipped with the structure of an Abelian group such that the composition maps

$$\operatorname{Hom}_{\mathscr{C}}(X,Y) \times \operatorname{Hom}_{\mathscr{C}}(Y,Z) \longrightarrow \operatorname{Hom}_{\mathscr{C}}(X,Z)$$

are bilinear.

We say that $\mathscr{C}$ is an *additive category* if it is preadditive and the following condition holds. [TohokuI, §1.3] [Wei1994, A.4.1]

(2) Finite products (including the empty product!) of objects in $\mathscr{C}$ exist.

By comparing the definition of products and final objects, we see that the empty product is the final object of a preadditive category $\mathscr{C}$. We prove that in a preadditive category, final objects are automatically initial, and vice versa, and hence are zero objects as well.

[Bor1994b, Prop. 1.2.3]

Proposition 1.3.19. *Let $\mathscr{C}$ be a preadditive category. Then, the following are equivalent.*

(i) *$\mathscr{C}$ has an initial object.*

(ii) *$\mathscr{C}$ has a final object.*

(iii) *$\mathscr{C}$ has a zero object.*

When one of these equivalent conditions hold, zero morphisms are exactly the zero element for the group structure on Hom.

[Bor1994b, Prop. 1.2.2]

Proof. By definition, we know that $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$. Since the notion of a preadditive category is self-dual, it suffices to show that $(i) \Rightarrow (iii)$ because we can apply the same implication to $\mathscr{C}^{\mathrm{op}}$. Let 0 be the initial object. The set $\mathrm{Hom}_{\mathscr{C}}(0,0)$ consists of a single element, which proves that id_0 is the zero element in the Abelian group $\mathrm{Hom}_{\mathscr{C}}(0,0)$. Now given an object X of $\mathscr{C}$, the Abelian group $\mathrm{Hom}_{\mathscr{C}}(X,0)$ has at least one element, the zero element. Now if $f\colon X \to 0$ is any morphism, we have $f = \mathrm{id}_0 \circ f$. Since the composition law is bilinear and id_0 is the zero element of $\mathrm{Hom}_{\mathscr{C}}(0,0)$, this implies f is the zero element of $\mathrm{Hom}_{\mathscr{C}}(X,0)$. Thus, 0 is a final object as well.

Finally, we want to show that zero morphisms are the zero elements in Hom. We consider the composition law

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X,0) \times \mathrm{Hom}_{\mathscr{C}}(0,Y) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(X,Y) \\ (0,0) &\longmapsto 0 \end{aligned}$$

where the unique map $X \to 0$ is the zero element of the group $\mathrm{Hom}_{\mathscr{C}}(X,0)$ as shown in the previous paragraph, and the unique map $0 \to Y$ is the zero element by the same argument applied to $\mathscr{C}^{\mathrm{op}}$. By bilinearity, the composition $X \to 0 \to Y$ is the zero element of the group $\mathrm{Hom}_{\mathscr{C}}(X,Y)$. ☐

[Har1977, Exer. II.1.9]

Proposition 1.3.20. *Let $(X,\mathscr{O}_X)$ be a ringed space. The category $\mathsf{Mod}(\mathscr{O}_X)$ of $\mathscr{O}_X$-modules is additive.*

Proof. This is Homework 2, Problem 1(b). ☐

1.3.3 Abelian categories

To define Abelian categories, we need to define kernels, cokernels, coimages, and images.

[TohokuI, §1.3]
[Wei1994, A.1.6]
[Bor1994b, Def. 1.1.5]

Definition 1.3.21. Let $\mathscr{C}$ be a preadditive category. Let $f\colon X \to Y$ be a morphism.

(1) A *kernel* for f is an object $\ker(f)$ together with a morphism $i\colon \ker(f) \to X$ such that $f \circ i = 0$ and such that for every commutative diagram

$$\begin{tikzcd}
\ker(f) \arrow[dr, "i"] \arrow[drr, bend left, "0"] & & \\
 & X \arrow[r, "f"] & Y \\
S \arrow[uu, dashed, "\exists!"] \arrow[ur] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

there is a unique dashed morphism making the diagram commute.

(2) A *cokernel* for f is an object $\operatorname{coker}(f)$ together with a morphism $p\colon Y \to \operatorname{coker}(f)$ such that $p \circ f = 0$ and such that for every commutative diagram

$$\begin{tikzcd}
 & & \operatorname{coker}(f) \arrow[dd, dashed, "\exists!"] \\
X \arrow[r, "f"] \arrow[urr, bend left, "0"] \arrow[drr, bend right, "0"'] & Y \arrow[ur, "p"] \arrow[dr] & \\
 & & T
\end{tikzcd}$$

there is a unique dashed morphism making the diagram commute.

(3) If a kernel and cokernel for f exist, then a *coimage* for f is a cokernel for $\ker(f) \to X$, and is denoted $X \to \operatorname{coim}(f)$.

(4) If a kernel and cokernel for f exist, then an *image* for f is a kernel for $Y \to \operatorname{coker}(f)$, and is denoted $\operatorname{im}(f) \to Y$.

We can finally define Abelian categories. The definition below is due to Grothendieck [TohokuI] but an equivalent definition due to Buchsbaum [Buc1955] appeared earlier.

Definition 1.3.22. ([TohokuI, §1.4]) Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ is an *Abelian category* if it is additive and the following conditions hold. [Buc1955, Pt. I, §1] [Har1977, p. 202] [Wei1994, Def. A.4.2] [Bor1994b, Def. 1.4.1]

(AB1) All kernels and cokernels of morphisms in $\mathscr{C}$ exist.

(AB2) For every morphism f in $\mathscr{C}$, the morphism

$$\operatorname{coim}(f) \longrightarrow \operatorname{im}(f)$$

induced by the universal property is an isomorphism.

We say that a sequence [TohokuI, p. 128] [Wei1994, p. 426] [Bor1994b, Def. 1.8.1] [God1973, II.2.5]

$$\cdots \longrightarrow X^{i-1} \xrightarrow{f^{i-1}} X^i \xrightarrow{f^i} X^{i+1} \longrightarrow \cdots$$

in an Abelian category is *exact* if $f^i \circ f^{i-1} = 0$ for every i and the morphism

$$\operatorname{im}(f^{i-1}) \longrightarrow \ker(f^i)$$

induced by the universal property of kernels is an isomorphism for every i.

[Wei1994, Ex. A.4.4] **Example 1.3.23.** Here are some examples of Abelian categories.

[Bor1994b, Ex. 1.4.6.a]

[Har1977, Ex. III.1.0.1] (1) The category Ab of Abelian groups.

[Har1977, Ex. III.1.0.2] (2) The category $\mathsf{Mod}(R)$ of modules over a ring R.

1.3.4 Sheaves of modules form an Abelian category

For this class, the following example of an Abelian category is (one of) the most important.

[Har1977, Ex. III.1.0.3] **Theorem 1.3.24.** ([TohokuI, Prop. 3.1.1]) *Let $(X, \mathcal{O}_X)$ be a ringed space. The category $\mathsf{Mod}(\mathcal{O}_X)$ of $\mathcal{O}_X$-modules is Abelian.*

We break up this theorem into parts. We denote by $\varphi\colon \mathscr{F} \to \mathscr{G}$ a morphism in $\mathsf{Mod}(\mathcal{O}_X)$. The point of the proof is although we have defined kernels, cokernels, and images of morphisms in $\mathsf{Mod}(\mathcal{O}_X)$, we need to justify why they satisfy the universal properties in Definition 1.3.21.

[Har1977, Exer. II.1.2(a)] **Claim 1.3.25.** *The kernel presheaf* $\ker(\varphi) \coloneqq \{U \mapsto \ker(\varphi(U))\}$ *is a sheaf and satisfies* $\ker(\varphi_P) \cong \ker(\varphi)_P$ *for every $P \in X$. Moreover,* $\ker(\varphi)$ *satisfies the universal property of the kernel.*

Proof. We first show that the kernel presheaf is a sheaf. Suppose that $U = \bigcup_i V_i$ is an open covering of an open set U. For sheaf condition (3), let $s, t \in \ker(\varphi(U))$ such that $s|_{V_i} = t|_{V_i}$ for all i. Considering s, t as elements of $\mathscr{F}(U)$, we see that $s = t$ in $\mathscr{F}(U)$ by the sheaf condition (3) for $\mathscr{F}$, and hence $s = t$ in $\ker(\varphi(U))$. For sheaf condition (4), let $s_i \in \ker(\varphi(V_i))$ be such that

$$s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$$

for all i, j. Considering the s_i as elements of $\mathscr{F}(V_i)$, we see that there exists an element $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$ in $\mathscr{F}(V_i)$ for all i by the sheaf condition (4) for $\mathscr{F}$. It remains to show that $s \in \ker(\varphi(U))$. Consider the commutative diagram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d] & \mathscr{G}(U) \arrow[d] \\
\mathscr{F}(V_i) \arrow[r, "\varphi(V_i)"] & \mathscr{G}(V_i)
\end{tikzcd}$$

where the vertical maps are the restriction maps. For each i, we see that

$$\varphi(U)(s)|_{V_i} = \varphi(V_i)(s_i) = 0$$

by the commutativity of the diagram. By the sheaf condition (3) for $\mathscr{G}$, we see that $\varphi(U)(s) = 0$.

We now show that $\ker(\varphi_P) \cong \ker(\varphi)_P$ for every $P \in X$. For every open set U, we have the exact sequence

$$0 \longrightarrow \ker\big(\varphi(U)\big) \longrightarrow \mathscr{F}(U) \xrightarrow{\varphi(U)} \mathscr{G}(U).$$

These exact sequences fit into the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker\big(\varphi(U)\big) \arrow[r] \arrow[d] & \mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d] & \mathscr{G}(U) \arrow[d] \\
0 \arrow[r] & \ker\big(\varphi(V)\big) \arrow[r] & \mathscr{F}(V) \arrow[r, "\varphi(V)"] & \mathscr{G}(V)
\end{tikzcd}$$

for all inclusions of open sets $V \subseteq U$, where the vertical maps are the restriction maps. Taking direct limits over all open sets containing P, we have the exact sequence

$$0 \longrightarrow \ker(\varphi)_P \longrightarrow \mathscr{F}_P \xrightarrow{\varphi_P} \mathscr{G}_P$$

since filtered direct limits are exact by Theorem 1.1.17. By the universal property of kernels in the category of $\mathscr{O}_{X,P}$-modules, we therefore have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker(\varphi)_P \arrow[r] \arrow[d, dashed, "\exists!"'] & \mathscr{F}_P \arrow[r, "\varphi_P"] \arrow[d, equal] & \mathscr{G}_P \arrow[d, equal] \\
0 \arrow[r] & \ker(\varphi_P) \arrow[r] & \mathscr{F}_P \arrow[r, "\varphi_P"] & \mathscr{G}_P
\end{tikzcd}$$

with exact rows. Since kernels of isomorphic maps of modules are isomorphic, we see that $\ker(\varphi)_P \to \ker(\varphi_P)$ is an isomorphism.

We now show that $\ker(\varphi)$ satisfies the universal property of the kernel. Let $i\colon \ker(\varphi) \to \mathscr{F}$ be the morphism defined by the inclusions $\ker(\varphi(U)) \subseteq \mathscr{F}(U)$ on every open subset U. We then see that $\varphi \circ i = 0$ since $(\varphi \circ i)(U) = 0$ for every open subset U. Now consider the commutative diagram

$$\begin{tikzcd}
\ker(\varphi) \arrow[dr, "i"'] \arrow[drr, bend left, "0"] & & \\
& \mathscr{F} \arrow[r, "\varphi"] & \mathscr{G}. \\
\mathscr{S} \arrow[uu, dashed, "\psi", "\exists!"'] \arrow[ur] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

We need to show that for every $\mathscr{S}$ fitting into this commutative diagram, there is a unique map $\mathscr{S} \to \ker(\varphi)$ fitting into this diagram making the diagram commute. On every open subset U, we have the commutative diagram

$$
\begin{tikzcd}
\ker\big(\varphi(U)\big) \arrow[dr, "i(U)"] \arrow[drr, bend left, "0"] & & \\
 & \mathscr{F}(U) \arrow[r, "\varphi(U)"] & \mathscr{G}(U). \\
\mathscr{S}(U) \arrow[uu, dashed, "\psi(U)", "\exists!"'] \arrow[ur] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}
$$

By the universal property of kernels in the category of $\mathscr{O}_X(U)$-modules, there is a unique map $\psi(U)\colon \mathscr{S}(U) \to \ker(\varphi(U))$ making the diagram commute. We then consider the diagram

$$
\begin{tikzcd}
 & \ker\big(\varphi(U)\big) \arrow[dr, "i(U)"] \arrow[drr, bend left, "0"] \arrow[ddd] & & \\
 & & \mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[ddd] & \mathscr{G}(U) \arrow[ddd] \\
\mathscr{S}(U) \arrow[uur, "\psi(U)"] \arrow[urr] \arrow[urrr, bend right, "0"'] \arrow[ddd] \arrow[dr, dashed, red] & & & \\
 & \ker\big(\varphi(V)\big) \arrow[dr, "i(V)"] \arrow[drr, bend left, "0"] & & \\
 & & \mathscr{F}(V) \arrow[r, "\varphi(V)"] & \mathscr{G}(V) \\
\mathscr{S}(V) \arrow[uur, "\psi(V)"] \arrow[urr] \arrow[urrr, bend right, "0"'] & & &
\end{tikzcd}
$$

where the vertical maps are the restriction maps and where every triangle or square commutes except possibly for the leftmost square face involving $\ker(\varphi)$ and $\mathscr{S}$. We need to show that the compositions along this leftmost square face from $\mathscr{S}(U)$ to $\ker(\varphi(V))$ are equal. It suffices to note that the composition $\mathscr{S}(U) \to \mathscr{G}(V)$ is the 0 map, and hence the universal property of $\ker(\varphi(V))$ implies that there is a unique map $\mathscr{S}(U) \to \ker(\varphi(V))$ shown as a dashed red arrow in the diagram above compatible with the map $\mathscr{S}(U) \to \mathscr{F}(V)$. We therefore see that the leftmost square face involving $\ker(\varphi)$ and $\mathscr{S}$ commutes. $\square$

[Har1977, Exer. II.1.6(a)]

Claim 1.3.26. *Let $\mathscr{F}'$ be a sub-$\mathscr{O}_X$-module of $\mathscr{F}$. Then, the natural morphism $\mathscr{F} \to \mathscr{F}/\mathscr{F}'$ is surjective with kernel $\mathscr{F}'$, and hence there is a short exact sequence*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}/\mathscr{F}' \longrightarrow 0.$$

Proof. Denote by $\mathscr{G}$ the quotient presheaf $U \mapsto \mathscr{F}(U)/\mathscr{F}'(U)$. For every inclusion $V \subseteq U$ of open sets, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}'(U) \arrow[r] \arrow[d] & \mathscr{F}(U) \arrow[r] \arrow[d] & \mathscr{G}(U) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}'(V) \arrow[r] & \mathscr{F}(V) \arrow[r] & \mathscr{G}(V) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Taking direct limits over all open sets containing P, we obtain the short exact sequence

$$0 \longrightarrow \mathscr{F}'_P \longrightarrow \mathscr{F}_P \longrightarrow \mathscr{G}_P \longrightarrow 0$$

since filtered direct limits are exact by Theorem 1.1.17.

We now consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}' \arrow[r] \arrow[d, equal] & \mathscr{F} \arrow[r] \arrow[d, equal] & \mathscr{G} \arrow[r] \arrow[d, "\theta"] & 0 \\
0 \arrow[r] & \mathscr{F}' \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{F}/\mathscr{F}' \arrow[r] & 0.
\end{tikzcd}$$

Taking stalks, we obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}'_P \arrow[r] \arrow[d, equal] & \mathscr{F}_P \arrow[r] \arrow[d, equal] & \mathscr{G}_P \arrow[r] \arrow[d, "\theta", "\wr"'] & 0 \\
0 \arrow[r] & \mathscr{F}'_P \arrow[r] & \mathscr{F}_P \arrow[r] & (\mathscr{F}/\mathscr{F}')_P \arrow[r] & 0
\end{tikzcd}$$

where the top row is exact and the right vertical map is an isomorphism since sheafification induces an isomorphism on stalks. We therefore see that the bottom row is exact, and hence the sequence

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}/\mathscr{F}' \longrightarrow 0.$$

is exact. Exactness at $\mathscr{F}/\mathscr{F}'$ says that $\pi \colon \mathscr{F} \to \mathscr{F}/\mathscr{F}'$ is surjective. To show that $\mathscr{F}'$ is the kernel of π, consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}' \arrow[r] \arrow[d, dashed, "\exists !"] & \mathscr{F} \arrow[r] \arrow[d, equal] & \mathscr{F}/\mathscr{F}' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker(\pi) \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{F}/\mathscr{F}' \arrow[r] & 0
\end{tikzcd}$$

where the top row is exact and the left dashed map is induced by the universal property of kernels from Claim 1.3.25. The bottom row is also exact by the fact that $\ker(\pi)_P \to \ker(\pi_P)$ is an isomorphism by Claim 1.3.25. Finally, since on every U, kernels of isomorphic maps of $\mathscr{O}_X(U)$-modules are isomorphic, we see that the map $\mathscr{F}' \to \ker(\pi)$ is an isomorphism. $\square$

[Har1977, Exer. II.1.7(a)]

Claim 1.3.27. (The first isomorphism theorem in $\mathsf{Mod}(\mathcal{O}_X)$) *We have*
$$\operatorname{im}(\varphi) \cong \mathscr{F}/\ker(\varphi).$$

Soluion. Recall that $\operatorname{im}(\varphi)$ is defined to be the sheafification of the presheaf $U \mapsto \operatorname{im}(\varphi(U))$. For every inclusion $V \subseteq U$ of open sets, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker\bigl(\varphi(U)\bigr) \arrow[r] \arrow[d] & \mathscr{F}(U) \arrow[r] \arrow[d] & \operatorname{im}\bigl(\varphi(U)\bigr) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker\bigl(\varphi(V)\bigr) \arrow[r] & \mathscr{F}(V) \arrow[r] & \operatorname{im}\bigl(\varphi(V)\bigr) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Taking direct limits over all open sets containing a point $P \in X$, we obtain the top exact sequence in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker(\varphi)_P \arrow[r] \arrow[d, equal] & \mathscr{F}_P \arrow[r] \arrow[d, equal] & \varinjlim_{U \ni P} \operatorname{im}\bigl(\varphi(U)\bigr) \arrow[r] \arrow[d, "\theta_P", "\wr"'] & 0 \\
0 \arrow[r] & \ker(\varphi)_P \arrow[r] & \mathscr{F}_P \arrow[r] & \operatorname{im}(\varphi)_P \arrow[r] & 0
\end{tikzcd}$$

since filtered direct limits are exact. The right vertical map is an isomorphism since sheafification induces an isomorphism on stalks. Thus, the bottom row is exact for every $P \in X$, and hence
$$0 \longrightarrow \ker(\varphi) \longrightarrow \mathscr{F} \longrightarrow \operatorname{im}(\varphi) \longrightarrow 0$$
is exact. We therefore see that $\operatorname{im}(\varphi) \cong \mathscr{F}/\ker(\varphi)$ by Claim 1.3.26. □

[Har1977, Exer. II.1.7(b)]

Claim 1.3.28. *We have* $\operatorname{coker}(\varphi) \cong \mathscr{G}/\operatorname{im}(\varphi)$. *Moreover, the cokernel sheaf* $\operatorname{coker}(\varphi)$ *satisfies the universal property of the cokernel.*

Proof. Recall that $\operatorname{coker}(\varphi)$ is the sheafification of the presheaf $\operatorname{coker}_{\mathscr{P}}(\varphi)$ defined by $U \mapsto \operatorname{coker}(\varphi(U))$. For every inclusion $V \subseteq U$ of open sets, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im}\bigl(\varphi(U)\bigr) \arrow[r] \arrow[d] & \mathscr{G}(U) \arrow[r] \arrow[d] & \operatorname{coker}\bigl(\varphi(U)\bigr) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{im}\bigl(\varphi(V)\bigr) \arrow[r] & \mathscr{G}(V) \arrow[r] & \operatorname{coker}\bigl(\varphi(V)\bigr) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Taking direct limits over all open sets containing a point $P \in X$, we obtain the top exact sequence in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim_{U \ni P} \operatorname{im}\bigl(\varphi(U)\bigr) \arrow[r] \arrow[d, "\theta", "\wr"'] & \mathscr{G}_P \arrow[r] \arrow[d, equal] & \varinjlim_{U \ni P} \operatorname{coker}\bigl(\varphi(U)\bigr) \arrow[r] \arrow[d, "\theta", "\wr"'] & 0 \\
0 \arrow[r] & \operatorname{im}(\varphi)_P \arrow[r] & \mathscr{G}_P \arrow[r] & \operatorname{coker}(\varphi)_P \arrow[r] & 0
\end{tikzcd}$$

since filtered direct limits are exact. The outer vertical maps are isomorphisms since sheafification induces isomorphisms on stalks. We therefore see that the bottom row is exact for every $P \in X$, and hence

$$0 \longrightarrow \operatorname{im}(\varphi) \longrightarrow \mathscr{G} \longrightarrow \operatorname{coker}(\varphi) \longrightarrow 0$$

is exact. We therefore see that $\operatorname{coker}(\varphi) \cong \mathscr{G}/\operatorname{im}(\varphi)$ by Claim 1.3.26.

We now show that $\operatorname{coker}(\varphi)$ satisfies the universal property of the cokernel. Let $p\colon \mathscr{G} \to \operatorname{coker}(\varphi)$ be the morphism defined by taking the sheafification of the map defined by $\mathscr{G}(U) \to \operatorname{coker}(\varphi(U))$ on presheaves. We then see that $p \circ \varphi = 0$ since $(p \circ \varphi)(U) = 0$ for every open subset U. Now consider the commutative diagram

$$\begin{tikzcd} & & \operatorname{coker}(\varphi) \arrow[dd, dashed, "\exists!"', "\psi"] \\ \mathscr{F} \arrow[r, "\varphi"] \arrow[urr, bend left, "0"] \arrow[drr, bend right, "0"'] & \mathscr{G} \arrow[ur, "p"'] \arrow[dr] & \\ & & \mathscr{T}. \end{tikzcd} \tag{1.3.29}$$

We need to show that for every $\mathscr{T}$ fitting into this commutative diagram, there is a unique map $\operatorname{coker}(\varphi) \to \mathscr{T}$ fitting into this diagram making the diagram commute. On every open subset U, we have the commutative diagram

$$\begin{tikzcd} & & \operatorname{coker}\big(\varphi(U)\big) \arrow[dd, dashed, "\exists!"', "\psi(U)"] \\ \mathscr{F}(U) \arrow[r, "\varphi"] \arrow[urr, bend left, "0"] \arrow[drr, bend right, "0"'] & \mathscr{G}(U) \arrow[ur, "p"'] \arrow[dr] & \\ & & \mathscr{T}(U). \end{tikzcd}$$

By the universal property of cokernels in the category of $\mathcal{O}_X(U)$-modules, there is a unique map $\psi(U)\colon \operatorname{coker}(\varphi(U)) \to \mathscr{T}(U)$ making the diagram commute.

We then consider the diagram

$$\begin{tikzcd}
 & & & \operatorname{coker}\bigl(\varphi(U)\bigr) \arrow[ddl, "\psi(U)"'] \arrow[ddd] \arrow[dddddl, dashed, red] \\
\mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[urrr, bend left, "0"] \arrow[drr, bend right, "0"'] \arrow[ddd] & \mathscr{G}(U) \arrow[urr, "p(U)"] \arrow[dr] \arrow[ddd] & & \\
 & & \mathscr{T}(U) \arrow[ddd] & \\
 & & & \operatorname{coker}\bigl(\varphi(V)\bigr) \arrow[ddl, "\psi(V)"'] \\
\mathscr{F}(V) \arrow[r, "\varphi(V)"] \arrow[urrr, bend left, "0"] \arrow[drr, bend right, "0"'] & \mathscr{G}(V) \arrow[urr, "p(V)"] \arrow[dr] & & \\
 & & \mathscr{T}(V) &
\end{tikzcd}$$

where the vertical maps are the restriction maps and where every triangle or square commutes except possibly for the rightmost square face involving $\operatorname{coker}(\varphi)$ and $\mathscr{T}$. We need to show that the compositions along this rightmost square face from $\operatorname{coker}(\varphi(U))$ to $\mathscr{T}(V)$ are equal. It suffices to note that the composition $\mathscr{F}(U) \to \mathscr{T}(V)$ is the 0 map, and hence the universal property of $\operatorname{coker}(\varphi(U))$ implies that there is a unique map $\operatorname{coker}(\varphi(U)) \to \mathscr{T}(V)$ shown as a dashed red arrow in the diagram above compatible with the map $\mathscr{G}(U) \to \mathscr{T}(V)$. We therefore see that the rightmost square face involving $\operatorname{coker}(\varphi)$ and $\mathscr{T}$ commutes. Finally, taking sheafifications, we see that there is a unique morphism $\psi^{\#} \colon \operatorname{coker}(\varphi) \to \mathscr{T}$ making the diagram (1.3.29) commute since sheafification induces a bijection

$$\operatorname{Hom}_{\mathsf{PMod}(\mathscr{O}_X)}\bigl(\operatorname{coker}_{\mathscr{P}}(\varphi), \mathscr{T}\bigr) \xrightarrow{\sim} \operatorname{Hom}_{\mathsf{Mod}(\mathscr{O}_X)}\bigl(\operatorname{coker}(\varphi), \mathscr{T}\bigr)$$

by Proposition 1.2.16 (note that the proof can be performed in the category of (pre-)sheaves of $\mathscr{O}_X$-modules since the commutative diagram in the proof consists of maps of $\mathscr{O}_X(U)$-modules). □

[Har1977, Exer. II.1.2(a)] **Claim 1.3.30.** *We have* $\operatorname{im}(\varphi_P) \cong \operatorname{im}(\varphi)_P$ *for every* $P \in X$*. Moreover, the image sheaf* $\operatorname{im}(\varphi)$ *satisfies the universal property of the image.*

Proof. By the universal property of kernels in the category of $\mathscr{O}_X$-modules, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im}(\varphi) \arrow[r] \arrow[d, dashed, "\exists!"'] & \mathscr{G} \arrow[r, "p"] \arrow[d, equal] & \operatorname{coker}(\varphi) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker(p) \arrow[r] & \mathscr{G} \arrow[r, "p"] & \operatorname{coker}(\varphi) \arrow[r] & 0
\end{tikzcd}$$

where the top row is the exact sequence obtained by combining Claim 1.3.26 and Claim 1.3.28, and the bottom row is the exact sequence obtained by combining Claim 1.3.26 and Claim 1.3.27. Taking stalks, we obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im}(\varphi)_P \arrow[r] \arrow[d, dashed, "\exists!"'] & \mathscr{G}_P \arrow[r, "p_P"] \arrow[d, equal] & \operatorname{coker}(\varphi)_P \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker(p)_P \arrow[r] & \mathscr{G}_P \arrow[r, "p_P"] & \operatorname{coker}(\varphi)_P \arrow[r] & 0
\end{tikzcd}$$

with exact rows. We therefore see that the left vertical map is an isomorphism since kernels of isomorphic maps of $\mathcal{O}_{X,P}$-modules are isomorphic, and hence $\operatorname{im}(\varphi) \to \ker(p)$ is an isomorphism by Proposition 1.2.12. Since $\ker(p)_P \to \ker(p_P)$ is an isomorphism and both satisfy the universal property of $\operatorname{im}(\varphi_P)$, we therefore see that $\operatorname{im}(\varphi)_P \to \operatorname{im}(\varphi_P)$ is an isomorphism. □

We can finally show that Theorem 1.3.24.

Claim 1.3.31. $\mathsf{Mod}(\mathcal{O}_X)$ *is Abelian.* [TohokuI, Prop. 3.1.1] [Har1977, Ex. III.1.0.3]

Proof. By Homework 11, Problem 2 from last semester, the category $\mathsf{Mod}(\mathcal{O}_X)$ is additive. By Claim 1.3.25 and Claim 1.3.28, $\mathsf{Mod}(\mathcal{O}_X)$ has kernels and cokernels. It remains to show that for every morphism $\varphi\colon \mathscr{F} \to \mathscr{G}$, the morphism $\operatorname{coim}(\varphi) \to \operatorname{im}(\varphi)$ induced by universal properties is an isomorphism.

We first construct this map. We have the commutative diagram

$$\begin{tikzcd}
& & \operatorname{coim}(\varphi) \arrow[dr] \arrow[drr, bend left, "0"] \arrow[dd, dashed, "\exists!\,\psi"'] & & \\
\ker(\varphi) \arrow[r] \arrow[urr, bend left, "0"] \arrow[drr, bend right, "0"'] & \mathscr{F} \arrow[ur] \arrow[dr] \arrow[rr, "\varphi"] & & \mathscr{G} \arrow[r] & \operatorname{coker}(\varphi). \\
& & \operatorname{im}(\varphi) \arrow[ur] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

Since $\operatorname{im}(\varphi) \to \operatorname{coker}(\varphi)$ (resp. $\ker(\varphi) \to \operatorname{coim}(\varphi)$) is the 0 map, there exists a unique dashed map $\operatorname{coim}(\varphi) \to \operatorname{im}(\varphi)$ that makes the diagram commute by the universal property of the coimage (resp. image). Taking stalks, we obtain the commutative diagram

$$\begin{tikzcd}
& & \operatorname{coim}(\varphi_P) \arrow[dr] \arrow[drr, bend left, "0"] \arrow[dd, "\psi_P"] & & \\
\ker(\varphi_P) \arrow[r] \arrow[urr, bend left, "0"] \arrow[drr, bend right, "0"'] & \mathscr{F}_P \arrow[ur] \arrow[dr] \arrow[rr, "\varphi_P"] & & \mathscr{G}_P \arrow[r] & \operatorname{coker}(\varphi_P) \\
& & \operatorname{im}(\varphi_P) \arrow[ur] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

where $\ker(\varphi)_P \xrightarrow{\sim} \ker(\varphi_P)$ by Claim 1.3.25, $\operatorname{coim}(\varphi)_P \xrightarrow{\sim} \operatorname{coim}(\varphi_P)$ by Claim 1.3.26 and Claim 1.3.27, $\operatorname{coker}(\varphi)_P \xrightarrow{\sim} \operatorname{coker}(\varphi_P)$ by Claim 1.3.26 and 1.3.28, and $\operatorname{im}(\varphi)_P \xrightarrow{\sim} \operatorname{im}(\varphi_P)$ by Claim 1.3.30. The map ψ_P is the same as that induced by the universal property of coimages (or images) of $\mathscr{O}_{X,P}$-modules by the uniqueness part of the universal property. Since the category of $\mathscr{O}_{X,P}$-modules is Abelian, the map ψ_P is an isomorphism. □

1.3.5 Grothendieck's axioms

As we saw in the proof of Theorem 1.3.24, the specific Abelian categories that we like to work with satisfy additional properties. Grothendieck identified these properties as useful and defined the following additional axioms (among others) for Abelian categories, and used them to give a general criterion for the existence of "enough" injectives.

[Wei1994, A.4.3] **Definition 1.3.32.** ([TohokuI, §1.5]) Let $\mathscr{A}$ be an Abelian category. We consider the following axiom:

(AB3) For every family of objects $\{A_i\}$ in $\mathscr{A}$, the coproduct $\coprod_i A_i$ exists.

Traditionally, in an Abelian category coproducts are called *direct sums* and are denoted $\bigoplus_i X_i$.

We also formulate the dual axiom:

(AB3*) For every family of objects $\{A_i\}$ in $\mathscr{A}$, the product $\prod_i A_i$ exists.

If $\mathscr{A}$ is an Abelian category satisfying AB3, then it has all direct limits.

[Wei1994, Exer. A.5.2] **Proposition 1.3.33.** ([TohokuI, Proposition 1.8]) *Let $\mathscr{A}$ be an Abelian category satisfying* AB3*. Then, $\mathscr{A}$ has arbitrary direct limits (not necessarily filtered).*

Proof. Consider the objects

$$S = \bigoplus_{i\in I} A_i \qquad\qquad T = \bigoplus_{j\in I}\bigoplus_{i\leq j} A_i$$

with insertion maps $v_i\colon A_i \to S$ and $w_{ij}\colon A_i \to T$. We then have two maps $d, e\colon T \to S$:

$$\begin{tikzcd} A_i \arrow[d, "w_{ij}"'] \arrow[dr, "v_i"] & \\ T \arrow[r, dashed, "d", "\exists!"'] & S \end{tikzcd} \qquad\qquad \begin{tikzcd} A_i \arrow[r, "f_{ij}"] \arrow[d, "w_{ij}"'] & A_j \arrow[d, "v_j"] \\ T \arrow[r, dashed, "e", "\exists!"'] & S. \end{tikzcd}$$

The cokernel for $d - e$ is the direct limit we wanted. □

We can ask whether filtered direct limits are exact, which was the case for $\mathsf{Mod}(R)$.

Definition 1.3.34. ([TohokuI, §1.5]) Let $\mathscr{A}$ be an Abelian category. We consider the following axiom: [Wei1994, A.4.6]

(AB5) The axiom AB3 holds and filtered direct limits in $\mathscr{A}$ are exact.

We also formulate the dual axiom:

(AB5*) The axiom AB3* holds and cofiltered inverse limits in $\mathscr{A}$ are exact.

Remark 1.3.35. AB5 is not Grothendieck's original formulation. Our version is equivalent to Grothendieck's by [TohokuI, Proposition 1.8].

There are also axioms AB4 and AB6 which we will not define or use.

We note that AB5* is not necessarily true in $\mathsf{Mod}(R)$.

Example 1.3.36. We show that cofiltered inverse limits are not necessarily exact in $\mathsf{Ab} = \mathsf{Mod}(\mathbf{Z})$. $\boxed{1/27}$ [Mur2015]

(1) Consider the commutative diagram [Mat1989, pp. 272–273]

$$\begin{tikzcd}
 & \vdots \arrow[d, "p"] & \vdots \arrow[d, "p"] & \vdots \arrow[d, "p"] & \\
0 \arrow[r] & \mathbf{Z} \arrow[r, "n"] \arrow[d, "p"] & \mathbf{Z} \arrow[r] \arrow[d, "p"] & \mathbf{Z}/(n) \arrow[r] \arrow[d, "p"] & 0 \\
 & \vdots \arrow[d, "p"] & \vdots \arrow[d, "p"] & \vdots \arrow[d, "p"] & \\
0 \arrow[r] & \mathbf{Z} \arrow[r, "n"] \arrow[d, "p"] & \mathbf{Z} \arrow[r] \arrow[d, "p"] & \mathbf{Z}/(n) \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & \mathbf{Z} \arrow[r, "n"] & \mathbf{Z} \arrow[r] & \mathbf{Z}/(n) \arrow[r] & 0
\end{tikzcd}$$

where n and p are coprime. The inverse limit of the inverse systems in the left and middle columns is

$$\begin{aligned}
&\lim\left(\mathbf{Z} \xleftarrow{p} \mathbf{Z} \xleftarrow{p} \cdots\right) \\
&\qquad = \left\{ (a_0, a_1, \ldots, a_n, \ldots) \in \prod_{i=0}^{\infty} \mathbf{Z} \;\middle|\; p^{j-i} a_j = a_i \text{ for all } j \geq i \right\} \\
&\qquad = 0
\end{aligned}$$

using the description of inverse limits in Ab from Example 1.1.16 or [AM1969, p. 103]. Since n and p are coprime, the multiplication by p map on $\mathbf{Z}/(n)$ is an isomorphism. Thus, the inverse limit of the exact sequences above is

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbf{Z}/(n) \longrightarrow 0,$$

which is not exact.

[AM1969, Exer. 10.2]
[Mat1989, p. 272]

(2) Consider the commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d, "p"] & \vdots \arrow[d, equal] & \vdots \arrow[d] & \\
0 \arrow[r] & \mathbf{Z} \arrow[r, "p^n"] \arrow[d, "p"] & \mathbf{Z} \arrow[r] \arrow[d, equal] & \mathbf{Z}/(p^n) \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d, "p"] & \vdots \arrow[d, equal] & \vdots \arrow[d] & \\
0 \arrow[r] & \mathbf{Z} \arrow[r, "p^2"] \arrow[d, "p"] & \mathbf{Z} \arrow[r] \arrow[d, equal] & \mathbf{Z}/(p^2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbf{Z} \arrow[r, "p"] & \mathbf{Z} \arrow[r] & \mathbf{Z}/(p) \arrow[r] & 0.
\end{tikzcd}$$

The inverse limit of these exact sequences is

$$0 \longrightarrow 0 \longrightarrow \mathbf{Z} \longrightarrow \hat{\mathbf{Z}}_p \longrightarrow 0,$$

which is not exact.

1.3.6 Additive functors and exact functors

Recall our goal is to describe functors between Abelian categories like sheaf cohomology and how to work with them. One way to think about this is that now that we know $\mathsf{Mod}(\mathscr{O}_X)$ forms an Abelian category, we want a good way to extract invariants from it with values in Abelian groups, vector spaces, modules, or even in another category (for our discussion, another Abelian category). For those of you taking the algebraic topology class, this is the idea underlying homology and cohomology.

We first define classes of functors that preserve additivity and exactness properties that exist in an Abelian category.

[Har1977, pp. 203–204]
[Wei1994, Def. 1.6.6]
[Bor1994b, Def. 1.11.1]
[TohokuI, p. 126]

Definition 1.3.37. Let $F\colon \mathscr{A} \to \mathscr{B}$ be a covariant functor between Abelian categories. We say that F is

(*i*) *additive* if, for every pair of objects A, A' in $\mathscr{A}$, the induced map

$$\operatorname{Hom}(A, A') \longrightarrow \operatorname{Hom}(FA, FA')$$

is a map of Abelian groups. (This makes sense even if $\mathscr{A}$ and $\mathscr{B}$ are preadditive.)

[TohokuI, p. 128]

(*ii*) *left exact* (resp. *right exact*, *middle exact*) if it is additive and if it sends short exact sequences

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

to exact sequence of the form

$$\begin{aligned} 0 \longrightarrow FA' \longrightarrow FA \longrightarrow FA'' \\ FA' \longrightarrow FA \longrightarrow FA'' \longrightarrow 0 \\ FA' \longrightarrow FA \longrightarrow FA'' \end{aligned}$$

respectively.

(*iii*) *exact* if it is both left and right exact.

We use the same terminology for contravariant functors $F\colon \mathscr{A}^{\mathrm{op}} \to \mathscr{B}$. (The way to remember which one is left or right is to think about what the final result looks like—this is also why I prefer to denote contravariant functors by $\mathscr{A}^{\mathrm{op}} \to \mathscr{B}$ instead of $\mathscr{A} \to \mathscr{B}^{\mathrm{op}}$.)

Let us see some examples. We first show that taking global sections is left exact. You will prove the analogous statement for $\Gamma_Z(X, -)$ on Homework 3.

Proposition 1.3.38. *Let $(X, \mathscr{O}_X)$ be a ringed space and let $U \subseteq X$ be an open subset. The functor* [Har1977, Exer. II.1.8] [God1973, p. 133]

$$\begin{aligned} \mathsf{Mod}(\mathscr{O}_X) &\longrightarrow \mathsf{Mod}\big(\Gamma(U, \mathscr{O}_X)\big) \\ \mathscr{F} &\longmapsto \Gamma(U, \mathscr{F}) \end{aligned}$$

is left exact.

Proof. Consider an exact sequence

$$0 \longrightarrow \mathscr{F}' \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}''.$$

We then have the sequence

$$0 \longrightarrow \Gamma(U, \mathscr{F}') \xrightarrow{\varphi(U)} \Gamma(U, \mathscr{F}) \xrightarrow{\psi(U)} \Gamma(U, \mathscr{F}'')$$

that we need to show is exact. The sequence is automatically exact at $\Gamma(U, \mathscr{F}')$ since a morphism of sheaves is injective if and only if it is injective on sections by Proposition 1.2.12(*a*).

By functoriality, we know that the composition $\psi(U) \circ \varphi(U)$ is the 0 map (you can check the map is 0 on stalks since you can check that the image is 0 on stalks). We therefore have the inclusion

$$\mathrm{im}(\varphi)(U) \subseteq \ker(\psi)(U)$$

as sub-$\mathscr{O}_X(U)$-modules of $\Gamma(U, \mathscr{F})$. It remains to show the opposite inclusion $\mathrm{im}(\varphi)(U) = \ker(\psi)(U)$. We have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(U, \mathscr{F}') \arrow[r, "\varphi(U)"] \arrow[d] & \Gamma(U, \mathscr{F}) \arrow[r, "\psi(U)"] \arrow[d] & \Gamma(U, \mathscr{F}'') \arrow[d] \\ 0 \arrow[r] & \mathscr{F}'_P \arrow[r, "\varphi_P"] & \mathscr{F}_P \arrow[r, "\psi_P"] & \mathscr{F}''_P \end{tikzcd}$$

where the bottom row is exact. Suppose $t \in \ker(\psi)(U)$. Then, $\psi(U)(t) = 0$, and hence

$$\psi(U)(t)_P = \psi_P(t_P) = 0$$

for all $P \in U$. Since the bottom row is exact, there exists $s_P \in \mathscr{F}'_P$ such that $\varphi_P(s_P) = t_P$. We claim we can lift s_P to some $s \in \mathscr{F}'(U)$. For each $P \in U$, pick an open set $V_P \ni P$ and $r_P \in \mathscr{F}'(V_P)$ such that $s_P = \langle V_P, r_P \rangle$. For all $P \in U$, since $\varphi(V_P)(r_P)_P = t_P$, we know there exists an open subset $W_P \subseteq V_P$ containing P such that $\varphi(V_P)(r_P)|_{W_P} = t|_{W_P}$. Thus, for all $P, Q \in U$, we have

$$\varphi(W_P \cap W_Q)(r_P)|_{W_P \cap W_Q} = t|_{W_P \cap W_Q} = \varphi(W_P \cap W_Q)(r_Q)|_{W_P \cap W_Q}.$$

Since $\varphi(W_P \cap W_Q)$ is injective, we conclude that

$$r_P|_{W_P \cap W_Q} = r_Q|_{W_P \cap W_Q}.$$

By the sheaf property (4), the r_P therefore glue to form a section $s \in \mathscr{F}(U)$ such that $\varphi(U)(s) = t$. This shows that $\ker(\psi)(U) \subseteq \operatorname{im}(\varphi)(U)$. □

Another important example for us is:

[Har1977, Ex. III.1.0.8]
[TohokuI, p. 128]
[CE1956, Prop. II.4.4]

Proposition 1.3.39. *Let $\mathscr{A}$ be an Abelian category and let M be an object in $\mathscr{A}$. Then,* $\operatorname{Hom}_{\mathscr{A}}(M, \cdot\,)$ *is a covariant left exact functor* $\mathscr{A} \to \mathsf{Ab}$ *and* $\operatorname{Hom}_{\mathscr{A}}(\,\cdot\,, M)$ *is a contravariant left exact functor* $\mathscr{A}^{\mathrm{op}} \to \mathsf{Ab}$*.*

Proof. We can apply the first statement to $\mathscr{A}^{\mathrm{op}}$ to prove the second statement. Here, we are using the fact that the opposite category of an Abelian category is Abelian. This is because additivity (by Homework 2, Problem 1(a)) and the axioms AB1 and AB2 are self-dual! It therefore suffices to show the first statement.

Consider a short exact sequence

$$0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{p} A'' \longrightarrow 0.$$

We want to show that

$$0 \longrightarrow \operatorname{Hom}_{\mathscr{A}}(M, A') \xrightarrow{i\circ -} \operatorname{Hom}_{\mathscr{A}}(M, A) \xrightarrow{p\circ -} \operatorname{Hom}_{\mathscr{A}}(M, A'')$$

is exact.

We show exactness on the left. Suppose $i \circ g = 0$ for $g \in \operatorname{Hom}_{\mathscr{A}}(M, A')$. Since $\ker(i) = 0$, the universal property of kernels yields the commutative diagram

$$\begin{tikzcd}
\ker(i) \arrow[dr, "0"] \arrow[drr, bend left, "0"] & & \\
& A' \arrow[r, "i"] & A. \\
M \arrow[uu, dashed, "\exists!"] \arrow[ur, "g"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

This shows that g factors through 0, and is therefore the zero morphism.

It remains to show that

$$\operatorname{im}\Big(\operatorname{Hom}_{\mathscr{A}}(M, A') \to \operatorname{Hom}_{\mathscr{A}}(M, A)\Big) = \ker\Big(\operatorname{Hom}_{\mathscr{A}}(M, A) \to \operatorname{Hom}_{\mathscr{A}}(M, A'')\Big).$$

The inclusion $\subseteq$ holds since $p \circ i = 0$. Conversely, suppose $f \in \operatorname{Hom}_{\mathscr{A}}(M, A)$ satisfies $p \circ f = 0$. By the universal property of kernels, we obtain the commutative diagram

$$\begin{tikzcd} & & M \arrow[dl, dashed, "g"', "\exists!"] \arrow[d, "f"] \arrow[dr, "0"] & & \\ 0 \arrow[r] & A' \arrow[r, "i"'] & A \arrow[r, "p"'] & A'' \arrow[r] & 0. \end{tikzcd}$$

By the commutativity of the diagram, we have $f = i \circ g$. ☐

Proposition 1.3.39 motivates the following:

Definition 1.3.40. ([Bae1940, p. 804; CE1956, p. 6 and p. 8]) Let $\mathscr{A}$ be an Abelian category.

[Wei1994, p. 33, p. 37]
[TohokuI, §1.10]
[Har1977, p. 204]

(*i*) We say that an object M of $\mathscr{A}$ is *injective* if the functor $\operatorname{Hom}_{\mathscr{A}}(\,\cdot\,, M)$ is exact. In other words, given the solid commutative diagram below where $0 \to A' \to A$ is exact, there is a dashed map (not necessarily unique) making the diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[dl, dashed, "\exists"] \\ & M & \end{tikzcd}$$

that is, $\operatorname{Hom}(A, M) \to \operatorname{Hom}(A', M) \to 0$ is exact.

(*ii*) We say that an object M of $\mathscr{A}$ is *projective* if the functor $\operatorname{Hom}_{\mathscr{A}}(M, \,\cdot\,)$ is exact. In other words, given the solid commutative diagram below where $A \to A'' \to 0$ is exact, there is a dashed map (not necessarily unique) making the diagram commute:

$$\begin{tikzcd} P \arrow[d, dashed, "\exists"'] \arrow[dr] & & \\ A \arrow[r] & A'' \arrow[r] & 0 \end{tikzcd}$$

that is, $\operatorname{Hom}(M, A) \to \operatorname{Hom}(M, A'') \to 0$ is exact.

The theme for a while will be: How can we fix the non-exactness of these two functors? For Hom, what we want to do is to approximate a given object A by an injective (resp. projective) one. For Γ, it turns out that injective objects will still give us the correct answer. (We will show this later.)

In an arbitrary Abelian category, you cannot always surject onto objects with a projective object. But there *is* a general recipe to find injective objects due to Grothendieck if the Abelian category satisfies some additional hypotheses. These hypotheses hold for $\mathsf{Mod}(\mathscr{O}_X)$. This is why we will focus on injective objects.

1.3.7 Examples of projective modules (not covered in class)

[MurCA, Lem. 7.8.2]
[AK2021, (5.22)]
[Hoc2017, p. 90]

Lemma 1.3.41. *Let R be a ring. An R-module is projective if and only if it is a direct summand of a free module.*

Proof. $\Rightarrow$. Let $\beta\colon F \twoheadrightarrow P$ be a surjection from a free module. Applying $\operatorname{Hom}_R(P,-)$, we obtain a surjection

$$\operatorname{Hom}_R(P,F) \twoheadrightarrow \operatorname{Hom}_R(P,P),$$

and hence there exists a map $s\colon P \to F$ such that $\beta \circ s = \mathrm{id}_P$.

$\Leftarrow$. Let $P \subseteq F$ be the map realizing P as a direct summand of a free module. It suffices to show that for every surjection $M \twoheadrightarrow N$, the map $\operatorname{Hom}_R(P,M) \to \operatorname{Hom}_R(P,N)$ is surjective. But this fits into the commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_R(P,M) \arrow[r] & \operatorname{Hom}_R(P,N) \\
\operatorname{Hom}_R(F,M) \arrow[r] \arrow[u, two heads] & \operatorname{Hom}_R(F,N) \arrow[u, two heads]
\end{tikzcd}$$

where the vertical arrows are surjective by the fact that they have a section. If the bottom arrow is surjective, then the top arrow is also surjective. ☐

[MurCA, Ex. 7.8.3]
[Hoc2017, p. 88]

Example 1.3.42. Not all projective modules are free: If $R = R_1 \times R_2$, each factor R_i is a projective module, but is not free.

We give the following more interesting example of a projective module that is not free, for which the important input is topological.

[MurCA, Ex. 7.9.1]
[Hoc2017, pp. 88–89]

Example 1.3.43. (Kaplansky; see [Swa1962, Example 1]**)** Let

$$A = \frac{\mathbf{R}[X,Y,Z]}{(X^2+Y^2+Z^2-1)}.$$

We denote the images of X, Y, and Z by x, y, and z, respectively. The elements of A can be considered as $\mathbf{R}$-valued polynomial functions on the unit 2-sphere centered at the origin on $\mathbf{R}^3$, and hence give continuous functions on the 2-sphere S^2.

Consider the A-linear map

$$f\colon A^{\oplus 3} \xrightarrow{(x\ y\ z)} A.$$

We have a map

$$g\colon A \xrightarrow{\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)} A^{\oplus 3}$$

and we set $u = \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)$. The composition $f \circ g$ is the matrix whose single entry is $x^2+y^2+z^2 = 1$, and so $f \circ g$ is the identity on A. We therefore see that

$$Q = \ker(f) \subseteq A^{\oplus 3}$$

is a projective A-module.

We claim that Q is not free. We proceed by contradiction. Base changing to the fraction field $K = \operatorname{Frac}(A)$, we see that $K^{\oplus 3} \cong K \oplus (K \otimes_A Q)$, and hence Q is a free module generated by two elements. It therefore suffices to show that $Q \cong A^{\oplus 2}$ yields a contradiction.

Suppose that Q has a free basis consisting of column vectors v and w in $A^{\oplus 3}$. Consider the 3×3 matrix

$$M = \begin{pmatrix} | & | & | \\ u & v & w \\ | & | & | \end{pmatrix}.$$

Since u, v, w span $A^{\oplus 3}$, there cannot be any linear relation on them, since their images in $K^{\oplus 3}$ form a basis as a K-vector, and hence cannot have any linear relations over K, either. Thus, u, v, w are a basis for $A^{\oplus 3}$, and the matrix M gives an automorphism of $A^{\oplus 3}$ with inverse matrix N. Computing determinants, we have

$$\det(M)\det(N) = 1,$$

and hence $\det(M)$ is a unit $\alpha \in A$. We can then multiply the second column of M by α^{-1}:

$$\begin{pmatrix} | & | & | \\ u & \alpha^{-1}v & w \\ | & | & | \end{pmatrix}.$$

Thus, we see that u is the first column of a 3×3 matrix over A with determinant 1.

We claim no such 3×3 matrix can exist, even with entries being continuous functions on S^2. If the third column is $w = \left(\begin{smallmatrix} f \\ g \\ h \end{smallmatrix}\right)$, then the vector-valued function

$$V = \begin{pmatrix} f \\ g \\ h \end{pmatrix} - (xf + yg + zh) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

is a continuous vector-valued function on S^2 that does not vanish on S^2 since for every point $(a, b, c) \in S^2$, the two vectors u and w are linearly independent. Moreover, the value of V is orthogonal to the unit vector (a, b, c) for every $(a, b, c) \in S^2$, since the dot product vanishes. This means V is a everywhere non-vanishing continuous tangent vector field on S^2. This contradicts the hedgehog theorem [Lee2013, Problem 16-6]!

1.3.8 Monomorphisms, epimorphisms, and generators

To make sense of Grothendieck's result, we need a few more pieces of terminology.

[TohokuI, p. 122]

Definition 1.3.44. Let $\mathscr{C}$ be a category and consider a morphism $u\colon A \to B$ in $\mathscr{C}$. We say that u is a *monomorphism*, write $A \hookrightarrow B$, and say A is a *subobject* of B (resp. *epimorphism*, write $A \twoheadrightarrow B$, and say B is a *quotient* of A) if

$$\begin{aligned} \operatorname{Hom}_{\mathscr{C}}(C, A) &\xrightarrow{u\circ -} \operatorname{Hom}_{\mathscr{C}}(C, B) \\ v &\longmapsto u \circ v \end{aligned}$$

(resp. if

$$\begin{aligned} \operatorname{Hom}_{\mathscr{C}}(B, C) &\xrightarrow{-\circ u} \operatorname{Hom}_{\mathscr{C}}(A, C) \\ w &\longmapsto w \circ u \end{aligned}$$

) is injective for every object C in $\mathscr{C}$.

Monomorphisms are the categorical versions of injections: In the diagram

$$C \underset{v'}{\overset{v}{\rightrightarrows}} A \xrightarrow{u} B$$

if the two compositions are equal, then $v = v'$. For an Abelian category, this is the same as saying that $\ker(u) = 0$ or $0 \to A \to B$ is exact.

Similarly, epimorphisms are the categorical versions of surjections: In the diagram

$$A \xrightarrow{u} B \underset{w'}{\overset{w}{\rightrightarrows}} C$$

if the two compositions are equal, then $w = w'$. For an Abelian category, this is the same as saying that $\operatorname{coker}(u) = 0$ or $A \to B \to 0$ is exact.

We now define generators.

Definition 1.3.45. ([TohokuI, p. 134]) Let $\mathscr{C}$ be a category. We say that U is a *generator of* $\mathscr{C}$ if, for every object A in $\mathscr{C}$ and every monomorphism $B \hookrightarrow A$ that is not the identity, the injective map

$$\operatorname{Hom}_{\mathscr{C}}(U, B) \hookrightarrow \operatorname{Hom}_{\mathscr{C}}(U, A)$$

is not surjective, that is, there exists a morphism $u\colon U \to A$ that does not come from a morphism $U \to B$.

1.3.9 Grothendieck Abelian categories have enough injectives

We can now formulate Grothendieck's result. The case for Abelian groups is due to Baer [Bae1940, Theorem 3].

[Stacks, Tag 079H]

Theorem 1.3.46. (Grothendieck [TohokuI, Théorème 1.10.1]**)** *Let $\mathscr{A}$ be an Abelian category satisfying* AB5 *and admitting a generator. Then, for every object A in $\mathscr{A}$, there exists a monomorphism $A \hookrightarrow M$ where M is an injective object.*

In fact, we will construct a *functorial* embedding of objects into injective objects: a functor $A \mapsto M(A)$ from $\mathscr{A}$ to $\mathscr{A}$ and a natural transformation $T\colon \mathrm{id}_{\mathscr{A}} \Rightarrow M$ such that for every object A in $\mathscr{A}$, $M(A)$ is injective and $T_A : A \hookrightarrow M(A)$ is a monomorphism:

$$\begin{tikzcd} A \arrow[r, hook, "T_A"] \arrow[d, "f"'] & M(A) \arrow[d, "M(f)"] \\ B \arrow[r, hook, "T_B"] & M(B). \end{tikzcd}$$

Theorem 1.3.46 motivates the following:

Definition 1.3.47. An Abelian category satisfying AB5 and admitting a generator is called a *Grothendieck Abelian category*. An Abelian category *has enough injective objects* if every objects admits a monomorphism into an injective object. [Har1977, p. 204]

To prove Theorem 1.3.46, we need to make sure we are working with sets. The existence of a generator will ensure that nothing we are working with is "too large." The following is a spelled-out version of the parenthetical remark in [TohokuI, p. 136]. The *cofinality* of an ordinal α is the smallest ordinal that is the order type of a cofinal subset of α [Jec2003, p. 31].

Lemma 1.3.48. *Let $\mathscr{A}$ be an Abelian category with a generator U. Let A be an object of $\mathscr{A}$. Let κ be the cardinality of* $\mathrm{Hom}_{\mathscr{A}}(U, A)$. [Stacks, Tag 0E8N] I did not do (ii) in class.

(i) *There does not exist a strictly increasing (or strictly decreasing) chain of subobjects of A indexed by a cardinal larger than κ.*

(ii) *If α is an ordinal of cofinality $> \kappa$ then any increasing (or decreasing) sequence of subobjects of A indexed by α is eventually constant.*

(iii) *The cardinality of the set of subobjects of A is $\leq 2^\kappa$.*

Proof. (i). Suppose $\kappa' > \kappa$ is a cardinal and assume $(A_i)_{i\in\kappa'}$ is strictly increasing. By definition of a generator, for each i, there exists $\phi_i \in \mathrm{Hom}_{\mathscr{A}}(U, A)$ such that ϕ_i factors through A_{i+1} but not A_i. The morphisms ϕ_i are distinct, contradicting the definition of κ.

(ii) follows from the definition of cofinality and (i).

(iii). For any subobject $B \hookrightarrow A$, let $S_B \in \mathcal{P}(\mathrm{Hom}_{\mathscr{A}}(U, A))$, the power set of $\mathrm{Hom}_{\mathscr{A}}(U, A)$, be the subset

$$S_B = \bigl\{\phi \in \mathrm{Hom}_{\mathscr{A}}(U, A) \bigm| \phi \text{ factors through } B\bigr\}.$$

Then, $B = B'$ if and only if $S_B = S_{B'}$ because U is a generator. Thus, the cardinality of the set of subobjects is at most the cardinality of this power set. $\square$

1/29

We now prove the following result, which is a version of Baer's criterion [Bae1940, Theorem 1] but for arbitrary Grothendieck Abelian categories. Baer's criterion is the special case when $\mathscr{A} = \mathsf{Mod}(R)$ and $U = R$.

Lemma 1.3.49. (Baer's criterion for Grothendieck Abelian categories [TohokuI, Lemme 1]**)** *Let $\mathscr{A}$ be an Abelian category satisfying* AB5 *and admitting a generator U. Consider an object M of $\mathscr{A}$. Then, M is injective if and only if for every subobject $V \hookrightarrow U$, any morphism $V \to M$ can be extended to a morphism $U \to M$.*

Proof. $\Rightarrow$ follows by definition of injectivity.

$\Leftarrow$. We want to construct an extension

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, "u"'] & A \arrow[dl, dashed] \\ & M & \end{tikzcd}$$

of a morphism $u\colon B \to M$. Consider the set P of extensions of u to subobjects of A containing B, partially ordered by saying that

$$\begin{aligned}(v_1\colon B_1 \longrightarrow M) &\le (v_2\colon B_2 \longrightarrow M) \\ &\iff B_1 \hookrightarrow B_2 \text{ and } v_2|_{B_1} = v_1.\end{aligned}$$

Note that P is a set since there is a set of subobjects of A by Lemma 1.3.48.

We want to apply Zorn's lemma [Kur1922; Zor1935] to P. Let T be a totally ordered subset of P. By AB5, we can then take the direct limit over elements in T to obtain an upper bound (in P) for the totally ordered set T. By Zorn's lemma, there exists a maximal element in P. Replacing u by this maximal element, we may assume that $u\colon B \to M$ is maximal.

We want to show that $B = A$. Suppose not. Since U is a generator, there exists a morphism $j\colon U \to A$ that does not come from a morphism $U \to B$. Consider the exact sequence

$$0 \longrightarrow V \xrightarrow{\varphi'} U \oplus B \xrightarrow{(j\ \mathrm{id})} A.$$

Note that $V \to U \oplus B \to U$ is a monomorphism because of the snake lemma [KS2006, Lemma 12.1.1] applied to the commutative diagram

$$\begin{tikzcd} & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[dddll, rounded corners] & \\ & 0 \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & V \arrow[r] & U \oplus B \arrow[r] & A & \\ & V \arrow[r] & U \arrow[r] & A/B. & \end{tikzcd}$$

Let

$$\begin{aligned} B' &= \operatorname{im}(U \oplus B \to A) \\ &\coloneqq \ker\Bigl(A \to \operatorname{coker}(U \oplus B \to A)\Bigr) \lhook\joinrel\longrightarrow A, \end{aligned}$$

which is a subobject of A strictly containing B because the diagram

$$\begin{tikzcd} U \oplus B \arrow[r, "(j\ \mathrm{id})"] & A \\ B \arrow[u, hook, "\iota_2"] \arrow[ur, hook] \end{tikzcd}$$

commutes and $j\colon U \to A$ does not come from a morphism $U \to B$. By the assumption that $\mathscr{A}$ is Abelian, we know that

$$\begin{aligned} B' &\cong \operatorname{coker}\Bigl(\ker(U \oplus B \to A) \to U \oplus B\Bigr) \\ &= \operatorname{coker}(V \to U \oplus B). \end{aligned}$$

We therefore have the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r, "\varphi'"] & U \oplus B \arrow[r, "\varphi"] \arrow[d, dashed, "w"'] & B' \arrow[r] \arrow[dl, dashed, "v"] & 0 \\ & & M. \end{tikzcd}$$

By the universal property of cokernels, to define a morphism $v\colon B' \to M$, it suffices to define a morphism $w\colon U \oplus B \to M$ such that $w \circ \varphi' = 0$.

Let k be an extension to U of $u \circ \pi_2 \circ \varphi'$:

$$\begin{tikzcd} V \arrow[r, hook, "\varphi'"] \arrow[dd, hook] & U \oplus B \arrow[d, "\pi_2"] \\ & B \arrow[d, "u"] \\ U \arrow[r, dashed, "k", "\exists"'] & M \end{tikzcd}$$

which exists by hypothesis. Let w be the morphism $(-k, u)\colon U \oplus B \to M$. By the commutative diagram above, the two compositions $V \to M$ through k and u are equal, and hence $w \circ \varphi' = 0$. By the previous paragraph, we have an extension $v\colon B' \to M$ of u such that $B \hookrightarrow B'$ is not an isomorphism, contradicting the maximality of u. $\square$

We can now prove Grothendieck's Theorem 1.3.46. A key technique in the proof is *transfinite induction* [Sup1972, pp. 195–197; Jec2003, Theorem 2.14].

Proof of Theorem 1.3.46. Let A be an object of $\mathscr{A}$. Consider the short exact sequence

$$0 \longrightarrow \bigoplus_{V \hookrightarrow U} \bigoplus_{u \in \operatorname{Hom}(V,A)} V \xrightarrow{\begin{pmatrix} \sum u \\ -\bigoplus(V \hookrightarrow U) \end{pmatrix}} A \oplus \bigoplus_{V \hookrightarrow U} \bigoplus_{u \in \operatorname{Hom}(V,A)} U \longrightarrow M_1(A) \longrightarrow 0 \tag{1.3.50}$$

where the first morphism is a monomorphism because of the factorization

$$\begin{tikzcd} \bigoplus_{V \hookrightarrow U} \bigoplus_{u \in \operatorname{Hom}(V,A)} V \arrow[r, "\begin{pmatrix} \sum u \\ -\bigoplus(V \hookrightarrow U) \end{pmatrix}"] \arrow[dr, hook, "-\bigoplus(V \hookrightarrow U)"'] & A \oplus \bigoplus_{V \hookrightarrow U} \bigoplus_{u \in \operatorname{Hom}(V,A)} U \arrow[d, "\pi_2"] \\ & \bigoplus_{V \hookrightarrow U} \bigoplus_{u \in \operatorname{Hom}(V,A)} U. \end{tikzcd}$$

Note that the direct sums indexed by subobjects $V \hookrightarrow U$ are indexed by a set by Lemma 1.3.48 and that $\bigoplus(V \hookrightarrow U)$ is a monomorphism by AB5. (Infinite direct sums are filtered direct limits by taking finitely many direct summands at a time and taking the direct limit.) We denote the composition

$$f_1(A)\colon A \overset{\iota_1}{\hookrightarrow} A \oplus \bigoplus_{V \hookrightarrow U} \bigoplus_{u \in \operatorname{Hom}(V,A)} \twoheadrightarrow M_1(A)$$

by $f_1(A)$.

We claim that every morphism $u\colon V \to A$ can be extended to a morphism $U \to M_1(A)$. By definition of $M_1(A)$ as the cokernel in (1.3.50), we know that the right square in the diagram

$$\begin{tikzcd} V \arrow[r, hook, "\iota_{(V,u)}"'] \arrow[d, hook] \arrow[rr, bend left, "u"] & \bigoplus_{V \hookrightarrow U} \bigoplus_{u \in \operatorname{Hom}(V,A)} V \arrow[r, "\sum u"'] \arrow[d, hook, "\bigoplus(V \hookrightarrow U)"'] & A \arrow[d, "f_1(A)"] \\ U \arrow[r, hook, "\iota_{(V,u)}"] & \bigoplus_{V \hookrightarrow U} \bigoplus_{u \in \operatorname{Hom}(V,A)} U \arrow[r] & M_1(A). \end{tikzcd}$$

commutes. The left square commutes by definition of the direct sum. This commutative diagram shows that every morphism $u\colon V \to A$ can be extended to a morphism $U \to M_1(A)$ because the composition along the top row is u.

We want to show that $f_1(A)$ is a monomorphism. Let $K \coloneqq \ker(f_1(A))$. By the universal property of kernels, we have the dashed arrow in the commutative diagram

$$\begin{tikzcd}
 & & K \arrow[d, hook] \arrow[ddl, dashed, bend right, "\exists!"'] \arrow[ddr, bend left, "0"] & & \\
 & & A \arrow[d, hook, "\iota_1"'] \arrow[dr, "f_1(A)"] & & \\
0 \arrow[r] & \bigoplus\limits_{V\hookrightarrow U}\bigoplus\limits_{u\in\mathrm{Hom}(V,A)} V \arrow[r] \arrow[dr, hook, "-\bigoplus(V\hookrightarrow U)"'] & A \oplus \bigoplus\limits_{V\hookrightarrow U}\bigoplus\limits_{u\in\mathrm{Hom}(V,A)} U \arrow[r] \arrow[d, "\pi_2"'] & M_1(A) \arrow[r] & 0 \\
 & & \bigoplus\limits_{V\hookrightarrow U}\bigoplus\limits_{u\in\mathrm{Hom}(V,A)} U. & &
\end{tikzcd}$$

Note that the composition along the middle column is 0 by Homework 2, Problem 1(a). Since $-\bigoplus(V \hookrightarrow U)$ is a monomorphism, this shows that

$$K \longrightarrow \bigoplus_{V\hookrightarrow U}\bigoplus_{u\in\mathrm{Hom}(V,A)} V$$

is 0, and hence

$$K \longrightarrow A \oplus \bigoplus_{V\hookrightarrow U}\bigoplus_{u\in\mathrm{Hom}(V,A)} U$$

is also 0. Since ι_1 is a monomorphism, we conclude that $K \hookrightarrow A$ is 0.

We now repeat this construction transfinitely many times. Let α be an ordinal.

- (Base case) If $\alpha = 0$, we set $M_0(A) \coloneqq A$.
- (Successor ordinals) Given $M_\alpha(A)$, we set $M_{\alpha+1}(A) \coloneqq M_1(M_\alpha(A))$.
- (Limit ordinals) Let β be a limit ordinal. We set

$$M_\beta(A) \coloneqq \lim_{\alpha<\beta} M_\alpha(A).$$

By transfinite induction [Sup1972, §7.1, Theorem Schema 4; Jec2003, Theorem 2.14], we have now constructed functorial morphisms $f_\alpha(A)\colon A \to M_\alpha(A)$ for every ordinal α. The functoriality comes from the functoriality of the short exact sequence (1.3.50). The morphisms f_α are monomorphisms by AB5.

To finish, let κ be the smallest ordinal whose cardinality is strictly larger than that of the set of subobjects of U, which exists by Lemma 1.3.48. We claim that

$$f_\kappa(A)\colon A \longhookrightarrow M_\kappa(A)$$

works. We need to show that $M_\kappa(A)$ is injective. By Lemma 1.3.49, we need to show that in the diagram

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] \arrow[d, "v"'] & U \arrow[dl, dashed, "u"] \\ & M_\kappa(A) & \end{tikzcd}$$

where the top row is exact, there exists a morphism $u\colon U \to M_\kappa(A)$ making the diagram commute. In the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & v^{-1}\big(M_\alpha(A)\big) \arrow[r] \arrow[d] & V \oplus M_\alpha(A) \arrow[r, "(v\ \mathrm{id})"] \arrow[d, hook] & M_\alpha(A) \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & V \arrow[r, "\binom{-\mathrm{id}}{v}"] & V \oplus M_\kappa(A) \arrow[r, "(v\ \mathrm{id})"] & M_\kappa(A) \arrow[r] & 0 \end{tikzcd}$$

with exact rows, the direct limit of the top rows over $\alpha \leq \kappa$ is the bottom row by AB5 (which we apply to say that filtered direct limits preserve kernels). We therefore know that

$$V \xleftarrow{\sim} \lim_{\alpha \leq \kappa} v^{-1}\big(M_\alpha(A)\big).$$

Note that the left vertical map is a monomorphism by the snake lemma. See [KS2006, Lemma 12.1.1] for a proof of the snake lemma that works in an arbitrary Abelian category.

We claim that V factors through some $M_\alpha(A)$. The inverse images

$$v^{-1}\big(M_\alpha(A)\big) \hookrightarrow V$$

form an increasing chain of subobjects of V indexed by a cardinal strictly larger than the cardinality of the set of subobjects of U by Lemma 1.3.48(*iii*). By Lemma 1.3.48(*i*), this means that the sequence of subobjects $v^{-1}(M_\alpha(A)) \hookrightarrow V$ is eventually stationary, and hence must equal V for some α. Finally, the morphism $v\colon V \to M_\alpha(A)$ is extended by $u\colon U \to M_{\alpha+1}(A)$ by what we proved in the first half of this proof. □

1.3.10 The category of sheaves of modules has enough injectives

We now apply Grothendieck's Theorem 1.3.46. Our convention for rings is that rings are always commutative and have an identity element 1.

Proposition 1.3.51. *Let R be a ring. Then,* $\mathsf{Mod}(R)$ *has enough injectives.*

Proof using Theorem 1.3.46. By Theorem 1.3.46, we need to check that $\mathsf{Mod}(R)$ is a Grothendieck Abelian category: AB5 holds by Theorem 1.1.17, and R is a generator. □

We also give another proof, which has the advantage of being more concrete. Note, however, that there are many categories that one comes across where there is no obvious way to make a concrete construction like this.

Alternative proof. We first construct *one* injective module for $R = \mathbf{Z}$. We claim that $\mathbf{Q}/\mathbf{Z}$ is injective. By Baer's criterion (Lemma 1.3.49), since R is a generator, it suffices to show that in the diagram

$$\begin{tikzcd} 0 \arrow[r] & n\mathbf{Z} \arrow[r] \arrow[d] & \mathbf{Z} \arrow[dl, dashed, "\exists?"] \\ & \mathbf{Q}/\mathbf{Z} \end{tikzcd}$$

the dashed map exists and makes the diagram commute. If $\bar{a} \in \mathbf{Q}/\mathbf{Z}$ is the image of n, then mapping $1 \mapsto \overline{a/n} \in \mathbf{Q}/\mathbf{Z}$ works.

We now consider the general case. First, $\operatorname{Hom}_{\mathbf{Z}}(R, \mathbf{Q}/\mathbf{Z})$ is an injective R-module since [Wei1994, Exer. 2.3.5] [God1973, Thm. I.1.2.2]

$$\operatorname{Hom}_R\Bigl(\,\cdot\,, \operatorname{Hom}_{\mathbf{Z}}(R, \mathbf{Q}/\mathbf{Z})\Bigr) \cong \operatorname{Hom}_{\mathbf{Z}}(\,\cdot\,, \mathbf{Q}/\mathbf{Z})$$

is an exact functor, where the isomorphism holds by tensor–Hom adjunction (the version in [BouAlgI, Chapter II, §4, no. 1, Proposition 1], for example). Now consider

$$I(M) \coloneqq \prod_{\varphi \in \operatorname{Hom}_R(M, \operatorname{Hom}_{\mathbf{Z}}(R, \mathbf{Q}/\mathbf{Z}))} \operatorname{Hom}_{\mathbf{Z}}(R, \mathbf{Q}/\mathbf{Z}).$$

This is injective since it is a direct product of injective modules. Moreover,

$$\begin{aligned} M &\longrightarrow I(M) \\ m &\longmapsto \bigl(\varphi(m)\bigr)_{\varphi} \end{aligned}$$

is injective. $\square$

Proposition 1.3.52. ([TohokuI, Proposition 3.1.1 and Corollaire]) *Let $(X, \mathcal{O}_X)$ be a ringed space. Then, $\mathsf{Mod}(\mathcal{O}_X)$ has enough injectives.*

Proof using Theorem 1.3.46. By Theorem 1.3.46, we need to check that $\mathsf{Mod}(\mathcal{O}_X)$ is a Grothendieck Abelian category: AB5 holds by Theorem 1.1.17 since exactness can be checked on stalks. To construct a generator, let

$$j_U \colon U \hookrightarrow X$$

be all possible open inclusions. We claim the object

$$\bigoplus_{U \subseteq X} (j_U)_! \mathcal{O}_U$$

is a generator. Consider a monomorphism $\mathscr{F} \hookrightarrow \mathscr{G}$ that is not an isomorphism. After replacing $\mathscr{F}$ by its image, we may assume that $\mathscr{F} \subseteq \mathscr{G}$. Let $U \subseteq X$ be an open subset such that $\mathscr{F}(U) \subsetneq \mathscr{G}(U)$. Then, we can consider a morphism $\mathscr{O}_U(U) \to \mathscr{G}(U)$ mapping $1 \in \mathscr{O}_U(U)$ to an element not in the image. We obtain a morphism $\mathscr{O}_U \to \mathscr{G}|_U$ and applying $(j_U)_!$, the composition

$$(j_U)_!\mathscr{O}_U \longrightarrow (j_U)_!\Big(\mathscr{G}|_U\Big) \longrightarrow \mathscr{G}$$

is a morphism not factoring through $\mathscr{F}$. Here, the injection on the right holds by [Har1977, Exercise II.1.19(c)] (Homework 1, Problem 4(c)). Combining this map with 0 maps for $U' \neq U$, we see that $\bigoplus_{U\subseteq X}(j_U)_!\mathscr{O}_U$ is a generator. $\square$

We can give a direct proof due to Godement [God1973, Chapitre II, Théorème 7.1.1] (see also [TohokuI, p. 156]).

[Har1977, Prop. III.2.2]

Alternative proof. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. For every $x \in X$, let $\mathscr{F}_x \hookrightarrow I_x$ be an injection into an injective $\mathscr{O}_{X,x}$-module. Denote

$$i_x\colon \Big(\{x\}, \underline{\mathscr{O}_{X,x}}\Big) \longrightarrow (X, \mathscr{O}_X)$$

and consider the sheaf

$$I(\mathscr{F}) \coloneqq \prod_{x\in X}(i_x)_*I_x.$$

We have a morphism $\mathscr{F} \to I(\mathscr{F})$ since

$$\begin{aligned}
\operatorname{Hom}_{\mathscr{O}_X}\Big(\mathscr{F}, I(\mathscr{F})\Big) &\cong \prod_{x\in X}\operatorname{Hom}_{\mathscr{O}_X}\Big(\mathscr{F}, (i_x)_*I_x\Big)\\
&\cong \prod_{x\in X}\operatorname{Hom}_{\mathscr{O}_{X,x}}(i_x^*\mathscr{F}, I_x)\\
&= \prod_{x\in X}\operatorname{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, I_x)
\end{aligned}$$

where the isomorphism hold by definition of products and by [Har1977, p. 110] (Homework 3, Problem 5), respectively. The morphisms $\mathscr{F}_x \hookrightarrow I_x$ therefore glue together to give a morphism $\mathscr{F} \to I(\mathscr{F})$, which is injective since it is injective on stalks. Finally, to show that $I(\mathscr{F})$ is injective, it suffices to show that each $(i_x)_*I_x$ is injective since direct products of injective objects are injective (direct products are exact for the category Ab of Abelian groups). But this holds since

$$\operatorname{Hom}_{\mathscr{O}_X}\Big(\,\cdot\,, (i_x)_*I_x\Big) \cong \operatorname{Hom}_{\mathscr{O}_{X,x}}\Big((i_x)^*\,\cdot\,, I_x\Big)$$

is the composition of the stalks functor, which is exact, followed by the exact functor obtained by Hom-ing into I_x. Note that here, we are considering the space $\{x\}$ with the structure sheaf $i_x^{-1}\mathscr{O}_X$, which is just the constant sheaf associated to $\mathscr{O}_{X,x}$ on $\{x\}$. $\square$

1.4 Derived functors and sheaf cohomology

1.4.1 Complexes

To define derived functors and cohomology, we need to discuss complexes.

Definition 1.4.1. ([CE1956, pp. 58–59]) Let $\mathscr{A}$ be an Abelian category. A *(cochain) complex* $A^\bullet$ in $\mathscr{A}$ is a collection of objects A^i together with a collection of *coboundary maps* $d^i\colon A^i \to A^{i+1}$ in $\mathscr{A}$, both indexed by $i \in \mathbf{Z}$, such that $d^{i+1} \circ d^i = 0$. We visualize a complex as a sequence of objects and maps: [Wei1994, pp. 2–3, p. 7] [TohokuI, §1.7, Ex. c] [Har1977, p. 203]

$$\cdots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \longrightarrow \cdots.$$

If the A^i are only specified in a certain range (e.g., $i \geq 0$), then we set $A^i = 0$ for the other i. A *morphism of complexes* $f\colon A^\bullet \to B^\bullet$ is a collection of morphisms $f^i\colon A^i \to B^i$ for each i that commute with the coboundary maps d^i, i.e., such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "d_A^{i-2}"] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, "f^{i-1}"] & A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"] & A^{i+1} \arrow[r, "d_A^{i+1}"] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r, "d_B^{i-2}"] & B^{i-1} \arrow[r, "d_B^{i-1}"] & B^i \arrow[r, "d_B^i"] & B^{i+1} \arrow[r, "d_B^{i+1}"] & \cdots.
\end{tikzcd} \tag{1.4.2}$$

Complexes form a category $\mathsf{Ch}(\mathscr{A})$.

Remark 1.4.3. By a *complex* in this course, we will mean cochain complexes. This is the "algebraic" or "cohomological" convention—indices on complexes go *up* as you compose maps in a complex. In topology, the opposite, "homological" convention with *chain complexes* is often used—indices go *down* as you compose maps in a complex. Traditionally, chain complexes are denoted with subscripts $A_\bullet$.

Proposition 1.4.4. *Let $\mathscr{A}$ be an additive category. Then, $\mathsf{Ch}(\mathscr{A})$ is an additive category. If $\mathscr{A}$ satisfies* AB*1,* AB*2,* AB3*, or* AB5*, then $\mathsf{Ch}(\mathscr{A})$ satisfies the same axiom.* [TohokuI, Prop. 1.6.1]

Proof idea. Compute kernels, cokernels, and coproducts termwise. Check they satisfy the desired universal properties. For AB5, the fact that kernels and cokernels are computed termwise means that exactness can be checked termwise. □

As a result, we can make sense of exact sequences of complexes.

The following is an intermediate definition. Ultimately, we will define sheaf cohomology as the i-th cohomology module of a certain complex obtained from a sheaf.

[Wei1994, p. 3, p. 7]
[Har1977, p. 203]
I prefer to use a boldface $\mathbf{h}^i$ instead of Hartshorne's notation h^i because the latter is often used to denote the k-vector space dimension of the sheaf cohomology of a projective variety over k.

Definition 1.4.5. ([CE1956, pp. 53–54, 58–59]) Let $\mathscr{A}$ be an Abelian category. The *i-th cocycles* of a complex $A^\bullet$ are

$$Z^i(A^\bullet) \coloneqq \ker(d_A^i)$$

and the *i-th coboundaries* of $A^\bullet$ are

$$B^i(A^\bullet) \coloneqq \operatorname{im}(d_A^{i-1}).$$

The *i-th cohomology object* of $A^\bullet$ is

$$\mathbf{h}^i(A^\bullet) \coloneqq \frac{Z^i(A^\bullet)}{B^i(A^\bullet)}.$$

(The way to remember the indices is that $\mathbf{h}^i(A^\bullet)$ is a quotient of a subobject of A^i.)

If $f\colon A^\bullet \to B^\bullet$ is a morphism of complexes, then f induces a map

$$\mathbf{h}^i(f)\colon \mathbf{h}^i(A^\bullet) \longrightarrow \mathbf{h}^i(B^\bullet)$$

by the commutativity of Definition 1.4.2. We therefore obtain an additive functor

$$\mathbf{h}^i\colon \mathsf{Ch}(\mathscr{A}) \longrightarrow \mathscr{A}.$$

[Wei1994, Exer. 1.1.2]

Explicitly, the map is obtained as follows. First, we look at the following square in Definition 1.4.2:

$$\begin{tikzcd} A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"'] & A^{i+1} \arrow[d, "f^{i+1}"] \\ B^i \arrow[r, "d_B^i"] & B^{i+1}. \end{tikzcd}$$

We now consider the kernels of the d^i to obtain the commutative diagram of solid maps below:

$$\begin{tikzcd} Z^i(A^\bullet) \arrow[r] \arrow[d, dashed, "\exists!"'] & A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"'] & A^{i+1} \arrow[d, "f^{i+1}"] \\ Z^i(B^\bullet) \arrow[r] & B^i \arrow[r, "d_B^i"] & B^{i+1}. \end{tikzcd}$$

By definition of $Z^i(A^\bullet) \coloneqq \ker(d_A^i)$, the composition $Z^i(A^\bullet) \to A^i \to A^{i+1}$ is 0. Thus, the composition $Z^i(A^\bullet) \to B^{i+1}$ is 0. By the universal property of $Z^i(B^\bullet) \coloneqq \ker(d_B^i)$, there is a unique dashed map $Z^i(A^\bullet) \to Z^i(B^\bullet)$ making the diagram commute. Now the fact that $A^\bullet$ and $B^\bullet$ are complexes means that the maps $B^i(A^\bullet) \to A^i$ and $B^i(B^\bullet) \to B^i$ factor through $A^i(A^\bullet)$ and $Z^i(B^\bullet)$, i.e., we can enlarge the commutative diagram to get the commutative diagram of solid maps below:

$$\begin{tikzcd} A^{i-1} \arrow[r] \arrow[d, "f^{i-1}"'] & B^i(A^\bullet) \arrow[r] \arrow[d, dashed, "\exists!"'] & Z^i(A^\bullet) \arrow[r] \arrow[d] & A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"'] & A^{i+1} \arrow[d, "f^{i+1}"] \\ B^{i-1} \arrow[r] & B^i(B^\bullet) \arrow[r] & Z^i(B^\bullet) \arrow[r] & B^i \arrow[r, "d_B^i"] & B^{i+1}. \end{tikzcd}$$

By the universal property of $B^i(A^\bullet) \coloneqq \operatorname{im}(d_A^{i-1})$, there is a unique dashed map $B^i(A^\bullet) \to B^i(B^\bullet)$ making the diagram commute. Finally, by the universal property of

$$\mathbf{h}^i(A^\bullet) \coloneqq \operatorname{coker}\Big(B^i(A^\bullet) \to Z^i(A^\bullet)\Big),$$

we obtain the map $\mathbf{h}^i(f)$. The fact that this defines a functor can be seen by imagining running through this argument with *three* rows in each commutative diagram for a composition of morphisms

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet.$$

Using universal properties, the maps induced by $g \circ f$ on cocycles, coboundaries, and cohomology objects coincide with the maps obtained by composing those obtained for g and f. Thus,

$$\mathbf{h}^i(g \circ f) = \mathbf{h}^i(g) \circ \mathbf{h}^i(f).$$

Lemma 1.4.6. **(**[CE1956, Chapter IV, Theorem 1.1]**)** *Let $\mathscr{A}$ be an Abelian category and consider a short exact sequence* [Har1977, p. 203] [Wei1994, Thm. 1.3.1]

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

in $\mathsf{Ch}(\mathscr{A})$. Then, there are maps $\delta^i \colon \mathbf{h}^i(C^\bullet) \to \mathbf{h}^{i+1}(A^\bullet)$ which yield a long exact sequence

$$\cdots \longrightarrow \mathbf{h}^{i-1}(C^\bullet) \xrightarrow{\delta^{i-1}} \mathbf{h}^i(A^\bullet) \longrightarrow \mathbf{h}^i(B^\bullet) \longrightarrow \mathbf{h}^i(C^\bullet) \xrightarrow{\delta^i} \mathbf{h}^{i+1}(A^\bullet) \longrightarrow \cdots$$

natural in $A^\bullet, B^\bullet, C^\bullet$.

We use the snake lemma below. See [KS2006, Lemma 12.1.1] for a proof of the snake lemma that works in an arbitrary Abelian category.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Z^i(A^\bullet) \arrow[r] \arrow[d] & Z^i(B^\bullet) \arrow[r] \arrow[d] & Z^i(C^\bullet) \arrow[d] & \\
0 \arrow[r] & A^i \arrow[r] \arrow[d, "d"'] & B^i \arrow[r] \arrow[d, "d"'] & C^i \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & A^{i+1} \arrow[r] \arrow[d] & B^{i+1} \arrow[r] \arrow[d] & C^{i+1} \arrow[r] \arrow[d] & 0 \\
 & \dfrac{A^{i+1}}{dA^i} \arrow[r] \arrow[d] & \dfrac{B^{i+1}}{dB^i} \arrow[r] \arrow[d] & \dfrac{C^{i+1}}{dC^i} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

The snake lemma implies that the top and bottom rows are exact. The rows in the commutative diagram

$$\begin{tikzcd}
 & \dfrac{A^i}{dA^{i-1}} \arrow[r] \arrow[d, "d"'] & \dfrac{B^i}{dB^{i-1}} \arrow[r] \arrow[d, "d"'] & \dfrac{C^i}{dC^{i-1}} \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & Z^{i+1}(A^\bullet) \arrow[r, "f"] & Z^{i+1}(B^\bullet) \arrow[r, "g"] & Z^{i+1}(C^\bullet) &
\end{tikzcd}$$

are therefore exact. Applying the snake lemma again, we obtain the commutative diagram

$$\begin{tikzcd}
 & \mathbf{h}^i(A^\bullet) \arrow[r] \arrow[d] & \mathbf{h}^i(B^\bullet) \arrow[r] \arrow[d] & \mathbf{h}^i(C^\bullet) \arrow[d] \arrow[dddll, rounded corners] & \\
 & \dfrac{A^i}{dA^{i-1}} \arrow[r] \arrow[d, "d"'] & \dfrac{B^i}{dB^{i-1}} \arrow[r] \arrow[d, "d"'] & \dfrac{C^i}{dC^{i-1}} \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & Z^{i+1}(A^\bullet) \arrow[r, "f"] \arrow[d] & Z^{i+1}(B^\bullet) \arrow[r, "g"] \arrow[d] & Z^{i+1}(C^\bullet) \arrow[d] & \\
 & \mathbf{h}^{i+1}(A^\bullet) \arrow[r] & \mathbf{h}^{i+1}(B^\bullet) \arrow[r] & \mathbf{h}^{i+1}(C^\bullet). &
\end{tikzcd}$$

The snake map in the diagram is the morphism δ^i. The long exact sequence is natural in $A^\bullet, B^\bullet, C^\bullet$ because we can perform the same proof with two copies of each commutative diagram connected to each other via the maps induced by $A^\bullet \to A'^\bullet$, $B^\bullet \to B'^\bullet$, $C^\bullet \to C'^\bullet$, respectively. ☐

For later use, we also define the following:

[Har1977, p. 203]
[Wei1994, Def. 1.4.4]

Definition 1.4.7. **(**[CE1956, p. 59]**)** Let $\mathscr{A}$ be an Abelian category. Two morphisms of complexes $f, g\colon A^\bullet \to B^\bullet$ are *homotopic* (written $f \sim g$) if there is a collection of morphisms $k^i\colon A^i \to B^{i-1}$ for each i (which need not commute with the d^i) such that

$$f^i - g^i = d_B^{i-1} \circ k^i + k^{i+1} \circ d_A^i$$

for all i, which we abbreviate by

$$f - g = dk + kd.$$

The collection of morphisms $k = (k^i)_{i \in \mathbf{Z}}$ is called a *homotopy operator*. We visualize the homotopy operator as diagonal maps in the (not commutative) diagram

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r] \arrow[d] & A^i \arrow[r] \arrow[d] \arrow[dl, "k^i"'] & A^{i+1} \arrow[r] \arrow[d] \arrow[dl, "k^{i+1}"'] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r] & B^i \arrow[r] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

Lemma 1.4.8. ([CE1956, Chapter V, Proposition 1.1a]) *Let $\mathscr{A}$ be an Abelian category. Consider two morphisms* [Wei1994, Lem. 1.4.5]

$$f, g\colon A^\bullet \longrightarrow B^\bullet.$$

If $f \sim g$, then f and g induce the same *morphism $\mathbf{h}^i(A^\bullet) \to \mathbf{h}^i(B^\bullet)$ on cohomology objects for every i.*

Proof. Since taking cohomology is an additive functor, it suffices to show that if $f - g \sim 0$, then $\mathbf{h}^i(A^\bullet) \to \mathbf{h}^i(B^\bullet)$ is the zero map. We change notation and replace $f - g$ with f.

We consider the following commutative diagram defining map $\mathbf{h}^i(f)$:

$$\begin{tikzcd}
A^{i-1} \arrow[r] \arrow[d, "f^{i-1}"'] & B^i(A^\bullet) \arrow[r] \arrow[d] & Z^i(A^\bullet) \arrow[r] \arrow[d] & A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"'] & A^{i+1} \arrow[d, "f^{i+1}"] \\
B^{i-1} \arrow[r] & B^i(B^\bullet) \arrow[r] \arrow[dr, "0"'] & Z^i(B^\bullet) \arrow[r] \arrow[d] & B^i \arrow[r, "d_B^i"] & B^{i+1} \\
& & \mathbf{h}^i(B^\bullet) & &
\end{tikzcd}$$

On $Z^i(A^\bullet)$, $d^{i-1}k^i + k^{i+1}d^i$ induces the same map as $d^{i-1}k^i$. Post-composing the map $d^{i-1}k^i$ with the map $Z^i(B^\bullet) \to \mathbf{h}^i(B^\bullet)$ yields the zero map since $Z^i(B^\bullet) \to \mathbf{h}^i(B^\bullet)$ kills the image of d^{i-1}. $\square$

1.4.2 Right derived functors and the definition of sheaf cohomology

We are now ready to define right derived functors.

Definition 1.4.9. ([CE1956, p. 78]) Let $\mathscr{A}$ be an Abelian category. A *right resolution* of an object A of $\mathscr{A}$ consists of a complex [TohokuI, p. 143n3]

$$C^\bullet = \left\{C^0 \longrightarrow C^1 \longrightarrow \cdots\right\}$$

indexed by the non-negative integers together with a *augmentation morphism* $\varepsilon\colon A \to C^\bullet$ such that

$$0 \longrightarrow A \xrightarrow{\varepsilon} C^0 \longrightarrow C^1 \longrightarrow \cdots$$

is exact. An *injective resolution* of A is a right resolution such that each C^i is injective. If $\mathscr{A}$ has enough injectives, then injective resolutions always exist by applying the definition of "enough injectives" repeatedly to $\operatorname{coker}(\varepsilon)$ and

$\operatorname{coker}(d^i)$:

$$\begin{tikzcd}
 & & & \operatorname{coker}(\varepsilon) \arrow[dr, hook] & & & \\
0 \arrow[r] & A \arrow[r, "\varepsilon"] & I^0 \arrow[ur, two heads] \arrow[rr, "d^0"] & & I^1 \arrow[rr, "d^1"] \arrow[dr, two heads] & & I^2 \arrow[r] & \cdots \\
 & & & & & \operatorname{coker}(d^0). \arrow[ur, hook] & &
\end{tikzcd}$$

If $\mathscr{A}$ is Grothendieck Abelian, the construction of injective resolutions is *functorial* by Theorem 1.3.46.

$\boxed{2/3}$

If $\mathscr{A}$ is a Grothendieck Abelian category, then given any morphism $A \to B$, we can construct a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & I^0 \arrow[r] \arrow[d] & I^1 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & B \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & \cdots
\end{tikzcd}$$

where each line is an injective resolution. Such a diagram exists more generally.

[Wei1994, Thm. 2.2.6 and Thm. 2.3.7]

Theorem 1.4.10. ([CE1956, Chapter V, Proposition 1.2*a*]) *Let $\mathscr{A}$ be an Abelian category. Let $0 \to A \to I^\bullet$ be a resolution and let $0 \to B \to J^\bullet$ be a chain complex such that the J^i are injective. Then, for every morphism $f' : A \to B$, we can construct a morphism $f \colon I^\bullet \to J^\bullet$ of chain complexes lifting f', i.e., we can find a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\varepsilon"] \arrow[d, "f'"'] & I^0 \arrow[r, "d_A^0"] \arrow[d, "f^0"'] & I^1 \arrow[r, "d_A^1"] \arrow[d, "f^1"'] & \cdots \\
0 \arrow[r] & B \arrow[r, "\eta"] & J^0 \arrow[r, "d_B^0"] & J^1 \arrow[r, "d_B^1"] & \cdots .
\end{tikzcd} \tag{1.4.11}$$

Moreover, the morphism $f^\bullet$ is unique up to homotopy equivalence.

Proof. We construct each f^i by induction on i. We denote $I^{-1} = A$, $J^{-1} = B$, and $f^{-1} = f' : A \to B$. Suppose that f^j is constructed for every $j \leq i$. By the universal property of cokernels, we have the commutative diagram

$$\begin{tikzcd}
I^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, "f^{i-1}"'] & I^i \arrow[r] \arrow[d, "f^i"'] & \operatorname{coker}(d_A^{i-1}) \arrow[r] \arrow[d, dashed, "\exists!"] & 0 \\
J^{i-1} \arrow[r, "d_B^{i-1}"] & J^i \arrow[r] & J^{i+1} &
\end{tikzcd}$$

with exact rows. Since $0 \to A \to I^\bullet$ is a resolution, the cokernel of d_A^{i-1} is the kernel of d_A^{i+1}. We therefore see that the top row in the diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{coker}(d_A^{i-1}) \arrow[r] \arrow[dr] & I^{i+1} \arrow[d, dashed, "\exists"', "f^{i+1}"] \\ & & J^{i+1} \end{tikzcd}$$

is exact. Since J^{i+1} is injective, we obtain the dashed map.

To show that f is unique up to homotopy equivalence, suppose there exists another chain map $g\colon I^\bullet \to J^\bullet$ that makes the diagram commute. We will show that $f - g \sim 0$ by constructing morphisms $k^i\colon A^i \to B^{i-1}$ such that

$$(f^j - g^j) - d_B^{j-1} \circ k^j = k^{j+1} \circ d_A^j$$

for all $j < i$ by induction on $i \geq 0$. For the base case, we set $k^0 = 0$. Then,

$$(f^{-1} - g^{-1}) - d_B^{-1} \circ k^{-1} = 0 = k^0 \circ d_A^{-1}$$

because $f^{-1} - g^{-1} = f' - f' = 0$ and $d_A^{-1} = d_B^{-1} = 0$. For the inductive case, consider the morphism $(f^i - g^i) - d_B^{i-1} \circ k^i\colon I^i \to J^i$. Then,

$$\begin{aligned}
&\Big((f^i - g^i) - d_B^{i-1} \circ k^i\Big) \circ d_A^{i-1} \\
&\quad = (f^i - g^i) \circ d_A^{i-1} - d_B^{i-1} \circ k^i \circ d_A^{i-1} \\
&\quad = (f^i - g^i) \circ d_A^{i-1} - d_B^{i-1} \circ \Big((f^{i-1} - g^{i-1}) - d_B^{j-2} \circ k^{i-1}\Big) \\
&\quad = (f^i - g^i) \circ d_A^{i-1} - d_B^{i-1} \circ (f^{i-1} - g^{i-1}) \\
&\quad = 0.
\end{aligned}$$

By the universal property of cokernels, there is a factorization of $(f^i - g^i) - d_B^{i-1} \circ k^i$ through $\operatorname{coker}(d_A^{i-1})$:

$$\begin{tikzcd} I^i \arrow[r] \arrow[d, "(f^i - g^i) - d_B^{i-1} \circ k^i"'] & \operatorname{coker}(d_A^{i-1}) \arrow[r, hook] \arrow[dl, dashed, "\exists!"] & I^{i+1} \arrow[dll, dashed, bend left, "\exists"', "k^{i+1}"] \\ J^i & & \end{tikzcd}$$

where the monomorphism $\operatorname{coker}(d_A^{i-1}) \hookrightarrow I^{i+1}$ holds by the fact that $0 \to A \to I^\bullet$ is a resolution. Since J^i is injective, we obtain the dashed map k^{i+1}, which satisfies

$$(f^i - g^i) - d_B^{i-1} \circ k^i = k^{i+1} \circ d_A^i. \quad \square$$

Remark 1.4.12. This theorem implies any two injective resolutions are homotopy equivalent: Given two injective resolutions $0 \to A \to I^\bullet$ and $0 \to A \to J^\bullet$, there are morphisms

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & \cdots
\end{array}$$

such that the composition from the first row to the third row and from the second row to the fourth row are homotopy equivalent to the identity.

Definition 1.4.13. ([TohokuI, §2.3]) Let $F\colon \mathscr{A} \to \mathscr{A}'$ be an additive (covariant) functor between two Abelian categories. Suppose that $\mathscr{A}$ has enough injectives. For every object A in $\mathscr{A}$, the *right derived functors* of F are defined by setting

$$R^iF(A) := \mathbf{h}^i\Big(F(I^\bullet)\Big)$$

for an injective resolution $A \to I^\bullet$. The action of R^iF on morphisms is determined by applying F to the diagram (1.4.11) after deleting A and B.

As a definition, it is not clear $R^iF(A)$ is well-defined: Why is the definition independent of the injective resolution chosen? There are of course more things we want to show about R^iF. For now, let us state the two main examples we will study in this course.

Definition 1.4.14. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathscr{F}$ be an $\mathcal{O}_X$-module.

[TohokuI, p. 157] [Har1977, p. 207, Exer. III.2.3]

(*i*) Consider the global sections functor with support in a closed subset $Z \subseteq X$

$$\Gamma_Z(X, \cdot\,)\colon\ \mathsf{Mod}(\mathcal{O}_X) \longrightarrow \mathsf{Mod}\Big(\mathcal{O}_X(X)\Big)$$

defined by $\Gamma_Z(X, \cdot\,) := \Gamma(X, \mathscr{H}^0_Z(\,\cdot\,))$. The *i-th sheaf cohomology group of $\mathscr{F}$ with support in Z* is

$$H^i_Z(X, \mathscr{F}) := (R^i\Gamma_Z)(\mathscr{F}).$$

If $Z = X$, we call this module the *i-th sheaf cohomology group of $\mathscr{F}$* and write $H^i(X, \mathscr{F})$. These modules are also called the *i-th local cohomology groups of $\mathscr{F}$ with support in Z.*

[TohokuII, §4.2] [Har1977, p. 233]

(*ii*) The functors $\mathrm{Ext}^i(\mathscr{F}, \cdot\,)$ are the right derived functors of $\mathrm{Hom}(\mathscr{F}, \cdot\,)$ and the functors $\mathscr{E}xt^i(\mathscr{F}, \cdot\,)$ are the right derived functors of $\mathscr{H}om(\mathscr{F}, \cdot\,)$. We will discuss these more later.

Here is the main existence result.

Theorem 1.4.15. ([CE1956, Chapter V, Propositions 2.3, 4.3, and 5.1]) *Let $F\colon \mathscr{A} \to \mathscr{A}'$ be an additive functor between Abelian categories such that $\mathscr{A}$ has enough injectives.* [Har1977, Thm. III.1.1A] [Wei1994, Thm. 2.4.6]

(*a*) *For every $i \geq 0$, R^iF as defined above is an additive functor $\mathscr{A} \to \mathscr{A}'$. Furthermore, it is independent (up to isomorphism of functors) of the choices of injective resolutions made.* [Wei1994, Lem. 2.4.1]

(*b*) *If F is left exact, there is an isomorphism of functors $F \cong R^0F$.*

(*c*) *For every short exact sequence $0 \to A' \to A \to A'' \to 0$ and for every $i \geq 0$, there is a morphism $\delta^i\colon R^iF(A'') \to R^{i+1}F(A')$ natural in the short exact sequence fitting in the long exact sequence*

$$\cdots \longrightarrow R^iF(A') \longrightarrow R^iF(A) \longrightarrow R^iF(A'') \xrightarrow{\delta^i} R^{i+1}F(A') \longrightarrow R^{i+1}F(A) \longrightarrow \cdots.$$

(*d*) *For every commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d, "f'"'] & A \arrow[r] \arrow[d, "f"'] & A'' \arrow[r] \arrow[d, "f''"'] & 0 \\
0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0
\end{tikzcd}$$

with exact rows, the δ's fit into the commutative diagrams

$$\begin{tikzcd}
R^iF(A'') \arrow[r, "\delta^i"] \arrow[d] & R^{i+1}F(A') \arrow[d] \\
R^iF(B'') \arrow[r, "\delta^i"] & R^{i+1}F(B').
\end{tikzcd}$$

(*e*) *For every injective object I of $\mathscr{A}$ and for every $i > 0$, we have $R^iF(I) = 0$.*

Proof. For (*a*), we compare two injective resolutions $I^\bullet, J^\bullet$ using Theorem 1.4.10 for $A = B$ to obtain a chain map $f\colon I^\bullet \to J^\bullet$. Using Theorem 1.4.10 again, we can find a chain map $g\colon J^\bullet \to I^\bullet$. The compositions $f \circ g$ and $g \circ f$ are both homotopic to the identity by Theorem 1.4.10, and hence $R^iF(A)$ is a well-defined object of $\mathscr{A}'$. Applying Theorem 1.4.10 to an arbitrary morphism $f'\colon A \to B$, any chain map connecting injective resolutions of A and B are unique up to homotopy equivalence. By Lemma 1.4.8, any two homotopy equivalent chain maps connecting injective resolutions of A and B induce the same map on $\mathbf{h}^i$. This shows that the action of R^iF on morphisms is well-defined.

(*e*) follows from (*a*) by letting I be its own injective resolution. (*b*) holds by left-exactness.

(*c*) and (*d*) follow from the "Horseshoe Lemma":

[Wei1994, Lem. 2.2.8]

Lemma 1.4.16. (Horseshoe Lemma [CE1956, Chapter V, Proposition 2.2]**)** *Let $\mathscr{A}$ be an Abelian category with enough injectives. Consider a commutative diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & I'^0 \arrow[r] & I'^1 \arrow[r] & \cdots \\
0 \arrow[r] & A \arrow[d] & & & \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & I''^0 \arrow[r] & I''^1 \arrow[r] & \cdots \\
 & 0 & & &
\end{tikzcd}$$

where the column is exact and the top and bottom rows are injective resolutions. Set $I^i = I'^i \oplus I''^i$. Then, $0 \to A \to I^\bullet$ forms an injective resolution and there is a split short exact sequence

$$0 \longrightarrow I'^\bullet \xrightarrow{\iota} I^\bullet \xrightarrow{\pi} I''^\bullet \longrightarrow 0$$

of injective resolutions, where ι is induced by the insertion maps and π is induced by the projection maps for the direct sums $I^i = I'^i \oplus I''^i$.

Proof of Lemma 1.4.16. Set $I^{-2} = 0$ and $I^{-1} = A$. We construct the coboundary maps $d^i \colon I^i \to I^{i+1}$ and that the sequence

$$0 \longrightarrow \operatorname{coker}(d'^{i-1}) \longrightarrow \operatorname{coker}(d^{i-1}) \longrightarrow \operatorname{coker}(d''^{i-1}) \longrightarrow 0$$

is a short exact sequence by induction on $i \geq -2$. For the base case, we set $d^{-2} = 0$. For the inductive case, consider the commuative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \operatorname{coker}(d'^{i-1}) \arrow[r, "d'^{i-2}"] \arrow[d] & I'^{i-1} \arrow[r] \arrow[d] & \operatorname{coker}(d'^{i-2}) \arrow[d] \\
0 \arrow[r] & \operatorname{coker}(d^{i-1}) \arrow[r, "d^{i-2}"] \arrow[d] & I^{i-1} \arrow[r] \arrow[d] & \operatorname{coker}(d'^{i-2}) \arrow[d] \\
0 \arrow[r] & \operatorname{coker}(d''^{i-1}) \arrow[r, "d''^{i-2}"] \arrow[d] & I''^{i-1} \arrow[r] \arrow[d] & \operatorname{coker}(d'^{i-2}) \arrow[d] \\
 & 0 & 0 & 0
\end{tikzcd}$$

where the rows are exact and the two left columns are exact. By the snake lemma, the right column is also exact. We then consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{coker}(d'^{i-1}) \arrow[r] \arrow[d] & I'^i \\
0 \arrow[r] & \operatorname{coker}(d^{i-1}) \arrow[d] \arrow[ur, dashed, "\exists !"'] & \\
0 \arrow[r] & \operatorname{coker}(d''^{i-1}) \arrow[r] \arrow[d] & I''^i \\
 & 0. &
\end{tikzcd}$$

Since I'^i is injective, there exists a map $\operatorname{coker}(d^{i-1}) \to I'^i$ making the diagram commute. By the universal property of direct sums, we can find a dashed map making the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \operatorname{coker}(d'^{i-1}) \arrow[r] \arrow[d] & I'^i \arrow[d] \\
 & \operatorname{coker}(d^{i-1}) \arrow[r, dashed] \arrow[d] & I'^i \oplus I''^i \arrow[d] \\
0 \arrow[r] & \operatorname{coker}(d''^{i-1}) \arrow[r] \arrow[d] & I''^i \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

commute, where the middle row is exact by the snake lemma. Note that $I'^i \oplus I''^i$ is injective because it coincides with $I'^i \times I''^i$, which is injective. $\square$

Returning to the proof of Theorem 1.4.15, by Lemma 1.4.16, the sequence

$$0 \longrightarrow F(I'^{\bullet}) \longrightarrow F(I^{\bullet}) \longrightarrow F(I''^{\bullet}) \longrightarrow 0$$

is a short exact sequence. We then apply Lemma 1.4.6 to prove (c). Finally, to prove (d), we want to construct injective resolutions for all the objects involved such that

$$\begin{tikzcd}
0 \arrow[r] & I'^{\bullet} \arrow[r] \arrow[d, "F'"'] & I^{\bullet} \arrow[r] \arrow[d, "F"'] & I''^{\bullet} \arrow[r] \arrow[d, "F''"'] & 0 \\
0 \arrow[r] & J'^{\bullet} \arrow[r] & J^{\bullet} \arrow[r] & J''^{\bullet} \arrow[r] & 0
\end{tikzcd}$$

commutes, and then apply the naturality in Lemma 1.4.6. We can construct the two exact rows using the Horseshoe Lemma 1.4.16. We can construct the left and right vertical chain maps by Theorem 1.4.10.

Applying Theorem 1.4.10 again would yield a chain map $I^\bullet \to J^\bullet$ that makes the two squares commute only *up to homotopy*. See [Wei1994, Theorem 2.4.6] for a proof. □

1.4.3 Delta functors

We want to give a universal property of derived functors. To do this, we introduce a more general notion.

[Har1977, p. 205]

Definition 1.4.17. ([TohokuI, §2.1]**; cf.** [CE1956, Chapter III]**)** Let $\mathscr{A}$ be an Abelian category and let $\mathscr{A}'$ be an additive category. Let $a, b \in \mathbf{Z} \cup \{\pm\infty\}$ such that $a + 1 < b$. A *(covariant) δ-functor $T\colon \mathscr{A} \to \mathscr{A}'$ defined in degrees $a < i < b$* is a sequence of covariant additive functors

$$T = \Bigl(T^i : \mathscr{A} \longrightarrow \mathscr{A}'\Bigr)_{i \in (a,b)}$$

and *connecting morphisms*

$$\delta^i : T^i(A'') \longrightarrow T^{i+1}(A')$$

for each short exact sequence $0 \to A' \to A \to A'' \to 0$ and $a < i < b - 1$ satisfying the following axioms:

(i) For every commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0 \end{tikzcd}$$

with exact rows, the corresponding diagram

$$\begin{tikzcd} T^i(A'') \arrow[r, "\delta^i"] \arrow[d] & T^{i+1}(A') \arrow[d] \\ T^i(B'') \arrow[r, "\delta^i"] & T^{i+1}(B') \end{tikzcd}$$

commutes.

(ii) For every short exact sequence $0 \to A' \to A \to A'' \to 0$, the sequence

$$\cdots \longrightarrow T^i(A') \longrightarrow T^i(A) \longrightarrow T^i(A'') \xrightarrow{\delta^i} T^{i+1}(A') \longrightarrow \cdots \qquad (1.4.18)$$

is a complex.

Suppose that $\mathscr{A}'$ is also Abelian. We say that T is *exact* if the sequence Definition 1.4.18 is exact for every short exact sequence $0 \to A' \to A \to A'' \to 0$. A *cohomological functor* is an exact δ-functor defined for all degrees.

If T and T' are two δ-functors defined for the same degrees, a *natural transformation* $f\colon T \Rightarrow T'$ is a sequence of natural transformations

$$f^i : T^i \Longrightarrow T'^i$$

such that for every short exact sequence $0 \to A' \to A \to A'' \to 0$, the diagrams

$$\begin{tikzcd} T^i(A'') \arrow[r, "\delta^i"] \arrow[d] & T^{i+1}(A') \arrow[d] \\ T'^i(A'') \arrow[r, "\delta^i"] & T'^{i+1}(A') \end{tikzcd}$$

commute for all i.

Definition 1.4.19. ([TohokuI, §2.2]) Let $\mathscr{A}, \mathscr{A}'$ be Abelian categories. Let $T = (T^i)_{0 \le i \le a}$ be a covariant δ-functor from $\mathscr{A}$ to $\mathscr{A}'$ where $a > 0$. We say that T is a *universal δ-functor* if for every δ-functor $T' = (T'^i)_i$ defined for the same degrees and every natural transformation $f^0 : T^0 \Rightarrow T'^0$, there exists a unique natural transformation $f\colon T \Rightarrow T'$ reducing to f^0 in degree 0.

[Har1977, pp. 205–206] See [McL2010, p. 368] for some discussion of the set-theoretic subtleties in this definition.

Since universal δ-functors are defined using a universal property, a universal δ-functor (if it exists!) is unique.

[TohokuI, p. 140] [Har1977, Rem. III.1.2.1]

To prove universal δ-functors exist, we use the following notion.

Definition 1.4.20. ([TohokuI, p. 141]) Let $F\colon \mathscr{A} \to \mathscr{A}'$ be an additive functor between Abelian categories. We say that F is *effaceable* if, for every object A in $\mathscr{A}$, there is a monomorphism $u\colon A \hookrightarrow M$ such that $F(u) = 0$. If $\mathscr{A}$ has enough injectives, this is equivalent to saying that $F(M) = 0$ for every injective object M.

[Har1977, p. 206]

Proposition 1.4.21. ([TohokuI, Proposition 2.2.1]) *Let $\mathscr{A}, \mathscr{A}'$ be two Abelian categories and consider an exact δ-functor*

$$T = (T^i)_{i \in [0,a)} : \mathscr{A} \longrightarrow \mathscr{A}'$$

where $a > 1$. If T^i is effaceable for every $i > 0$, then T is a universal δ-functor.

[Har1977, Thm. III.1.3A]

The converse holds if every object A in $\mathscr{A}$ admits an injective effacement [TohokuI, Théorème 2.2.2], in particular if enough injectives exist.

Proof. By induction and relabeling, it suffices to show that if (T'^0, T'^1) is a δ-functor defined in degrees $0, 1$ and f^0 is a natural transformation $T^0 \Rightarrow T'^0$, then there exists a unique natural transformation $(f^0, f^1)\colon (T^0, T^1) \Rightarrow (T'^0, T'^1)$ reducing to f^0 in degree 0. Let A be an object of $\mathscr{A}$ and consider a short exact sequence

$$0 \longrightarrow A \xrightarrow{u} M \longrightarrow A' \longrightarrow 0$$

such that $T^1(u) = 0$. Consider the commutative diagram

$$\begin{tikzcd}
T^0(M) \arrow[r] \arrow[d] & T^0(A') \arrow[r] \arrow[d, "f^0(A')"] & T^1(A) \arrow[r, "0"] \arrow[d, dashed, "f^1(A)"] & T^1(M) \\
T'^0(M) \arrow[r] & T'^0(A') \arrow[r] & T'^1(A)
\end{tikzcd} \tag{1.4.22}$$

where the first row is exact. There exists a dashed morphism making the diagram commute by the universal property of cokernels since the composition $T^0(M) \to T'^1(A)$ is 0 by the fact that the bottom row is a complex. Since the first row is exact, $T^0(A') \to T^1(A)$ is an epimorphism. Thus, the morphism $f^1(A)$ is unique.

Next, we show that $f^1(A)$ does not depend on the choice of effacement $u\colon A \hookrightarrow M$. If we have another effacement $v\colon A \hookrightarrow N$, then consider

$$v\colon A \longrightarrow M \oplus N$$

which is still an effacement by additivity of the functor T^1. We can then consider the commutative diagram

$$\begin{tikzcd}
T^0(M \oplus N) \arrow[r] \arrow[d] & T^0(A'_\oplus) \arrow[r] \arrow[d] & T^1(A) \arrow[r, "0"] \arrow[d, equal] & T^1(M \oplus N) \\
T^0(M) \arrow[r] \arrow[d] & T^0(A') \arrow[r] \arrow[d, "f^0(A')"] & T^1(A) \arrow[r, "0"] \arrow[d, "f^1(A)"] & T^1(M) \\
T'^0(M) \arrow[r] & T'^0(A') \arrow[r] & T'^1(A) \\
T'^0(M \oplus N) \arrow[r] \arrow[u] & T'^0(A'_\oplus) \arrow[r] \arrow[u] & T'^1(A). \arrow[u, equal]
\end{tikzcd}$$

The composition $T^1(A) \to T'^1(A)$ in the third column is unique by using the universal property of cokernels as before.

Finally, it remains to show that f^1 defined as above commutes with connecting morphisms. Consider a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Let $B \hookrightarrow M$ be an effacement. Then, $A \to B \to M$ is also an effacement. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, hook] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & M \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Constructing the f^1 maps as before, we have the diagram

$$\begin{tikzcd}
T^0(C) \arrow[rr] \arrow[dd, "f^0(C)"'] \arrow[dr] & & T^1(A) \arrow[dd, "f^1(A)"] \\
& T^0(C') \arrow[ur] \arrow[dd, "f^0(C')" near start] & \\
T'^0(C) \arrow[rr] \arrow[dr] & & T'^1(A) \\
& T'^0(C') \arrow[ur] &
\end{tikzcd}$$

where the front left face commutes by the functoriality of f^0, the front right face commutes by the construction of $f^1(A)$ in Equation (1.4.22), and the top and bottom faces commute by the definition of δ-functors. This shows the back face commutes. □

The following is a special case of [TohokuI, Théorème 2.2.2].

Theorem 1.4.23. ([TohokuI, Théorème 2.2.2]) *Let $F\colon \mathscr{A} \to \mathscr{A}'$ be a left exact functor between Abelian categories such that $\mathscr{A}$ has enough injectives. Then, the derived functors R^iF exist and form a universal δ-functor with $F \cong R^0F$. Conversely, if $T = (T^i)_{i\geq 0}$ is a universal δ-functor, then T^0 is left exact, and the T^i are isomorphic to R^iT^0 for all $i \geq 0$.* [Har1977, Cor. III.1.4]

Proof. The derived functors R^iF exist by Theorem 1.4.15. Since $R^iF(I) = 0$ for all $i > 0$ and $\mathscr{A}$ has enough injectives, we see that R^iF is effaceable for every $i > 0$, and hence the R^iF form a universal δ-functor.

For the converse, T^0 is left exact by the definition of δ-functors. Thus, the derived functors R^iT^0 exist by Theorem 1.4.15, and also form a universal δ-functor. Since $R^0T^0 = T^0$, we see that $R^iT^0 \cong T^i$ for all i by uniqueness. □

1.4.4 Computing derived functors using acyclic objects

One difficult aspect of our theory so far is: How can we actually compute derived functors in practice? Injective resolutions are hard to compute, so it is not easy to use them in practice.

We will explore multiple ways to compute sheaf cohomology. When we get back to talking about schemes, we will explore different conditions on schemes that make sheaf cohomology computable and satisfy finiteness properties. For now, we want to find a way to use flasque sheaves to compute sheaf cohomology. This is the definition taken in [God1973, Chapitre II, §4].

We start with a general definition.

Definition 1.4.24. ([TohokuI, Théorème 2.4.1, Remarque 3]) Consider a left exact functor $F\colon \mathscr{A} \to \mathscr{A}'$ between Abelian categories with right derived functors R^iF. An object I of $\mathscr{A}$ is *F-acyclic* if $R^iF(I) = 0$ for all $i > 0$. [Har1977, p. 205] [Wei1994, 2.4.3]

[Har1977, Prop. III.1.2A]
[Wei1994, Exer. 2.4.3]

Proposition 1.4.25. ([TohokuI, Théorème 2.4.1, Remarque 3]) *Consider a left exact functor $F\colon \mathscr{A} \to \mathscr{A}'$ between Abelian categories with right derived functors R^iF. Suppose there is a right resolution*

$$0 \longrightarrow A \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots \tag{1.4.26}$$

where each J^i is F-acyclic for all $i \geq 0$. For every $i \geq 0$, there is an isomorphism

$$R^iF(A) \cong \mathbf{h}^i\Big(F(J^\bullet)\Big).$$

A resolution of the form Proposition 1.4.26 is called an *F-acyclic resolution*. The proof method below is called "dimension shifting" in [Wei1994, Exercise 2.4.3].

Proof. We proceed by induction on i (where A is allowed to vary!). For $i = 0$, we consider the exact sequence

$$0 \longrightarrow A \longrightarrow J^0 \longrightarrow J^1.$$

Applying F, we obtain

$$0 \longrightarrow F(A) \longrightarrow F(J^0) \longrightarrow F(J^1).$$

We therefore see that $F(A)$ is isomorphic to $\mathbf{h}^0(F(J^\bullet))$.

For $i > 0$, we consider the exact sequence

$$0 \longrightarrow A \longrightarrow J^0 \longrightarrow A' \longrightarrow 0.$$

Note that $0 \to A' \to J^1 \to \cdots$ is an F-acyclic resolution. Looking at the long exact sequence for R^iF, we obtain the isomorphisms

$$\mathbf{h}^{i-1}\Big(F(J^1 \to J^2 \to \cdots)\Big) \cong R^{i-1}F(A') \xrightarrow{\sim} R^iF(A)$$

for every $i \geq 1$, where the isomorphism on the left holds by inductive hypothesis. We are now done since

$$\mathbf{h}^{i-1}\Big(F(J^1 \to J^2 \to \cdots)\Big) = \mathbf{h}^i\Big(F(J^0 \to J^1 \to J^2 \to \cdots)\Big). \quad \square$$

1.4.5 Flasque sheaves are acyclic

Our goal now is to show that flasque sheaves are acyclic for the global sections functor $\Gamma(X, \cdot)$.

[Har1977, Lem. III.2.4]

Lemma 1.4.27. ([God1973, Chapitre II, Lemme 7.3.2]) *Let $(X, \mathcal{O}_X)$ be a ringed space. Then, every injective $\mathcal{O}_X$-module $\mathscr{I}$ is flasque.*

Proof. Let $\mathscr{I}$ be an injective $\mathcal{O}_X$-module and let $V \subseteq U$ be open sets. We then have the inclusion

$$0 \longrightarrow (j_V)_! \mathcal{O}_V \longrightarrow (j_U)_! \mathcal{O}_U$$

of sheaves of $\mathcal{O}_X$-modules, where $j_V \colon V \hookrightarrow X$ and $j_U \colon U \hookrightarrow X$ are the respective inclusion maps. Applying $\operatorname{Hom}(\,\cdot\,, \mathscr{I})$, we obtain the surjection

$$\operatorname{Hom}\Big((j_U)_! \mathcal{O}_U, \mathscr{I}\Big) \longrightarrow \operatorname{Hom}\Big((j_V)_! \mathcal{O}_V, \mathscr{I}\Big) \longrightarrow 0.$$

This surjection is isomorphic to $\mathscr{I}(U) \to \mathscr{I}(V)$, and hence $\mathscr{I}$ is flasque. □

We use dimension shifting again to show the following:

Proposition 1.4.28. ([TohokuI, Proposition 3.3.2, Corollaire]) *Let $(X, \mathcal{O}_X)$ be a ringed space. Suppose $\mathscr{F}$ is a flasque $\mathcal{O}_X$-module. Then, $H^i(X, \mathscr{F}) = 0$ for all $i > 0$.* [Har1977, Prop. III.2.5]

Proof. We induce on i (where $\mathscr{F}$ is allowed to vary). Since $\mathsf{Mod}(\mathcal{O}_X X)$ has enough injectives, we can find an injective $\mathcal{O}_X$-module $\mathscr{I}$ and a short exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I} \longrightarrow \mathscr{G} \longrightarrow 0.$$

Then, $\mathscr{F}$ is flasque by hypothesis, $\mathscr{I}$ is flasque by Lemma 1.4.27, and $\mathscr{G}$ is flasque by [Har1977, Exercise II.1.16(c)] (Homework 2, Problem 3(c)).

For the case $i = 1$, since $\mathscr{F}$ is flasque, we have the short exact sequence

$$0 \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{I}) \longrightarrow \Gamma(X, \mathscr{G}) \longrightarrow 0$$

by [Har1977, Exercise II.1.16(b)] (Homework 2, Problem 3(b)). Since $\mathscr{I}$ is injective, we have $H^i(X, \mathscr{I}) = 0$ for all $i > 0$. Looking at the associated long exact sequence, the sequence

$$\Gamma(X, \mathscr{I}) \longrightarrow \Gamma(X, \mathscr{G}) \xrightarrow{0} H^1(X, \mathscr{F}) \longrightarrow H^1(X, \mathscr{I}) = 0$$

is exact, and hence $H^1(X, \mathscr{F}) = 0$.

For the case $i \geq 2$, the long exact sequence yields the isomorphisms

$$H^{i-1}(X, \mathscr{G}) \xrightarrow{\sim} H^i(X, \mathscr{F})$$

for every $i \geq 2$. By the inductive hypothesis, the left-hand side is 0. □

In particular, we have the following result, which says that to compute sheaf cohomology of an $\mathcal{O}_X$-module, it does not matter whether we think of it as an Abelian sheaf or an $\mathcal{O}_X$-module.

Proposition 1.4.29. ([TohokuI, Théorème 2.4.1, Remarque 3]) *Let $(X, \mathcal{O}_X)$ be a ringed space. We have the following commutative diagram of categories and functors* [Har1977, Prop. III.2.6]

$$\begin{tikzcd}
\mathsf{Mod}(\mathcal{O}_X) \arrow[r, "R^i\Gamma(X,\,\cdot\,)"] \arrow[d, "\mathsf{Forget}"'] & \mathsf{Mod}\Big(\Gamma(X, \mathcal{O}_X)\Big) \arrow[d, "\mathsf{Forget}"] \\
\mathsf{Ab}(X) \arrow[r, "H^i(X,\,\cdot\,)"] & \mathsf{Ab}
\end{tikzcd}$$

where in the first row, $R^i\Gamma(X, \cdot)$ is computed using injective resolutions in $\mathsf{Mod}(\mathscr{O}_X)$.

As a result, there is no ambiguity when we use the notation $H^i(X, \mathscr{F})$.

Proof. We calculate the derived functors as $\mathscr{O}_X$-modules using injective resolutions in $\mathsf{Mod}(\mathscr{O}_X)$. But injective sheaves are flasque (Lemma 1.4.27), and flasque sheaves are acyclic (Proposition 1.4.28). We are done by Proposition 1.4.25. ☐

We can also compute sheaf cohomology of sheaves of the form $i_*\mathscr{F}$ where i is the inclusion of a closed set.

[Har1977, Lem. III.2.10]

Lemma 1.4.30. ([God1973, Chapitre II, Théorème 4.9.1, Corollaire; TohokuI, Théorème 3.5.1]) *Let Y be a closed subset of a topological space X with inclusion map $i\colon Y \hookrightarrow X$. Let $\mathscr{F}$ be an Abelian sheaf on Y. Then, we have isomorphisms*

$$H^i(X, i_*\mathscr{F}) \cong H^i(Y, \mathscr{F}).$$

Proof. Let $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ be an injective resolution on Y. The pushforward $0 \to i_*\mathscr{F} \to i_*\mathscr{I}^\bullet$ is a flasque resolution on X: Exactness holds by [Har1977, Exercise II.1.19(a)] and flasqueness holds by [Har1977, Exercise II.1.16(d)]. Both these exercises were solved on Homework 2. ☐

1.4.6 Grothendieck's vanishing theorem

We now come to the main application of the theory so far: Sheaf cohomology vanishes beyond the Krull dimension of a Noetherian topological space.

[Har1977, Thm. III.2.7]

Theorem 1.4.31. (Grothendieck vanishing [TohokuI, Théorème 3.6.5]**)** *Let X be a Noetherian topological space of dimension $\leq n$. Then, $H^i(X, \mathscr{F}) = 0$ for all $i > n$ and every Abelian sheaf $\mathscr{F}$ on X.*

When X is an algebraic variety and $\mathscr{F}$ is *coherent* (we will define this soon!), Theorem 1.4.31 is due to Serre. See [FAC, n° 53, Proposition 4] for the case of curves, see [FAC, n° 52, Proposition 3] for the case of quasi-projective varieties, and see [Ser1957, Théorème 2] for the case of algebraic varieties (in the sense of [FAC, n° 34, Définition]) in general.

Proving Theorem 1.4.31 requires some work. We start by proving how derived functors behave under direct limits.

[Har1977, Prop. III.2.9]

Proposition 1.4.32. ([TohokuI, Proposition 3.6.3(1)]) *Let $\mathscr{A}$ and $\mathscr{A}'$ be two Abelian categories satisfying* AB5. *Suppose that $\mathscr{A}$ admits a generator. Let $T\colon \mathscr{A} \to \mathscr{A}'$ be a left exact functor. For the right derived functors R^iT to commute with filtered direct limits, it suffices for the following to hold:*

(a) *T commutes with filtered direct limits.*

(b) *If $M = \lim M_i$ is a filtered direct limit of injective objects in $\mathscr{A}$, then M is T-acyclic.*

Proof. Let $\{A_i\}_{i\in I}$ be a filtered direct system in $\mathscr{A}$ and let A be its direct limit. We want to show that

$$\lim R^pT(A_i) \longrightarrow R^pT(A)$$

is an isomorphism.

We claim there exists a direct system $\{C_i^\bullet\}_{i\in I}$ of complexes (with values in $\mathscr{A}$) and an augmentation map $\{A_i\} \to \{C_i^\bullet\}$ such that for every $i \in I$, $A_i \to C_i^\bullet$ is an injective resolution of A_i. To show this, consider the category $\mathsf{ind}_I(\mathscr{A})$ consisting of all direct systems of objects of $\mathsf{Ab}(X)$ indexed by I. If U is a generator for $\mathscr{A}$, then

$$\bigoplus_{i\in I} \varepsilon_i(U)$$

is a generator for $\mathsf{ind}_I(\mathscr{A})$, where $\varepsilon_i(U)$ is the inductive system that is U at i and 0 elsewhere (see [TohokuI, Proposition 1.9.2]). Now by Theorem 1.3.46, we see that $\mathsf{ind}_I(\mathscr{A})$ has enough injectives. Since exactness is computed entrywise, we see that if $\{M_i\}$ is an injective object, then each M_i is injective. Now we can construct the desired direct system $\{C_i^\bullet\}_{i\in I}$ by taking an injective resolution in $\mathsf{ind}_I(\mathscr{A})$.

Now by hypothesis, each $C^j \coloneqq \lim\limits_i C_i^j$ is acyclic. AB5 ensures that $A \to C^\bullet$ is an acyclic resolution. Since T commutes with filtered direct limits by the hypothesis, the exactness of lim on $\mathsf{ind}_I(\mathscr{A}')$ implies

$$\mathbf{h}^p\Big(T(C^\bullet)\Big) = \lim_{i\in I} \mathbf{h}^p\Big(T(C_i^\bullet)\Big) = \lim_{i\in I} R^pT(A_i). \quad \square$$

In order to apply Proposition 1.4.32, we need to verify the two hypotheses. We start with $\Gamma(X, \cdot\,)$ permuting with direct limits.

Lemma 1.4.33. **(**[God1973, Chapitre II, Théorème 3.10.1]**)** *Let X be a topological space and consider a filtered direct system $\{\mathscr{F}_i\}_{i\in I}$ of sheaves of sets on X. Consider the direct limit presheaf* [Har1977, Exer. II.1.11]

$$\lim_{i\in I}{}^{\mathcal{P}} \mathscr{F}_i \coloneqq \left(U \longmapsto \lim_{i\in I} \mathscr{F}_i(U)\right)$$

and its sheafification

$$\lim_{i\in I} \mathscr{F}_i \coloneqq \left(U \longmapsto \lim_{i\in I} \mathscr{F}_i(U)\right)^{\#}.$$

Suppose that X is Noetherian. Then, the sheafification map

$$\lim_{i\in I}{}^{\mathcal{P}} \mathscr{F}_i \longrightarrow \lim_{i\in I} \mathscr{F}_i$$

is an isomorphism. In particular,

$$\lim_{i\in I} \Gamma(X, \mathscr{F}_i) \longrightarrow \Gamma\left(X, \lim_{i\in I} \mathscr{F}_i\right)$$

is an isomorphism.

Remark 1.4.34. The forgetful functors

$$\mathsf{Mod}(\mathscr{O}_X) \longrightarrow \mathsf{Ab}(X) \longrightarrow \mathsf{Sh}(X)$$

reflect isomorphisms, that is, a morphism in one of these categories is an isomorphism if it is an isomorphism after applying a forgetful functor. This is because in the category of modules or of Abelian groups, bijective morphisms are isomorphisms. Thus, Lemma 1.4.33 applies to $\mathscr{O}_X$-modules or Abelian sheaves as well.

Proof of Lemma 1.4.33. Set $\mathscr{F} \coloneqq \lim \mathscr{F}_i$. For injectivity, suppose that

$$s', s'' \in \lim_{i \in I} \mathscr{F}_i(U)$$

map to the same section in $\mathscr{F}(U)$. Since a presheaf and its sheafification have the same stalks, there exists an open covering $U = \bigcup_j U_j$ such that

$$s'|_{U_j} = s''|_{U_j}$$

for all j. Since X is Noetherian, U is quasi-compact, and hence we can take a finite subcover of the $\{U_j\}$ to assume that there are only finitely many U_j. Now let i_0 be an index for which s', s'' are represented by elements

$$s'_0, s''_0 \in \mathscr{F}_{i_0}(U).$$

For each j, we can find an index $i_j \geq i_0$ such that $s'_0|_{U_j}, s''_0|_{U_j}$ are equal after mapping to $\mathscr{F}_{i_j}(U_j)$. We then see that s', s'' are represented by equal elements in $\mathscr{F}_{\max\{i_j\}}(U)$.

For surjectivity, let $s \in \mathscr{F}(U)$. Since sheafification induces bijections on stalks (Proposition 1.2.16(i)), for each $P \in U$, there exists a section

$$s^P \in \mathscr{F}_{i_P}(U_P)$$

representing the germ $s_P \in \mathscr{F}_P$ of s at P. Replacing U_P by smaller subsets and using the definition of stalks as a direct limit, we may assume that the image of s^P in $\mathscr{F}(U_P)$ and $s|_{U_P}$ coincide. Since X is Noetherian, U is quasi-compact, and hence we can take a finite subcover of the $\{U_P\}$ to assume that there are only finitely many U_P, which we call $U_1, U_2, \dots, U_n$ with corresponding sections

$$s_1, s_2, \dots, s_n \in \mathscr{F}_j(U_j).$$

Replacing the s_j by their images in $\mathscr{F}_{\max\{j\}}(U_j)$, we can then glue the s_j to a section in $\mathscr{F}_{\max\{j\}}(U)$. This section maps to $s \in \mathscr{F}(U)$ by definition of the direct limit. ∎

As a consequence, we obtain:

Lemma 1.4.35. ([TohokuI, Lemme 3.6.4; God1973, p. 163]) *Let X be a Noetherian topological space. Then, filtered direct limits of flasque sheaves of sets on X are flasque.* [Har1977, Lem. III.2.8]

Proof. Let $V \subseteq U$ be an inclusion of open subsets of X. This follows from the commutative diagram

$$\begin{tikzcd} \lim\limits_{i\in I} \mathscr{F}_i(U) \arrow[r, "\sim"] \arrow[d, two heads] & \Bigl(\lim\limits_{i\in I} \mathscr{F}_i\Bigr)(U) \arrow[d] \\ \lim\limits_{i\in I} \mathscr{F}_i(V) \arrow[r, "\sim"] & \Bigl(\lim\limits_{i\in I} \mathscr{F}_i\Bigr)(V) \end{tikzcd}$$

where the left vertical map is surjective by the fact that each $\mathscr{F}_i$ is flasque and the horizontal maps are bijective by Lemma 1.4.33. □

Corollary 1.4.36. ([TohokuI, Proposition 3.6.3(1)]) *Let X be a Noetherian topological space. Then, $H^i(X, \cdot)$ commutes with filtered direct limits of Abelian sheaves.* [Har1977, Prop. III.2.9]

Proof. The two hypotheses in Proposition 1.4.32 hold by Lemma 1.4.33 and by Lemma 1.4.35 since injective sheaves are flasque by Lemma 1.4.27. □

We are now ready to prove Grothendieck's Vanishing Theorem 1.4.31. The proof is a good example of how lots of proofs in algebraic geometry go: One uses induction with respect to closed subsets to reduce to the irreducible case, and then limit arguments to reduce to the finitely generated case. Eventually, we will reduce to the case of considering sheaves of the form $\mathbf{Z}_U$.

Proof of Grothendieck's vanishing Theorem 1.4.31. We fix some notation. If $Y \subseteq X$ is a closed subset with complement U, denote

$$U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Y.$$

For any sheaf $\mathscr{F}$ on X, we set $\mathscr{F}_Y \coloneqq i_*(\mathscr{F}|_Y)$ and $\mathscr{F}_U \coloneqq j_!(\mathscr{F}|_U)$, in which case we have the short exact sequence

$$0 \longrightarrow \mathscr{F}_U \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}_Y \longrightarrow 0$$

by [Har1977, Exercise II.1.19(c)].

We proceed by induction on n.

Step 1. *The case $n = 0$.*

If $n = 0$, then every $\mathscr{F}$ is flasque, and hence we are done by Proposition 1.4.28.

2/5 **Step 2.** *The case $n \geq 1$. Reduction to the irreducible case.*

Let X_k be the irreducible components of X [MurAGI, Proposition 1.1.32]. Setting $\mathscr{F}_k \coloneqq \mathscr{F}_{X_k}$, we have the short exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \bigoplus_k \mathscr{F}_k \longrightarrow \mathscr{R} \longrightarrow 0,$$

where $\mathscr{R}$ has support contained in $Y = \bigcup_{k \neq l} X_k \cap X_l$, which is of dimension $\leq n-1$. Here, the injection on the left holds by considering how functions glue on a closed covering of an open subset $U \subseteq X$ and thinking about a sheaf in terms of their associated espace étalé. We then have the exact sequence

$$H^{i-1}(X, \mathscr{R}) \longrightarrow H^i(X, \mathscr{F}) \longrightarrow \bigoplus_k H^i(X, \mathscr{F}_k).$$

If $i > n$, then $i - 1 > n - 1$, and hence

$$H^{i-1}(X, \mathscr{R}) \cong H^{i-1}(Y, \mathscr{R}|_Y) = 0$$

by the inductive hypothesis, since $\mathscr{R} \cong i_*(\mathscr{R}|_Y)$ by looking at the short exact sequence

$$0 \longrightarrow \mathscr{R}_{X-Y} \longrightarrow \mathscr{R} \longrightarrow \mathscr{R}_Y \longrightarrow 0$$

and noting that $\mathscr{R}_{X-Y} = 0$. It therefore suffices to show that $H^i(X, \mathscr{F}_k) = 0$ for all $i > n$. Since

$$H^i(X, \mathscr{F}_k) \cong H^i\Big(X_k, \mathscr{F}|_{X_k}\Big)$$

and each X_k is of dimension $\leq n$, we may work one irreducible component at a time to assume that X is irreducible.

[TohokuI, Prop. 3.6.1] **Step 3.** *The case $n \geq 1$. The irreducible case.*

To simplify notation, we introduce the following:

Definition 1.4.37. ([TohokuI, p. 167]) Let $\mathscr{F}$ be an Abelian sheaf on a topological space X. Let $\{f_i \in \mathscr{F}(U_i)\}_{i \in I}$ be a family of sections on open subsets $U_i \subseteq X$. Each f_i defines a morphism $\mathbf{Z}_{U_i} \to \mathscr{F}$, and hence the family $\{f_i\}$ induces a morphism

$$\bigoplus_{i \in I} \mathbf{Z}_{U_i} \longrightarrow \mathscr{F}. \tag{1.4.38}$$

We say that the family $\{f_i\}$ is a *system of generators of $\mathscr{F}$* if the morphism Definition 1.4.38 is a surjection.

Taking $\{f_i\} = \bigcup_{U \subseteq X} \mathscr{F}(U)$, we see that every Abelian sheaf $\mathscr{F}$ is generated by some family of sections, and that every Abelian sheaf is the filtered direct limit of subsheaves generated by a *finite* system of generators. By Lemma 1.4.33, we may therefore assume that $\mathscr{F}$ is generated by a finite system of generators $\{f_i\}_{1 \leq i \leq m}$.

We now induce on the number of generators m.

Substep 3.1. The inductive step.

Let $\mathscr{F}_j$ be the subsheaf of $\mathscr{F}$ generated by $f_1, f_2, \dots, f_j$ for each $j \leq m$. The long exact sequence associated to the short exact sequence

$$0 \longrightarrow \mathscr{F}_{m-1} \longrightarrow \mathscr{F} \longrightarrow \frac{\mathscr{F}}{\mathscr{F}_{m-1}} \longrightarrow 0$$

together with the inductive hypothesis and the $m = 1$ case proves the inductive case.

Substep 3.2. The case $m = 1$.

We have the short exact sequence

$$0 \longrightarrow \mathscr{R} \longrightarrow \mathbf{Z}_U \longrightarrow \mathscr{F} \longrightarrow 0.$$

By the long exact sequence on cohomology, it suffices to show that $H^i(X, \mathbf{Z}_U) = 0$ and that $H^{i+1}(X, \mathscr{R}) = 0$ for all $i > n$. The vanishing $H^i(X, \mathbf{Z}_U) = 0$ for $i > n$ holds by considering the long exact sequence associated to

$$0 \longrightarrow \mathbf{Z}_U \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Z}_Y \longrightarrow 0$$

and induction on $n = \dim(X)$ since $\mathbf{Z}$ is flasque [Har1977, Exercise II.1.16(a)] (Homework 3, Problem 3(a)) and $\dim(Y) < n$ by the irreducibility of X. The vanishing $H^{i+1}(X, \mathscr{R}) = 0$ holds if $\mathscr{R} = 0$, and hence it remains to consider the case when $\mathscr{R} \neq 0$. Let d be the least positive integer that occurs in any of the groups $\mathscr{R}_x$ as $x \in X$ varies. Then, there is a nonempty open subset $V \subseteq U$ such that

$$\mathscr{R}|_V \cong d \cdot \mathbf{Z}|_V \subseteq \mathbf{Z}|_V.$$

We therefore see that $\mathscr{R}_V \cong \mathbf{Z}_V$ and we have a short exact sequence

$$0 \longrightarrow \mathbf{Z}_V \xrightarrow{d} \mathscr{R} \longrightarrow \frac{\mathscr{R}}{\mathbf{Z}_V} \longrightarrow 0.$$

The sheaf $\mathscr{R}/\mathbf{Z}_V$ is supported on $\overline{(U - V)} \subseteq X$, which is a closed subset of dimension $< n$ since X is irreducible. Thus, by the induction on $n = \dim(X)$ and the vanishing for $\mathbf{Z}_V$ shown above, we are done. $\square$

Chapter 2

Schemes

2.1 Schemes

We are now ready to define schemes. The idea is to associate a topological space to a commutative ring A, and use A to construct a sheaf of rings on it. The resulting object is the *spectrum* $\operatorname{Spec}(A)$. As a topological space, this is a familiar object from commutative algebra: $\operatorname{Spec}(A)$ is the set of prime ideals in A. This means we have more points than just maximal ideals, which correspond to points on the corresponding variety when A is of finite type over an algebraically closed field.

We will then glue these objects together to form a general scheme, in the same way that one glues together copies of $\mathbf{R}^n$ to build a manifold. An important class of examples is the class of schemes of the form $\operatorname{Proj}(S)$ where S is a graded ring. When S is a graded domain of finite type over an algebraically closed field k such that $S_0 = k$, these schemes will be the scheme-theoretic version of projective varieties.

2.1.1 Spectra of rings

We start with spectra of rings.

Definition 2.1.1. ([EGAI, (1.1); EGAInew, (1.1)]) Let A be a ring. As a set, the *spectrum* $\operatorname{Spec}(A)$ of A is the set of prime ideals in A. As a topological space, the closed sets in $\operatorname{Spec}(A)$ are sets of the form [Har1977, p. 70]

$$V(I) := \{\mathfrak{p} \in \operatorname{Spec}(A) \mid I \subseteq \mathfrak{p}\}$$

for every ideal $I \subseteq A$. This topology is called the *Zariski topology*.

When we think of $\operatorname{Spec}(A)$ geometrically, we sometimes denote its points by x, in which case the corresponding prime ideals are denoted by $\mathfrak{p}_x$.

Let us see some examples.

Example 2.1.2. $\operatorname{Spec}(k)$ for a field k consists of one point (0). [Mum1967, Ex. A, p. 135] [Har1977, Ex. II.2.3.1]

[Mum1967, Ex. B, p. 136]
[Har1977, Ex. II.2.3.3, Exer. II.2.10]

Example 2.1.3. (The affine line) $\mathbf{A}^1_k \coloneqq \operatorname{Spec}(k[x])$ is the *affine line over* k. This space consists of two types of points: The *generic point* $\eta \coloneqq (0)$ and closed points (f), where $f \in k[x]$ is an irreducible polynomial. If k is algebraically closed, for example when $k = \mathbf{C}$, these points are $(x - \alpha)$ where $\alpha \in k$ by the Nullstellensatz [MurCA, Theorem 5.3.3]. However, if k is not algebraically closed, there are more points. For example, if $k = \mathbf{R}$, then the set of closed points in $\operatorname{Spec}(k[x])$ corresponds to the closed upper half plane in $\mathbf{C}$ because roots of irreducible quadratic polynomials come in conjugate pairs.

[Mum1967, Ex. C, p. 137]
[Har1977, Exer. II.2.5]

Example 2.1.4. ($\operatorname{Spec}(\mathbf{Z})$**)** The. ring $\mathbf{Z}$ is a PID, and we can visualize $\operatorname{Spec}(\mathbf{Z})$ as the "number line" together with the generic point (0).

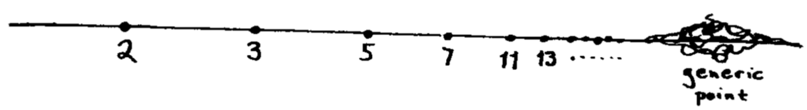

Figure 2.1: $\operatorname{Spec}(\mathbf{Z})$. From [Mum1967, p. 137].

[Mum1967, Ex. D, pp. 137–138]
[Har1977, Ex. II.2.3.2]
The letter T is used to stand for the French word *trait* [EGAInew, (5.5.1)]. Some example translations are line, stroke, or dash, but the word also refers to connections as in "avoir trait à."

Example 2.1.5. (A *trait*) If R is a DVR (i.e., a discrete rank 1 valuation ring), then $T = \operatorname{Spec}(R)$ consists of two points $t_0 \coloneqq (\varpi)$ and $t_1 \coloneqq (0)$, where ϖ is a uniformizer for R. The topology is such that t_0 is closed and $\overline{\{t_1\}} = \operatorname{Spec}(R)$. This relationship can be summarized as

$$t_1 \rightsquigarrow t_0,$$

which is read as "t_0 is a *specialization* of t_1" or "t_1 is a *generization* of t_0."

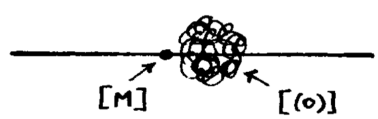

Figure 2.2: A trait, i.e., the spectrum of a DVR. From [Mum1967, p. 138].

[Mum1967, Ex. E, pp. 138–139, Ex. H, pp. 141–143]
[Har1977, Ex. II.2.3.4]

Example 2.1.6. (Polynomial rings over PIDs) For R a PID, we have the following description of $\operatorname{Spec}(R[x])$:

[AK2021, (2.26)]
[Rei1995, (1.5)]

Theorem 2.1.7. *Let R be a PID and consider the polynomial ring $R[x]$ in one variable over R. Let $P \subseteq R[x]$ be a prime ideal.*

(i) $P = (0)$, *or* $P = (f)$ *with* f *prime, or* P *is maximal.*

(ii) *If P is maximal, then either $P = (f)$ with f prime, or $P = (p, g)$ for $p \in R$ prime and $g \in R[x]$ such that its image in $(R/(p))[x]$ is prime.*

Proof. Suppose $P \neq (0)$ and P is not principal. Then, there exist two polynomials $f_1, f_2 \in P$ with no common factor. After possibly replacing f_1 and f_2 by prime factors (which lie in P by the assumption that P is prime), we may assume that f_1 and f_2 are prime. Set K to be the fraction field of R, i.e., the field obtained from R by adjoining an inverse for every nonzero element in R. Gauss's lemma implies that f_1 and f_2 are relatively prime in $K[x]$. Since $K[x]$ is a PID, there exist $h_1, h_2 \in K[x]$ such that $h_1 f_1 + h_2 f_2 = 1$. Clearing denominators gives $P \cap R \neq 0$. Since R is a PID, we have $P \cap R = (p)$ for a prime element $p \in R$.

Now set $k \coloneqq R/(p)$, which is a field. Set

$$Q = P \cdot \Big(R[x]/(p)\Big) \subseteq R[x]/(p) \cong k[x].$$

We then have

$$k[x]/Q \cong R[x]/P.$$

Now since P is prime, these rings are domains, and hence we have $Q = (g')$, where $g' \in k[x]$ is prime. Moreover, $k[x]/Q$ is a field since Q is in fact a maximal ideal by the fact that $k[x]$ is a PID. Now choosing $g \in R[x]$ mapping to g' under the quotient map $R[x] \to k[x]$, we are done. $\square$

For $R = k[y]$ and $R = \mathbf{Z}$, Mumford draws particularly nice pictures. See Figure 2.3. Note that the case when $P = (f)$ is maximal actually occurs: If (V, ϖ) is a DVR, then

$$(\varpi x - 1) \subseteq V[x]$$

is a maximal ideal.

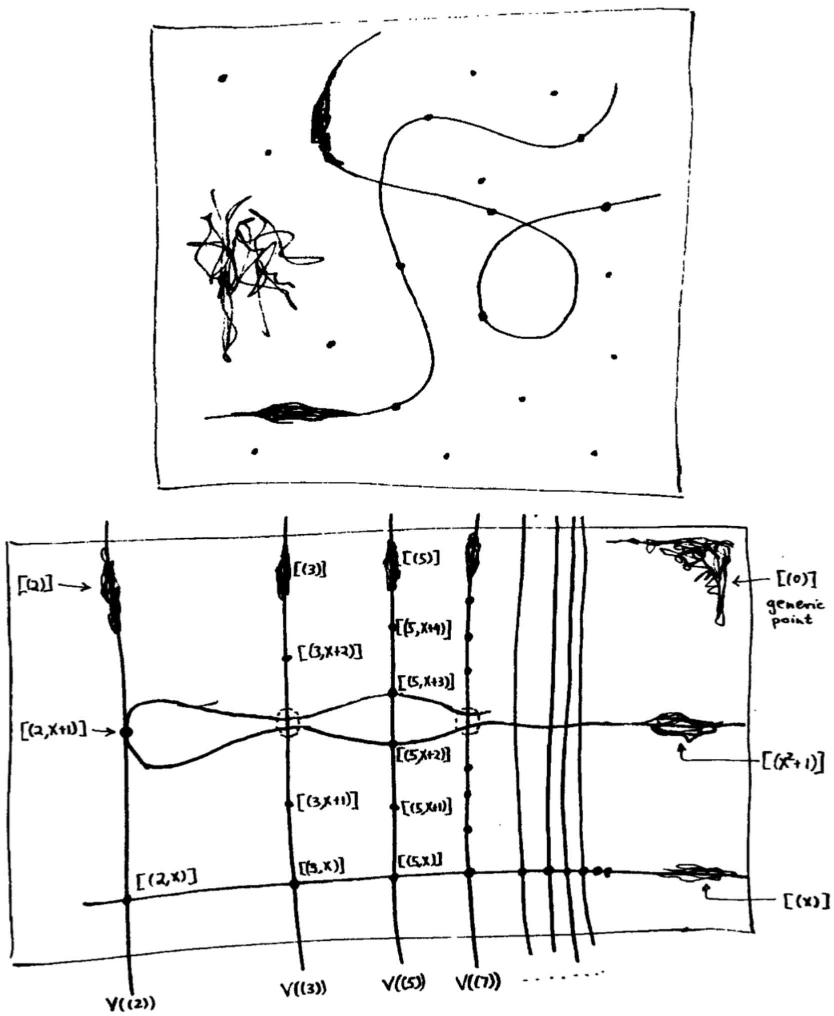

Figure 2.3: Spec($k[x, y]$) and Spec($\mathbf{Z}[x]$). From [Mum1967, p. 139, p. 141].

2.1.2 Sheaves associated to modules

To define the structure sheaf on Spec(A), we will construct sheaves on Spec(A) from A-modules. The *distinguished open sets*

$$D(f) := \operatorname{Spec}(A) - V(f)$$

form a basis for the Zariski topology on Spec(A) since

$$V(I) = \bigcap_{f \in I} V(f) \implies X - V(I) = \bigcup_{f \in I} D(f).$$

The basic idea of this definition is that we know from commutative algebra that

$$\operatorname{Spec}(A_f) \overset{\text{1-1}}{\longleftrightarrow} D(f) \subseteq \operatorname{Spec}(A)$$

since prime ideals not containing f are in bijection with prime ideals in A_f. There is therefore a nice candidate for what the value of $\mathcal{O}_{\operatorname{Spec}(A)}(D(f))$ should be: It should be A_f! Using [EGAI, (**0**, 3.2); EGAInew, (**0**, 3.2)] (which you studied in Homework 1, Problem 2), we can make the following definition.

Definition 2.1.8. **(**[EGAInew, Définition 1.3.4]**)** Let A be a ring and let M be an A-module. The *sheaf $\tilde{M}$ associated to M* is the sheafification of the presheaf defined by the basis of principal open sets by sending

$$D(f) \longmapsto M_f$$

for every principal open set $D(f)$. The *structure sheaf* for $X \coloneqq \operatorname{Spec}(A)$ is the sheaf $\mathcal{O}_X \coloneqq \tilde{A}$. We therefore consider $\operatorname{Spec}(A)$ as a ringed space $(\operatorname{Spec}(A), \tilde{A})$.

In [Har1977, p. 70, p. 110], Hartshorne gives a different (but equivalent) definition in terms of the espace étalé.

By definition in [EGAI, (**0**, 3.2); EGAInew, (**0**, 3.2)], we have

$$\tilde{M} \coloneqq \left(V \longmapsto \lim_{D(f) \subseteq V} M_f\right)^{\#}.$$

Note that by definition, the presheaf in the parentheses on the right maps to $\tilde{M}$, and hence we have maps

$$\begin{aligned} \theta_f\colon\ A_f &\longrightarrow \Gamma\bigl(D(f), \tilde{A}\bigr)\\ \theta_f\colon M_f &\longrightarrow \Gamma\bigl(D(f), \tilde{M}\bigr) \end{aligned}$$

of A_f-algebras and A_f-modules, respectively.

We prove some properties of the sheaf $\tilde{M}$.

Proposition 2.1.9. **(**[EGAI, p. 85, (**0**, 3.2.4); EGAInew, p. 198, (**0**, 3.2.4)]**)** *Let A be a ring with spectrum $X \coloneqq \operatorname{Spec}(A)$ and let $x \in X$ be a point. For every A-module M, we have*

$$\tilde{A}_x \cong A_{\mathfrak{p}_x} \qquad \text{and} \qquad \tilde{M}_x \cong M_{\mathfrak{p}_x}$$

as $A_{\mathfrak{p}_x}$-modules.

[Har1977, Props. II.2.2(a), II.5.1(b)]

Proof. Before we proceed with the proof, we note that

$$D(g) \subseteq D(f) \iff V(g) \subseteq V(f) \iff \sqrt{(g)} \subseteq \sqrt{(f)} \iff \bar{S}_g \supseteq \bar{S}_f$$

where the bars denote saturation [AK2021, (3.25)]. Thus, in this situation, we have an induced map $M_f \to M_g$.

We first show the statement for M. For open sets $V \subseteq U$ and principal open sets $D(f) \subseteq U$, $D(g) \subseteq V$ such that $D(g) \subseteq D(f)$, we have the commutative diagram

$$\begin{tikzcd}
 & \tilde{M}\big(D(f)\big) \arrow[d, "\rho^{D(f)}_{D(g)}"'] & M_f \arrow[l, "\theta_f"'] \arrow[d] \arrow[ddr, dashed] & M \arrow[l] \arrow[dd] \\
 & \tilde{M}\big(D(g)\big) \arrow[d] & M_g \arrow[l, "\theta_g"'] \arrow[d] \arrow[dr, dashed] & \\
\tilde{M}_x & \varinjlim_{D(f)\ni x} \tilde{M}\big(D(f)\big) \arrow[l, "\sim"'] & \varinjlim_{f\notin\mathfrak{p}_x} M_f \arrow[l, "\sim"'] \arrow[r, dashed, "\exists!"'] & T
\end{tikzcd}$$

where the bottom left map is an isomorphism since the directed system of sections on open sets containing x and the drected system of sections on *principal* open sets containing x are cofinal, and the bottom middle map is an isomorphism since sheafification does not affect stalks.

We claim that the direct limit in the right column satisfies the universal property of the localization $M_{\mathfrak{p}_x}$. Consider an A-module T on which all $f \notin \mathfrak{p}_x$ act as units. Consider a map $M \to T$. Then, since $f, g \notin \mathfrak{p}_x$ act as units on T, the universal property of localization of modules shows the diagonal dashed maps exist and are unique. By the universal property of direct limits, the bottom right map exists and is unique.

Finally, the case $M = A$ follows since letting T be an A-algebra on which all $f \notin \mathfrak{p}_x$ act as units, the maps in the diagram above are all ring maps. □

[Har1977, Prop. II.5.2]

Proposition 2.1.10. ([EGAI, Proposition 1.3.5; EGAInew, Proposition 1.3.5]) *Let A be a ring. The functor*

$$\begin{aligned} \mathsf{Mod}(A) &\longrightarrow \mathsf{Mod}(\tilde{A}) \\ M &\longmapsto \tilde{M} \end{aligned}$$

is exact.

Proof. This pretty much follows from Proposition 2.1.9 by the exactness of localization. However, since we were not careful about how $M \mapsto \tilde{M}$ acts on *maps*, we prove this more carefully.

Suppose we have an exact sequence $M' \to M \to M''$. Consider the commutative diagram

$$\begin{tikzcd}
M'_f \arrow[r] \arrow[d] & M_f \arrow[r] \arrow[d] & M''_f \arrow[d] \\
M'_g \arrow[r] \arrow[d] & M_g \arrow[r] \arrow[d] & M''_g \arrow[d] \\
\widetilde{M'}_x \arrow[r] & \tilde{M}_x \arrow[r] & \widetilde{M''}_x
\end{tikzcd}$$

for each $f, g \notin \mathfrak{p}_x$ such that $D(f) \supseteq D(g)$. By the exactness of localization, the top two rows are exact. By the proof of Proposition 2.1.9, the direct limit of the top two rows is the bottom row, and is therefore exact by the exactness of filtered direct limits of modules [AK2021, (7.9)] (the M_f are filtered since M_f and M_g both map to M_{fg}). $\square$

Proposition 2.1.11. ([EGAI, Proposition 1.3.6; EGAInew, Proposition 1.3.6]) *Let A be a ring and let M be an A-module. Consider an element $f \in A$. Then,* [Har1977, Exer. II.2.1]

$$\widetilde{M_f} \cong \tilde{M}|_{D(f)}.$$

Proof. These two sheaves come from the same presheaf $D(f) \mapsto M_f$ defined on principal open sets. The result now follows since restricting to open subsets does not require sheafification. (Even if it did, we get a map from left to right that induces an isomorphism on stalks.) $\square$

Our first goal for today is to show that $\tilde{A}$ satisfies what we wanted last time: On each principal open set, $\Gamma(D(f), \tilde{A})$ is just A_f. To prove this, we need some preliminaries on $\mathrm{Spec}(A)$.

Lemma 2.1.12. ([EGAI, Proposition 1.1.2]) *Let A be a ring.* [Har1977, Lem. II.2.1] [BouCA, II.4.3]

(a) *If I and J are two ideals of A, then $V(IJ) = V(I) \cup V(J)$.*

(b) *If I_i is a set of ideals of A, then $V(\sum I_i) = \bigcap_i V(I_i)$.*

(c) *If I and J are two ideals, then $V(I) \subseteq V(J)$ if and only if $\sqrt{I} \supseteq \sqrt{J}$.*

Proof. (a). $\supseteq$. If $P \supseteq I$ or $P \supseteq J$, then $P \supseteq IJ$. $\subseteq$. If $P \supseteq IJ$ and $P \not\supseteq J$, then there exists $j \in J$ such that $j \notin P$. Now for any $i \in I$, we have $ij \in P$, so by primeness we must have $i \in P$. We therefore see that $P \supseteq I$.

(b). We have $P \supseteq \sum_i I_i$ if and only if $P \supseteq I_i$ since by definition, $\sum_i I_i$ is the smallest ideal containing all the I_i.

(c). By the Scheinnullstellensatz [AK2021, (3.27)], the radical $\sqrt{I}$ of I is the intersection of all prime ideals containing I. So, we have $\sqrt{I} \supseteq \sqrt{J}$ if and only if $V(I) \subseteq V(J)$. $\square$

Proposition 2.1.13. ([EGAI, Proposition 1.1.10(ii)]) *Let A be a ring. Then,* $\mathrm{Spec}(A)$ *is quasi-compact.* [Har1977, p. 72] [BouCA, II.4.3, Prop. 12] [Hoc2017, p. 30]

Proof. Let $\operatorname{Spec}(A) = \bigcup_{i\in I} U_i$ be an open cover. Since the principal open sets form a basis, we can write

$$U_i = \bigcup_{j\in J_i} D(f_{ij})$$

and hence

$$\operatorname{Spec}(A) = \bigcup_{\substack{i\in I \\ j\in J_i}} D(f_{ij}).$$

Taking complements, we have $V(1) = V((f_{ij})_{i,j})$, and hence there exist $n_{ij} \in \mathbf{N}$ such that we have a "partition of unity"

$$\sum_{i,j} g_{ij} f_{ij}^{n_{ij}} = 1$$

where $n_{ij} = 0$ for all but finitely many (i,j). Let K be the set of pairs (i,j) such that $n_{i,j} > 0$. Taking these indices (i,j) yields a finite cover

$$\operatorname{Spec}(A) = \bigcup_{(i,j)\in K} D(f_{ij}) = \bigcup_{\{i \mid (i,j)\in K\ \exists j\}} U_i$$

as needed. □

[Har1977, Props. II.2.2(b),(c), II.5.1(c),(d)]

Theorem 2.1.14. ([EGAI, Théorème 1.3.7; EGAInew, Théorème 1.3.7]) *Let A be a ring and let M be an A-module. For every $f \in A$, the map*

$$\theta_f \colon M_f \longrightarrow \Gamma\Big(D(f), \tilde{M}\Big)$$

is bijective, and hence the presheaf $D(f) \mapsto M_f$ defined on principal open sets is in fact a sheaf. In particular,

$$\theta_1 \colon M \xrightarrow{\sim} \Gamma\Big(\operatorname{Spec}(A), \tilde{M}\Big).$$

We note that if $M = A$, then θ_f is a ring map. Thus, if we identify A_f with $\Gamma(D(f), \tilde{A})$ using θ_f for A, the maps θ_f will be isomorphisms of A_f-modules.

Proof. We first show that θ_f is injective. For each $x \in D(f)$, consider the commutative diagram

$$\begin{tikzcd}
M_f \arrow[r, "\theta_f"] \arrow[d] & \Gamma\Big(D(f), \tilde{M}\Big) \arrow[d, "\rho^{D(f)}_{D(g)}"] \\
M_g \arrow[r, "\theta_g"] \arrow[d] & \Gamma\Big(D(g), \tilde{M}\Big) \arrow[d] \\
M_{\mathfrak{p}_x} \arrow[r, "\sim"] & \tilde{M}_x
\end{tikzcd}$$

where the bottom row is the direct limit of the first two rows. If $\theta_f(\xi) = 0$, then ξ maps to 0 in $\tilde{M}_x$. By the commutativity of the diagram, this means that

ξ maps to 0 in $M_{\mathfrak{p}_x}$ for every $x \in D(f)$. By [MurCA, Proposition 3.5.1], this implies $\xi = 0$ in M_f.

We now show that θ_f is surjective. By Proposition 2.1.11, we may replace A by A_f to reduce to the case when $f = 1$. Let $s \in \Gamma(\mathrm{Spec}(A), \tilde{M})$. Since the stalks of $\tilde{M}$ are $M_{\mathfrak{p}_x}$, for each $x \in \mathrm{Spec}(A)$, there exists a principal open set $D(f_x) \ni x$ over which $s|_{D(f_x)}$ is the image of $\xi_x \in M_f$. Since $\mathrm{Spec}(A)$ is quasi-compact (Proposition 2.1.13), there exists a finite open covering

$$\mathrm{Spec}(A) = \bigcup_{i=1}^{s} D(f_i)$$

such that $s_i \coloneqq s|_{D(f_i)}$ is of the form $\theta_{f_i}(\xi_i)$ for some $\xi_i \in M_{f_i}$. For each $i, j \in I$, we know that

$$\xi_i = \xi_j \tag{2.1.15}$$

in $M_{f_i f_j}$ since they map to the same section $(s_i)|_{D(f_i f_j)} = (s_j)|_{D(f_i f_j)}$ under $\theta_{f_i f_j}$. By definition, for each $i \in I$, we can write

$$\xi_i = \frac{z_i}{f_i^{n_i}}$$

for $z_i \in M$. Since I is finite, after multiplying each z_i by a power of f_i, we may assume that the n_i are all equal to some n. The equations (2.1.15) show that there exist $m_{ij} \geq 0$ such that

$$(f_i f_j)^{m_{ij}}(f_j^n z_i - f_i^n z_j) = 0.$$

Replacing the m_{ij} by their maximum, we may assume that these relations hold for all i, j for a uniform m. Replacing the z_i by $f_i^m z_i$, we now can write

$$\xi_i = \frac{z_i}{f_i^n}$$

for $z_i \in M$ such that

$$f_j^n z_i = f_i^n z_j \tag{2.1.16}$$

for all $i, j \in I$. Now since the $D(f_i^n) = D(f_i)$ cover $\mathrm{Spec}(A)$, the f_i^n generate the unit ideal in A. Thus, there are elements $g_i \in A$ such that

$$\sum_{i \in I} g_i f_i^n = 1.$$

Consider the element

$$z = \sum_{i \in I} g_i z_i \in M.$$

By (2.1.16), we have

$$f_i^n z = f_i^n \sum_{j \in I} g_j z_j = \sum_{j \in I} g_j (f_i^n z_j) = \sum_{j \in I} g_j (f_j^n z_i) = z_i \sum_{j \in I} g_j f_j^n = z_i,$$

and hence $\xi_i = z/1$ in M_{f_i}. We therefore see that $s_i = \theta_1(z)|_{D(f_i)}$ for all i, and hence θ_1 is surjective. $\square$

2.1.3 Locally ringed spaces and affine schemes

$\boxed{2/10}$

Now that we have defined a ringed space structure on $\operatorname{Spec}(A)$, we want to glue them together to form other schemes. Schemes live in the following larger category:

[Har1977, pp. 70–71]

Definition 2.1.17. (Locally ringed spaces [EGAI, (**0**, 5.5.1); EGAInew, (**0**, 4.1.9, 4.1.12)]**)** A ringed space $(X, \mathcal{O}_X)$ is a *locally ringed space* if $\mathcal{O}_{X,x}$ is a local ring for every point $P \in X$. In this situation, we denote by $\mathfrak{m}_{X,P}$ or $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,P}$.

To define a morphism of locally ringed spaces, let $P \in X$ be a point and let $V \subseteq Y$ be a neighborhood of $f(P) \in Y$. As V ranges over all open neighborhoods of $f(P)$, we see that $f^{-1}(V)$ ranges over a subset of the open neighborhoods of P. We then obtain the commutative diagram

$$\begin{tikzcd}
\mathcal{O}_Y(U) \arrow[r, "f^{\#}(U)"] \arrow[d] & \mathcal{O}_X\big(f^{-1}(U)\big) \arrow[d] \\
\mathcal{O}_Y(V) \arrow[r, "f^{\#}(V)"] & \mathcal{O}_X\big(f^{-1}(V)\big).
\end{tikzcd}$$

Taking direct limits over each column, we get the commutative diagram

$$\begin{tikzcd}
\mathcal{O}_Y(U) \arrow[r, "f^{\#}(U)"] \arrow[d] & \mathcal{O}_X\big(f^{-1}(U)\big) \arrow[d] \arrow[drr, bend left] & & \\
\mathcal{O}_Y(V) \arrow[r, "f^{\#}(V)"] \arrow[d] & \mathcal{O}_X\big(f^{-1}(V)\big) \arrow[d] \arrow[drr, bend left] & & \\
\mathcal{O}_{Y,f(P)} \arrow[r] & \displaystyle\lim_{f^{-1}(V)\ni P} \mathcal{O}_X\big(f^{-1}(V)\big) \arrow[rr, dashed, "\exists!"] & & \mathcal{O}_{X,P}.
\end{tikzcd}$$

We denote the composition along the bottom row as

$$f_P^{\#} \colon \mathcal{O}_{Y,f(P)} \longrightarrow \mathcal{O}_{X,P}.$$

A *morphism* of locally ringed spaces is a morphism $(f, f^{\#})$ of ringed spaces such that $f_P^{\#}$ is a *local map* for every $P \in X$, that is, $f_P^{\#}(\mathfrak{m}_{Y,f(P)}) \subseteq \mathfrak{m}_{X,P}$. This forms a category LRS.

An *isomorphism* of locally ringed spaces is a morphism with a two-sided inverse. Thus, a morphism $(f, f^{\#})$ is an isomorphism if and only if f is a homeomorphism of the underlying topological spaces and $f^{\#}$ is an isomorphism of sheaves.

[EGAI, (**0**, 5.5.2)]
[EGAInew, (**0**, 4.1.9)]

For a locally ringed space $(X, \mathcal{O}_X)$ and a section $f \in \Gamma(X, \mathcal{O}_X)$, we set

$$X_f \coloneqq \big\{x \in X \mid f_x \notin \mathfrak{m}_x\big\} \subseteq X.$$

Lemma 2.1.18. **(**[EGAI, (**0**, 5.5.2); EGAInew, (**0**, 4.1.9); Stacks, Tag 01HZ]**)** *Let $(X, \mathcal{O}_X)$ be a locally ringed space and let $f \in \Gamma(X, \mathcal{O}_X)$. Then, X_f is open in X. Moreover, $f|_{X_f}$ is invertible in $\Gamma(X_f, \mathcal{O}_X)$.* [Har1977, Exer. II.2.16(a)]

Proof. We first show that X_f is open. For every $x \in X_f$, we have $f_x \notin \mathfrak{m}_x$, and hence there exists an inverse $g(x) \in \mathcal{O}_{X,x}$ for f_x. The germ $g(x)$ lifts to a section$\tilde{g}(x)$ defined on an open neighborhood V_x of x. Since $f_x g(x) = 1$ in $\mathcal{O}_{X,x}$, after replacing V_x by a smaller open neighborhood of x, we have $f|_{V_x} \tilde{g}(x) = 1$ in $\mathcal{O}_X(V_x)$, and hence $x \in V_x \subseteq X_f$.

We now show that the inverses $\tilde{g}(x)$ glue. Let $y \in X_f$ be another point. On $V_x \cap V_y$, we have

$$f|_{V_x \cap V_y} \tilde{g}(x)|_{V_x \cap V_y} = 1 = f|_{V_x \cap V_y} \tilde{g}(y)|_{V_x \cap V_y}.$$

At every $z \in V_x \cap V_y$, we therefore have

$$f_z \tilde{g}(x)_z = 1 = f_z \tilde{g}(y)_z.$$

In the local ring $\mathcal{O}_{X,z}$, we therefore have $\tilde{g}(x)_z = \tilde{g}(y)_z$. By the sheaf condition (3) for $\mathcal{O}_X$ on $V_x \cap V_y$, we have

$$\tilde{g}(x)|_{V_x \cap V_y} = \tilde{g}(y)|_{V_x \cap V_y}.$$

By the sheaf condition (4) for $\mathcal{O}_X$ on X_f, we therefore see that the sections $\{\tilde{g}(x)\}_{x \in X_f}$ glue to a global section $g \in \Gamma(X_f, \mathcal{O}_X)$. □

Note that $(\operatorname{Spec}(A), \tilde{A})$ is a locally ringed space by Proposition 2.1.9. We can now define the analogue of an affine variety and the affine coordinate ring.

Definition 2.1.19. **(Affine scheme** [EGAI, Définition 1.7.1; EGAInew, Définition 1.6.1]**)** A locally ringed space $(X, \mathcal{O}_X)$ is an *affine scheme* if it is isomorphic to a locally ringed space of the form $(\operatorname{Spec}(A), \tilde{A})$ where A is a ring. In this case, we say that the ring [Har1977, p. 74]

$$A(X) \coloneqq \Gamma(X, \mathcal{O}_X),$$

which can be identified with A by Theorem 2.1.14, is the *ring* of the affine scheme.

Our next goal is show that $\operatorname{Spec}(-)$ defines a contravariant functor $\mathsf{Ring}^{\mathrm{op}} \to \mathsf{LRS}$. We start with describing the map on topological spaces associated to a ring map.

Proposition 2.1.20. **(**[EGAI, Corollaire 1.2.3; EGAInew, Corollaire 1.2.3]**)** *If $\varphi\colon A \to B$ is a ring map, then φ induces a continuous map* [Har1977, Prop. II.2.3(b)]

$$^a\varphi\colon \operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(A).$$

This association defines a contravariant functor

$$\begin{aligned} \operatorname{Spec}\colon \mathsf{Ring}^{\mathrm{op}} &\longrightarrow \mathsf{Top} \\ A &\longmapsto \operatorname{Spec}(A) \\ \varphi &\longmapsto {}^a\varphi. \end{aligned}$$

Proof. We define the map by

$$^{a}\varphi(P) = \varphi^{-1}(P)$$

for a prime ideal $P \in \operatorname{Spec}(B)$, where we recall that the contraction of a prime ideal under a ring map is prime. If $f \in A$ is an element, then

$$^{a}\varphi^{-1}\Big(D(f)\Big) = D\Big(\varphi(f)\Big)$$

because $f \in \varphi^{-1}(P)$ if and only if $\varphi(f) \in P$. Since the $D(f)$ form a basis for the topology on $\operatorname{Spec}(A)$, we see that $^{a}\varphi$ is continuous. □

We now note that if we have a morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, taking global sections on the map $f^{\#}$ yields a ring map $\Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$. This gives a map

$$\rho\colon \operatorname{Hom}_{\mathsf{RS}}(X, Y) \longrightarrow \operatorname{Hom}_{\mathsf{Ring}}\Big(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X)\Big).$$

Restricting to LRS, we obtain the analogue of [MurAGI, Proposition 1.3.22] for the category of locally ringed spaces.

[Har1977, Exer. II.2.4] **Proposition 2.1.21. (Greenberg, Tate** [Gre1961, Proposition 1; EGAII, Errata et addenda, Proposi *Let $(S, \mathcal{O}_S)$ be an affine scheme and let $(X, \mathcal{O}_X)$ be a locally ringed space. Then, the restriction of ρ to morphisms in LRS:*

$$\rho_{\mathsf{LRS}}\colon \operatorname{Hom}_{\mathsf{LRS}}(X, S) \longrightarrow \operatorname{Hom}_{\mathsf{Ring}}\Big(A(S), \Gamma(X, \mathcal{O}_X)\Big)$$

is a bijection natural in X and S.

Proof. The naturality follows from the definition of ρ: it associates to a map of locally ringed spaces the corresponding map on global sections.

We want to define an inverse

$$\sigma\colon \operatorname{Hom}_{\mathsf{Ring}}\Big(A(S), \Gamma(X, \mathcal{O}_X)\Big) \longrightarrow \operatorname{Hom}_{\mathsf{LRS}}(X, S).$$

Set $A \coloneqq A(S) = \Gamma(S, \mathcal{O}_S)$ and consider a ring map

$$\varphi\colon A \longrightarrow \Gamma(X, \mathcal{O}_X).$$

We first define a continuous function $^{a}\varphi\colon X \to S$. For every $x \in X$, we can consider the composition

$$\varphi_x\colon A \longrightarrow \Gamma(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{X,x}.$$

Then, for each $x \in X$, we set

$$^{a}\varphi(x) \coloneqq \varphi_x^{-1}(\mathfrak{m}_x) = \Big\{ f \in A \;\Big|\; \varphi_x(f) \in \mathfrak{m}_x \Big\} \subseteq A. \tag{2.1.22}$$

The ideal $\varphi_x^{-1}(\mathfrak{m}_x)$ is a prime ideal since it is the contraction of $\mathfrak{m}_x$. To show this assignment is continuous, let $D(f) \subseteq \operatorname{Spec}(A)$ be an open subset. Then,

$$^{a}\varphi^{-1}\Big(D(f)\Big) = X_{\varphi(f)}, \tag{2.1.23}$$

which is open by what we showed in Definition 2.1.17.

We now define a morphism

$$\tilde{\varphi}\colon \mathcal{O}_S \longrightarrow {}^a\varphi_*\mathcal{O}_X$$

making $({}^a\varphi, \tilde{\varphi})\colon (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ a morphism of locally ringed spaces. Consider the ring maps

$$\begin{tikzcd}
A \arrow[r, "\sim"] \arrow[d] & \Gamma(S, \mathcal{O}_S) \arrow[r, "\varphi"] \arrow[d] & \Gamma(X, \mathcal{O}_X) \arrow[d] \\
A_f \arrow[r, "\sim"] & \Gamma(S_f, \mathcal{O}_S) \arrow[r, dashed, "\varphi_f", "\exists!"'] & \Gamma(X_{\varphi(f)}, \mathcal{O}_X).
\end{tikzcd}$$

By the universal property of localization, to show that the dashed map exists (uniquely), it suffices to note that f maps to the element $\varphi(f) \in \Gamma(X_{\varphi(f)}, \mathcal{O}_X)$, which is invertible by Lemma 2.1.18.

Since the map $\Gamma(S_f, \mathcal{O}_S) \to \Gamma(X_{\varphi(f)}, \mathcal{O}_X)$ is the unique map making the diagram commute, we have the commutative diagrams

$$\begin{tikzcd}
A_f \arrow[r, "\sim"] \arrow[d] & \Gamma(S_f, \mathcal{O}_S) \arrow[r, "\varphi_f"] \arrow[d] & \Gamma(X_f, \mathcal{O}_X) \arrow[d] \\
A_{fg} \arrow[r, "\sim"] & \Gamma(S_{fg}, \mathcal{O}_S) \arrow[r, dashed, "\varphi_{fg}"] & \Gamma(X_{\varphi(fg)}, \mathcal{O}_X) \\
A_g \arrow[r, "\sim"] \arrow[u] & \Gamma(S_g, \mathcal{O}_S) \arrow[r, "\varphi_g"] \arrow[u] & \Gamma(X_g, \mathcal{O}_X) \arrow[u]
\end{tikzcd}$$

for all $f, g \in A$. Thus, since the $D(f)$ form a basis, we have a well-defined sheaf morphism $\tilde{\varphi}\colon \tilde{A} \to {}^a\varphi_*\mathcal{O}_X$, since we can specify maps of sheaves on a basis by using the formula with inverse limits in the homework problem on specifying sheaves on a basis. Moreover, this map yields a morphism of locally ringed spaces since $f \in \mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ if and only if $\varphi_x(f) \in \mathfrak{m}_{f(x)} \subseteq \mathcal{O}_{S,f(x)}$ by definition in (2.1.22). We therefore obtain a map

$$\sigma\colon \operatorname{Hom}_{\mathsf{Ring}}\Big(A(S), \Gamma(X, \mathcal{O}_X)\Big) \longrightarrow \operatorname{Hom}_{\mathsf{LRS}}(X, S).$$

It remains to show that ρ_{LRS} and σ are mutually inverse. We know that $\rho_{\mathsf{LRS}} \circ \sigma = \mathrm{id}$ since $\Gamma(S, \tilde{\varphi}) = \varphi$, and hence ρ_{LRS} is surjective. To show that $\sigma \circ \rho_{\mathsf{LRS}} = \mathrm{id}$, we consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \mathcal{O}_{S,f(x)} \arrow[d, "f_x^\#"] \\
\Gamma(X, \mathcal{O}_X) \arrow[r] & \mathcal{O}_{X,x}.
\end{tikzcd}$$

The right vertical map is local and hence the inverse of $\mathfrak{m}_x$ must be $\mathfrak{m}_{f(x)}$. Thus, the underlying continuous map for $(\sigma \circ \rho_{\mathsf{LRS}})(f)$ is f. The map on sheaves for $(\sigma \circ \rho_{\mathsf{LRS}})(f)$ is $f^\sharp$ because of the universal property of localization. $\square$

As a result, we obtain the following:

Corollary 2.1.24. ([EGAI, Théorème 1.7.3; EGAInew, (1.6.5)]) *There is a anti-equivalence of categories*

$$\begin{aligned} \mathrm{Spec}\colon \mathsf{Ring}^{\mathrm{op}} &\longrightarrow \mathsf{AffSch} \\ A &\longmapsto \big(\mathrm{Spec}(A), \tilde{A}\big) \\ \varphi &\longmapsto ({}^a\varphi, \tilde{\varphi}). \end{aligned}$$

Proof. The functor $\Gamma(-)$ of taking global sections is an inverse for Spec by Proposition 2.1.21. ☐

Corollary 2.1.25. ([EGAI, p. 103; EGAInew, p. 226]) $\mathrm{Spec}(\mathbf{Z})$ *is the final object in the category of locally ringed spaces.*

2.1.4 Schemes

We have one more corollary of Proposition 2.1.21:

Corollary 2.1.26. *Let A be a ring. The ring map $A \to A_f$ induces a morphism* $\mathrm{Spec}(A_f) \to \mathrm{Spec}(A)$ *that induces an isomorphism*

$$\big(\mathrm{Spec}(A_f), \widetilde{A_f}\big) \xrightarrow{\sim} \big(D(f), \mathcal{O}_{\mathrm{Spec}(A)}|_{D(f)}\big).$$

Proof. Looking at the proof of Proposition 2.1.21, the continuous map

$$\mathrm{Spec}(A_f) \longrightarrow \mathrm{Spec}(A)$$

is the map inducing the bijection $\mathrm{Spec}(A_f) \to D(f)$ in [MurCA, Proposition 3.2.10(*ii*)]. The structure sheaves match by the description of $\tilde{A}$. ☐

To simplify notation, we use the following notation.

[Har1977, Exer. II.2.2]

Definition 2.1.27. ([EGAInew, (**0**, 4.1.2)]) Let $(X, \mathcal{O}_X)$ be a ringed space. If $j\colon U \hookrightarrow X$ is an open subset, we set $\mathcal{O}_U \coloneqq \mathcal{O}_X|_U$ and call $(U, \mathcal{O}_U)$ an *open subspace* of $(X, \mathcal{O}_X)$ with the *canonical inclusion morphism*

$$j\colon (U, \mathcal{O}_U) \longhookrightarrow (X, \mathcal{O}_X)$$

where $j^{\#}\colon \mathcal{O}_X \to j_*\mathcal{O}_U$ is the restriction map.

Remark 2.1.28. Note that this notation clashes with the notation $\mathscr{F}_U$ from the proof of Grothendieck's Vanishing Theorem 1.4.31! Whenever structure sheaves are involved, you should use this new definition and not the old one.

We now define schemes.

Definition 2.1.29. (Scheme [EGAI, Définition 2.1.2; EGAInew, Définition 2.1.2]**)** A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ in which every point has a neighborhood U such that $(U, \mathcal{O}_X|_U)$ is an affine scheme, called an *affine open neighborhood*. We call X the *underlying topological space* of the scheme. If we want to forget the scheme structure on a scheme (or ringed space) we write $\mathrm{sp}(X)$, which we read as "the space of X". A *morphism* of schemes is a morphism of locally ringed spaces. In other words, the category Sch of schemes is a full subcategory of the category LRS of locally ringed spaces.

[Har1977, p. 74] Before [EGAInew], schemes were called *preschemes*.

To construct interesting examples of schemes, we prove the following.

Lemma 2.1.30. (Gluing Lemma [EGAI, (**0**, 4.1.7), (2.3.1); EGAInew, (**0**, 4.1.7), (2.4.1)]**)** *Let $\{X_i\}$ be a possibly infinite family of schemes. Suppose for each i, j, we are given an open subset $U_{ij} \subseteq X_i$, which we think of as a scheme with structure sheaf $\mathcal{O}_X|_{U_{ij}}$. Suppose also for each i, j, we are given an isomorphism of schemes*

[Har1977, Exer. II.2.12]

$$\varphi_{ij} \colon U_{ij} \xrightarrow{\sim} U_{ji}$$

such that

(1) *For each i, j, we have $\varphi_{ji} = \varphi_{ij}^{-1}$.*

(2) *For each i, j, k, we have*

$$\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk},$$
$$\varphi_{ik}|_{U_{ij} \cap U_{ik}} = (\varphi_{jk} \circ \varphi_{ij})|_{U_{ij} \cap U_{ik}}.$$

There exists a scheme X together with morphisms $\psi_i \colon X_i \to X$ for each i such that

(1) *$\psi_i(X_i)$ is an open subset of X and ψ_i induces an isomorphism*

$$(X_i, \mathcal{O}_{X_i}) \xrightarrow{\sim} \big(\psi_i(X_i), \mathcal{O}_X|_{\psi_i(X_i)}\big).$$

(2) *The $\psi_i(X_i)$ cover X.*

(3) *$\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$.*

(4) *$\psi_i|_{U_{ij}} = (\psi_j \circ \varphi_{ij})|_{U_{ij}}$.*

Before proving Lemma 2.1.30, we define some terminology and write down some examples.

Definition 2.1.31. **(**[EGAI, (2.3.1; EGAInew, (2.4.1)]**)** In the situation of Lemma 2.1.30, we say that X is obtained by *gluing* the schemes X_i along the isomorphisms φ_{ij}.

[Har1977, Exer. II.2.12]

Example 2.1.32. When $U_{ij} = \varnothing$ and $\varphi_{ij} \colon \varnothing \to \varnothing$ for every i, j, the scheme X is called the *disjoint union* of the X_i, and is denoted $\coprod_i X_i$.

[Har1977, Exer. II.2.12]

Figure 2.4: Affine line with two origins. From [Har1977, p. 76].

[Har1977, Ex. II.2.3.6]

Example 2.1.33. (The affine line with two origins I) Let k be a field, let $X_1 = X_2 = \mathbf{A}^1_k$, let $U_1 = U_2 = \mathbf{A}^1_k - \{P\}$ where P is the point corresponding to $(x) \subseteq k[x]$, and let $\varphi \colon U_1 \to U_2$ be the identity map. Let X be the gluing of X_1 and X_2 along U_1 and U_2 via φ. Then, X is the *affine line with two origins*.

This is a scheme that is not an affine scheme! To see this, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X, \mathcal{O}_X) \arrow[r, "\binom{1}{1}"] \arrow[d, "\rho^X_{X_1}", "\wr"'] & k[x] \oplus k[x] \arrow[r, "(1\ -1)"] \arrow[d, "(1\ 0)"] & k[x, x^{-1}] \\
0 \arrow[r] & \Gamma(X_1, \mathcal{O}_X) \arrow[r, "\sim"] & k[x] \arrow[r] & 0
\end{tikzcd}$$

Calculating the kernel in the first row shows that $\rho^X_{X_1}$ is an isomorphism.

Now suppose that X were affine. Then, Corollary 2.1.24 shows that $\psi_1 \colon X_1 \to X$ is an isomorphism of affine schemes since it induces an isomorphism on global sections. This is a contradiction because ψ_1 is not surjective.

We also spell out this last part of the argument more explicitly. Suppose X were affine. Then, ψ_1 maps to the isomorphism $\Gamma(\psi_1) = \rho^X_{X_1}$. Letting $\varphi_1 \coloneqq \Gamma(\psi_1)^{-1}$ be the inverse map in Rings, we have

$$\varphi_1 \circ \Gamma(\psi_1) = \mathrm{id}_{\Gamma(X,\mathcal{O}_X)}$$

and hence

$$\psi_1 \circ {}^a\varphi_1 = \mathrm{id}_X.$$

This is a contradiction because ψ_1 is not surjective.

Proof of Lemma 2.1.30. The underlying topological space of X is obtained by taking the set

$$X \coloneqq \coprod_i X_i \Big/ \varphi_{ij}(x) \sim x \text{ for all } i \le j$$

and giving it the quotient topology for the map $\coprod_i X_i \twoheadrightarrow X$. With this quotient topology, the maps $\psi \colon X_i \to X$ are open embeddings. We can then glue the sheaves $\mathcal{O}_{X_i}$ from each X_i together to obtain the structure sheaf $\mathcal{O}_X$. □

2.2 Quasi-coherent and coherent sheaves

Now that we have defined affine schemes and schemes, we want to understand the $\mathcal{O}_X$-modules on a scheme X. A temporary goal is to answer the following:

Question 2.2.1. *Let $X = \mathrm{Spec}(A)$ be an affine scheme. What are the $\mathcal{O}_X$-modules that are obtained by the functor $M \mapsto \tilde{M}$ for A-modules M?*

A good reason to want to answer this question is that this will give us a way to get a handle of many modules on schemes X as those that locally are obtained in this manner. Another reason is sheaf cohomology: We have seen so far (especially on Homework 4) that it is pretty difficult to compute sheaf cohomology using just what we have done so far. We will see that on sufficiently nice schemes, the sheaf cohomology of a sheaf that is locally of the form $\tilde{M}$ can be computed in a simpler way (using Čech cohomology) and satisfy nice finiteness properties on projective varieties or, more generally, relative to proper morphisms.

The answer ends up being the class of *quasi-coherent* sheaves. Coherent modules will correspond to *coherent* sheaves. We will see that on sufficiently nice schemes, the associated sheaf cohomology sheaves can be computed in a simpler way (using Čech cohomology) and satisfy nice finiteness properties on projective varieties or, more generally, relative to proper morphisms. Since these notions make sense on arbitrary ringed spaces, we will define them in general.

2.2.1 Quasi-coherent sheaves

We start by defining quasi-coherent sheaves. We will eventually show that on schemes, this is the class of $\mathcal{O}_X$-modules that "come from commutative algebra," that is, they are sheaves associated to modules on affine open subsets.

Definition 2.2.2. ([EGAInew, (**0**, 5.1.1)]) Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. To give a morphism $u\colon \mathcal{O}_X \to \mathscr{F}$ of $\mathcal{O}_X$-modules is equivalent to giving a section $s = u(1) \in \Gamma(X, \mathscr{F})$. This is because if s is given, then for every section $t \in \Gamma(U, \mathcal{O}_X)$, we must have $u(t) = t \cdot (s|_U)$. We say that u is the morphism *defined by the global section s*. If I is an indexing set, consider the direct sum $\mathcal{O}_X^{(I)}$ with injection maps $h_i\colon \mathcal{O}_X \to \mathcal{O}_X^{(I)}$. We see that there is a bijection [Har1977, p. 121]

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_X}\left(\mathcal{O}_X^{(I)}, \mathscr{F}\right) &\xleftrightarrow{\text{1-1}} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathscr{F})^I \\ u &\longmapsto (u \circ h_i)_{i \in I} \end{aligned}$$

where the superscript on the right denotes the direct product. We therefore see that morphisms $u\colon \mathcal{O}_X^{(I)} \to \mathscr{F}$ are in one-to-one correspondence with families $(s_i)_{i \in I}$ of global sections of $\mathscr{F}$.

We say that $\mathscr{F}$ is *generated by the family $(s_i)_{i \in I}$* if the morphism

$$\mathcal{O}_X^{(I)} \longrightarrow \mathscr{F} \tag{2.2.3}$$

defined by the family is surjective. We say that $\mathscr{F}$ is *globally generated* if it is generated by $\Gamma(X, \mathscr{F})$, which by the previous paragraph is equivalent to saying that there exists some surjective morphism of the form (2.2.3).

[Har1977, p. 111]
This definition is equivalent to the definition in [Har1977]. See [Har1977, Exer. II.5.4].

Definition 2.2.4. ([EGAInew, (**0**, 5.1.3)]) Let $(X, \mathscr{O}_X)$ be a ringed space and let $\mathscr{F}$ be an $\mathscr{O}_X$-module. We say that $\mathscr{F}$ is *quasi-coherent* if, for every $x \in X$, there exists an open neighborhood U of x and a morphism $u\colon \mathscr{O}_U^{(I)} \to \mathscr{O}_U^{(J)}$ for some indexing sets I and J such that

$$\mathscr{F}|_U \cong \operatorname{coker}\Big(\mathscr{O}_U^{(I)} \xrightarrow{u} \mathscr{O}_U^{(J)}\Big). \tag{2.2.5}$$

Equivalently, $\mathscr{F}$ is quasi-coherent if, for every $x \in X$, there exists an open neighborhood U of x and an exact sequence of the form

$$\mathscr{O}_U^{(I)} \xrightarrow{u} \mathscr{O}_U^{(J)} \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

[Har1977, Prop. II.5.8(a)]

for some indexing sets I and J. Quasi-coherence is preserved under f^* by taking the pullback of the map (2.2.5) [EGAInew, (**0**, 5.1.4)].

For simplicity, we say *quasi-coherent sheaf on X* instead of *quasi-coherent $\mathscr{O}_X$-module*. Quasi-coherent $\mathscr{O}_X$-modules form a full subcategory of $\mathsf{Mod}(\mathscr{O}_X)$, which we denote by $\mathsf{QCoh}(\mathscr{O}_X)$.

[Har1977, Ex. II.5.2.1]

Example 2.2.6. ([EGAInew, (**0**, 5.1.3)]) Let $(X, \mathscr{O}_X)$ be a ringed space. Then, $\mathscr{O}_X$ and finite direct sums thereof are quasi-coherent.

Let us see one example of a non-quasi-coherent sheaf. When setting up the theory for sheaf cohomology, we used the construction $j_!$ many times. This example shows why the extra flexibility of working in $\mathsf{Mod}(\mathscr{O}_X)$ is so useful despite the fact that not all $\mathscr{O}_X$-modules "come from commutative algebra."

2/12
[Har1977, Ex. II.5.2.3]

Example 2.2.7. (An $\mathscr{O}_X$-module that is not quasi-coherent) Let X be a quasi-projective variety in the sense of [Har1977, Chapter I, §2] and let $j\colon U \hookrightarrow X$ be a nonempty open subset such that $U \neq X$. Note that the existence of such a U implies $\dim(X) > 0$.

We claim that $\mathscr{F} = j_!(\mathscr{O}_U)$ is not quasi-coherent. Let $x \in X - U$. We will show that there cannot be an open neighborhood V of x such that $\mathscr{F}|_V$ is the cokernel of a morphism

$$\mathscr{O}_V^{(I)} \longrightarrow \mathscr{O}_V^{(J)}.$$

By the discussion in Definition 2.2.2, it suffices to show that $\mathscr{F}|_V$ is not globally generated for all $V \ni x$.

We know that $\Gamma(V, j_!(\mathscr{O}_U)) = 0$. Thus, the only morphism $\mathscr{O}_V \to \mathscr{F}|_V$ is the zero morphism. This morphism is not surjective: Letting $y \in U \cap V$, we see that $\mathscr{O}_{V,y} \to \mathscr{F}_y$ is the zero morphism. Since $\mathscr{F}_y \cong \mathscr{O}_{X,y} \neq 0$, we see that $\mathscr{O}_V \to \mathscr{F}|_V$ is not surjective.

Remark 2.2.8. The notion of quasi-coherence defined in Definition 2.2.4 works well for schemes but not always for arbitrary ringed spaces.

For rigid-analytic spaces, see [Con2006, §2.1].[1] An example of Gabber [Con2006, Example 2.1.6] shows that defining quasi-coherence in the rigid-analytic setting as in Definition 2.2.4 would make the rigid-analytic version

[1] Technically, rigid-analytic spaces are "G-ringed spaces" and not ringed spaces, but one can still try to define quasi-coherence in the same way for such spaces. Also, Conrad's discussion applies to other types of non-Archimedean analytic spaces that are actually ringed spaces.

of Cartan's Theorem B false (cf. the statement for schemes in Theorem 2.3.7 below), and that not all quasi-coherent sheaves on rigid-analytic spaces are direct limits of coherent sheaves. This is why the definition of quasi-coherence in [Con2006, Definition 2.1.1] is different.

In complex analytic geometry, the same issues as those raised in [Con2006, §2.1] apply. See [KReiser2018] for the version of Gabber's example in this setting. Furthermore, one can use Gabber's construction to show that quasi-coherence is not preserved under infinite direct sums [Sta2023] (see [Liu2025b, Example 2.2]). On the other hand, the notion of *good* $\mathscr{O}_X$-modules due to Kashiwara [Kas2003, Definition 4.22] is a good analogue for the category of quasi-coherent sheaves on a scheme. For example, Liu recently proved that for a complete complex variety X_0, good $\mathscr{O}_X$-modules on the corresponding complex analytic space X correspond to quasi-coherent sheaves on X_0 [Liu2025a]. Nevertheless, quasi-coherent sheaves on a complex analytic space form an Abelian category [Liu2025b, Theorem 1.2]. A different notion of quasi-coherence is defined by Ramis and Ruget [RR1974, §2] and is more analytic in nature. According to [Ram2015], this notion was used previously by Kiehl and Verdier [KV1971] in their proof of the Grauert direct image theorem.

2.2.2 Sheaves of finite type and of finite presentation

In [MurCA], we saw that over Noetherian rings, various finiteness properties of modules coincide, like finite generation (also called finite type) and finite presentation [MurCA, Theorem 7.5.2]. For $\mathscr{O}_X$-modules, we have to deal with the same ambiguity. The possible lack of quasi-coherence also causes issues.

Definition 2.2.9. ([FAC, n° 12, Définition 1; TohokuII, p. 185]) Let $(X, \mathscr{O}_X)$ be a ringed space and let $\mathscr{F}$ be an $\mathscr{O}_X$-module. We say that $\mathscr{F}$ is *of finite type* if, for every $x \in X$, there exists an open neighborhood U of x such that $\mathscr{F}|_U$ is generated by a *finite* family of sections on U, i.e., $\mathscr{F}|_U$ is isomorphic to a quotient of a sheaf of the form $\mathscr{O}_U^p$ for a non-negative integer p. Equivalently, $\mathscr{F}$ is of finite type if, for every $x \in X$, there exists an open neighborhood U of x such that $\mathscr{F}|_U$ fits into an exact sequence of the form [EGAInew, (**0**, 5.2.1)]

$$\mathscr{O}_U^p \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

for a non-negative integer p. "Finite type" is preserved under f^* by taking the pullback of the surjection $\mathscr{O}_U^p \twoheadrightarrow \mathscr{F}|_U$ [EGAInew, (**0**, 5.2.4)].

Remark 2.2.10. An $\mathscr{O}_X$-module of finite type is not necessarily quasi-coherent. For example, consider the quotient $\mathscr{O}_X/\mathscr{F}$, where $\mathscr{F} = (j_U)_!\mathscr{O}_U$ for a proper nonempty open subset $U \subsetneq X$ (see Example 2.2.7). This is because we have a short exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_X/\mathscr{F} \longrightarrow 0$$

and if $\mathscr{O}_X/\mathscr{F}$ were quasi-coherent, then $\mathscr{F}$ would be by the fact that kernels of morphisms of quasi-coherent sheaves are quasi-coherent (shown in Corollary 2.3.17(i) below).

We prove some properties of sheaves of finite type for all ringed spaces. While usually stated for coherent sheaves in algebraic geometry, this shows how the tools of commutative algebra are useful even if you are not working on a scheme.

Proposition 2.2.11. ([FAC, n° 12, Proposition 1; EGAInew, **0**, Proposition 5.2.2, Corollaire 5.2.2.1, a
Let $(X, \mathcal{O}_X)$ be a ringed space, let $\mathscr{F}, \mathscr{G}$ be $\mathcal{O}_X$-modules, and let $x \in X$ be a point. Suppose that $\mathscr{F}$ is of finite type.

(*i*) *Suppose $(s_i)_{1\le i\le n}$ are sections of $\mathscr{F}$ on an open subset $U \subseteq X$. Suppose the $s_{i,x}$ generate $\mathscr{F}_x$. Then, there exists an open subset $V \subseteq U$ containing x such that the $s_{i,y}$ generate $\mathscr{F}_y$ for all $y \in V$.*

(*ii*) *If $u\colon \mathscr{F} \to \mathscr{G}$ is a morphism such that $u_x = 0$, then there exists an open neighborhood U of x such that $u|_U = 0$.*

(*iii*) *If $v\colon \mathscr{G} \to \mathscr{F}$ is a morphism such that v_x is surjective, then there exists an open neighborhood V of x such that $v|_V$ is surjective.*

[Har1977, Exer. II.5.6(c)] (*iv*) *The support* Supp($\mathscr{F}$) *is closed.*

(*v*) *If $(X, \mathcal{O}_X)$ is a locally ringed space, then*

$$\operatorname{Supp}(\mathscr{F}) = \bigl\{x \in X \bigm| \mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x \neq 0\bigr\}.$$

(*vi*) *If $(X, \mathcal{O}_X)$ is a locally ringed space and $\mathscr{G}$ is of finite type, then*

$$\operatorname{Supp}(\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}) = \operatorname{Supp}(\mathscr{F}) \cap \operatorname{Supp}(\mathscr{G}).$$

Proof. (i). Let $(t_j)_{1\le j\le m}$ be sections of $\mathscr{F}$ on an open set $U' \subseteq U$ containing x that generate $\mathscr{F}|_{U'}$. Since the $s_{i,x}$ generate $\mathscr{F}_x$, there exist sections a_{ij} on an open subset $U'' \subseteq U'$ containing x such that

$$t_{j,x} = \sum_{i=1}^{n} a_{ij,x}\, s_{i,x}$$

for all $1 \le j \le m$. We conclude that there exists an open subset $V \subseteq U''$ containing x such that for all $y \in V$, we have

$$t_{j,y} = \sum_{i=1}^{n} a_{ij,y}\, s_{i,y}$$

for all $1 \le j \le m$, in which case the $s_{i,y}$ generate $\mathscr{F}_y$ for all $y \in V$.

(iv) holds by applying (i) to the case $n = 1$ and $s_{1,x} = 0$. (iv) implies (iii) and (ii) since coker(v) and im(u) are of finite type.

(v) holds by the NAK lemma [MurCA, Lemma 2.3.8], and implies (vi) since the tensor product of nonzero $\kappa(x)$-vector spaces is nonzero. ☐

Lemma 2.2.12. ([EGAInew, (**0**, 5.2.9)]) *Let $(X, \mathscr{O}_X)$ be a ringed space. Consider a short exact sequence* [Har1977, Prop. II.5.7]

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

of $\mathscr{O}_X$-modules.

(i) *If $\mathscr{G}$ is of finite type, then $\mathscr{H}$ is of finite type.*

(ii) *If $\mathscr{F}$ and $\mathscr{H}$ are of finite type, then $\mathscr{G}$ is of finite type.*

Proof. (i) holds by definition.

(ii). Since the question is local, after replacing X by an open subset, we may assume that $\mathscr{F}$ and $\mathscr{H}$ are generated by finitely many sections $(s_i)_{1 \le i \le n}$ and $(t_j)_{1 \le j \le m}$ on X, respectively, and that the sections t_j lift to sections t'_j of $\mathscr{G}$. Then, $\mathscr{G}$ is generated by the t'_j and the images of the s_i. □

Next, we define sheaves of finite presentation. These are called *pseudo-coherent* in [TohokuII, p. 185], but the term pseudo-coherent often means something different today (see [SGA6, Exposé I, Définition 2.1]).

Definition 2.2.13. ([TohokuII, p. 185]) Let $(X, \mathscr{O}_X)$ be a ringed space and let $\mathscr{F}$ be an $\mathscr{O}_X$-module. We say that $\mathscr{F}$ is *of finite presentation* if, for every $x \in X$, there exists an open neighborhood U of x and a morphism $u \colon \mathscr{O}_U^p \to \mathscr{O}_U^q$ for p, q non-negative integers such that [EGAInew, (**0**, 5.2.5)]

$$\mathscr{F}|_U \cong \operatorname{coker}\Bigl(\mathscr{O}_U^p \xrightarrow{u} \mathscr{O}_U^q\Bigr). \tag{2.2.14}$$

Equivalently, $\mathscr{F}$ is of finite presentation if, for every $x \in X$, there exists an open neighborhood U of x such that $\mathscr{F}|_U$ fits into an exact sequence of the form

$$\mathscr{O}_U^p \xrightarrow{u} \mathscr{O}_U^q \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

for some p, q non-negative integers. Such an $\mathscr{O}_X$-module is both of finite type and is quasi-coherent. Finite presentation is preserved under f^* by taking the pullback of the map in (2.2.14).

A useful fact about sheaves of finite type and finite presentation is the following result, which for affine schemes is a special case of "Hom commutes with flat base change" [MurCA, Proposition 7.7.4].

Proposition 2.2.15. ([TohokuII, Proposition 4.1.1]) *Let $(X, \mathscr{O}_X)$ be a ringed space and let $\mathscr{F}, \mathscr{G}$ be $\mathscr{O}_X$-modules. For every $x \in X$, there is a natural morphism* [Har1977, Prop. III.6.8]

$$\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})_x \longrightarrow \operatorname{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x).$$

This morphism is injective (resp. bijective) if $\mathscr{F}$ is of finite type (resp. finitely presented).

Proof. For every open neighborhood U of x, there is a natural morphism

$$\operatorname{Hom}_{\mathscr{O}_U}\Big(\mathscr{F}|_U, \mathscr{G}|_U\Big) \longrightarrow \operatorname{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x).$$

Taking direct limits over all $U \ni x$ on the left-hand side, we obtain the morphism.

Since the question is local, after replacing X by an open neighborhood of x, we may assume that $\mathscr{F}$ has a presentation

$$\mathscr{O}_X^{(J)} \longrightarrow \mathscr{F} \longrightarrow 0$$

where J is finite in the finite type case and

$$\mathscr{O}_X^{(I)} \longrightarrow \mathscr{O}_X^{(J)} \longrightarrow \mathscr{F} \longrightarrow 0$$

where both I and J are finite in the finite presentation case. Applying $\mathscr{H}om_{\mathscr{O}_X}(\,\cdot\,, \mathscr{G})$, which is left exact (since it is on sections over each U), we obtain the left exact sequence in the top row of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})_x \arrow[r] \arrow[d] & \mathscr{H}om_{\mathscr{O}_X}\Big(\mathscr{O}_X^{(J)}, \mathscr{G}\Big)_x \arrow[r] \arrow[d, "\wr"] & \mathscr{H}om_{\mathscr{O}_X}\Big(\mathscr{O}_X^{(I)}, \mathscr{G}\Big)_x \arrow[d] \\
0 \arrow[r] & \operatorname{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x) \arrow[r] & \operatorname{Hom}_{\mathscr{O}_{X,x}}\Big(\mathscr{O}_{X,x}^{(J)}, \mathscr{G}_x\Big) \arrow[r] & \operatorname{Hom}_{\mathscr{O}_{X,x}}\Big(\mathscr{O}_{X,x}^{(I)}, \mathscr{G}_x\Big)
\end{tikzcd}$$

where in the finite type case, we only work with the left square. The middle vertical map is an isomorphism in either case since J is finite and both modules represent the same data as specifying $|J|$ elements of $\mathscr{G}_x$. When $\mathscr{F}$ is of finite presentation, the right vertical map is an isomorphism since I is finite. $\square$

Corollary 2.2.16. ([EGAInew, (**0**, 5.2.7)]) *Let $(X, \mathscr{O}_X)$ be a ringed space and let $\mathscr{F}, \mathscr{G}$ be $\mathscr{O}_X$-modules of finite presentation. Let $x \in X$ be a point such that $\mathscr{F}_x \cong \mathscr{G}_x$. Then, there exists an open neighborhood U of x such that $\mathscr{F}|_U \cong \mathscr{G}|_U$.*

Proof. If $\varphi\colon \mathscr{F}_x \to \mathscr{G}_x$ and $\psi\colon \mathscr{G}_x \to \mathscr{F}_x$ are mutually inverse, then Proposition 2.2.15 implies there exists an open neighborhood $V \ni x$ and sections u and v of $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ and $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{G}, \mathscr{F})$ over V such that $u_x = \varphi$ and $v_x = \psi$. Since $(u \circ v)_x = \mathrm{id}_{\mathscr{G}_x}$ and $(v \circ u)_x = \mathrm{id}_{\mathscr{F}_x}$, by Proposition 2.2.15, there exists an open subset $U \subseteq V$ containing x such that $(u \circ v)|_U = \mathrm{id}_{\mathscr{F}|_U}$ and $(v \circ u)|_V = \mathrm{id}_{\mathscr{G}|_U}$. $\square$

2.2.3 Coherent sheaves

We can now define coherent sheaves. The first version of this definition is from complex analytic geometry [SHC51/52, Exposé 15, Définition 3 and Exposé 18, Définition 1]. The fact that the sheaf of holomorphic functions on a complex manifold is coherent is a deep result known as Oka's coherence theorem [Oka1950].

Serre first applied the definition to algebraic varieties in [FAC, nº 12, Définition 2]. It is important to remember that in algebraic geometry, we are pretty spoiled: The fact that our spaces will usually come from gluing spectra of Noetherian rings means that the algebraic analogue of Oka's coherence theorem basically follows from definition of Noetherianity. As an aside, this shows how far-reaching and powerful Noether's original insights are!

Definition 2.2.17. ([FAC, nº 12, Définition 2; EGAInew, Chapitre 0, (5.3.1)]) Let $(X, \mathscr{O}_X)$ be a ringed space and let $\mathscr{F}$ be an $\mathscr{O}_X$-module. We say that $\mathscr{F}$ is *coherent* if it satisfies the following two conditions:

This is not equivalent to the definition in [Har1977, p. 111] in general. They *are* equivalent when X is a locally Noetherian scheme. See [Har1977, Exer. II.5.4].

(a) $\mathscr{F}$ is of finite type.

(b) For every open subset $U \subseteq X$, every integer $n \geq 0$, and every morphism
$$u\colon \mathscr{O}_U^n \longrightarrow \mathscr{F}|_U,$$
the kernel of u is of finite type.

By definition, every sub-$\mathscr{O}_X$-module *of finite type* of a coherent $\mathscr{O}_X$-module is coherent. By definition, every coherent $\mathscr{O}_X$-module is of finite presentation.

For simplicity, we say *coherent sheaf on X* instead of *coherent $\mathscr{O}_X$-module*. Coherent $\mathscr{O}_X$-modules form a full subcategory of $\mathsf{QCoh}(\mathscr{O}_X)$, which we denote by $\mathsf{Coh}(\mathscr{O}_X)$.

The four classes of $\mathscr{O}_X$-modules are related in the following manner:

$$\begin{array}{ccccc} \text{coherent} & \Longrightarrow & \text{finite presentation} & \Longrightarrow & \text{quasi-coherent} \\ & & \Big\Downarrow & & \\ & & \text{finite type.} & & \end{array}$$

Note that the three classes on the right are preserved under f^*. This is not the case for coherence without extra hypotheses.

So far, we have seen that finite type sheaves satisfy some nice formal properties with respect to exact sequences (Lemma 2.2.12). The following result shows why coherent sheaves are even nicer to work with.

Proposition 2.2.18. ("2 out of 3" property for coherent sheaves [FAC, nº 13, Théorème 1; EGAInew, Chap
Let $(X, \mathscr{O}_X)$ be a ringed space and consider a short exact sequence

[Har1977, Prop. II.5.7]

$$0 \longrightarrow \mathscr{F} \xrightarrow{u} \mathscr{G} \xrightarrow{v} \mathscr{H} \longrightarrow 0$$

be a short exact sequence of $\mathscr{O}_X$-modules. If two of the three sheaves are coherent, then the third sheaf is also coherent.

Below, we use the snake lemma [KS2006, Lemma 12.1.1] a few times, which you showed as Homework 4, Problem 1. If you are worried about the fact that we are working with $\mathscr{O}_X$-modules and cannot chase elements, we are using the snake lemma to check surjectivity of morphisms, and hence one could also pass to stalks and apply the usual snake lemma for modules.

Proof. We proceed in steps.

Step 1. *If $\mathscr{G}$ and $\mathscr{H}$ are coherent, then $\mathscr{F}$ is coherent.*

Since the question is local, after possibly replacing X by an open subset, we may assume there exists a surjective morphism $w\colon \mathscr{O}_X^p \to \mathscr{G}$. Consider the following commutative diagram:

$$
\begin{tikzcd}
0 \arrow[r] & \ker(v \circ w) \arrow[r] \arrow[d, dashed, two heads] & \mathscr{O}_X^p \arrow[r, "v \circ w"] \arrow[d, "w"', two heads] & \mathscr{H} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{F} \arrow[r, "u"] & \mathscr{G} \arrow[r, "v"] & \mathscr{H} \arrow[r] & 0
\end{tikzcd}
$$

Since $\mathscr{H}$ is coherent, $\ker(v \circ w)$ is of finite type. Now by the snake lemma [KS2006, Lemma 12.1.1], since w is surjective, the map $\mathscr{I} \to \mathscr{F}$ induced by the universal property of the kernel is also surjective. Thus, $\mathscr{F}$ is of finite type, and hence coherent.

Step 2. *If $\mathscr{F}$ and $\mathscr{G}$ are coherent, then $\mathscr{H}$ is coherent.*

Since $\mathscr{G}$ is of finite type, we know that $\mathscr{H}$ is also of finite type by Lemma 2.2.12(i). It remains to show that $\mathscr{H}$ satisfies the condition (b). Consider a morphism $f\colon \mathscr{O}_U^n \to \mathscr{H}|_U$ and let $(s_i)_{1 \le i \le n}$ be the corresponding sections of $\mathscr{H}$ over U. Since the question of whether $\ker(f)$ is of finite type is local, after replacing X by an open subset of U, we may assume that there exist n sections s_i' of $\mathscr{G}$ over U such that $s_i = v(s_i')$, and that $\mathscr{F}|_U$ is generated by sections $(t_j)_{1 \le j \le m}$ over U. Now consider the morphism

$$
w\colon \mathscr{O}_U^{n+m} \longrightarrow \mathscr{G}|_U
$$

defined by the n sections s_i' and the m sections $u(t_j)$. These morphisms fit into the commutative diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \ker(w) \arrow[r] \arrow[d] & \ker(f) \arrow[r, dashed] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{O}_U^m \arrow[r] \arrow[d, two heads] & \mathscr{O}_U^{n+m} \arrow[r, "p"] \arrow[d, "w"'] & \mathscr{O}_U^n \arrow[r] \arrow[d, "f"'] & 0 \\
0 \arrow[r] & \mathscr{F}|_U \arrow[r, "u"] & \mathscr{G}|_U \arrow[r, "v"] & \mathscr{H}|_U \arrow[r] & 0.
\end{tikzcd}
$$

By the snake lemma [KS2006, Lemma 12.1.1], we see that $\ker(w) \to \ker(f)$ is surjective. Since $\ker(w)$ is of finite type, this shows that $\ker(f)$ is of finite type.

Step 3. *If $\mathscr{F}$ and $\mathscr{H}$ are coherent, then $\mathscr{G}$ is coherent.*

By Lemma 2.2.12(ii), we know that $\mathscr{G}$ is of finite type. It remains to show that $\mathscr{G}$ satisfies the condition (b). Consider a morphism $f\colon \mathcal{O}_U^n \to \mathscr{G}|_U$ and consider the commutative diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \ker(f) \arrow[d] & & \\
0 \arrow[r] & \ker(v \circ f) \arrow[r] \arrow[d] & \mathcal{O}_U^n \arrow[r, "v\circ f"] \arrow[d, "f"] & \mathscr{H}|_U \arrow[d, equal] & \\
0 \arrow[r] & \mathscr{F}|_U \arrow[r, "u"] & \mathscr{G}|_U \arrow[r, "v"] & \mathscr{H}|_U \arrow[r] & 0
\end{tikzcd}
$$

where the morphism $\ker(v\circ f) \to \mathscr{F}|_U$ exists by the universal property of kernels. Since $\mathscr{H}$ is coherent, $\ker(v \circ f)$ is of finite type. Since the question of whether $\ker(f)$ is of finite type is local, after possibly replacing U by a smaller subset, we may assume that $\ker(v \circ f)$ is generated by finitely many sections over U. Let $\mathcal{O}_U^m \twoheadrightarrow \ker(v \circ f)$ be the corresponding surjection. Note that the image of $\mathcal{O}_U^m$ in $\mathcal{O}_U^n$ is $\ker(v \circ f)$. We therefore have the commutative diagram

$$
\begin{tikzcd}
 & \ker(w) \arrow[r] \arrow[d] & \ker(f) \arrow[r] \arrow[d] & 0 \arrow[d] & \\
 & \mathcal{O}_U^m \arrow[r] \arrow[d, "w"'] & \mathcal{O}_U^n \arrow[r, "v\circ f"] \arrow[d, "f"] & \operatorname{im}(v \circ f) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}|_U \arrow[r, "u"] & \mathscr{G}|_U \arrow[r, "v"] & \mathscr{H}|_U \arrow[r] & 0
\end{tikzcd}
$$

where the middle row is exact. By the snake lemma [KS2006, Lemma 12.1.1], the top row is exact. Since $\ker(w)$ is of finite type by the coherence of $\mathscr{F}$, we see that $\ker(f)$ is of finite type as desired. ☐

Corollary 2.2.19. ([FAC, nº 13, Théorème 1, Corollaire and Théorème 2, nº 14, Propositions 4 and 6; EGAInew, [Har1977, Prop. II.5.7]
Let $(X, \mathcal{O}_X)$ *be a ringed space.*

(i) *Finite direct sums of coherent* $\mathcal{O}_X$*-modules are coherent.*

(ii) *Suppose* $u\colon \mathscr{F} \to \mathscr{G}$ *is a morphism of coherent* $\mathcal{O}_X$*-modules. Then,* $\operatorname{im}(u)$*,* $\ker(u)$*, and* $\operatorname{coker}(u)$ *are coherent, and hence* $\mathsf{Coh}(\mathcal{O}_X)$ *is an Abelian category.*

(iii) *Suppose* $\mathscr{F}$ *and* $\mathscr{G}$ *are coherent* $\mathcal{O}_X$*-modules. Then,* $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}$ *and* $\mathscr{H}\!om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$ *are coherent.* [Har1977, Exer. III.6.3(a)]

Proof. (i) follows immediately from Proposition 2.2.18.

(*ii*). By Lemma 2.2.12(*i*), $\operatorname{im}(u)$ is of finite type. Since it is a subsheaf of a coherent sheaf, we see that $\operatorname{im}(u)$ is coherent. For $\ker(u)$ and $\operatorname{coker}(u)$, we apply Proposition 2.2.18 to the short exact sequences

$$0 \longrightarrow \ker(u) \longrightarrow \mathscr{F} \longrightarrow \operatorname{im}(u) \longrightarrow 0,$$
$$0 \longrightarrow \operatorname{im}(u) \longrightarrow \mathscr{G} \longrightarrow \operatorname{coker}(u) \longrightarrow 0.$$

(*iii*). Since the questions are local, after possibly replacing X by an open subset, we have the right exact sequence

$$\mathscr{O}_X^p \longrightarrow \mathscr{O}_X^q \longrightarrow \mathscr{F} \longrightarrow 0.$$

Since tensor products are right exact, we obtain the right exact sequence

$$\mathscr{O}_X^p \otimes_{\mathscr{O}_X} \mathscr{G} \longrightarrow \mathscr{O}_X^q \otimes_{\mathscr{O}_X} \mathscr{G} \longrightarrow \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} \longrightarrow 0.$$

The left and middle sheaves are isomorphic to $\mathscr{G}^p$ and $\mathscr{G}^q$, respectively. By (*i*) and Proposition 2.2.18, we are done. For $\mathscr{H}\!om$, we use the left exactness of Hom and Proposition 2.2.15 to obtain the left exact sequence

$$0 \longrightarrow \mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \longrightarrow \mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{O}_X^q, \mathscr{G}) \longrightarrow \mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{O}_X^p, \mathscr{G}).$$

The middle and right sheaves are isomorphic to $\mathscr{G}^q$ and $\mathscr{G}^p$, respectively. By (*i*) and Proposition 2.2.18, we are done. □

To guarantee nice behavior under pullbacks, we have the following:

Definition 2.2.20. ([FAC, n° 15, Définition 3; EGAInew, (**0**, 5.3.9)]) Let $(X, \mathscr{O}_X)$ be a ringed space. We say that $\mathscr{O}_X$ is a *coherent sheaf of rings* if $\mathscr{O}_X$ is coherent as a $\mathscr{O}_X$-module.

If $\mathscr{O}_X$ is a coherent sheaf of rings, every $\mathscr{O}_X$-module $\mathscr{F}$ of finite presentation is coherent: Locally, $\mathscr{F}$ is the cokernel of a map between direct sums of finitely many copies of $\mathscr{O}_X$, which are coherent by Corollary 2.2.19(*i*). Cokernels of such maps are coherent by Corollary 2.2.19(*ii*).

We add this last comment to the chart of implications from before:

$$\begin{tikzcd}
\text{coherent} \arrow[r, Rightarrow] & \text{finite presentation} \arrow[r, Rightarrow] \arrow[l, dashed, Rightarrow, bend right, "\mathscr{O}_X\text{ coh.}"'] \arrow[d, Rightarrow] & \text{quasi-coherent} \\
& \text{finite type.} &
\end{tikzcd}$$

Example 2.2.21. Let X be a complex manifold. Then, the sheaf $\mathscr{O}_X$ of germs of holomorphic functions on X is coherent by Oka's coherence theorem [Oka1950]. This is a deep theorem from several complex variables. For a textbook account, see [Hör1990, Theorem 6.4.1].

We can also say something about pullbacks.

Lemma 2.2.22. ([EGAInew, (**0**, 5.3.14)]) *Let $f\colon X \to Y$ be a morphism of ringed spaces and suppose that $\mathcal{O}_X$ is coherent. Then, for every finitely presented $\mathcal{O}_Y$-module $\mathcal{G}$, the pullback $f^*\mathcal{G}$ is coherent. In particular, if $\mathcal{G}$ is coherent, then $f^*\mathcal{G}$ is coherent.* [Har1977, Prop. II.5.8(b)]

Proof. Pullbacks of finitely presented sheaves are finitely presented and the coherence of $\mathcal{O}_X$ implies finitely presented $\mathcal{O}_X$-modules are coherent. □

2.3 Quasi-coherent sheaves on schemes

2.3.1 Serre's equivalence for affine schemes

Our next goal is to show that $M \mapsto \tilde{M}$ is an equivalence of categories $\mathsf{Mod}(A) \to \mathsf{QCoh}(\mathcal{O}_{\mathrm{Spec}(A)})$. We first show that the functor $M \mapsto \tilde{M}$ is fully faithful.

Corollary 2.3.1. ([EGAInew, Corollaire 1.3.8]) *Let A be a ring and let M, N be an A-module. Then, the homomorphism* [Har1977, Exer. III.6.7]

$$\begin{aligned} \mathrm{Hom}_A(M,N) &\longrightarrow \mathrm{Hom}_{\tilde{A}}(\tilde{M},\tilde{N}) \\ u &\longmapsto \tilde{u} \end{aligned}$$

is bijective. In particular, $M = 0$ if and only if $\tilde{M} = 0$.

Proof. Consider the canonical map

$$\begin{aligned} \mathrm{Hom}_{\tilde{A}}(\tilde{M},\tilde{N}) &\longrightarrow \mathrm{Hom}_{\Gamma(\tilde{A})}\Big(\Gamma(\tilde{M}),\Gamma(\tilde{N})\Big) \\ v &\longmapsto \Gamma(v). \end{aligned}$$

By Theorem 2.1.14, the A-module on the right is naturally isomorphic to $\mathrm{Hom}_A(M,N)$. We want to show that $u \mapsto \tilde{u}$ and $v \mapsto \Gamma(v)$ are mutually inverse. We know that $\Gamma(\tilde{u}) = u$ by definition of $\tilde{u}$. Conversely, given $v \in \mathrm{Hom}_{\tilde{A}}(\tilde{M},\tilde{N})$, the induced map

$$w\colon \Gamma\Big(D(f),\tilde{M}\Big) \longrightarrow \Gamma\Big(D(f),\tilde{N}\Big)$$

fits into the commutative diagram

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d] & N \arrow[d] \\ M_f \arrow[r, "w"] & N_f \end{tikzcd}$$

where the vertical maps are the localization maps by Theorem 2.1.14. By the universal property of localization [MurCA, Theorem 10.2.6], we know that $w = u_f$ for every $f \in A$. By the definition of the action of $M \mapsto \tilde{M}$ on maps, we see that $(\Gamma(v))^{\sim} = v$. □

We now prove the following theorem, which explains why quasi-coherent sheaves are sheaves that "come from algebra." Condition (a) is the definition of quasi-coherence in [Har1977, p. 111].

[Har1977, p. 111, Exer. II.5.4]

Theorem 2.3.2. ([EGAInew, Théorème 1.4.1]) *Let A be a ring, let $X = \operatorname{Spec}(A)$, and let $\mathscr{F}$ be an $\mathscr{O}_X$-module. The following three conditions are equivalent.*

(a) *There exists an A-module M such that $\mathscr{F} \cong \tilde{M}$.*

[Har1977, Exer. II.5.4]

(b) *The sheaf $\mathscr{F}$ is quasi-coherent.*

[Har1977, Lem. II.5.3]

(c) *The following two properties hold:*

$(c1)$ *For every $f \in A$ and every section $s \in \Gamma(D(f), \mathscr{F})$, there exists an integer $n \geq 0$ such that $f^n s$ extends to a section of $\mathscr{F}$ on X.*

$(c2)$ *For every $f \in A$ and every section $t \in \Gamma(X, \mathscr{F})$ such that $t|_{D(f)} = 0$, there exists an integer $n \geq 0$ such that $f^n t = 0$.*

Proof. $(a) \Rightarrow (b)$. Let M be an A-module such that $\mathscr{F} \cong \tilde{M}$. Then, M has a presentation

$$A^{(I)} \longrightarrow A^{(J)} \longrightarrow M \longrightarrow 0.$$

Applying the functor $\tilde{(\cdot)}$, which is exact by Proposition 2.1.10, we obtain the presentation

$$\mathscr{O}_X^{(I)} \longrightarrow \mathscr{O}_X^{(J)} \longrightarrow \mathscr{F} \longrightarrow 0$$

which shows that $\mathscr{F}$ is quasi-coherent.

$(b) \Rightarrow (c)$. We proceed in steps.

Step 1. *The case when $\mathscr{F} = \tilde{M}$ for an A-module M.*

By Proposition 2.1.11 and Theorem 2.1.14, we have the isomorphism

$$M_f \xrightarrow{\sim} \Gamma\Big(D(f), \tilde{M}\Big).$$

We first show $(c1)$. A section $s \in \Gamma(D(f), \tilde{M})$ corresponds to an element $z/f^n \in M_f$ where $z \in M$. We then see that $f^n s$ corresponds to the element $z/1 \in M$ under this isomorphism. The element $z \in M$ gives a section of $\mathscr{F} = \tilde{M}$ on X extending $f^n s$, proving $(c1)$.

We now show $(c2)$. A section $t \in \Gamma(X, \tilde{M})$ corresponds to an element $z' \in M$, and the restriction of z' to $D(f)$ corresponds to the image $z'/1$ of z' in M_f. By the definition of localization, $z'/1 = 0$ in M_f implies $f^n z' = 0$ in M for some $n \geq 0$. We see that for the same n, we have $f^n t = 0$.

2/17

Step 2. *The general case.*

Since X is quasi-compact (Proposition 2.1.13) and distinguished open sets $D(g)$ form a basis for the Zariski topology on X, the definition of quasi-coherence implies there exist finitely many elements $g_i \in A$ such that $X = \bigcup_i D(g_i)$ and we have presentations

$$\tilde{A}_{g_i}^{(I_i)} \longrightarrow \tilde{A}_{g_i}^{(J_i)} \longrightarrow \mathscr{F}|_{D(g_i)} \longrightarrow 0.$$

In this case, the exactness (Proposition 2.1.10) and the fully faithfulness (Corollary 2.3.1) of the functor $(\tilde{\cdot})$ shows that $\mathscr{F}|_{D(g_i)} \cong \tilde{N}_i$ for an A_{g_i}-module N_i. By Step 1, $\mathscr{F}|_{D(g_i)}$ satisfies $(c1)$ and $(c2)$. Step 1 also shows that

$$\mathscr{F}|_{D(g_i)\cap D(g_j)} = \mathscr{F}|_{D(g_i g_j)}$$

satisfies $(c1)$ and $(c2)$.

We first show $(c2)$. Since $D(f) \cap D(g_i) = D(fg_i)$, for every i, there exists an integer n_i such that $(fg_i)^{n_i} t|_{D(g_i)} = 0$. Since the image of g_i in A_{g_i} is invertible, we have $f^{n_i} t|_{D(g_i)} = 0$. Setting $n = \max\{n_i\}$, we see that $f^n t = 0$ on X.

To prove $(c1)$ for $\mathscr{F}$, we apply $(c1)$ to $\mathscr{F}|_{D(g_i)}$. For every i, there exists an integer $n_i \geq 0$ and a section $s_i' \in \Gamma(D(g_i), \mathscr{F})$ extending $(fg_i)^{n_i} s|_{D(fg_i)}$. Since the image of g_i in A_{g_i} is invertible, there exists a section $s_i \in \Gamma(D(g_i), \mathscr{F})$ such that $s_i' = g_i^{n_i} s_i$ and such that s_i extends $f^{n_i} s|_{D(fg_i)}$. After possibly replacing the n_i by $n = \max\{n_i\}$, we may assume the n_i are equal to a fixed number n. By construction,

$$(s_i - s_j)|_{D(f)\cap D(g_i)\cap D(g_j)} = (s_i - s_j)|_{D(fg_i g_j)} = 0.$$

By $(c2)$ applied to $\mathscr{F}|_{D(g_i g_j)}$, there exists an integer $m_{ij} \geq 0$ such that

$$(fg_i g_j)^{m_{ij}}(s_i - s_j)|_{D(g_i g_j)} = 0.$$

Since the image of $g_i g_j$ in $A_{g_i g_j}$ is invertible, we see that

$$f^{m_{ij}}(s_i - s_j)|_{D(g_i g_j)} = 0.$$

After possibly replacing the m_{ij} by $m = \max\{m_{ij}\}$, we see there exists a section $s' \in \Gamma(X, \mathscr{F})$ extending the $f^m s_i$. This section extends $f^{m+n} s$, proving $(c1)$.

$(c) \Rightarrow (a)$. We first describe an auxiliary construction. Consider the morphism [Har1977, Exer. II.5.3] [EGAInew, (1.7.1)]

$$i\colon \Big(\operatorname{Spec}(A), \tilde{A}\Big) \longrightarrow \Big(\{*\}, \underline{A}\Big)$$

of ringed spaces. We then have the isomorphism

$$\operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathscr{F}) \xrightarrow{\sim} \operatorname{Hom}_A\Big(M, \Gamma(X, \mathscr{F})\Big) \tag{2.3.3}$$

by [Har1977, p. 110] (Homework 3, Problem 5) since $\tilde{M} \cong i^* M$. Plugging in $M = \Gamma(X, \mathscr{F})$, we obtain a morphism $u\colon \tilde{M} \to \mathscr{F}$ of $\tilde{A}$-modules.

We describe this morphism explicitly. Since the image of f in A_f is invertible, the universal property of localization [MurCA, Theorem 10.2.6] yields the commutative diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d] & \Gamma(X,\mathscr{F}) \arrow[d, "\rho^X_{D(f)}"] \\ M_f \arrow[r, "u_f"] & \Gamma\big(D(f),\mathscr{F}\big). \end{tikzcd}$$

The compatibility of the morphisms u_f follows by the uniqueness part of the universal property, and hence we obtain a morphism $\tilde{M} \to \mathscr{F}$.

We now use property $(c1)$ (resp. $(c2)$) to show that the u_f are surjective (resp. injective), which will show (a). If $s \in \Gamma(D(f),\mathscr{F})$, then by $(c1)$, there exists an integer $n \geq 0$ such that $f^n s$ extends to a section $z \in M$. Since $u_f(z/f^n) = s$, this shows that u_f is surjective. If $z \in M$ is such that $u_f(z/1) = 0$, then $z|_{D(f)} = 0$. By $(c2)$, there exists an integer $n \geq 0$ such that $f^n z = 0$, and hence $z/1 = 0$ in M_f. This shows that u_f is injective. □

We can restate what we have shown as an equivalence of categories. I am used to calling it Serre's equivalence for affine schemes, although this terminology does not seem to be widespread. Serre proved this statement for coherent sheaves on affine varieties [FAC, n° 49, Théorème 1].

[Har1977, Cor. II.5.5] **Corollary 2.3.4.** ([EGAInew, Corollaire 1.4.2 and Corollaire 1.4.3]) *Let A be a ring and let $X = \operatorname{Spec}(A)$. We have an exact equivalence of categories*

$$\begin{aligned} \mathsf{Mod}(A) &\overset{\sim}{\longleftrightarrow} \mathsf{QCoh}(\mathscr{O}_X) \\ M &\longmapsto \tilde{M} \\ \Gamma(X,\mathscr{F}) &\longleftarrow\!\shortmid \mathscr{F}. \end{aligned}$$

In particular, the functor $\Gamma(X,\,\cdot\,)$ is exact on $\mathsf{QCoh}(X)$. Moreover, $\tilde{M}$ is of finite type (resp. of finite presentation) if and only if M is of finite type (resp. of finite presentation).

Proof. The equivalence of categories follows by combining Corollary 2.3.1 and Theorem 2.3.2.

It remains to show the "Moreover" statement. The direction $\Leftarrow$ follows by choosing a presentation for M and using the exactness of the functor $M \mapsto \tilde{M}$ (Proposition 2.1.10).

[BouCA, II.5.2, Prop. 3, Cor.] For the direction $\Rightarrow$, since X is quasi-compact, we know there exist finitely many elements $f_1, f_2, \ldots, f_r \in A$ such that $D(f_i)$ cover X and such that for every i, the localization M_{f_i} is an A_{f_i}-module of finite type (resp. of finite presentation). In the finite type case, let $z_{i1}, z_{i2}, \ldots, z_{is_i}$ be a set of generators for M_{f_i}. Take n large enough such that $f_i^n z_{ij} \in M$ for all i,j. The collection $\{f_i^n z_{ij}\}_{i,j}$ generates M since

$$A^{\sum_i s_i} \xrightarrow{(f_1^n z_{11} \,\cdots\, f_1^n z_{1s_1}\ f_2^n z_{21} \,\cdots\, f_r^n z_{rs_r})} M$$

is surjective after localizing at every prime ideal $\mathfrak{p} \subseteq A$. For finite presentation, we do the same thing once to obtain a surjection $A^q \twoheadrightarrow M$ and another time to obtain a surjection $A^p \twoheadrightarrow \ker(A^q \to M)$. Here, we use [MurCA, Lemma 7.10.2] to say that the kernel of $A^q_{f_i} \to M_{f_i}$ is finitely generated. $\square$

As a consequence, we obtain the following:

Corollary 2.3.5. ([EGAInew, Corollaire 1.3.12(i),(ii) and Corollaire 1.7.7]) *Let A be a ring and let M, N be two A-modules.*

(i) *There is a natural isomorphism* [Har1977, Prop. II.5.2(b)]

$$(M \otimes_A N)^\sim \xrightarrow{\sim} \tilde{M} \otimes_{\tilde{A}} \tilde{N}.$$

(ii) *If M is of finite presentation, then there is a natural isomorphism* [Har1977, Exer. III.6.7]

$$\Big(\mathrm{Hom}_A(M, N)\Big)^\sim \xrightarrow{\sim} \mathscr{H}om_{\tilde{A}}(\tilde{M}, \tilde{N}).$$

(iii) *Let $\varphi\colon A \to B$ be a ring map and let $u\colon X \to S$ be the morphism of corresponding affine schemes. For every A-module M, we have* [Har1977, Prop. II.5.2(d),(e)]

$$u^*\tilde{M} \xrightarrow{\sim} (M \otimes_A B)^\sim$$

and for every B-module N, we have

$$\Big(N_{[\varphi]}\Big)^\sim \xrightarrow{\sim} u_*\tilde{N},$$

where $(\,\cdot\,)_{[\varphi]}$ denotes restriction of scalars.

Proof. Briefly, (i) and (ii) hold because tensor products commute with localization and Hom commutes with localization when the first argument is of finite presentation. We describe what maps are used in each of the arguments below.

(ii). The sheaf $\mathscr{G} = (\mathrm{Hom}_A(M, N))^\sim$ is the sheaf associated to the presheaf

$$D(f) \longmapsto \mathscr{G}\Big(D(f)\Big) = \mathrm{Hom}_{A_f}(M_f, N_f)$$

on principal open sets $U = D(f) \subseteq \mathrm{Spec}(A)$. On the other hand, $\mathscr{G}(D(f))$ is naturally isomorphic to

$$\Big(\mathrm{Hom}_A(M, N)\Big)_f \xrightarrow{\sim} \Gamma\Big(D(f), \mathscr{G}\Big)$$

by the fact that Hom commutes with flat base change [MurCA, Proposition 7.7.4], where the isomorphism is from Theorem 2.1.14. Since the construction of these isomorphisms is compatible with restriction in Proposition 2.1.11 and Theorem 2.1.14, we are done.

(i). Ideally one can just observe that they satisfy the same universal property using bilinear maps and then apply the equivalence in Corollary 2.3.4. However, because the universal property of tensor products talks about bilinear maps, and the equivalence as stated does not know about these, we need to proceed differently.

One proof is via tensor–Hom adjunction, which you show in [Har1977, Exercise II.5.1(c)] (Homework 3, Problem 1(c)). We have

$$\begin{aligned}
\operatorname{Hom}_{\mathscr{O}_X}\Bigl((M \otimes_A N)^\sim, \,\cdot\,\Bigr) &\xleftarrow{\sim} \operatorname{Hom}_A\Bigl(M \otimes_A N, \Gamma(X, \,\cdot\,)\Bigr)\\
&\xrightarrow{\sim} \operatorname{Hom}_A\Bigl(M, \operatorname{Hom}_A\Bigl(N, \Gamma(X, \,\cdot\,)\Bigr)\Bigr)\\
&\xrightarrow{\sim} \operatorname{Hom}_{\mathscr{O}_X}\Bigl(\tilde{M}, \mathscr{H}\!om_{\mathscr{O}_X}(\tilde{N}, \,\cdot\,)\Bigr)\\
&\xleftarrow{\sim} \operatorname{Hom}_{\mathscr{O}_X}(\tilde{M} \otimes_{\mathscr{O}_X} \tilde{N}, \,\cdot\,)
\end{aligned}$$

where the first and third isomorphisms hold by Corollary 2.3.1, the third isomorphism uses (ii), and the second and fourth isomorphisms hold by tensor–Hom adjunction for modules [MurCA, Theorem 7.6.1] and $\mathscr{O}_X$-modules [Har1977, Exercise II.5.1(c)] (Homework 3, Problem 6(a)), respectively. By the Yoneda lemma [MurCA, Corollary 4.6.3], we are done.

Another proof uses the definition of $\tilde{(\cdot)}$. The sheaf $\mathscr{F} = (M \otimes_A N)^\sim$ is the sheaf associated to the presheaf

$$U \longmapsto \mathscr{F}(U) = \Gamma(U, \tilde{M}) \otimes_{\Gamma(U,\tilde{A})} \Gamma(U, \tilde{N})$$

on principal open sets $U = D(f) \subseteq \operatorname{Spec}(A)$. On the other hand, $\mathscr{F}(D(f))$ is isomorphic (via the universal property of localization) to $M_f \otimes_{A_f} N_f$ by Proposition 2.1.11. In turn, this module is naturally isomorphic to

$$(M \otimes_A N)_f \xrightarrow{\sim} \Gamma\Bigl(D(f), \mathscr{F}\Bigr)$$

by [MurCA, Lemma 7.12.7], where the isomorphism is from Theorem 2.1.14. Since the construction of these isomorphisms is compatible with restriction in Proposition 2.1.11 and Theorem 2.1.14, we are done.

[EGAI, Prop. 1.6.3] (iii). For direct images, the left-hand side is the sheaf associated to

$$D(f) \longmapsto \Bigl(N_{[\varphi]}\Bigr)_f \cong \Bigl(N_{\varphi(f)}\Bigr)_{[\varphi_f]} = \Gamma\Bigl(D\Bigl(\varphi(f)\Bigr), \tilde{N}\Bigr)_{[\varphi_f]}$$

[EGAI, Prop. 1.6.5] on principal open sets $D(f) \subseteq \operatorname{Spec}(A)$. For inverse images, we show that the two sides represent the same functor. We have

$$\begin{aligned}
\operatorname{Hom}_{\mathscr{O}_X}(u^*\tilde{M}, \,\cdot\,) &\xrightarrow{\sim} \operatorname{Hom}_{\mathscr{O}_S}(\tilde{M}, u_* \,\cdot\,)\\
&\xrightarrow{\sim} \operatorname{Hom}_A\Bigl(M, \Gamma(S, u_* \,\cdot\,)\Bigr)\\
&\xrightarrow{\sim} \operatorname{Hom}_A\Bigl(M, \Gamma(X, \,\cdot\,)_{[\varphi]}\Bigr)\\
&\xleftarrow{\sim} \operatorname{Hom}_B\Bigl(M \otimes_A B, \Gamma(X, \,\cdot\,)\Bigr)\\
&\xleftarrow{\sim} \operatorname{Hom}_{\mathscr{O}_X}\Bigl((M \otimes_A B)^\sim, \,\cdot\,\Bigr)
\end{aligned}$$

where the first isomorphism holds by [Har1977, p. 110] (Homework 3, Problem 5), the second and fifth isomorphisms hold by the adjunction in (2.3.3), the third isomorphism holds by definition of direct images, and the fourth isomorphism holds by tensor–Hom adjunction [MurCA, Corollary 7.6.2]. By the Yoneda lemma [MurCA, Corollary 4.6.3], we are done. □

In the Noetherian case, we obtain Hartshorne's definition for coherent sheaves.

Theorem 2.3.6. ([EGAInew, Théorème 1.5.1]) *Let A be a* Noetherian *ring, let $X = \mathrm{Spec}(A)$, and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. The following conditions are equivalent.* [Har1977, Exer. II.5.4]

(*a*) *$\mathscr{F}$ is coherent.*

(*b*) *$\mathscr{F}$ is quasi-coherent and of finite type.*

(*c*) *There exists a finite type A-module M such that $\mathscr{F} \cong \tilde{M}$.*

Proof. This follows from Theorem 2.3.2 and Corollary 2.3.4 using the fact that for A Noetherian, sub-modules of the Noetherian module A^q are finitely generated [MurCA, Theorem 7.5.2]. □

2.3.2 Cartan's Theorem B for affine schemes

One consequence of Corollary 2.3.4 is that taking global sections is exact for exact sequences of quasi-coherent sheaves on $\mathrm{Spec}(A)$. In fact, we can say something stronger: For a short exact sequence

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0,$$

of $\mathcal{O}_{\mathrm{Spec}(A)}$-modules on $\mathrm{Spec}(A)$ *and assuming that $\mathscr{F}'$ is quasi-coherent*, taking global sections is exact. This exactness result will be a consequence of the cohomological result proved below. See [Har1977, Proposition II.5.6] for a direct proof of this exactness result using techniques from this course. See [EGAInew, Proposition 1.4.6] for an alternative proof using Yoneda Ext as defined in [Yon1954] instead of sheaf cohomology.

In the complex analytic case, the following result is due to Cartan [SHC51/52, Exposé 18, Théorème B], who showed that complex Stein manifolds have vanishing cohomology for *coherent* sheaves. Because of the name of Cartan's theorem in [SHC51/52, Exposé XVIII], Cartan's result is known as *Cartan's Theorem B*. Serre [FAC, nº 46, Théorème 1, Corollaire 1] proved Cartan's Theorem B for affine varieties. For another analogue of Cartan's Theorem B, see [Kie1967, Satz 2.4.2], where Kiehl shows Cartan's Theorem B for coherent sheaves on affinoid spaces (or more generally, quasi-Stein spaces) in rigid analytic geometry.

Theorem 2.3.7. (Cartan's Theorem B for affine schemes [EGAIII_1, Théorème 1.3.1]**)** *Let A be a ring and let $X = \mathrm{Spec}(A)$. Consider a quasi-coherent sheaf $\mathscr{F}$ on X. Then, we have* [Har1977, Prop. II.5.6, Thm. III.3.5, Rem. III.3.5.1]

$$H^i(X, \mathscr{F}) = 0$$

for all $i > 0$.

We prove Theorem 2.3.7 following [Kem1980, §1]. If $j\colon U \hookrightarrow X$ is an open inclusion, we set

$$_U\mathscr{F} \coloneqq j_*\bigl(\mathscr{F}|_U\bigr).$$

We start with the following preliminary result, which is another "dimension shifting" type argument.

Proposition 2.3.8. ([Kem1980, Proposition 1]) *Let X be a topological space and let $\mathscr{F}$ be an Abelian sheaf. Suppose X has a basis $\mathscr{U}$ such that for some positive integer $n > 0$, we have*

$$H^i\bigl(U, \mathscr{F}|_U\bigr) = 0$$

for all $0 < i < n$ and for all $U \in \mathscr{U}$. Given any element $\alpha \in H^n(X, \mathscr{F})$, there exists a covering $\mathscr{V}$ of X by members of $\mathscr{U}$ such that the image of α in $H^n(X, {}_V\mathscr{F})$ is 0 for all $V \in \mathscr{V}$.

Proof. Let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be a short exact sequence where $\mathscr{G}$ is flasque. Then, for every open subset $U \subseteq X$, we have $H^i(U, \mathscr{G}|_U) = 0$ for all $i > 0$. By the long exact sequence on sheaf cohomology, we have the exact sequence

$$0 \longrightarrow \Gamma\bigl(U, \mathscr{F}|_U\bigr) \longrightarrow \Gamma\bigl(U, \mathscr{G}|_U\bigr) \longrightarrow \Gamma\bigl(U, \mathscr{H}|_U\bigr) \longrightarrow H^1\bigl(U, \mathscr{F}|_U\bigr) \longrightarrow 0 \tag{2.3.9}$$

and the isomorphisms

$$H^i\bigl(U, \mathscr{H}|_U\bigr) \xrightarrow{\sim} H^{i+1}\bigl(U, \mathscr{F}|_U\bigr) \tag{2.3.10}$$

for all $i > 0$.

For any open subset $V \subseteq X$, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{G} \arrow[r] \arrow[d] & \mathscr{H} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & {}_V\mathscr{F} \arrow[r] & {}_V\mathscr{G} \arrow[r] & {}_V\mathscr{H}
\end{tikzcd}$$

with exact rows, where ${}_V\mathscr{G}$ is still flasque. The image $\mathscr{K}$ of $\mathscr{H} \to {}_V\mathscr{H}$ and of ${}_V\mathscr{G} \to {}_V\mathscr{H}$ coincide since they are both the sheafification of the presheaf

$$W \longmapsto \operatorname{im}\Bigl(\mathscr{G}(W \cap V) \to \mathscr{H}(W \cap V)\Bigr) = \operatorname{im}\Bigl(\mathscr{G}(V) \to \mathscr{H}(W \cap V)\Bigr)$$

where the equality holds by the flasqueness of $\mathscr{G}$. Replacing ${}_V\mathscr{H}$ by $\mathscr{K}$, the same long exact sequence argument as in the previous paragraph yields the exact sequence

$$0 \longrightarrow \Gamma\bigl(X, {}_V\mathscr{F}\bigr) \longrightarrow \Gamma\bigl(X, {}_V\mathscr{G}\bigr) \longrightarrow \Gamma\bigl(X, \mathscr{K}\bigr) \longrightarrow H^1\bigl(X, {}_V\mathscr{F}\bigr) \longrightarrow 0 \tag{2.3.11}$$

and the isomorphisms

$$H^i(X, \mathscr{K}) \xrightarrow{\sim} H^{i+1}\bigl(X, {}_V\mathscr{F}\bigr) \tag{2.3.12}$$

for all $i > 0$.

We proceed by induction on n. Suppose that $n = 1$. We consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(X, \mathscr{F}) & \longrightarrow & \Gamma(X, \mathscr{G}) & \longrightarrow & \Gamma(X, \mathscr{H}) & \xrightarrow{\delta} & H^1(X, \mathscr{F}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Gamma\big(X, {}_V\mathscr{F}\big) & \longrightarrow & \Gamma\big(X, {}_V\mathscr{G}\big) & \longrightarrow & \Gamma\big(X, \mathscr{K}\big) & \longrightarrow & H^1\big(X, {}_V\mathscr{F}\big) & \longrightarrow & 0 \end{array}$$

with exact rows obtained by combining the exact sequences (2.3.9) for $U = X$ and (2.3.11). By exactness in the top row, $\alpha \in H^1(X, \mathscr{F})$ is $\delta(\beta)$ for some $\beta \in \Gamma(X, \mathscr{H})$. By exactness in the bottom row, the image of α in $H^1(X, {}_V\mathscr{F})$ is 0 if and only if the image of β in

$$\Gamma(X, \mathscr{K}) \subseteq \Gamma\big(X, {}_V\mathscr{H}\big) = \Gamma(V, \mathscr{H})$$

lifts to an element of $\Gamma(X, {}_V\mathscr{G}) = \Gamma(V, \mathscr{G})$. Since $\mathscr{U}$ contains arbitrarily small open subsets, we can find a suitable covering $\mathscr{V}$ of X because $\mathscr{G} \to \mathscr{H} \to 0$ is exact.

Now suppose that $n > 1$. If V and W are members of $\mathscr{U}$, the sequence (2.3.9) for $U = V \cap W$ shows that

$$\Gamma\big(U, \mathscr{G}|_U\big) \longrightarrow \Gamma\big(U, \mathscr{H}|_U\big) \longrightarrow 0$$

is exact. By definition, we know that $\Gamma(U, \mathscr{G}|_U) = \Gamma(U, {}_V\mathscr{G})$ and $\Gamma(U, \mathscr{H}|_U) = \Gamma(U, {}_V\mathscr{H})$ since $U \subseteq V$. We therefore see that

$$\Gamma\big(U, {}_V\mathscr{G}\big) \longrightarrow \Gamma\big(U, {}_V\mathscr{H}\big) \longrightarrow 0$$

is surjective for every open set of the form $U = V \cap W$, and hence $\mathscr{K} = {}_V\mathscr{H}$. Moreover, (2.3.10) shows that $\mathscr{H}$ satisfies the assumption in the proposition for the integer $n - 1$. For the element $\alpha \in H^n(X, \mathscr{F})$, we consider its image under the top isomorphism

$$\begin{array}{ccc} H^{n-1}\big(X, {}_V\mathscr{H}\big) & \xrightarrow{\sim} & H^n\big(X, {}_V\mathscr{F}\big) \\ \downarrow & & \downarrow \\ H^{n-1}\big(X, {}_V\mathscr{H}\big) & \xrightarrow{\sim} & H^n\big(X, {}_V\mathscr{F}\big) \end{array}$$

from (2.3.12). By the inductive hypothesis applied to $\mathscr{H}$, there exists a covering $\mathscr{V}$ of X such that the image of α in $H^{n-1}(X, {}_V\mathscr{H})$ is 0 for every $V \in \mathscr{V}$. Since the bottom horizontal map is also an isomorphism, this covering $\mathscr{V}$ works for $\mathscr{F}$. $\square$

We now show Cartan's Theorem B for affine schemes (Theorem 2.3.7).

[Kem1980, Thm. 2]

Proof of Theorem 2.3.7. Let $\mathscr{U}$ be the basis of X consisting of principal open subsets $D(f)$. We will show by induction on n that $H^i(X, \mathscr{F}) = 0$ for all $0 < i < n$ and for all quasi-coherent sheaves $\mathscr{F}$ on spectra of rings. The hypothesis of Proposition 2.3.8 holds for $\mathscr{U}$. Thus, given any element $\alpha \in H^n(X, \mathscr{F})$, there exists a covering $X = \bigcup_{j=1}^p V_i$ by members of $\mathscr{U}$ such that the image of α in

$$H^n\left(X, \bigoplus_{j=1}^p {}_{V_j}\mathscr{F}\right)$$

is 0. The long exact sequence on sheaf cohomology associated to

$$0 \longrightarrow \mathscr{F} \longrightarrow \bigoplus_{j=1}^p {}_{V_j}\mathscr{F} \longrightarrow \mathscr{G} \longrightarrow 0$$

shows that α is in the image of $\delta(H^{n-1}(X, \mathscr{G}))$. Note that ${}_{V_j}\mathscr{F}$ is quasi-coherent by the fact that each V_j is a distinguished open set, and hence is an affine scheme. The cokernel $\mathscr{G}$ is quasi-coherent by Serre's equivalence (Corollary 2.3.4).

If $n = 1$, then $\delta = 0$ because $\Gamma(X, \cdot\,)$ is exact for quasi-coherent sheaves by Corollary 2.3.4. If $n > 1$, then $H^{n-1}(X, \mathscr{G}) = 0$ by inductive hypothesis. In either case, we see that $\alpha = 0$. $\square$

As an application, we revisit the affine line with two origins.

Example 2.3.13. (The affine line with two origins II) Consider the affine line with two origins X from Example 2.1.33. We give another proof that X is not affine. We claim that $H^1(X, \mathscr{O}_X) \neq 0$, which shows that X is not affine by Cartan's Theorem B for affine schemes (Theorem 2.3.7). We have the left exact sequence

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{\binom{1}{1}} {}_{X_1}\mathscr{O}_X \oplus {}_{X_2}\mathscr{O}_X \xrightarrow{(1\ -1)} {}_{X_1 \cap X_2}\mathscr{O}_X \longrightarrow 0 \tag{2.3.14}$$

which we show is exact on the right by taking stalks. For $x \in X_1 \cap X_2$, we have the short exact sequence

$$0 \longrightarrow \mathscr{O}_{X,x} \xrightarrow{\binom{1}{1}} \mathscr{O}_{X,x} \oplus \mathscr{O}_{X,x} \xrightarrow{(1\ -1)} \mathscr{O}_{X,x} \longrightarrow 0.$$

For the first origin $x_1 \in X_1 - X_2$, we have the short exact sequence

$$0 \longrightarrow \mathscr{O}_{X,x} \xrightarrow{\binom{1}{1}} \mathscr{O}_{X,x} \oplus \mathscr{K}_{X,x} \xrightarrow{(1\ -1)} \mathscr{K}_{X,x} \longrightarrow 0$$

and for the second origin $x_2 \in X_2 - X_1$, we have the short exact sequence

$$0 \longrightarrow \mathscr{O}_{X,x} \xrightarrow{\binom{1}{1}} \mathscr{K}_{X,x} \oplus \mathscr{O}_{X,x} \xrightarrow{(1\ -1)} \mathscr{K}_{X,x} \longrightarrow 0$$

where $\mathscr{K}_X = \underline{k(t)}$ is the constant sheaf of rational functions on X. We now take global sections in (2.3.14):

$$\begin{array}{ccc} \Gamma(X_1, \mathcal{O}_X) \oplus \Gamma(X_2, \mathcal{O}_X) & \xrightarrow{(1\ -1)} & \Gamma(X_1 \cap X_2, \mathcal{O}_X) \\ \| & & \| \\ k[t] \oplus k[t] & \xrightarrow{(1\ -1)} & k[t, t^{-1}]. \end{array}$$

The bottom map is not surjective. Since the sequence

$$\Gamma(X_1, \mathcal{O}_X) \oplus \Gamma(X_2, \mathcal{O}_X) \longrightarrow \Gamma(X_1 \cap X_2, \mathcal{O}_X) \xrightarrow{\neq 0} H^1(X, \mathcal{O}_X)$$

is exact, we see that $H^1(X, \mathcal{O}_X) \neq 0$. Thus, X is not affine by Cartan's Theorem B (Theorem 2.3.7).

Remark 2.3.15. A subtle point in this computation is that we do not know (yet) that the middle and right terms in (2.3.14) are quasi-coherent. Thus, we actually need Cartan's Theorem B (Theorem 2.3.7) instead of Corollary 2.3.4.

2.3.3 Quasi-coherent sheaves on schemes

Using what we know in the affine case, we immediately have the following. This is the definition of quasi-coherence in [Har1977, p. 111].

Proposition 2.3.16. ([EGAInew, Proposition 2.2.1]) *Let X be a scheme and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. Then, $\mathscr{F}$ is quasi-coherent if and only if, for every affine open subset $V = \operatorname{Spec}(A) \subseteq X$, we have $\mathscr{F}|_V \cong \tilde{M}$ for some A-module M.* [Har1977, Exer. II.5.4]

Proof. The definition of quasi-coherence is local. We can therefore apply Theorem 2.3.2. □

Corollary 2.3.17. ([EGAInew, Corollaire 2.2.2]) *Let X be a scheme.*

(*i*) ("2 out of 3" property for quasi-coherent sheaves on schemes) *Consider an exact sequence* [Har1977, Prop. II.5.7]

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

of $\mathcal{O}_X$-modules. If two of the three sheaves are quasi-coherent, then the third sheaf is also quasi-coherent.

(*ii*) *Tensor products and $\mathscr{H}\!om$'s of quasi-coherent $\mathcal{O}_X$-modules are quasi-coherent.*

(*iii*) *Images, kernels, and cokernels of maps of quasi-coherent $\mathcal{O}_X$-modules are quasi-coherent.* [Har1977, Prop. II.5.7]

(*iv*) *Direct limits and (possibly infinite) direct sums of quasi-coherent $\mathcal{O}_X$-modules are quasi-coherent.*

Proof. By Proposition 2.3.16, we reduce to the affine case.

(i). If $\mathscr{F}$ and $\mathscr{G}$ (resp. $\mathscr{G}$ and $\mathscr{H}$) are quasi-coherent, then this is also the case for $\mathscr{H}$ (resp. $\mathscr{F}$) because kernels and cokernels can be computed in terms of their associated A-modules. Now suppose that $\mathscr{F}$ and $\mathscr{H}$ are quasi-coherent. The long exact sequence on sheaf cohomology yields the exact sequence

$$0 \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{G}) \longrightarrow \Gamma(X, \mathscr{H}) \longrightarrow H^1(X, \mathscr{F}) = 0$$

by Theorem 2.3.7. Taking associated sheaves, we obtain the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(X, \mathscr{F})^\sim \arrow[r] \arrow[d] & \Gamma(X, \mathscr{G})^\sim \arrow[r] \arrow[d] & \Gamma(X, \mathscr{H})^\sim \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{G} \arrow[r] & \mathscr{H} \arrow[r] & 0 \end{tikzcd}$$

with exact rows, where the vertical maps are those constructed during the proof of (c) $\Rightarrow$ (a) in Theorem 2.3.2 (see (2.3.3)). Since the outer maps are isomorphisms, the middle map is an isomorphism by the snake lemma [KS2006, Lemma 12.1.1].

(ii) holds by Corollary 2.3.5. The last two statements hold since $M \mapsto \tilde{M}$ is an equivalence of categories (Corollary 2.3.4). □

2.4 Properties of schemes and morphisms of schemes

2.4.1 Noetherian schemes and coherent sheaves on Noetherian schemes

To get a nice description of coherent sheaves, we introduce the following class of schemes.

[Har1977, Exer. II.2.13(b)]
[Har1977, p. 83]

Definition 2.4.1. ([EGAInew, Définition 2.7.1 and Proposition 2.7.2]) Let X be a scheme. We say that X is *quasi-compact* if $\mathrm{sp}(X)$ is quasi-compact. We say that X is *locally Noetherian* if there exists an affine open covering

$$X = \bigcup_i \mathrm{Spec}(A_i)$$

of X such that A_i is Noetherian for every i. We say that X is *Noetherian* if X is locally Noetherian and quasi-compact.

[Har1977, Caution II.3.1.1, Exer. II.2.13(d)]

Example 2.4.2. Noetherian schemes have Noetherian underlying topological spaces because it is a finite union of Noetherian spaces. The converse does not hold! Here are some examples. Below, k is a field.

(1) $\mathrm{Spec}(k[x_1, x_2, ...]/(x_1^2, x_2^2, ...)) = \{*\}$.

(2) The spectrum of a finite-dimensional non-Noetherian valuation ring. One example of such a ring is from last semester [MurAGI, Example 1.6.2]: The ring [Har1977, Exer. II.4.12(b)(3)]
$$k\left[x, y, \frac{x}{y}, \frac{x}{y^2}, \dots\right]_{\mathfrak{m}} \subseteq k(x,y)$$
where $\mathfrak{m}$ is generated by all the generators on the left-hand side is a non-Noetherian valuation ring. This is the valuation ring of $k(x,y)$ with value group $\mathbf{Z}^2$ with the lexicographic order, where
$$v(x^i y^j) = (i,j).$$
This ring is non-Noetherian since $\mathfrak{m}$ cannot be generated by finitely many elements: The set of powers of y appearing in the denominators of elements in $\mathfrak{m}$ is bounded below.

We want to characterize Noetherianity in terms of having a property for *every* affine open subset.

Proposition 2.4.3. ([EGAInew, Proposition 2.7.3]) *Let X be a scheme. The following are equivalent.* [Har1977, Prop. II.3.2]

(i) *X is locally Noetherian.*

(ii) *For* every *affine open subset* $\operatorname{Spec}(B) \subseteq X$*, the ring B is Noetherian.*

In particular, an affine scheme $\operatorname{Spec}(A)$ *is Noetherian if and only if A is Noetherian.*

Proof. (ii) $\Rightarrow$ (i) holds since we can choose an arbitrary affine open cover.

(i) $\Rightarrow$ (ii). Consider an affine open subset $\operatorname{Spec}(B) \subseteq X$ and a chain of ideals
$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$$
in B. Taking associated sheaves yields an ascending chain of quasi-coherent subsheaves of $\mathcal{O}_{\operatorname{Spec}(B)}$. Now let
$$\operatorname{Spec}(B) \cap \operatorname{Spec}(A_i) = \bigcup_j \operatorname{Spec}\left(A_{i,f_{ij}}\right)$$
be an affine open cover, which exists since principal opens form a basis. Since $\operatorname{Spec}(B)$ is quasi-compact (Proposition 2.1.13), the resulting affine open cover
$$\operatorname{Spec}(B) = \bigcup_{i,j} \operatorname{Spec}\left(A_{i,f_{ij}}\right)$$
has a finite subcover. For each i, j, the ascending chains
$$\tilde{\mathfrak{a}}_1\big|_{\operatorname{Spec}(A_{i,f_{ij}})} \subseteq \tilde{\mathfrak{a}}_2\big|_{\operatorname{Spec}(A_{i,f_{ij}})} \subseteq \cdots$$
of quasi-coherent subsheaves of $\mathcal{O}_{\operatorname{Spec}(B)}$ stabilize at some index n_{ij} since the localizations $(A_i)_{f_{ij}}$ are Noetherian and taking associated sheaves is an equivalence of categories (Corollary 2.3.4). Taking $n = \max\{n_{ij}\}$, the original chain stabilizes at n on $\operatorname{Spec}(B)$. ☐

Caution 2.4.4. It is not true that Noetherian schemes X have Noetherian rings of global sections $\Gamma(X, \mathcal{O}_X)$. In [Oja2008], Ojanguren constructs a quasi-projective example of this phenomenon. We will discuss this more after we define Proj.

[Har1977, Exer. II.5.4]

Proposition 2.4.5. ([EGAInew, p. 228]) *Let X be a locally Noetherian scheme and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. The following are equivalent.*

(*a*) *$\mathscr{F}$ is coherent.*

(*b*) *$\mathscr{F}$ is quasi-coherent and of finite type.*

Moreover, every quasi-coherent sub-$\mathcal{O}_X$-module or quotient $\mathcal{O}_X$-module of a coherent sheaf is coherent.

Proof. We can apply Theorem 2.3.6 to an arbitrary affine open cover of X using Proposition 2.4.3. □

Example 2.4.6. As we mentioned before (Remark 2.2.10), an $\mathcal{O}_X$-module of finite type is not necessarily quasi-coherent, even if X is Noetherian. For example, let $X = \mathbf{A}^1_k = \operatorname{Spec}(k[t])$ for a field k, let $\mathscr{F} = j_!(\mathcal{O}_X|_U)$ where $U = X - \{0\}$, and consider the quotient $\mathcal{O}_X/\mathscr{F}$. We showed that $\mathscr{F}$ is not quasi-coherent in Example 2.2.7. The short exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathscr{F} \longrightarrow 0$$

shows that $\mathcal{O}_X/\mathscr{F}$ is also not quasi-coherent, for otherwise $\mathscr{F}$ would be by Corollary 2.3.17(i).

2.4.2 Ideal sheaves, subschemes, and immersions

Before moving on to Serre's criterion for affineness, we have some preliminaries to take care of. So far, we have talked about schemes and sheaves on schemes. In commutative algebra, ideals are the first examples of modules we see. We define the analogous notion for ringed spaces and schemes, which already appeared in the proof of Proposition 2.4.3.

[Har1977, p. 109]

Definition 2.4.7. ([EGAInew, (4.1.3)]) Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of ideals* in $\mathcal{O}_X$ or *ideal sheaf* on X is a sub-$\mathcal{O}_X$-module of $\mathcal{O}_X$.

Example 2.4.8. Sheaves of ideals are not necessarily quasi-coherent, for example $j_!j^*\mathcal{O}_X \hookrightarrow \mathcal{O}_X$ for an open inclusion $j\colon U \hookrightarrow X$ (see Example 2.2.7).

Using ideal sheaves, we can construct closed subschemes of a given scheme.

[Har1977, p. 85, Ex. II.3.2.3, Exer. II.3.11(b), Prop. II.5.9, Cor. II.5.10]

Proposition 2.4.9. ([EGAInew, Proposition 4.1.1]) *Let X be a scheme and let $\mathscr{I}$ be a quasi-coherent ideal sheaf on X. The support $Y \coloneqq \operatorname{Supp}(\mathcal{O}_X/\mathscr{I})$ of the sheaf of rings $\mathcal{O}_X/\mathscr{I}$ is closed. Setting*

$$\mathcal{O}_Y \coloneqq \bigl(\mathcal{O}_X/\mathscr{I}\bigr)\big|_Y,$$

the locally ringed space $(Y, \mathcal{O}_Y)$ is a scheme.

Definition 2.4.10. ([EGAInew, p. 257]) With notation as in Proposition 2.4.9, we say that $(Y, \mathcal{O}_Y)$ is the *(closed) subscheme of* $(X, \mathcal{O}_X)$ *defined by the quasi-coherent ideal sheaf* $\mathscr{I}$.

Proof of Proposition 2.4.9. Since affine open subsets of X form a basis and one can check whether a locally ringed space is a scheme on an open cover, it suffices to prove the case when $X = \operatorname{Spec}(A)$ is affine. In this situation, $\mathscr{I} = \tilde{\mathfrak{a}}$ for an ideal $\mathfrak{a} \subseteq A$ by the fact that $M \mapsto \tilde{M}$ is an exact equivalence of categories (Corollary 2.3.4). By [Har1977, Exercise II.5.6(b)] (Homework 5, Problem 3(b)), we have $Y = \operatorname{Supp}(A/\mathfrak{a})$.

Consider the canonical ring map $\varphi \colon A \to A/\mathfrak{a}$. The induced map on affine schemes satisfies

$$\operatorname{Spec}(\varphi)_*\left(\widetilde{A/\mathfrak{a}}\right) \cong \tilde{A}/\tilde{\mathfrak{a}} = \mathcal{O}_X/\mathscr{I}$$

by Corollary 2.3.5(iii) and Corollary 2.3.4. By the 1-1 correspondence between ideals containing $\mathfrak{a}$ in A and ideals in $A/\mathfrak{a}$ [MurCA, Proposition 1.3.12], we know that $\operatorname{Spec}(\varphi)$ factors as

$$\begin{tikzcd}
\operatorname{Spec}(A/\mathfrak{a}) \arrow[rr, "\operatorname{Spec}(\varphi)"] \arrow[dr, "\sim"'] & & X \\
& Y \arrow[ur, hook] &
\end{tikzcd}$$

where $\operatorname{Spec}(A/\mathfrak{a}) \xrightarrow{\sim} Y$ is a homeomorphism. $\square$

The subspace Y in Definition 2.4.10 is a special case of the following notion.

Definition 2.4.11. ([EGAInew, Définition 4.1.2, (4.1.4)]) We say that a ringed space $(Y, \mathcal{O}_Y)$ is a *(locally closed) subscheme* of a scheme $(X, \mathcal{O}_X)$ if the following conditions hold. $\boxed{2/19}$ [Har1977, p. 85]

(1) Y is a locally closed subspace of X.

(2) If U denotes the largest open subset of X containing Y such that Y is closed in U (i.e., $U = (X - \bar{Y}) \cup Y$), then $(Y, \mathcal{O}_Y)$ is the subscheme of $(U, \mathcal{O}_U)$ defined by a quasi-coherent sheaf of ideals in $\mathcal{O}_U$.

The morphism $(Y, \mathcal{O}_Y) \hookrightarrow (X, \mathcal{O}_X)$ is called the *canonical inclusion morphism*. We say that a subscheme $(Y, \mathcal{O}_Y)$ of $(X, \mathcal{O}_X)$ is a *closed subscheme* of $(X, \mathcal{O}_X)$ if Y is closed in X. We say that a subscheme $(Y, \mathcal{O}_Y)$ of $(X, \mathcal{O}_X)$ is an *open subscheme* of $(X, \mathcal{O}_X)$ if it is an open subspace of $(X, \mathcal{O}_X)$ in the sense of Definition 2.1.27. In this situation, Y is open in X in (1) and the ideal sheaf in (2) is the 0 ideal.

Caution 2.4.12. In the literature, the word "subscheme" can mean "closed subscheme."

[EGAInew, p. 257]
[Har1977, Prop. II.5.9, Cor. II.5.10]

By Proposition 2.4.9 and Definition 2.4.11, we have the 1-1 correspondence

$$\begin{aligned} \left\{\begin{matrix}\text{quasi-coherent ideal sheaves}\\ \mathscr{I} \subseteq \mathcal{O}_X\end{matrix}\right\} &\overset{\text{1-1}}{\longleftrightarrow} \left\{\begin{matrix}\text{closed subschemes}\\ i\colon Y \hookrightarrow X\end{matrix}\right\} \\ \mathscr{I} &\longmapsto \Big(Y, \big(\mathcal{O}_X/\mathscr{I}\big)\big|_Y\Big) \\ \ker\Big(\mathcal{O}_X \to i_*\mathcal{O}_Y\Big) &\longleftarrow\!\shortmid\ (Y, \mathcal{O}_Y). \end{aligned} \tag{2.4.13}$$

quasi-coherent ideal sheaves in $\mathcal{O}_X$ and closed subschemes of X.

Example 2.4.14. Let $(Y, \mathcal{O}_Y)$ be an open subscheme of X. Then, Y is defined by the ideal $0 \subseteq \mathcal{O}_X|_Y$ in (2).

We now define immersions, which are morphisms that "look like" the canonical inclusion map for a subscheme.

[Har1977, p. 85, p. 120]

Definition 2.4.15. ([EGAInew, Définition 4.2.1]) Let $f\colon Y \to X$ be a morphism of schemes. We say that f is a (locally closed) *immersion* (resp. *closed immersion*, *open immersion*) if it factors as

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "g"', "\sim"] & & X \\ & Z \arrow[ur, hook, "j"'] & \end{tikzcd}$$

where g is an isomorphism and Z is a subscheme (resp. closed subscheme, open subscheme) of X and j is the canonical inclusion morphism.

Caution 2.4.16. In [Har1977, p. 120], Hartshorne defines an immersion to be a morphism with a factorization as above where j factors as an open immersion followed by a closed immersion. If we need to distinguish between the two notions, we will call immersion in our sense a *locally closed immersion* and call Hartshorne's definition an *H-immersion* (the latter terminology is not standard). These definitions are not equivalent in general [Stacks, Tag 01QW].

[Har1977, Exer. II.3.11(*b*)]

Remark 2.4.17. ([EGAInew, Remarque 4.2.1.1]) Let X be an affine scheme. By Proposition 2.4.9 and Proposition 2.1.21, a morphism $Y \to X$ of schemes is a closed immersion if and only if Y is an affine scheme and the homomorphism $\Gamma(X, \mathcal{O}_X) \to \Gamma(Y, \mathcal{O}_Y)$ is surjective.

Remark 2.4.18. Our definition of a closed immersion is equivalent to Hartshorne's [Har1977, p. 85], although this takes work to show. The key point is that a consequence of Hartshorne's definition is that the kernel of $\mathcal{O}_X \to i_*\mathcal{O}_Y$ is quasi-coherent. In the Stacks project, part of the definition of a closed immersion is that the ideal sheaf $\ker(\mathcal{O}_X \to i_*\mathcal{O}_Y)$ is locally generated by sections. We define closed immersions following [EGAInew] because it avoids some confusing aspects of the definition of a closed immersion in [Har1977]. For example, the fact that a closed immersion is affine is part of our definition since it is the canonical inclusion of the closed subscheme defined by a quasi-coherent ideal up to isomorphism. See [Stacks, Tag 01IM] for more discussion. See [Har1977, Exercise II.3.11(b)] for the relevant exercise in [Har1977].

2.4.3 Reduced schemes and the reduced subscheme structure

We also want a way to think of closed subsets of a scheme as a scheme itself. By the previous section, what we would like to do is to find an ideal that cuts out a closed subset exactly. On affine open subsets, this is easy to do: just take the radical ideal corresponding to a closed subset. We need to globalize this construction.

Definition 2.4.19. ([EGAInew, Chapitre 0, (4.1.4)]) Let X be a scheme (or more generally, a ringed space). We say that X is *reduced* if $\mathscr{O}_{X,x}$ is reduced for every $x \in X$. [Har1977, p. 82, Exer. II.2.3]

Note that if X is a scheme, then X is reduced if and only if $\mathscr{O}_X(U)$ is reduced for every affine open subset $U \subseteq X$ because being reduced is a local condition [MurCA, Proposition 3.5.2] and by Proposition 2.1.9.

Proposition 2.4.20. ([EGAInew, Proposition 4.5.1]) *Let $(X, \mathscr{O}_X)$ be a scheme and let $\mathscr{B}$ be a quasi-coherent $\mathscr{O}_X$-algebra. There exists a unique quasi-coherent ideal $\mathscr{N}$ in $\mathscr{B}$ such that the stalk $\mathscr{N}_x$ is the nilradical of $\mathscr{B}_x$ for every $x \in X$. If $X = \operatorname{Spec}(A)$ is affine and $\mathscr{B} = \tilde{B}$ for an A-algebra B, then $\mathscr{N} = \tilde{\mathfrak{N}}_B$, where $\mathfrak{N}_B$ is the nilradical of B.* [Har1977, Ex. II.3.0.1]

Proof. The question is local and hence it suffices to prove the affine case. The sheaf $\tilde{\mathfrak{N}}_B$ is a quasi-coherent $\mathscr{O}_X$-module by Corollary 2.3.4. Its stalks are equal to $(\mathfrak{N}_B)_{\mathfrak{p}_x}$ for every $x \in X$ by [MurCA, Proposition 3.5.2] and Proposition 2.1.9. $\square$

Definition 2.4.21. ([EGAInew, p. 268]) Let X be a scheme and let $\mathscr{B}$ be a quasi-coherent $\mathscr{O}_X$-algebra. The quasi-coherent ideal sheaf $\mathscr{N}$ in $\mathscr{B}$ constructed in Proposition 2.4.20 is the *nilradical* of $\mathscr{B}$. We denote by $\mathscr{N}_X$ the nilradical of $\mathscr{O}_X$.

Corollary 2.4.22. ([EGAInew, Corollaire 4.5.2]) *Let X be a scheme. The closed subscheme X_{red} defined by $\mathscr{N}_X$ is the unique reduced subscheme of X whose underlying space is X. It is the "smallest" subscheme of X whose underlying subspace is X, i.e., for every closed subscheme $Z \hookrightarrow X$ such that $\operatorname{sp}(Z) = \operatorname{sp}(X)$, we have $\mathscr{I}_Z \subseteq \mathscr{N}_X$, and hence the canonical inclusion morphism factors uniquely through the inclusion $X_{\mathrm{red}} \hookrightarrow X$:* [Har1977, Exer. II.2.3(b), Exer. II.3.11(c)]

$$\begin{tikzcd} Z \arrow[rr, hook] & & X \\ & X_{\mathrm{red}}. \arrow[ul, dashed, "\exists!"] \arrow[ur, hook] & \end{tikzcd}$$

Proof. For every $x \in X$, we have $\mathscr{N}_x \subseteq \mathfrak{m}_x \subsetneq \mathscr{O}_{X,x}$. We therefore see that $\mathrm{sp}(X_{\mathrm{red}}) = X$.

For the remaining assertions, let Z be a subscheme of X such that $\mathrm{sp}(Z) = \mathrm{sp}(X)$. Then, Z is the closed subscheme defined by a quasi-coherent ideal sheaf $\mathscr{I}$ in $\mathscr{O}_X$ such that $\mathscr{I}_x \subseteq \mathfrak{m}_x \subsetneq \mathscr{O}_{X,x}$ for every $x \in X$. We want to show that $\mathscr{I}_x \subseteq \mathscr{N}_{X,x}$ for every $x \in X$. This would imply that the map $\mathscr{I} \to \mathscr{O}_X/\mathscr{N}_X$ is the 0 map, and hence the inclusion $\mathscr{I} \hookrightarrow \mathscr{O}_X$ factors through $\mathscr{N}_X$ by the universal property of kernels:

$$\begin{tikzcd}
 & & \mathscr{I} \arrow[dl, dashed, "\exists !"'] \arrow[d, hook] \arrow[dr, "0"] & & \\
0 \arrow[r] & \mathscr{N}_X \arrow[r] & \mathscr{O}_X \arrow[r] & \mathscr{O}_X/\mathscr{N}_X \arrow[r] & 0.
\end{tikzcd}$$

Since we can check that $\mathscr{I}_x \subseteq \mathscr{N}_{X,x}$ locally, we can reduce to the affine case. Suppose $X = \mathrm{Spec}(A)$ and $\mathscr{I} = \tilde{I}$ for an ideal $I \subseteq A$. Since $\mathscr{I}_x \subseteq \mathfrak{m}_x$ for every $x \in X$, we see that I is contained in the intersection of all prime ideals in A, and hence $I \subseteq \mathfrak{N}_A$ by a special case of the Scheinnullstellensatz [MurCA, Corollary 1.5.6]. This shows that X_{red} is the smallest subscheme of X whose underlying subspace in X. □

[Har1977, Exer. II.2.3(b)] **Definition 2.4.23.** ([EGAInew, Définition 4.5.3]) Let X be a scheme. The *reduced scheme associated to* X is the scheme X_{red} constructed in Corollary 2.4.22.

Using the universal property in Corollary 2.4.22, we can put a scheme structure on any closed subset of a scheme.

[Har1977, Ex. II.3.2.6] **Proposition 2.4.24.** ([EGAInew, Proposition 4.6.1]) *Let X be a scheme. For every locally closed subset $Y \subseteq \mathrm{sp}(X)$, there exists a unique reduced subscheme of X whose underlying topological space is Y.*

Proof. Uniqueness holds by Corollary 2.4.22. It therefore suffices to show existence.

First consider the case when $X = \mathrm{Spec}(A)$ for a ring A and Y is closed in X. The largest ideal $I(Y)$ such that $V(I(Y)) = Y$ is a radical ideal by [MurCA, Proposition 1.6.3], and hence $A/I(Y)$ is reduced.

We now consider the general case. For every affine open subset $U \subseteq X$ such that $U \cap Y$ is closed in U, we consider the closed subscheme $Y_U \hookrightarrow U$ defined by the quasi-coherent ideal sheaf associated to $I(U \cap Y)$ of $A(U)$. The subscheme Y_U is reduced by the previous paragraph. For every open subset $V \subseteq U$, the open subscheme $Y_U \cap V \hookrightarrow Y_U$ is reduced with underlying topological space $Y \cap V$. Using the universal property in Corollary 2.4.22, we may therefore glue the scheme structures on the Y_U to obtain a scheme structure on Y. □

2.4.4 Irreducible and integral schemes

We collect here one more property of schemes that will appear often.

Definition 2.4.25. ([EGAInew, (2.1.8)]) Let X be a scheme. We say that X is *irreducible* (resp. *connected*) if $\mathrm{sp}(X)$ is irreducible (resp. connected). We say that X is *integral* if it is both irreducible and reduced. [Har1977, p. 82]

Reducedness and integrality can be characterized in terms of *all* open subsets. This is somewhat special: Other properties of schemes (like Noetherianity) will not have the same behavior.

Proposition 2.4.26. ([Har1977, Exerercise II.2.3(a) and Proposition II.3.1]) *Let X be a scheme. Then, X is reduced (resp. integral) if and only if $\mathcal{O}_X(U)$ is reduced (resp. a domain) for every open subset $U \subseteq X$.*

Proof. We start with reducedness. The direction $\Leftarrow$ holds by the fact that reducedness can be checked on affine opens: Being reduced is a local condition [MurCA, Proposition 3.5.2], and the stalks $\mathcal{O}_{X,x}$ are the local rings of the rings associated to affine open subsets by Proposition 2.1.9. For the direction $\Rightarrow$, let $U \subseteq X$ be an open subset. Since affine open subsets form a basis, there exists an open cover

$$U = \bigcup_{i \in I} \mathrm{Spec}(A_i)$$

by affine open subsets. We then have the injective ring map

$$\mathcal{O}_X(U) \hookrightarrow \prod_{i \in I} A_i$$

by the sheaf condition (3). Finally, $\prod_i A_i$ is reduced: If $(x_i)_{i \in I}$ satisfies

$$(x_i)_{i \in I}^n = (x_i^n)_{i \in I} = 0$$

for some $n > 0$, then the fact that each A_i is reduced implies $x_i = 0$.

We now prove the proposition for integrality. Suppose that X is integral and let $U \subseteq X$ be an open subset. Suppose there are elements $f, g \in \mathcal{O}_X(U)$ such that $fg = 0$. By Lemma 2.1.18,

$$\begin{aligned} Y &= U - U_f = \{x \in U \mid f_x \in \mathfrak{m}_x\} \\ Z &= U - U_g = \{x \in U \mid g_x \in \mathfrak{m}_x\} \end{aligned}$$

are closed subsets such that $U = Y \cup Z$. Since X is irreducible, U is irreducible, and hence either $U = Y$ or $U = Z$. After possibly switching the roles of Y and Z, we may assume that $U = Y$. But then, $f \in \mathcal{N}_X(\mathrm{Spec}(A)) = 0$ for every affine open subset $\mathrm{Spec}(A) \subseteq U$ because $f|_{\mathrm{Spec}(A)} \in \mathfrak{p}$ for every prime ideal $\mathfrak{p} \subseteq A$, and hence $f = 0$ by the sheaf condition (3).

Now suppose that $\mathcal{O}_X(U)$ is a domain for every open subset $U \subseteq X$. By the proposition for reducedness, we see that X is reduced. We proceed by contradiction. If X is not irreducible, then there exist two nonempty disjoint open subsets $U_1, U_2 \subseteq X$. By the sheaf conditions (3) and (4),

$$0 \longrightarrow \mathcal{O}_X(U_1 \cup U_2) \xrightarrow{\binom{1}{1}} \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2) \xrightarrow{(1\ -1)} \mathcal{O}_X(U_1 \cap U_2) = 0$$

is exact. Since $\mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$ is not a domain, we are done. □

2.4.5 Serre's criterion for affineness and the qcqs lemma

Cartan's Theorem B (Theorem 2.3.7) says that affine schemes have vanishing higher cohomology for quasi-coherent sheaves. We were able to use this to prove that certain schemes are not affine (Example 2.1.33). This raises the question: Does vanishing of higher cohomology for quasi-coherent sheaves characterize affine schemes? The answer is yes, by Serre's criterion for affineness below.

As was the case with Cartan's Theorem B, the complex analytic case predates the algebraic case: For complex manifolds, Serre showed that cohomology vanishing for coherent sheaves characterizes Stein manifolds [SHC51/52, Exposé XX, n° 2]. The case for algebraic varieties is also due to Serre [Ser1957, Théorème 1].

[Stacks, Tag 01XF]
[Har1977, Thm. III.3.7, Exer. II.2.17(b)]

Theorem 2.4.27. (Serre's criterion for affineness [EGAII, Théorème 5.2.1; EGAIV$_1$, (1.7.17)]**)** *Let X be a quasi-compact scheme. Set $A = \Gamma(X, \mathcal{O}_X)$. The following conditions are equivalent.*

(a) *X is affine.*

(b) *There exists a family of elements $f_\alpha \in A$ such that the open sets*

$$X_{f_\alpha} \coloneqq \left\{ x \in X \mid f_{\alpha,x} \notin \mathfrak{m}_x \right\}$$

are affine and the f_α generate the unit ideal in A.

(c) *The functor $\Gamma(X, \,\cdot\,)$ is exact on $\mathsf{QCoh}(\mathcal{O}_X)$.*

(c') *The functor $\Gamma(X, \,\cdot\,)$ is exact on short exact sequences*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

in $\mathsf{QCoh}(X)$ where $\mathscr{F}$ is a sub-$\mathcal{O}_X$-module of $\mathcal{O}_X^q$ for some non-negative integer q.

(d) *$H^i(X, \mathscr{F}) = 0$ for every quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$ and for every $i > 0$.*

(d') *$H^1(X, \mathscr{I}) = 0$ for every quasi-coherent ideal sheaf $\mathscr{I} \subseteq \mathcal{O}_X$.*

[Har1977, Exer. II.2.16(a)]

Note that the X_{f_α} are open by Lemma 2.1.18. Alternatively, since we know that Proposition 2.1.21 is true, the X_{f_α} are open because they are the preimages of

$$D(f_\alpha) \subseteq \operatorname{Spec}\bigl(\Gamma(X, \mathcal{O}_X)\bigr)$$

under the morphism $X \to \operatorname{Spec}(\Gamma(X, \mathcal{O}_X))$ from Proposition 2.1.21.

Remark 2.4.28. It is tempting to try to use the dimension shifting argument from Proposition 1.4.28 to show that (c) $\Leftrightarrow$ (d). This is not so easy because we have not shown that $\mathsf{QCoh}(\mathcal{O}_X)$ has enough injectives. Hartshorne gets around this issue by assuming that X is Noetherian (see [Har1977, Corollary III.3.6]).

It is known that $\mathsf{QCoh}(\mathcal{O}_X)$ has enough injectives because it is Grothendieck Abelian and has a generator. This is a relatively new result due to Gabber [Stacks, Tag 077K], first stated without proof in [Con2000, Lemma 2.1.7].

To prove Theorem 2.4.27 in the quasi-compact quasi-separated case, one can instead use [SGA6, Exposé II, Lemme 3.1], which says that $\mathsf{QCoh}(\mathcal{O}_X)$ is a Grothendieck Abelian category with a generator when X is quasi-compact and quasi-separated.

To prove Serre's criterion (Theorem 2.4.27), we need some preliminaries.

Definition 2.4.29. ([EGAInew, Définition 6.1.3 and Proposition 6.1.12]) Let X be a scheme. We say that X is *quasi-separated* if, for every pair U, V of quasi-compact open subsets of X, the intersection $U \cap V$ is quasi-compact.

We say that X is *qcqs* if it is both quasi-compact and quasi-separated.

Example 2.4.30. Let X be a scheme.

(1) If X is Noetherian, or more generally, if $\mathrm{sp}(X)$ is Noetherian, then X is qcqs.

(2) If X is affine, then X is quasi-separated. This holds because if U, V are quasi-compact open subsets of X, then we can find finite affine open covers $U = \bigcup D(f_i)$ and $V = \bigcup D(g_j)$, and

$$U \cap V = \bigcup_{i,j} D(f_i) \cap D(g_j) = \bigcup_{i,j} D(f_i g_j)$$

is a finite cover by affine opens, which are quasi-compact.

Remark 2.4.31. Definition 2.4.29 is a preliminary definition for quasi-separatedness. It is equivalent to the general topological definition in [$\mathrm{SGA4}_2$, Exposé VI, Définition 1.13] because affine open subsets form a basis for the Zariski topology on a scheme. The definition for schemes is usually stated in terms of the diagonal morphism (to be defined later).

There are at least three alternatives to the terminology "qcqs." In [$\mathrm{SGA4}_2$, Exposé VI, Exemple 1.22], Grothendieck and Verdier proposed the terminology *coherent*. In [Kem1980, p. 640], Kempf proposed the terminology *quasi-Noetherian*. In [ATJLL1997, p. 18], Alonso Tarrío, Jeremías López, and Lipman proposed the terminology *concentrated*.

Quasi-separatedness can be checked on a single open cover.

Lemma 2.4.32. ([Stacks, Tag 01KO]) *Let X be a scheme. The following are equivalent:* [EGAInew, Prop. 6.1.12]

(*a*) *X is quasi-separated, that is, for every pair U, V of quasi-compact open subsets of X, the intersection $U \cap V$ is quasi-compact.*

(*b*) *For every pair U, V of affine open subsets of X, the intersection $U \cap V$ is quasi-compact.*

(*c*) *There exists an affine open cover $X = \bigcup_\alpha X_\alpha$ such that $X_\alpha \cap X_\beta$ is quasi-compact for all α, β.*

Proof. $(a) \Rightarrow (b) \Rightarrow (c)$ are clear. It therefore suffices to show that $(c) \Rightarrow (a)$.

Consider a pair U, V of quasi-compact open subsets. For each $x \in U$, there exists an α and a distinguished open subset $D_{X_\alpha}(f_x) \subseteq X_\alpha$ such that $x \in D_{X_\alpha}(f_x)$. Since U is quasi-compact, we can find finitely many $x_1, x_2, \dots, x_m$ such that

$$U = \bigcup_{i=1}^{m} D_{X_{\alpha_i}}(f_i).$$

Similarly, we can write

$$V = \bigcup_{j=1}^{n} D_{X_{\beta_j}}(g_j).$$

The intersection $U \cap V$ is

$$U \cap V = \bigcup_{i,j} \Big(D_{X_{\alpha_i}}(f_i) \cap D_{X_{\beta_j}}(g_j) \Big).$$

The members of this intersection are affine because $D_{X_{\alpha_i}}(f_i) \cap D_{X_{\beta_j}}(g_j)$ is the distinguished open set defined by $g_j|_{D_{X_{\alpha_i}}(f_i)}$ in the affine scheme $D_{X_{\alpha_i}}(f_i)$ by (2.1.23). Thus, $U \cap V$ is the union of finitely many quasi-compact open sets, and is therefore quasi-compact. $\square$

The following theorem allows us to "clear denominators" even when we are not working on an affine scheme. This is called the "qcqs lemma" in [Vak2025, Lemma 6.2.9].

[Har1977, Lem. II.5.14]

Theorem 2.4.33. (Extending sections of quasi-coherent sheaves [EGAInew, Théorème 6.8.1]**)** *Let X be a scheme, let $\mathscr{L}$ be an invertible $\mathcal{O}_X$-module on X, let $f \in \Gamma(X, \mathscr{L})$, and let*

$$X_f \coloneqq \big\{ x \in X \;\big|\; f_x \notin \mathfrak{m}_x \mathscr{L}_x \big\}.$$

Consider a quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$ on X.

(i) Suppose that X is quasi-compact and let $s \in \Gamma(X, \mathscr{F})$ such that $s|_{X_f} = 0$. Then, for some $n > 0$, the section

$$s \otimes f^{\otimes n} \in \Gamma\Big(X, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n}\Big)$$

satisfies $s \otimes f^{\otimes n} = 0$.

(ii) Suppose that X is qcqs. For every section $t \in \Gamma(X_f, \mathscr{F})$, there exists $n > 0$ such that the section

$$t \otimes f^{\otimes n} \in \Gamma\Big(X_f, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n}\Big)$$

extends to a section $t' \in \Gamma(X, \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes n})$.

[EGAInew, Cor. 6.8.3]

Thus, if X is qcqs and $\mathscr{L} = \mathcal{O}_X$, then the map

$$\Gamma(X, \mathscr{F})_f \longrightarrow \Gamma\Big(X_f, \mathscr{F}\Big)$$

induced by the universal property of localization is bijective.

This generalizes one of the implications in Theorem 2.3.2 and also generalizes [Har1977, Exercise II.2.16(b),(c)].

Implicit in the statement of Theorem 2.3.2 is the following generalization of Lemma 2.1.18:

Lemma 2.4.34. ([EGAI, (**0**, 5.5.2); EGAInew, (**0**, 4.1.9); Stacks, Tag 01HZ]) *Let $(X, \mathscr{O}_X)$ be a locally ringed space, let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and let $f \in \Gamma(X, \mathscr{L})$. Then, X_f is open in X.* [Har1977, Exer. II.2.16(a)]

Proof. To show that X_f is open, it suffices to note that for every $x \in X$, there is an open neighborhood $U \ni x$ on which $\mathscr{L}|_U \cong \mathscr{O}_U$ by definition (see Homework 6, Problem 1). Now Lemma 2.1.18 implies $U \cap X_f$ is open. □

Proof of Theorem 2.4.33. (i). Since X is quasi-compact, we may cover X by finitely many affine open subsets U_i such that $\mathscr{L}|_{U_i} \cong \mathscr{O}_{U_i}$. We may therefore reduce to the case when X is affine and $\mathscr{L} = \mathscr{O}_X$, which was shown in Theorem 2.3.2.

(ii). Since X is qcqs, there exists a finite covering by affine open subsets U_i such that $U_i \cap U_j$ is quasi-compact for every i, j and such that $\mathscr{F}|_{U_i}$ is free for every i. By Theorem 2.3.2, there exists an integer n such that the sections $(t \otimes f^{\otimes n})|_{U_i \cap X_f}$ extend to sections

$$t_i \in \Gamma\Big(U_i, \mathscr{F} \otimes \mathscr{L}^{\otimes n}\Big)$$

for every i. Now set $t_{i|j} \coloneqq t_i|_{U_i \cap U_j}$. By definition, we have

$$\Big(t_{i|j} - t_{j|i}\Big)\big|_{X_f \cap U_i \cap U_j} = 0.$$

Since $U_i \cap U_j$ is quasi-compact, by (i), there exists an integer m (independent of i, j) such that

$$\Big(t_{i|j} - t_{j|i}\Big) \otimes f^{\otimes m} = 0$$

for all i, j. We can therefore glue the sections $t_i \otimes f^{\otimes m}$ together to obtain a section

$$t' \in \Gamma\Big(X, \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes (n+m)}\Big)$$

extending $t \otimes f^{\otimes (n+m)}$. □

Proof of Theorem 2.4.27. We proceed in steps.

Step 1. (a) $\Leftrightarrow$ (b). *We do not need quasi-compactness for this equivalence.* [Stacks, Tag 01QF]

The direction $\Rightarrow$ holds by choosing $\{f_\alpha\} = \{1\}$. For $\Leftarrow$, write

$$1 = \sum_\alpha g_\alpha f_\alpha$$

for some $g_\alpha \in \Gamma(X, \mathscr{O}_X)$. Then, the f_α that appear in this sum generate the unit ideal, and hence we may assume there are finitely many f_α. We have

$X = \bigcup_\alpha X_{f_\alpha}$ for otherwise, we would have $1 \in \mathfrak{m}_x$ for some $x \in X$. The X_{f_α} are quasi-compact since they are affine. Since X is a finite union of quasi-compact open sets, X is quasi-compact.

We now consider the canonical morphism

$$\begin{aligned} j\colon X &\longrightarrow \operatorname{Spec}\Bigl(\Gamma(X, \mathcal{O}_X)\Bigr) \\ x &\longmapsto \bigl\{ f \in \Gamma(X, \mathcal{O}_X) \bigm| f_x \in \mathfrak{m}_x \bigr\} \end{aligned} \tag{2.4.35}$$

from Proposition 2.1.21. Note that $j^{-1}(D(f_\alpha)) = X_{f_\alpha}$ by construction of the map in Proposition 2.1.21. We claim that to show that j is an isomorphism, it suffices to show that

$$j|_{X_{f_\alpha}} : X_{f_\alpha} \longrightarrow D(f_\alpha)$$

is an isomorphism for every α. First, the $D(f_\alpha)$ cover $\operatorname{Spec}(\Gamma(X, \mathcal{O}_X))$ because the f_α generate the unit ideal. Second, the inverses defined on each $D(f_\alpha)$ glue to a homeomorphism of X with $\operatorname{Spec}(\Gamma(X, \mathcal{O}_X))$. To check that the map $j^\#$ on structure sheaves is an isomorphism, it suffices to show it is an isomorphism on an open cover.

It remains to show that $j|_{X_{f_\alpha}}$ is an isomorphism for every α. Since both X_{f_α} and $D(f_\alpha)$ are affine, it suffices to show that

$$\varphi_\alpha \colon \Gamma(X, \mathcal{O}_X)_{f_\alpha} \longrightarrow \Gamma(X_{f_\alpha}, \mathcal{O}_X)$$

is an isomorphism by Corollary 2.1.24. This holds by Theorem 2.4.33 for $\mathscr{F} = \mathscr{L} = \mathcal{O}_X$. Note that for each α, β, the intersection

$$X_{f_\alpha} \cap X_{f_\beta} = D\Bigl(f_\beta|_{X_{f_\alpha}}\Bigr) \subseteq X_{f_\alpha}$$

of affine open sets is also affine and, in particular, quasi-compact. This shows that X is quasi-separated by Lemma 2.4.32.

Step 2. $(a) \Rightarrow (c) \Rightarrow (c') \Rightarrow (b)$.

$(a) \Rightarrow (c)$ holds by Corollary 2.3.4. $(c) \Rightarrow (c')$ holds since (c') is a special case of (c). It therefore suffices to show $(c') \Rightarrow (b)$. If $X = \varnothing$, we have $X = \operatorname{Spec}(0)$. We may therefore assume that X is nonempty.

[Stacks, Tag 005E]
[Sch2005, Prop. 4.1]

Substep 2.1. Let X be a nonempty quasi-compact topological space that is T_0. Then, X has a closed point. In particular, every point in a quasi-compact scheme has a closed point in its closure.

Consider the set

$$\mathcal{T} = \left\{ \overline{\{x\}} \subseteq X \;\middle|\; x \in X \right\}$$

partially ordered by inclusion, which is nonempty since X is nonempty. We claim that $\mathcal{T}$ has a minimal element. By Zorn's lemma [Kur1922; Zor1935], it suffices to show that every descending chain

$$Z_1 \supseteq Z_2 \supseteq \cdots \tag{2.4.36}$$

in $\mathcal{T}$ has a lower bound in $\mathcal{T}$. Let $Z = \bigcap_i Z_i$. We claim that $Z \neq \varnothing$, and hence choosing a point $z \in Z$, the closure $\overline{\{z\}} \in \mathcal{T}$ is a lower bound for the chain. Suppose that $Z = \varnothing$. Then,

$$X = \bigcup_i (X - Z_i)$$

is an open cover. Since X is quasi-compact, there is a finite subcover. Thus, the descending chain (2.4.36) stabilizes at some $Z_i = \overline{\{x_i\}}$, which is nonempty, a contradiction.

By Zorn's lemma [Kur1922; Zor1935], there exists a minimal element $\overline{\{x\}} \in \mathcal{T}$. We claim that $\{x\} = \overline{\{x\}}$, that is, x is closed in X. If not, choose $y \in \overline{\{x\}} - x$. Since X is T_0, either there exists an open neighborhood U_x of x that does not contain y, or there exists an open neighborhood U_y of y that does not contain x. In the first case, we have $\overline{\{x\}} \supsetneq \overline{\{y\}}$, contadicting minimality. The second case is impossible since it would imply $y \notin \overline{\{x\}}$.

The "in particular" statement holds by applying what we showed above to the closure of a given point.

Substep 2.2. Assume (c'). Then, for every closed point $x \in X$ and every open neighborhood U of x, there exists $f \in A = \Gamma(X, \mathcal{O}_X)$ such that $x \in X_f \subseteq U$.

By Proposition 2.4.24, there exist quasi-coherent ideal sheaves $\mathscr{I}, \mathscr{I}'$ defining $X - U$ and $(X - U) \cup \{x\}$ with the reduced scheme structures, respectively. Consider the short exact sequence

$$0 \longrightarrow \mathscr{I}' \longrightarrow \mathscr{I} \longrightarrow \mathscr{I}'' \longrightarrow 0$$

in $\mathsf{QCoh}(\mathcal{O}_X)$, where we note that $\operatorname{Supp}(\mathscr{I}'') = \{x\}$ and $\mathscr{I}''_x = k(x) \coloneqq \mathcal{O}_{X,x}/\mathfrak{m}_x$. By (c'), the map

$$\Gamma(X, \mathscr{I}) \longrightarrow \Gamma(X, \mathscr{I}'')$$

is surjective. Thus, there exists $f \in \Gamma(X, \mathscr{I}) \subseteq A$ such that $f(x) = 1$ and $f(y) = 0$ for all $y \in X - U$, where $f(\,\cdot\,)$ denotes the image of f in the residue field at a point.

Substep 2.3. $(c') \Rightarrow (b)$. 2/24

For every closed $x \in X$, which exist by Subtep 2.1, choose an affine open neighborhood U of x. By Subtep 2.2, we can find an open neighborhood X_{f_x} of x such that $x \in X_{f_x} \subseteq U$. Since U is affine, we see that X_{f_x} is affine. Setting

$$Z = \bigcup_{\substack{x \in X \\ \text{closed}}} X_{f_x},$$

we see that Z contains all closed points of X. We claim that $Z = X$. Since X is quasi-compact and T_0, the closed subspace $X - Z$ is also quasi-compact and T_0. If $Z \subsetneq X$, then $X - Z$ would contain a closed point by Subtep 2.1. Such a closed point would also be closed in X, which contradicts the construction of

Z. Next, since X is quasi-compact, there exist finitely many $f_i \in A$ such that $X = \bigcup_i X_{f_i}$. Consider the morphism

$$\mathcal{O}_X^n \longrightarrow \mathcal{O}_X$$

defined by these f_i. Since for every $x \in X$, at least one of the $f_{i,x}$ is a unit, we see that this morphism is surjective. Now consider the short exact sequence

$$0 \longrightarrow \mathscr{R} \longrightarrow \mathcal{O}_X^n \longrightarrow \mathcal{O}_X \longrightarrow 0$$

where $\mathscr{R}$ is a quasi-coherent sub-$\mathcal{O}_X$-module of $\mathcal{O}_X^n$. Then, (c') implies

$$\Gamma(X, \mathcal{O}_X^n) \xrightarrow{(\, f_1 \ f_2 \ \cdots \ f_n \,)} \Gamma(X, \mathcal{O}_X) = A$$

is surjective, showing (b).

Step 3. $(a) \Rightarrow (d) \Rightarrow (d') \Rightarrow (c')$.

$(a) \Rightarrow (d)$ is Cartan's Theorem B (Theorem 2.3.7), and $(d) \Rightarrow (d')$ holds since (d') is a special case of (d). It remains to show $(d') \Rightarrow (c')$.

Since $\mathscr{F}' \subseteq \mathscr{F} \subseteq \mathcal{O}_X^q$, we know that $\mathscr{F}'$ is a quasi-coherent sub-$\mathcal{O}_X$-module of $\mathcal{O}_X^q$. The filtration

$$0 \subseteq \mathcal{O}_X \subseteq \mathcal{O}_X^2 \subseteq \cdots \subseteq \mathcal{O}_X^q$$

defines a filtration $\mathscr{F}_k' = \mathscr{F}' \cap \mathcal{O}_X^k$ on $\mathscr{F}_k'$ where $k \in \{0, 1, \dots, q\}$. Each $\mathscr{F}_k'$ is quasi-coherent since it is the kernel of the morphism

$$\mathscr{F}' \longrightarrow \mathcal{O}_X^q / \mathcal{O}_X^k.$$

We claim that

$$\mathscr{F}_{k+1}' / \mathscr{F}_k' \hookrightarrow \mathcal{O}_X,$$

for all k, i.e., $\mathscr{F}_{k+1}' / \mathscr{F}_k'$ is an ideal sheaf for every k. This follows from applying the snake lemma to the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & \mathscr{F}' \arrow[r, "="] & \mathscr{F}' \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{O}^q/\mathcal{O}_k \arrow[r] & \mathcal{O}^q/\mathcal{O}_{k+1} \arrow[r] & 0
\end{tikzcd}$$

which yields the exact sequence

$$\mathscr{F}_k' \longrightarrow \mathscr{F}_{k+1}' \longrightarrow \mathcal{O}_X.$$

Now (d') implies that $H^1(X, \mathscr{F}_{k+1}' / \mathscr{F}_k') = 0$ for all k. By inducing on k and using the exact sequence

$$H^1(X, \mathscr{F}_k') \longrightarrow H^1(X, \mathscr{F}_{k+1}') \longrightarrow H^1(X, \mathscr{F}_{k+1}' / \mathscr{F}_k') = 0$$

this shows that $H^1(X, \mathscr{F}_k') = 0$ for all k. In particular, $H^1(X, \mathscr{F}') = 0$, showing (c'). □

Remark 2.4.37. In our discussion of Cartan's Theorem B (Theorem 2.3.7), I mentioned that there are versions of Cartan's Theorem B in other contexts, for example for rigid analytic spaces. For rigid analytic spaces, it is an *open question* whether there exists a cohomological characterization of being affinoid, which is the rigid analytic analogue of affineness. See, for example, [msteve2016]. Liu [Liu1988] constructed an example of an non-affinoid space such that $H^i(X, \mathscr{F}) = 0$ for all coherent $\mathscr{F}$ and all $i > 0$. Gabber [Con2006, Example 2.1.6] constructed an example of a quasi-coherent sheaf on an affinoid space such that $H^1(X, \mathscr{F}) \neq 0$, where quasi-coherence is defined as in [Con2006, Definition 2.1.1].

2.5 The Proj construction

Now that we have developed some tools that apply to schemes in general, it is time to turn our attention to more special constructions of schemes that will ultimately be our main interest for much of the rest of the course.

2.5.1 Proj as a topological space

Last semester, a very important construction was that of projective space and varieties in projective space. We want an analogous construction for schemes.

Definition 2.5.1. (Proj as a set [EGAII, (2.3.1)]**)** Let S be an $\mathbf{N}$-graded ring. The *homogeneous spectrum* or the *Proj* of S is the set

[Har1977, p. 76]

$$\operatorname{Proj}(S) \coloneqq \left\{ \mathfrak{p} \in \operatorname{Spec}(S) \;\middle|\; \begin{array}{c} \mathfrak{p} \text{ is homogeneous} \\ S_+ \not\subseteq \mathfrak{p} \end{array} \right\}$$

where we recall that $S_+ = \bigoplus_{d>0} S_d$.

To define the topology on $\operatorname{Proj}(S)$, we define certain subsets of $\operatorname{Proj}(S)$.

Definition 2.5.2. (The topology on $\operatorname{Proj}(S)$ [EGAII, (2.3.2)]**)** Let S be an $\mathbf{N}$-graded ring and let $E \subseteq S$ be a subset. We set

[Har1977, p. 76] Hartshorne omits the subscripts "+." I prefer to keep them in to distinguish between subsets of $\operatorname{Spec}(S)$ and of $\operatorname{Proj}(S)$.

$$V_+(E) \coloneqq \{\mathfrak{p} \in \operatorname{Proj}(S) \mid \mathfrak{p} \supseteq E\}.$$

Viewing $\operatorname{Proj}(S)$ as a subset of $\operatorname{Spec}(S)$, we see that

$$V_+(E) = V(E) \cap \operatorname{Proj}(S) \subseteq \operatorname{Spec}(S).$$

[Har1977, Lem. II.2.4]

By Lemma 2.1.12, we see that

$$\begin{aligned} V_+(0) &= \operatorname{Proj}(S) \\ V_+(S) &= V_+(S_+) = \varnothing \\ V_+\Big(\bigcup_\lambda E_\lambda\Big) &= \bigcap_\lambda V_+(E_\lambda) \\ V_+(EE') &= V_+(E) \cup V_+(E'). \end{aligned}$$

We give $\mathrm{Proj}(S)$ the subspace topology induced by $\mathrm{Spec}(S)$. In other words, the closed sets are the sets $V_+(E)$. The *distinguished open sets*

$$D_+(f) \coloneqq D(f) \cap \mathrm{Proj}(S) = \mathrm{Proj}(S) - V_+(f)$$

[EGAII, Prop. 2.3.4]
[Har1977, Prop. 2.5(b)]

for f homogeneous of degree $d \geq 1$ form a basis for this topology, and we have

$$D_+(fg) = D_+(f) \cap D_+(g).$$

To define sheaves on $\mathrm{Proj}(S)$, we identify certain principal open subsets with affine schemes. Recall that if $f \in S_+$ is a homogeneous element of degree $d > 0$, then

$$S_{(f)} = \left\{ \frac{x}{f^n} \,\middle|\, \deg(x) = dn,\, n \geq 0 \right\} \subseteq S_f.$$

[Har1977, Prop. II.2.5(b)]

Proposition 2.5.3. ([EGAII, (2.3.5) and Proposition 2.3.6]) *Let S be an $\mathbf{N}$-graded ring and let $f \in S_+$ be a homogeneous element of degree $d > 0$. Then, the map*

$$\begin{aligned} \psi_f \colon D_+(f) &\longrightarrow \mathrm{Spec}\big(S_{(f)}\big) \\ \mathfrak{p} &\longmapsto (\mathfrak{p}S_f) \cap S_{(f)} \end{aligned}$$

is a homeomorphism and fits into the commutative diagram

$$\begin{tikzcd} D_+(f) \arrow[r, "\psi_f"] & \mathrm{Spec}\big(S_{(f)}\big) \\ D_+(fg) \arrow[u, hook] \arrow[r, "\psi_{fg}"] & \mathrm{Spec}\big(S_{(fg)}\big) \arrow[u] \end{tikzcd} \tag{2.5.4}$$

for every homogeneous element $g \in S_+$. Here, the left vertical map is the inclusion and the right vertical map is induced by the ring map $S_{(f)} \to S_{(fg)}$.

Proof. We first show that the diagram (2.5.4) commutes. Suppose $\mathfrak{p} \in D_+(fg)$. We need to show that

$$(\mathfrak{p}S_f) \cap S_{(f)} = \big((\mathfrak{p}S_{fg}) \cap S_{(fg)}\big) \cap S_{(f)}.$$

The inclusion $\subseteq$ holds by definition of extension/contraction of ideals. Conversely, suppose

$$\frac{x}{f^n} \in \big((\mathfrak{p}S_{fg}) \cap S_{(fg)}\big) \cap S_{(f)}.$$

Since $\mathfrak{p} \in D_+(fg)$, we have $fg \notin \mathfrak{p}$. Thus,

$$\frac{g^n x}{(fg)^n} \in \big((\mathfrak{p}S_{fg}) \cap S_{(fg)}\big) \cap S_{(f)}.$$

Since $g^n x \in \mathfrak{p}$ but $g \notin \mathfrak{p}$, we have $x \in \mathfrak{p}$.

Next, we show that ψ_f is a homeomorphism. We will show that the ring maps [Stacks, Tag 00JP(6)]

$$S \longrightarrow S_f \longleftarrow S_{(f)}$$

induce homeomorphisms

$$D_+(f) \longleftarrow \{\mathfrak{p} \in \operatorname{Spec}(S_f) \mid \mathfrak{p} \text{ is } \mathbf{Z}\text{-graded}\} \longrightarrow \operatorname{Spec}\big(S_{(f)}\big).$$

The first map is a homeomorphism since it is obtained from the homeomorphism $D(f) \leftarrow \operatorname{Spec}(S_f)$ by restricting to homogeneous prime ideals, that is, prime ideals that are generated by homogeneous elements. For the second map, we claim that [Stacks, Tag 00JO]

$$\begin{aligned} \operatorname{Spec}\big(S_{(f)}\big) &\longrightarrow \{\mathfrak{p} \in \operatorname{Spec}(S_f) \mid \mathfrak{p} \text{ is } \mathbf{Z}\text{-graded}\} \\ \mathfrak{p}_0 &\longmapsto \sqrt{\mathfrak{p}_0 S_f} \end{aligned}$$

is a continuous inverse. First, we show that the ideal $\sqrt{\mathfrak{p}_0 S_f}$ is prime. If $ab \in \mathfrak{p}_0 S_f$ with a, b homogeneous, then

$$\frac{a^d b^d}{f^{\deg(a)+\deg(b)}} \in \mathfrak{p}_0.$$

Since $\mathfrak{p}_0$ is prime, we have either $a^d/f^{\deg(a)} \in \mathfrak{p}_0$ or $b^d/f^{\deg(b)} \in \mathfrak{p}_0$, and hence either $a^d \in \mathfrak{p}_0 S_f$ or $b^d \in \mathfrak{p}_0 S_f$. Thus, $\sqrt{\mathfrak{p}_0 S_f}$ is prime. Next, we note that

$$\sqrt{\mathfrak{p}_0 S_f} \cap S_{(f)} = \sqrt{\mathfrak{p}_0 S_f \cap S_{(f)}} = \mathfrak{p}_0$$

since $\mathfrak{p}_0$ is radical and $S_{(f)} \hookrightarrow S_f$ is the inclusion of the degree 0 direct summand of S_f. Thus, the map $\mathfrak{p}_0 \mapsto \sqrt{\mathfrak{p}_0 S_f}$ is an inverse.

It remains to show that $\mathfrak{p} \mapsto \mathfrak{p} \cap S_{(f)}$ is continuous. We claim the image of

$$\{\mathfrak{p} \in \operatorname{Spec}(S_f) \mid \mathfrak{p} \text{ is } \mathbf{Z}\text{-graded}\} \cap D(g)$$

is open for any $g \in S_f$. Write $g = \sum_i g_i$ for $\deg(g_i) = i$. The image of this set is

$$\bigcup_i D\left(\frac{g_i^d}{f^i}\right),$$

which is open. □

2.5.2 Sheaves associated to graded modules

We now discuss the projective analogue of the construction $M \mapsto \tilde{M}$ we used to define the structure sheaf on $\operatorname{Spec}(A)$.

Definition 2.5.5. ([EGAII, Définition 2.5.3]) Let S be an $\mathbf{N}$-graded ring and consider a $\mathbf{Z}$-graded S-module M. The *sheaf $\tilde{M}$ associated to M* is the sheaf associated to the presheaf defined on distinguished open sets $D_+(f)$ as [Har1977, p. 116]

$$D_+(f) \longmapsto M_{(f)}.$$

[Har1977, p. 76] The *structure sheaf* on $\operatorname{Proj}(S)$ is $\tilde{S}$. By construction, we see that $\tilde{M}|_{D_+(f)} \cong \widetilde{M_{(f)}}$, and hence $(\operatorname{Proj}(S), \tilde{S})$ is a scheme.

Proposition 2.5.6. ([EGAII, Proposition 2.4.6]) *Let A be a ring and let S be an $\mathbf{N}$-graded A-algebra, which means these exists a graded ring map $A \to S$ where A is concentrated in degree 0. Then, there is a natural morphism* $\operatorname{Proj}(S) \to \operatorname{Spec}(A)$.

In this situation (and similar ones in the future), we say that $\operatorname{Proj}(S)$ is a scheme *over* $\operatorname{Spec}(A)$ or *over* A for simplicity.

Proof. For homogeneous elements $f, g \in S$, we have the commutative diagram

$$\begin{tikzcd}
S_{(f)} \arrow[r] & S_{(fg)} & S_{(g)} \arrow[l] \\
& A. \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

The morphisms $\operatorname{Spec}(S_{(f)}) \to \operatorname{Spec}(A)$ therefore glue together to yield a morphism $\operatorname{Proj}(S) \to \operatorname{Spec}(A)$. $\square$

[Har1977, Ex. II.2.5.1] **Example 2.5.7. (Projective n-space)** Let A be a ring and let $n \geq 0$. The *projective n-space over A* is

$$\mathbf{P}^n_A \coloneqq \operatorname{Proj}\Big(A[x_0, x_1, \dots, x_n]\Big).$$

Using this construction, we can construct an example of a Noetherian scheme with non-Noetherian global sections. Below, the exact choices of A, B, L, D are not too relevant, but I have chosen specific ones for concreteness.

We use the language of regular functions to mean sections of the structure sheaf defined on some open set.

Example 2.5.8. ([Oja2008]) Let k be an algebraically closed field. Let $A = V_+(x)$ and $B = V_+(y)$ be two planes in $\mathbf{P}^3_k = \operatorname{Proj}(k[w, x, y, z])$ intersecting along $L = V_+(x, y)$. Let $X = A \cup B$. Let $D = V_+(x, z) \subseteq A$ be another line, distinct from L, such that $D \cap L = \{P\}$ for a unique closed point P. See Figure 2.5 for an illustration.

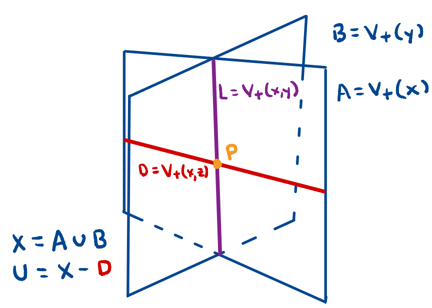

Figure 2.5: Ojanguren's example of a Noetherian scheme with non-Noetherian global sections [Oja2008].

We claim that $U = X - D$ is such that $\Gamma(U, \mathscr{O}_X)$ is non-Noetherian. Note that since U is a subscheme of the Noetherian scheme $\mathbf{P}^3_k$, it is Noetherian. Since $U = (A - D) \cup (B - D)$, we first describe what the restrictions of f to $A - D$ and $B - D$ are.

First, we consider $f|_{B-D}$. Every $f \in \Gamma(U, \mathscr{O}_U)$ is constant on

$$B - D = B - \{P\} \cong \mathbf{P}^2_{[w:x:z]} - \big\{[1:0:0]\big\}$$

since in the diagram

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] \\
\Gamma\Big(B - \{P\}, \mathscr{O}_X\Big) \arrow[rr, equal] \arrow[d, "\binom{1}{1}"'] & & \Gamma\Big(B - \{P\}, \mathscr{O}_X\Big) \arrow[d, "\binom{1}{1}"] \\
\Gamma\Big(B - V_+(x), \mathscr{O}_X\Big) \oplus \Gamma\Big(B - V_+(z), \mathscr{O}_X\Big) \arrow[rr, equal] \arrow[d, "(1\ -1)"'] & & k\left[\frac{w}{x}, \frac{z}{x}\right] \oplus k\left[\frac{w}{z}, \frac{x}{z}\right] \arrow[d, "(1\ -1)"] \\
\Gamma\Big(B - V_+(xz), \mathscr{O}_X\Big) \arrow[rr, hook] & & k\left(\frac{w}{x}, \frac{w}{z}\right)
\end{tikzcd}$$

the bottom right map $(1\ -1)$ is injective, and the intersection of $k[\frac{w}{x}, \frac{z}{x}]$ and $k[\frac{w}{z}, \frac{x}{z}]$ in $k(\frac{w}{x}, \frac{w}{z})$ is just k.

Next, we consdier $f|_{A-D}$. First,

$$A_0 \coloneqq A - D = V_+(x) - V_+(x, z) \cong \mathbf{P}^2_{[w:y:z]} - V_+(z) \cong \mathbf{A}^2_{\frac{w}{z}, \frac{y}{z}}.$$

The restriction of f to $L_0 \coloneqq A_0 \cap L \subseteq B - D$ is a constant by the previous paragraph. On the other hand, if f is a regular function on A_0 that is equal to a constant c on L_0, then it extends to a regular function on U. Thus, the ring $\Gamma(U, \mathscr{O}_U)$ is isomorphic to the ring [use2019]

$$\begin{aligned}
R &= \big\{ f(w, y) \in k[w, y] \;\big|\; f(w, 0) \equiv c \text{ for some } c \in k \big\} \\
&= \big\{ c + g(w, y) \;\big|\; c \in k,\ g(w, y) \in y \cdot k[w, y] \big\} \\
&= k + y \cdot k[w, y].
\end{aligned}$$

The k-algebra R is generated by the monomials $w^m y^{1+n}$ for $m, n \geq 0$. The ideal $I \subseteq R$ of polynomials vanishing along the line $L = V_+(w, y)$ is generated by these monomials. The ideal I cannot be generated by finitely many elements f_i because the f_i will have a term with largest power N on w, in which case

$$w^{N+1} y \in I - (f_i)_i.$$

Proposition 2.5.9. ([EGAII, Propositions 2.5.2, 2.5.4, and 2.5.5]) *Let S be an $\mathbf{N}$-graded ring and consider a $\mathbf{Z}$-graded S-module M.*

(i) *There exists on $X = \mathrm{Proj}(S)$ exactly one quasi-coherent $\mathcal{O}_X$-module $\tilde{M}$ such that for every homogeneous $f \in S_+$, we have*

$$\Gamma\Big(D_+(f), \tilde{M}\Big) = M_{(f)}$$

and for every $f, g \in S_+$, the diagram

$$\begin{tikzcd} \Gamma\Big(D_+(f), \tilde{M}\Big) \arrow[r, equal] \arrow[d] & M_{(f)} \arrow[d] \\ \Gamma\Big(D_+(fg), \tilde{M}\Big) \arrow[r, equal] & M_{(fg)} \end{tikzcd}$$

commutes.

(ii) *The functor*

$$\begin{aligned} {}^*\mathsf{Mod}(S) &\longrightarrow \mathsf{QCoh}(\mathcal{O}_X) \\ M &\longmapsto \tilde{M} \end{aligned}$$

I chose this notation to match the notation for graded Hom in [BH1998, p. 33].

from the category ${}^*\mathsf{Mod}(S)$ of graded *S-modules with* graded *homomorphisms is exact and commutes with direct limits and with direct sums.*

(iii) *For every $\mathfrak{p} \in X = \mathrm{Proj}(S)$, we have $\tilde{M}_{\mathfrak{p}} = M_{(\mathfrak{p})}$.*

[Har1977, Props. II.2.5(a), II.5.11(a)]

Proof. (i) holds by definition and the construction of sheaves defined on a basis. (ii) holds since exactness is computed stalkwise, and since direct limits and direct sums can be computed in terms of the behavior on each distinguished open set $D_+(f)$ because they will have the same associated sheaves. (iii) holds by computing the direct limit along the restriction maps in (i). $\square$

2.5.3 Serre twists and sheaves associated to tensor products, Hom, restriction of scalars, and extension of scalars

We want to define an "inverse" to the functor $M \mapsto \tilde{M}$. To do so, we use the following construction.

[Har1977, p. 117]

Definition 2.5.10. ([FAC, n° 54; EGAII, Notations 2.1.1 and (2.5.10)]) Let S be an $\mathbf{N}$-graded ring and let $X = \mathrm{Proj}(S)$. Let $n \in \mathbf{Z}$.

[Har1977, p. 50]

(i) Let M be a $\mathbf{Z}$-graded S-module. The $\mathbf{Z}$-graded S-module $M(n)$ satisfies

$$M(n)_d = M_{n+d}.$$

(*ii*) For every $n \in \mathbf{Z}$, *the n-th twisting sheaf of Serre* is

$$\mathscr{O}_X(n) \coloneqq \widetilde{S(n)}.$$

(*iii*) The *n-th twist* of an $\mathscr{O}_X$-module $\mathscr{F}$ is

$$\mathscr{F}(n) \coloneqq \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n).$$

Here are some important properties of the twisting sheaf.

Proposition 2.5.11. ([FAC, n° 58, Proposition 4; EGAII, Proposition 2.5.7, Corollaire 2.5.8, Corollaire 2.5.9, Coro
Let S be an $\mathbf{N}$-graded ring and let $X = \operatorname{Proj}(S)$. $\boxed{2/26}$ [Har1977, Prop. II.5.12(a),(b)]

(*i*) *Let $d > 0$ be an integer and let $f \in S_d$. Then, for all $n \in \mathbf{Z}$, we have*

$$\bigl(S(nd)\bigr)^{\sim}\big|_{D_+(f)} \cong \mathscr{O}_X|_{D_+(f)}.$$

In particular, ***if S is generated by S_1 as an S_0-algebra,*** *then $\mathscr{O}_X(n)$ is an invertible sheaf on X for every $n \in \mathbf{Z}$.*

(*ii*) *Let M and N be $\mathbf{Z}$-graded S-module. There are morphisms*

$$\lambda\colon \tilde{M} \otimes_{\mathscr{O}_X} \tilde{N} \longrightarrow (M \otimes_S N)^{\sim}$$
$$\mu\colon \Bigl({}^{*}\operatorname{Hom}_S(M,N)\Bigr)^{\sim} \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(\tilde{M},\tilde{N})$$

natural in M and N, where

$$\begin{aligned} {}^{*}\operatorname{Hom}_S(M,N) &\coloneqq \bigoplus_{n\in\mathbf{Z}} \bigl\{\varphi \in \operatorname{Hom}_S(M,N) \bigm| \varphi(M_d) \subseteq N_{d+n}\bigr\} \\ &\subseteq \operatorname{Hom}_S(M,N). \end{aligned}$$

Now ***assume that S is generated by S_1 as an S_0-algebra****. Then, λ is an isomorphism, $\tilde{M}(n) \cong (M(n))^{\sim}$, and*

$$\mathscr{O}_X(n) \otimes_{\mathscr{O}_X} \mathscr{O}_X(m) \cong \mathscr{O}_X(n+m).$$

If M is finitely generated (resp. has a graded *finite presentation), then μ is injective (resp. an isomorphism).*

Caution 2.5.12. The notation $\operatorname{Hom}_S(M,N)$ from [EGAII, (2.1.2)] is what we call ${}^{*}\operatorname{Hom}_S(M,N)$. The notation ${}^{*}\operatorname{Hom}_S(M,N)$ is from [BH1998, p. 33]. The inclusion

$${}^{*}\operatorname{Hom}_S(M,N) \subseteq \operatorname{Hom}_S(M,N)$$

is an equality when M is finitely generated [EGAII, (2.1.2)], but is not an equality in general [Jim2013].

Proof of Proposition 2.5.11. (i). This follows from the bijection

$$\begin{aligned}(S_f)_0 &\longleftrightarrow (S_f)_{nd}\\ \frac{s}{f^m} &\longmapsto \frac{sf^n}{f^m}\\ \frac{s}{f^{n+m}} &\longleftarrow\!\shortmid \frac{s}{f^m}.\end{aligned}$$

The "in particular" statement holds since

$$X - \bigcup_{f\in S_1} D_+(f) = \bigcap_{f\in S_1} V_+(f) = V_+(S_1) = V_+(S_+) = \varnothing$$

[EGAII, Cor. 2.3.14] and hence the $D_+(f)$ as f ranges over all $f \in S_1$ forms an open cover of X.

[EGAII, (2.5.11), (2.5.12)] (ii). We first construct the maps. For every $f \in S_d$ with $d > 0$, we construct a $S_{(f)}$-linear map

$$\lambda_f\colon M_{(f)} \otimes_{S_{(f)}} N_{(f)} \longrightarrow (M \otimes_S N)_{(f)}$$

functorial in M and N as the composition

$$M_{(f)} \otimes_{S_{(f)}} N_{(f)} \hookrightarrow M_f \otimes_{S_f} N_f \xrightarrow{\sim} (M \otimes_S N)_f$$

and then noting that homogeneous elements of degree 0 on the left-hand side map to homogeneous elements of degree 0 on the right-hand side:

$$\frac{x}{f^m} \otimes \frac{y}{f^n} \longmapsto \frac{(x \otimes y)}{f^{m+n}}.$$

With this definition, for every $g \in S_e$ for $e > 0$, we have the commutative diagram

$$\begin{tikzcd} M_{(f)} \otimes_{S_{(f)}} N_{(f)} \arrow[r, "\lambda_f"] \arrow[d] & (M \otimes_S N)_{(f)} \arrow[d] \\ M_{(fg)} \otimes_{S_{(fg)}} N_{(fg)} \arrow[r, "\lambda_{fg}"] & (M \otimes_S N)_{(fg)}. \end{tikzcd}$$

We therefore obtain the morphism

$$\lambda\colon \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N} \longrightarrow (M \otimes_S N)^{\sim}$$

natural in M and N. Similarly, we construct the $S_{(f)}$-linear map

$$\begin{aligned}\mu_f\colon \Big({}^*\mathrm{Hom}_S(M,N)\Big)_{(f)} &\longrightarrow \mathrm{Hom}_{S_{(f)}}\big(M_{(f)}, N_{(f)}\big)\\ \frac{u}{f^n} &\longmapsto \left(\frac{x}{f^m} \mapsto \frac{u(x)}{f^{m+n}}\right)\end{aligned}$$

where u is a map of degree nd and x is an element of degree md. For every $g \in S_e$ for $e > 0$, we have the commutative diagram

$$\begin{tikzcd}
\left({}^*\mathrm{Hom}_S(M,N)\right)_{(f)} \arrow[r, "\mu_f"] \arrow[d] & \mathrm{Hom}_{S_{(f)}}\left(M_{(f)}, N_{(f)}\right) \arrow[d] \\
\left({}^*\mathrm{Hom}_S(M,N)\right)_{(fg)} \arrow[r, "\mu_{fg}"] & \mathrm{Hom}_{S_{(fg)}}\left(M_{(fg)}, N_{(fg)}\right).
\end{tikzcd}$$

We therefore obtain the morphism

$$\mu\colon \left({}^*\mathrm{Hom}_S(M,N)\right)^{\sim} \longrightarrow \mathscr{H}om_{\mathcal{O}_X}(\tilde{M}, \tilde{N})$$

natural in M and N.

It remains to show that, assuming that S is generated by S_1, the maps λ, μ are isomorphisms where for μ, we also assume that M has a graded finite presentation. For every $f \in S_1$, we want to show that

$$\begin{aligned}
M_{(f)} \otimes_{S_{(f)}} N_{(f)} &\longrightarrow (M \otimes_S N)_{(f)} \\
\frac{x}{f^m} \otimes \frac{y}{f^n} &\longmapsto \frac{x \otimes y}{f^{m+n}}
\end{aligned}$$

is an isomorphism. Consider the S_0-bilinear map

$$\begin{aligned}
M_m \times N_n &\longrightarrow M_{(f)} \otimes_{S_{(f)}} N_{(f)} \\
(x, y) &\longmapsto \frac{x}{f^m} \otimes \frac{y}{f^n}.
\end{aligned}$$

By the universal property of tensor products, we have the dashed map in the commutative diagram

$$\begin{tikzcd}
M_m \times N_n \arrow[r] \arrow[dr] & M_m \otimes_{S_0} N_n \arrow[d, dashed, "\exists!"] \\
& M_{(f)} \otimes_{S_{(f)}} N_{(f)}.
\end{tikzcd}$$

Now let $s \in S_q$. Then, as m, n varies on the left-hand side these morphisms are compatible with multiplication by s in the sense that

$$(sx) \otimes y \longmapsto \frac{sx}{f^{q+m}} \otimes \frac{y}{f^n} = \frac{s}{f^q}\left(\frac{x}{f^m} \otimes \frac{y}{f^n}\right).$$

Thus, passing to the quotient

$$\bigoplus_{d \in \mathbf{Z}} \bigoplus_{m+n=d} M_m \otimes_{S_0} N_n \twoheadrightarrow M \otimes_S N$$

as in [EGAII, (2.1.2)], we obtain a di-homomorphism of modules

$$\gamma_f \colon M \otimes_S N \longrightarrow M_{(f)} \otimes_{S_{(f)}} N_{(f)}$$

over the map $S \to S_{(f)}$ mapping $s \mapsto s/f^q$ on homogeneous elements $s \in S_q$. Since the image 1 of f acts as a unit on the image of $M \otimes_S N$, the map γ_f factors through the localization at f to yield a map

$$(M \otimes_S N)_f \longrightarrow M_{(f)} \otimes_{S_{(f)}} N_{(f)}$$

compatible with the map $S_f \to S_{(f)}$. The map $S_f \to S_{(f)}$ mapping $s/f^m \mapsto s/f^q$ restricts to the identity on $S_{(f)}$. Thus, we can restrict to degree 0 parts to obtain the $S_{(f)}$-linear map

$$\begin{aligned} \lambda'_f \colon (M \otimes_S N)_{(f)} &\longrightarrow M_{(f)} \otimes_{S_{(f)}} N_{(f)} \\ \frac{x \otimes y}{f^{m+n}} &\longmapsto \frac{x}{f^m} \otimes \frac{y}{f^n}. \end{aligned}$$

Comparing the descriptions of λ_f and λ'_f, we see they are mutually inverse.

For μ, suppose that M has a graded finite presentation, i.e., there exists a right exact sequence

$$\bigoplus_{j=1}^n S(-d_j) \xrightarrow{A} \bigoplus_{i=1}^m S(-d_i) \longrightarrow M \longrightarrow 0$$

of graded S-modules, where $d_i, d_j \in \mathbf{Z}$ and where $A = (a_{ij})$ is an $m \times n$ matrix with $a_{ij} \in S_{d_i - d_j}$. Since ${}^*\mathrm{Hom}$ and $\mathscr{H}\!om$ are left exact, we have the commutative diagram

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] \\ \Big({}^*\mathrm{Hom}_S(M,N)\Big)^\sim \arrow[r, "\mu"] \arrow[d] & \mathscr{H}\!om_{\mathcal{O}_X}(\tilde{M},\tilde{N}) \arrow[d] \\ \displaystyle\bigoplus_{i=1}^m \Big({}^*\mathrm{Hom}_S\big(S(-d_i),N\big)\Big)^\sim \arrow[r, "\mu"] \arrow[d] & \displaystyle\bigoplus_{i=1}^m \mathscr{H}\!om_{\mathcal{O}_X}\big(\widetilde{S(-d_i)},\tilde{N}\big) \arrow[d] \\ \displaystyle\bigoplus_{j=1}^n \Big({}^*\mathrm{Hom}_S\big(S(-d_j),N\big)\Big)^\sim \arrow[r, "\mu"] & \displaystyle\bigoplus_{j=n}^m \mathscr{H}\!om_{\mathcal{O}_X}\big(\widetilde{S(-d_j)},\tilde{N}\big) \end{tikzcd} \tag{2.5.13}$$

with exact columns. Since isomorphic morphisms have isomorphic kernels, it suffices to show that the bottom two horizontal maps are isomorphisms. Changing notation, it suffices to prove the case when $M = S(d)$, i.e., it suffices to show

that
$$\mu\colon \Big({}^*\mathrm{Hom}_S\big(S(d),N\big)\Big)^{\sim} \longrightarrow \mathscr{H}\!om_{\mathcal{O}_X}\big(\widetilde{S(d)},\tilde{N}\big)$$
is an isomorphism. As before, since the open sets $D_+(f)$ for $f \in S_1$ form an open cover of X, it suffices to show that
$$\mu_f\colon {}^*\mathrm{Hom}_S\big(S(d),N\big)_{(f)} \longrightarrow \mathrm{Hom}_{S_{(f)}}\big(S(d)_{(f)},N_{(f)}\big)$$
is an isomorphism for every $f \in S_1$. We claim we have a commutative diagram
$$\begin{tikzcd}
{}^*\mathrm{Hom}_S\big(S(d),N\big)_{(f)} \arrow[rr, "\mu_f"] & & \mathrm{Hom}_{S_{(f)}}\big(S(d)_{(f)},N_{(f)}\big) \\
& \big(N(-d)\big)_{(f)} \arrow[ul, "\eta_f", "\sim"'] \arrow[ur, "\eta'_f"', "\sim"] &
\end{tikzcd}$$
where η_f,η'_f are isomorphisms. Let $z \in N(-d)$. Then, we have a homomorphism $S(d) \to N$ such that $u_z(1) = z$. The resulting map $N(-d) \to \mathrm{Hom}_S(S(d),N)$ is an isomorphism. We therefore obtain an isomorphism η_f. Likewise, if $z' \in (N(-d))_{(f)}$, then we define the map η'_f as
$$\begin{aligned}
\eta_{f'}\colon \big(N(-d)\big)_{(f)} &\longrightarrow \mathrm{Hom}_{S_{(f)}}\big(S(d)_{(f)},N_{(f)}\big)\\
\frac{z'}{f^m} &\longmapsto \left(\frac{s}{f^k} \mapsto \frac{sz'}{f^{m+k}}\right)
\end{aligned}$$
where on the left-hand side, $z' \in (N(-d))_m = N_{m-d}$ and on the right-hand side, $s \in (S(d))_k = S_{d+k}$. This map is an isomorphism as well.

Finally, if M is finitely generated, then by taking homogeneous components of the generators we see that M is generated by finitely many *homogeneous* elements. We can therefore use the top two rows in the diagram (2.5.13) to see that μ is injective. □ [EGAII, (2.1.2)]

Caution 2.5.14. The assumption that S is generated by S_1 as an S_0-algebra is very important! Without this assumption, these and many other facts about $\mathrm{Proj}(S)$ are false. See [Dol1982, §1.5].

We also have the analogue of Corollary 2.3.5(*iii*) for Proj.

Proposition 2.5.15. ([EGAII, Propositions 2.8.7 and 2.8.8]) *Let $\varphi\colon S \to T$ be a graded map of $\mathbf{N}$-graded rings. Consider the associated morphism* [Har1977, Prop. II.5.12(*c*)]
$$\Phi\colon U \longrightarrow \mathrm{Proj}(S)$$
of schemes, where
$$U \coloneqq \big\{\mathfrak{p} \in \mathrm{Proj}(T) \bigm| \mathfrak{p} \not\supseteq \varphi(S_+)\big\}.$$

(*i*) *Let N be a graded T-module. There exists an isomorphism*

$$\big(N_{[\varphi]}\big)^{\sim} \xrightarrow{\sim} \Phi_*\big(\tilde{N}|_U\big)$$

of $\mathcal{O}_{\mathrm{Proj}(S)}$-modules natural in N.

(*ii*) *Let M be a graded S-module. There exists a morphism*

$$\nu\colon \Phi^*\tilde{M} \longrightarrow (M \otimes_S T)^{\sim}\big|_U$$

of $\mathcal{O}_U$-modules natural in M. If S is generated by S_1 over S_0, then ν is an isomorphism.

Proof. (*i*). Let $f \in S_+$ be a homogeneous element and let $f' = \varphi(f)$. Then, there is an isomorphism

$$\big(N_{[\varphi]}\big)_{(f)} \xrightarrow{\sim} \big(N_{(f')}\big)_{[\varphi_{(f')}]}.$$

These isomorphisms correspond to the isomorphisms

$$\big(N_{[\varphi]}\big)^{\sim}\big|_{D_+(f)} \xrightarrow{\sim} (\Phi_{f'})_*\big(\tilde{N}|_{D_+(f')}\big)$$

under Serre's equivalence for affine schemes by Corollary 2.3.5(*iii*) and Proposition 2.5.9(*i*). Moreover, for homogeneous $g \in S_+$ with image $g' = \varphi(g)$, the diagrams

$$\begin{tikzcd}
\big(N_{[\varphi]}\big)^{\sim}\big|_{D_+(f)} \arrow[r, "\sim"] \arrow[d] & (\Phi_{f'})_*\big(\tilde{N}|_{D_+(f')}\big) \arrow[d] \\
\big(N_{[\varphi]}\big)^{\sim}\big|_{D_+(fg)} \arrow[r, "\sim"] & (\Phi_{f'g'})_*\big(\tilde{N}|_{D_+(f'g')}\big)
\end{tikzcd}$$

commute by the way the isomorphism in Corollary 2.3.5(*iii*) is defined. Thus, these isomorphisms glue to yield the isomorphism we wanted. The isomorphism is natural since the isomorphisms used to construct it on distinguished open subsets, Serre's equivalence for affine schemes (Corollary 2.3.4), and the commutative diagram above are natural in N.

(*ii*) (Sketch). Let $f \in S_d$ for $d > 0$, and set $f' = \varphi(f)$. We can then define a natural morphism

$$\nu_f\colon M_{(f)} \otimes_{S_{(f)}} T_{(f')} \longrightarrow (M \otimes_S T)_{(f')}$$

of $T_{(f')}$-modules as the map obtained by corestricting the composition

$$M_{(f)} \otimes_{S_{(f)}} T_{(f')} \hookrightarrow M_f \otimes_{S_f} T_{f'} \xrightarrow{\sim} (M \otimes_S T)_{f'}$$

to degree 0 components. One can check that ν_f fits into the commutative diagram

$$\begin{tikzcd} M_{(f)} \otimes_{S_{(f)}} T_{(f')} \arrow[r, "\nu_f"] \arrow[d] & (M \otimes_S T)_{(f')} \arrow[d] \\ M_{(fg)} \otimes_{S_{(fg)}} T_{(f'g')} \arrow[r, "\nu_{fg}"] & (M \otimes_S T)_{(f'g')} \end{tikzcd}$$

and hence we obtain the morphism ν. For the isomorphism statement, it suffices to show that ν_f is an isomorphism for all $f \in S_1$ since U is covered by the open sets $D_+(f)$ as f ranges over all elements of S_1. We write down an inverse for ν_f as follows. We have an S_0-bilinear map

$$\begin{aligned} M_m \times S_n &\longrightarrow M_{(f)} \otimes_{S_{(f)}} T_{(f')} \\ (x, s) &\longmapsto \frac{x}{f^m} \otimes \frac{s}{f^n}. \end{aligned}$$

As in the proof of Proposition 2.5.11(ii), we obtain a di-homomorphism

$$\eta_f \colon M \otimes_S T \longrightarrow M_{(f)} \otimes_{S_{(f)}} T_{(f')}$$

with respect to the ring map $S \to S_{(f)}$ mapping $s \mapsto s/f^q$ on homogeneous elements $s \in S_q$. One then checks that the maps ν_f and η_f are mutually inverse. □

2.5.4 Graded modules associated to sheaves

We now define an "inverse" functor to $M \mapsto \tilde{M}$.

Definition 2.5.16. ([FAC, n° 59; EGAII, (2.6.1) and (2.6.2)]) Let S be an $\mathbf{N}$-graded ring generated by S_1 as an S_0-algebra. Let $X = \operatorname{Proj}(S)$ and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. We then have a graded ring map [Har1977, p. 118]

$$S \longrightarrow \Gamma_*(\mathcal{O}_X) \coloneqq \bigoplus_{n \in \mathbf{Z}} \Gamma\Big(X, \mathcal{O}_X(n)\Big)$$

where the $\mathbf{N}$-grading on the right is obtained from the multiplication maps

$$\Gamma\Big(X, \mathcal{O}_X(n)\Big) \otimes_{\Gamma(X, \mathcal{O}_X)} \Gamma\Big(X, \mathcal{O}_X(d)\Big) \longrightarrow \Gamma\Big(X, \mathcal{O}_X(n+d)\Big).$$

The *graded* $\Gamma_*(\mathcal{O}_X)$*-module associated to* $\mathscr{F}$ is

$$\Gamma_*(\mathscr{F}) \coloneqq \bigoplus_{n \in \mathbf{Z}} \Gamma\Big(X, \mathscr{F}(n)\Big).$$

This is a graded $\Gamma_*(\mathcal{O}_X)$-module as follows. Consider a section $s \in \Gamma(X, \mathcal{O}_X(d))$. Then, for any $t \in \Gamma(X, \mathscr{F}(n))$, the product $s \cdot t \in \Gamma(X, \mathscr{F}(n+d))$ is defined by taking the tensor product

$$s \otimes t \in \Gamma\Big(X, \mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathscr{F}(n)\Big)$$

and using the isomorphism

$$\mathscr{O}_X(d) \otimes_{\mathscr{O}_X} \mathscr{F}(n) \cong \mathscr{F}(n+d).$$

Since $\mathscr{O}_X(d)$ is invertible, $\Gamma_*(\,\cdot\,)$ is a left exact functor.

[Har1977, Exer. II.5.9(a)]

If M is a graded S-module, we obtain a morphism

$$\alpha\colon M \longrightarrow \Gamma_*(\tilde{M})$$

of graded S-modules natural in M since an element $m \in M_n$ determines a section of $\tilde{M}(n)$ by working on the distinguished open sets $D_+(f)$.

We calculate one example of the graded module associated to an $\mathscr{O}_X$-module.

[Har1977, Prop. II.5.13]

Proposition 2.5.17. **(**[FAC, n$^{\mathrm{o}}$ 62, Proposition 2(a); EGAIII$_1$, Proposition 2.1.12(ii)]**)** *Let A be a ring, let $S = A[x_0, x_1, \dots, x_r]$ for $r \geq 1$, and let $X = \mathbf{P}^r_A = \mathrm{Proj}(S)$. Then,*

$$S \longrightarrow \Gamma_*(\mathscr{O}_X)$$

is an isomorphism.

Proof. We want to calculate the kernel in the exact sequence

$$0 \longrightarrow \Gamma\Big(X, \mathscr{O}_X(n)\Big) \longrightarrow \prod_{i=0}^{r} \Gamma\Big(D_+(x_i), \mathscr{O}_X(n)\Big) \longrightarrow \prod_{i,j} \Gamma\Big(D_+(x_i x_j), \mathscr{O}_X(n)\Big).$$

Taking the sum over all n, we want to compute the kernel in the exact sequence

$$0 \longrightarrow \Gamma_*(\mathscr{O}_X) \longrightarrow \prod_{i=0}^{r} S_{x_i} \longrightarrow \prod_{i,j} S_{x_i x_j}$$

where K consists of all tuples $(t_0, t_1, \dots, t_r)$ such that the images of t_i and t_j in $S_{x_i x_j}$ are the same. The x_i are nonzerodivisors on S, so the localization maps $S \to S_{x_i}$ and $S_{x_i} \to S_{x_i x_j}$ are all injective. Moreover, we can think of all rings involved as subrings of $S' = S_{x_0 x_1 \cdots x_r}$. We therefore see that

$$\Gamma_*(\mathscr{O}_X) = \bigcap_{i=0}^{r} S_{x_i} \subseteq S'.$$

Any homogeneous element of S' can be written as

$$x_0^{i_0} x_1^{i_1} \cdots x_r^{i_r} f(x_0, x_1, \dots, x_r)$$

where $i_j \in \mathbf{Z}$ and f is a homogeneous polynomial not divisible by any x_i. This element is in S_{x_i} if and only if $i_j \geq 0$ for all $j \neq i$. Thus, the intersection of all the S_{x_i} is just S. $\square$

Caution 2.5.18. It is not true in general that $S \to \Gamma_*(\mathscr{O}_X)$ is an isomorphism [Har1977, Exercise II.5.14].

2.5.5 Graded local cohomology and the kernel and cokernel of α

We can use local cohomology to understand Proposition 2.5.17 and the morphism α better. We will in fact show the following:

Theorem 2.5.19. ([EGAIII$_1$, Proposition 2.1.5]) *Let S be a Noetherian $\mathbf{N}$-graded ring generated by S_1 as an S_0-algebra, let M be a $\mathbf{Z}$-graded S-module, and let $X = \mathrm{Proj}(S)$. We have the exact sequence* [Eis2005, Cor. A1.12]

$$0 \longrightarrow H^0_{S_+}(M) \longrightarrow M \xrightarrow{\alpha} \Gamma_*(\tilde{M}) \longrightarrow H^1_{S_+}(M) \longrightarrow 0$$

of graded S-modules and the isomorphisms

$$\bigoplus_{n\in\mathbf{Z}} H^{i-1}\Bigl(X, \tilde{M}(n)\Bigr) \xrightarrow{\sim} H^i_{S_+}(M)$$

of graded S-modules for every $i \geq 2$.

Thus, $H^0_{S_+}(M)$ and $H^1_{S_+}(M)$ measure how far $\alpha\colon M \to \Gamma_*(M)$ is from being an isomorphism. Note that to even make sense of α, we are assuming that S is generated by S_1 as an S_0-algebra.

Since we are making statements about local cohomology in the setting of graded modules, we need to make sense of local cohomology as a functor on ${}^*\mathsf{Mod}(S)$. This is allowed by the following:

Lemma 2.5.20. *Let S be an $\mathbf{N}$-graded ring. Let $\mathfrak{a} \subseteq S$ be a homogeneous ideal.*

(*i*) *A graded S-module M is injective in ${}^*\mathsf{Mod}(S)$ if and only if* [BH1998, p. 136] [BS2013, Lem. 13.2.5]

$${}^*\mathrm{Hom}(\,\cdot\,, M)\colon {}^*\mathsf{Mod}(S) \longrightarrow {}^*\mathsf{Mod}(S)$$

is exact.

(*ii*) ${}^*\mathsf{Mod}(S)$ *is a Grothendieck Abelian category with a generator. Thus,* ${}^*\mathsf{Mod}(S)$ *has enough injective objects.* [BH1998, Exer. 1.5.19, Thm. 3.6.2] [BS2013, 13.1.7(*i*), Thm. 13.2.4(*i*)]

(*iii*) *We have a commutative diagram* [BH1998, p. 143] [BS2013, Ex. 13.3.3(*ii*)]

$$\begin{tikzcd} {}^*\mathsf{Mod}(S) \arrow[r, "\mathsf{Forget}"] \arrow[d, "\Gamma_{\mathfrak{a}}"'] & \mathsf{Mod}(S) \arrow[d, "\Gamma_{\mathfrak{a}}"] \\ {}^*\mathsf{Mod}(S) \arrow[r, "\mathsf{Forget}"] & \mathsf{Mod}(S). \end{tikzcd}$$

(*iv*) *The forgetful functor* [BS2013, Prop. 13.2.6, Thm. 13.4.2]

$$\mathsf{Forget}\colon {}^*\mathsf{Mod}(S) \longrightarrow \mathsf{Mod}(S)$$

maps injective objects in ${}^*\mathsf{Mod}(S)$ *to* $\Gamma_{\mathfrak{a}}$*-acyclic objects in* $\mathsf{Mod}(S)$*. Thus, the forgetful functor defines a natural transformation of* δ*-functors such that the diagram*

$$\begin{tikzcd} {}^*\mathsf{Mod}(S) \arrow[r, "\mathsf{Forget}"] \arrow[d, "H^i_{\mathfrak{a}}"'] & \mathsf{Mod}(S) \arrow[d, "H^i_{\mathfrak{a}}"] \\ {}^*\mathsf{Mod}(S) \arrow[r, "\mathsf{Forget}"] & \mathsf{Mod}(S) \end{tikzcd}$$

commutes for every i*. Here, we compute the right vertical functor* $H^i_{\mathfrak{a}}$ *using injective resolutions on* $\mathsf{Mod}(S)$*.*

Proof. (i). We have

$$\begin{aligned} {}^*\mathrm{Hom}(\,\cdot\,, M) &= \bigoplus_{n\in\mathbf{Z}} \mathrm{Hom}_{{}^*\mathsf{Mod}(S)}\big(\,\cdot\,, M(n)\big) \\ &\cong \bigoplus_{n\in\mathbf{Z}} \mathrm{Hom}_{{}^*\mathsf{Mod}(S)}\big(\,\cdot\,(-n), M\big) \end{aligned}$$

which is exact if and only if M is injective.

(ii). The category ${}^*\mathsf{Mod}(S)$ is Abelian because kernels and cokernels are computed degree-wise, where on each degree, they are computed as kernels and cokernels of morphisms of S_0-modules. Arbitrary direct sums exist and filtered direct limits are exact since direct sums and direct limits are computed degree-wise. Finally, a generator is given by

$$\bigoplus_{n\in\mathbf{Z}} S(n).$$

Thus, ${}^*\mathsf{Mod}(S)$ has enough injective objects by Theorem 1.3.46.

(iii). Let M be a graded S-module. Then, $m \in \Gamma_{\mathfrak{a}}(M)$ if and only if $\mathfrak{a}^n m = 0$ for some $n \geq 0$, which holds if and only if each graded component of $\mathfrak{a}^n m$ is equal to 0. Since $\mathfrak{a}$ and m are graded, this hold if and only if each component of m is in $\Gamma_{\mathfrak{a}}(M)$, and hence $\Gamma_{\mathfrak{a}}(M)$ is a graded submodule of M.

(iv). By dimension shifting (Proposition 1.4.25), it suffices to show that injective objects in ${}^*\mathsf{Mod}(S)$ are $\Gamma_{\mathfrak{a}}$-acyclic. To prove this, note that the composition

[BS2013, 13.1.8]

$$\Gamma_{\mathfrak{a}}(\,\cdot\,) \cong \lim_{n\geq 0} {}^*\mathrm{Hom}_S(S/\mathfrak{a}^n, \,\cdot\,) \hookrightarrow \lim_{n\geq 0} \mathrm{Hom}_S(S/\mathfrak{a}^n, \,\cdot\,) \cong \Gamma_{\mathfrak{a}}(\,\cdot\,)$$

is the identity (on the right-hand side, we think of $\Gamma_{\mathfrak{a}}$ as a functor on $\mathsf{Mod}(S)$) and hence the middle "inclusion" is an isomorphism. Thus, we can compute $H^i_{\mathfrak{a}}$ as

[BS2013, Rem. 13.4.6]

$$H^i_{\mathfrak{a}}(\,\cdot\,) \cong \lim_{n\geq 0} {}^*\mathrm{Ext}^i_S(S/\mathfrak{a}^n, \,\cdot\,) \cong \lim_{n\geq 0} \mathrm{Ext}^i_S(S/\mathfrak{a}^n, \,\cdot\,)$$

which is 0 for injective objects in ${}^*\mathsf{Mod}(S)$ by (i). $\square$

To prove Theorem 2.5.19, the idea is to compute local cohomology on the affine cone over $\mathrm{Proj}(S)$. Affine cones are defined as follows.

Definition 2.5.21. ([EGAII, (8.3.1) and Corollaire 8.3.6]) Let S be an $\mathbf{N}$-graded ring. The *affine cone* over $\mathrm{Proj}(S)$ is

$$C\Big(\mathrm{Proj}(S)\Big) \coloneqq \mathrm{Spec}(S).$$

We can define the *canonical projection morphism*

$$\pi\colon \mathrm{Spec}(S) - V(S_+) \longrightarrow \mathrm{Proj}(S) \tag{2.5.22}$$

as follows. Over each homogeneous element $f \in S$, we have a ring map

$$S_f \longleftarrow S_{(f)}$$

which yields the morphism

$$D(f) \longrightarrow D_+(f)$$

by applying Spec. These morphisms glue to give the morphism (2.5.22) since the diagram

$$\begin{tikzcd} S_f \arrow[d] & S_{(f)} \arrow[l, hook'] \arrow[d] \\ S_{fg} & S_{(fg)} \arrow[l, hook'] \\ S_g \arrow[u] & S_{(g)} \arrow[l, hook'] \arrow[u] \end{tikzcd}$$

commutes for all homogeneous $f, g \in S_+$. Note that π is surjective since, for every prime ideal $\mathfrak{p} \subseteq S_{(f)}$, there exists a prime ideal in S_f lying over $\mathfrak{p}$ because

$$0 \neq \frac{S_{(f)}}{\mathfrak{p}} \hookrightarrow \frac{S_f}{\mathfrak{p}S_f}.$$

Lemma 2.5.23. ([EGAII, (2.2.1) and Corollaire 8.3.6]) *Let S be an $\mathbf{N}$-graded ring and let $f \in S_d$ for $d > 0$. The monomials $(f/1)^h \in S_f$ form a free system of generators for $(S^{(d)})_f$ over $S_{(f)}$, and hence*

$$(S^{(d)})_f \cong S^{(d)}_{(f)}[T, T^{-1}] \cong S^{(d)}_{(f)} \otimes_{\mathbf{Z}} \mathbf{Z}[T, T^{-1}].$$

Thus, if

$$\pi\colon C\Big(\mathrm{Proj}(S)\Big) - V(S_+) \longrightarrow \mathrm{Proj}(S)$$

is the canonical projection morphism from the affine cone $C(\mathrm{Proj}(S))$, then for all nonozero $f \in S_1$, we have

$$\pi^{-1}\Big(D_+(f)\Big) \cong \mathrm{Spec}\Big(S_{(f)}[T, T^{-1}]\Big).$$

Proof. For the first statement, first note that since we are working in $(S^{(d)})_f$, the element f is a unit. Thus, a relation of the form

$$\sum_{h=-a}^{b} z_h \, (f/1)^h = 0$$

holds where $z_h = x_h/f^m$ and $x_h \in S_{md}$ if and only if a relation of the form

$$\sum_{h=-a}^{b} f^{h+k} x_h = 0$$

holds for some k. Since the terms in this equation are all in diferent degrees, a relation of this form holds if and only if $f^{h+k}x_h = 0$ for all h, which is equivalent to $z_h = 0$ for all h, again using the fact that f is a unit in $(S^{(d)})_f$.

The last statement of the lemma holds by setting $d = 1$ and looking at the definition of (2.5.22) on distinguished open sets. □

3/3

We can now prove Theorem 2.5.19.

[Dol1982, §1.4]

Proof of Theorem 2.5.19. Let $C(X) = \operatorname{Spec}(S)$ be the affine cone over $X = \operatorname{Proj}(S)$. Let $E = C(X) - V(S_+)$. Since $C(X)$ is affine, Cartan's Theorem B (Theorem 2.3.7) implies that the long exact sequence on local cohomology from [Har1977, Exercise III.2.3(e)] (Homework 5, Problem 4(e)) has the form

$$0 \longrightarrow H^0_{V(S_+)}\Big(C(X), \tilde{M}\Big) \longrightarrow M \longrightarrow H^0\Big(E, \tilde{M}|_E\Big) \longrightarrow H^1_{V(S_+)}\Big(C(X), \tilde{M}\Big) \longrightarrow 0$$

and contains the isomorphisms

$$H^{i-1}\Big(E, \tilde{M}|_E\Big) \xrightarrow{\sim} H^i_{V(S_+)}\Big(C(X), \tilde{M}\Big)$$

for every $i \geq 2$. Since S is Noetherian, [Har1977, Exercise III.3.3] (Homework 7, Problem 5(e)) implies that

For those who are curious, it suffices here to assume that S_+ is finitely generated as an ideal by [SGA2new, Exp. II, Prop. 5]. Another option is to assume that S_+ is generated by a *weakly proregular sequence* in the sense of [Sch2003].

$$H^i_{V(S_+)}\Big(C(X), \tilde{M}\Big) \cong H^i_{S_+}(M)$$

for every i. Finally, we claim we have

$$\begin{aligned} H^i\Big(E, \tilde{M}|_E\Big) &\cong H^i\Big(X, \pi_*\tilde{M}|_E\Big) \\ &\cong H^i\left(X, \bigoplus_{n\in\mathbf{Z}} \tilde{M}(n)\right) \\ &\cong \bigoplus_{n\in\mathbf{Z}} H^i\Big(X, \tilde{M}(n)\Big). \end{aligned}$$

The first isomorphism holds since π is affine by Lemma 2.5.23 and using [Har1977, Exercise III.4.1] (Homework 7, Problem 4(c)). To show the second isomorphism, note that

$$\Gamma\Big(D_+(f), \pi_*\tilde{M}|_E\Big) = \Gamma\Big(D(f), \tilde{M}\Big) = M_f \cong \bigoplus_{n\in\mathbf{Z}} \Big(M(n)\Big)_{(f)}$$

for every $f \in S_d$, and that the associated isomorphisms of sheaves on $D_+(f)$ under Serre's equivalence for affine schemes (Corollary 2.3.4) are compatible with restriction to distinguished open sets of the form $D_+(fg)$ for $g \in S_e$. □

2.5.6 Serre's equivalence for quasi-coherent sheaves on Proj

We have now constructed two functors

$$\begin{gathered} {}^{*}\mathsf{Mod}(S) \rightleftarrows \mathsf{QCoh}\left(\mathcal{O}_{\mathrm{Proj}(S)}\right) \\ M \longmapsto \tilde{M} \\ \Gamma_*(\mathscr{F}) \longleftarrow\!\shortmid \mathscr{F}. \end{gathered}$$

We want to understand to what extent the compositions are isomorphic to the identity functor.

Proposition 2.5.24. ([FAC, n° 59, Propositions 7 and 8; EGAII, (2.6.4) and Proposition 2.6.5]) *Let S be an $\mathbf{N}$-graded ring that is generated by S_1 as an S_0-algebra and let M be a graded S-module. Let $X = \mathrm{Proj}(S)$ and let $\mathscr{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then, there is a morphism*

$$\beta\colon \Gamma_*(\mathscr{F})^{\sim} \longrightarrow \mathscr{F}$$

natural in $\mathscr{F}$ such that the compositions

$$\tilde{M} \xrightarrow{\tilde{\alpha}} \Gamma_*(\tilde{M})^{\sim} \xrightarrow{\beta} \tilde{M} \tag{2.5.25}$$

$$\Gamma_*(\mathscr{F}) \xrightarrow{\alpha} \Gamma_*\left(\Gamma_*(\mathscr{F})^{\sim}\right) \xrightarrow{\Gamma_*(\beta)} \Gamma_*(\mathscr{F}) \tag{2.5.26}$$

are the identity maps.

Proof. We first define β. Let $f \in S_d$ for $d > 0$ and $g \in S_e$ for $e > 0$. Set $M = \Gamma_*(\mathscr{F})$. We then have the commutative diagrams

$$\begin{tikzcd} M_{(f)} \arrow[r, "\beta_f"] \arrow[d] & \Gamma\left(D_+(f), \mathscr{F}\right) \arrow[d] \\ M_{(fg)} \arrow[r, "\beta_{fg}"] & \Gamma\left(D_+(fg), \mathscr{F}\right). \end{tikzcd}$$

Since $\Gamma(D_+(f), \tilde{M}) = M_{(f)}$ by Proposition 2.5.9(i), these maps glue together to obtain the morphism β.

We now show (2.5.25). It suffices to show the composition is the identity locally. On each distinguished open $D_+(f)$, the morphism corresponds to

$$\Gamma\left(D_+(f), \tilde{M}\right) \xrightarrow{\Gamma(D_+(f),\tilde{\alpha})} M_{(f)} \xrightarrow{\beta_f} \Gamma\left(D_+(f), \tilde{M}\right)$$

which is the identity.

To show (2.5.26), we work one degree at a time. Set $M = \Gamma_*(\mathscr{F})$. Then, $M_n = \Gamma(X, \mathscr{F}(n))$ and

$$\left(\Gamma_*(\tilde{M})\right)_n = \Gamma\left(X, \tilde{M}(n)\right) = \Gamma\left(X, \widetilde{M(n)}\right).$$

If $f \in S_1$ and $z \in M_n$, then over each principal open set $D_+(f)$,

$$\alpha_n(z)|_{D_+(f)} = \frac{z}{1} = \left(\frac{f}{1}\right)^n \frac{z}{f^n} \in \Big(M(n)\Big)_{(f)}$$

which maps via β_f to the section

$$\Big(\alpha_1(f)^n|_{D_+(f)}\Big) \cdot \Big(z|_{D_+(f)}\Big) \cdot \Big(\alpha_1(f)^n|_{D_+(f)}\Big)^{-1} = z|_{D_+(f)}. \quad \square$$

Theorem 2.5.27. ([FAC, n° 65, Proposition 6; EGAII, Théorème 2.7.5 and Corollaire 2.7.7])

[Har1977, Prop. II.5.15] *Let S be an $\mathbf{N}$-graded ring that is* finitely *generated by S_1 as an S_0-algebra and let $X = \operatorname{Proj}(S)$. Then, for every quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$, the morphism*

$$\beta\colon \Gamma_*(\mathscr{F})^\sim \longrightarrow \mathscr{F}$$

is an isomorphism. As a consequence, the functor $M \mapsto \tilde{M}$ is essentially surjective.

Proof. First, S_+ is generated by finitely many elements $f_i \in S_1$, and hence we can write

$$X = \bigcup_i \operatorname{Spec}\Big(S_{(f_i)}\Big)$$

as a finite union of quasi-compact open subsets whose intersections are quasi-compact. That is, X is qcqs. By the qcqs lemma (Theorem 2.4.33), we see that for every $f \in S_d$ for $d > 0$, the morphisms

$$\Big(\Gamma_*(\mathscr{F})\Big)_{(\alpha_d(f))} \xrightarrow{\sim} \Gamma\Big(D_+(f), \mathscr{F}\Big)$$

are isomorphisms, where we consider f as a section in $\Gamma(X, \mathcal{O}(d))$. On the other hand, the left-hand side is equal to $(\Gamma_*(\mathscr{F}))_{(f)}$ by definition, and the resulting map is β_f. $\square$

As a consequence, we have Serre's equivalence for quasi-coherent sheaves on $\operatorname{Proj}(S)$.

Corollary 2.5.28. (Serre's equivalence for $\mathsf{QCoh}(\mathcal{O}_{\operatorname{Proj}(S)})$) *Let S be an $\mathbf{N}$-graded ring that is* finitely *generated by S_1 as an S_0-algebra and let $X = \operatorname{Proj}(S)$. We then have an equivalence of categories*

$$\left\{\begin{matrix} \text{graded } S\text{-modules } M \text{ such that} \\ \alpha\colon M \to \Gamma_*(\tilde{M}) \text{ is an isomorphism} \end{matrix}\right\} \underset{\Gamma_*}{\overset{\widetilde{(\cdot)}}{\rightleftarrows}} \mathsf{QCoh}(\mathcal{O}_X).$$

Proof. This follows from Proposition 2.5.24 and Theorem 2.5.27. $\square$

2.5.7 Finiteness conditions for graded modules

To understand the left-hand side of the equivalence in Corollary 2.5.28, we will restrict to quasi-coherent sheaves of finite type. On the module side, we introduce the following two finiteness conditions:

Definition 2.5.29. ([FAC, n° 56; EGAII, (2.7.2)]) Let S be an $\mathbf{N}$-graded ring and let M be a graded S-module. We consider the following two finiteness conditions on M.

(TF) There exists an integer n such that the sub-module [Har1977, Exer. II.5.9(c)]

$$\bigoplus_{k\geq n} M_k$$

is a finitely generated S-module.

(TN) There exists an integer n such that $M_k = 0$ for all $k \geq n$.

Note that if M satisfies (TN), then $M_{(f)} = 0$ for every homogeneous element $f \in S_+$, and hence $\tilde{M} = 0$.

Proposition 2.5.30. ([FAC, n° 58, Proposition 5; EGAII, Proposition 2.7.3]) *Let S be an $\mathbf{N}$-graded ring and let $X = \operatorname{Proj}(S)$. Let M be a graded S-module.*

(i) *If M satisfies* (TF)*, then $\tilde{M}$ is of finite type.*

(ii) *Assume that S_+ is a finitely generated ideal. Suppose that M satisfies* (TF)*. Then, $\tilde{M} = 0$ if and only if M satisfies* (TN)*.*

Proof. We have already seen $\Leftarrow$ in (ii).

(i). Let $M' = \bigoplus_{k\geq n} M_k$ be the sub-module that is finitely generated. The short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M/M' \longrightarrow 0$$

induces the short exact sequence

$$0 \longrightarrow \widetilde{M'} \overset{\sim}{\longrightarrow} \tilde{M} \longrightarrow \widetilde{M/M'} \longrightarrow 0$$

where $\widetilde{M/M'} = 0$ by the direction $\Leftarrow$ in (ii). Replacing M by M', it suffices to consider the case when M itself is finitely generated. Since being of finite type is a local condition, it suffices to show that $M_{(f)}$ is finitely generated as an $S_{(f)}$-module for every homogeneous element $f \in S_d$. Since M is finitely generated, there is a least common multiple e of the absolute values of the degrees of its generators. Then, $M^{(de)}$ is a finitely generated $S^{(de)}$-module, and we have surjection [EGAII, Prop. 2.2.5]

$$M^{(de)} \longrightarrow\!\!\!\!\to M_{(f^e)} \cong M_{(f)}$$

sending $z \in M_{den}$ to $z/(f^e)^n$.

It remains to show $\Rightarrow$ in (ii). With the same notation as above, we have $\widetilde{M'} = 0$, and the condition (TN) for M' holds if and only if it holds for M. Replacing M by M', it suffices to consider the case when M itself is finitely generated by homogeneous elements $\{x_i\}_{1\leq i\leq p}$. Let $\{f_j\}_{1\leq j\leq q}$ be a finite homogeneous set of generators for S_+. By hypothesis, we know that $M_{(f_j)} = 0$ for all j. Since both sets are finite, there exists an integer n such that

$$f_j^n x_i = 0$$

for all i, j. Set $n_j = \deg(f_j)$ and let $m = \max\{\sum_j r_j n_j\}$ where the r_j range over all tuples such that $\sum_j r_j \leq nq$. Then, for every $k > m$, we have $S_k x_i = 0$ for all i. If $h = \max\{\deg(x_i)\}$, then $M_k = 0$ for all $k > h + m$. □

[Har1977, Prop. II.5.15]

Corollary 2.5.31. ([FAC, n° 60, Théorème 2; EGAII, Corollaire 2.7.8]) *Let S be an $\mathbf{N}$-graded ring that is* finitely *generated by S_1 as an S_0-algebra and let $X = \operatorname{Proj}(S)$. Then, every quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$ of finite type is of the form $\tilde{N}$ for a graded S-module of finite type.*

Proof. By Theorem 2.5.27, we can write $\mathscr{F} = \tilde{M}$ for a graded S-module M. Write M as a direct limit of its finitely generated graded submodules M_λ. Let $f_j \in S_1$ be a finite set of generators for S over S_0. For each distinguished open set $D_+(f_j)$, a large enough submodule M_{λ_j} satisfies

$$\left(M_{\lambda_j}\right)_{(f_j)} = M_{(f_j)}$$

by the same argument as in Proposition 2.5.30(i). Now letting $\lambda' = \max\{\lambda_j\}$, we see that $\widetilde{M_{\lambda'}} = \tilde{M}$. □

2.5.8 Finiteness and vanishing theorems for graded local cohomology

Corollary 2.5.31 tells us that (under the same assumptions) we can corestrict the equivalence in Corollary 2.5.28 to obtain an essentially surjective functor

$$\widetilde{(\,\cdot\,)}\colon {}^*\mathsf{Mod}_{\mathsf{ft}}(S) \longrightarrow \mathsf{QCoh}_{\mathsf{ft}}(\mathcal{O}_X).$$

However, this does not give an equivalence of categories since, for example, any graded S-module satisfying (TN) will map to 0, and hence the composition $M \mapsto \tilde{M} \to \Gamma_*(\tilde{M})$ may not be isomorphic to the identity.

To prove the rest of Serre's equivalence, the key is the following:

[BS2013, Thm. 16.1.5]
[Har1977, Thm. II.5.19, Thm. III.5.2]

Theorem 2.5.32. ([FAC, n° 63, Proposition 3; EGAIII$_1$, Théorème 2.2.1]) *Let S be a Noetherian $\mathbf{N}$-graded ring. Let M be a finitely generated graded S-module.*

(i) *For all $i \geq 0$ and all $n \in \mathbf{Z}$, the S_0-module $H^i_{S_+}(M)_n$ is finitely generated.*

(ii) *For every $i \geq 0$, there exists $r \in \mathbf{Z}$ such that $H^i_{S_+}(M)_n = 0$ for all $n \geq r$.*

In fact, it turns out that r can be chosen independently of i in (*ii*). Proving this requires knowing that $H^i_{S_+}(\,\cdot\,)$ vanishes past a certain degree, which we will see later using Čech cohomology.

To prove Theorem 2.5.32, we need to prove one more result on local cohomology modules, namely, that S_+-power-torsion modules are Γ_{S_+}-acyclic. To prove this, we will use the following result. The Noetherian property is used to apply the Artin–Rees lemma [MurCA, Theorem 10.4.2].

Proposition 2.5.33. *Let R be a Noetherian ring and let I be an injective R-module. Consider an ideal $\mathfrak{a} \subseteq R$. Then, $\Gamma_{\mathfrak{a}}(I)$ is injective.*

[SGA2new, Exp. IV, Cor. 2.2]
[BS2013, Prop. 2.1.4]

Proof. By Baer's criterion (Lemma 1.3.49), it suffices to show that for every ideal $\mathfrak{b} \subseteq R$, every morphism $\mathfrak{b} \to \Gamma_{\mathfrak{a}}(I)$ can be extended to a morphism $R \to \Gamma_{\mathfrak{a}}(I)$:

$$\begin{tikzcd} \mathfrak{b} \arrow[r, hook] \arrow[d] & R \arrow[dl, dashed] \\ \Gamma_{\mathfrak{a}}(I). \end{tikzcd}$$

We put I into the diagram to obtain an extension of $\mathfrak{b} \to I$ to a morphism $R \to I$:

$$\begin{tikzcd} \mathfrak{b} \arrow[r, hook] \arrow[d, "h"'] & R \arrow[dl, dashed] \arrow[d, "1\mapsto w"] \\ \Gamma_{\mathfrak{a}}(I) \arrow[r, hook] & I. \end{tikzcd}$$

Since R is Noetherian, $\mathfrak{b}$ is finitely generated, and hence $\mathfrak{a}^t\, h(\mathfrak{b}) = h(\mathfrak{b})\, \mathfrak{a}^t = 0$ for some t. By the Artin–Rees lemma [MurCA, Theorem 10.4.2] for the inclusion $h(\mathfrak{b}) \subseteq Rw$ as submodules of I, there exists $c \geq 0$ such that, for all $n \geq c$,

$$\mathfrak{a}^n(Rw) \cap h(\mathfrak{b}) = \mathfrak{a}^{n-c}\Big(\mathfrak{a}^c(Rw) \cap h(\mathfrak{b})\Big).$$

Thus,

$$\mathfrak{a}^{t+c}(Rw) \cap H(\mathfrak{b}) \subseteq \mathfrak{a}^t\, h(\mathfrak{b}) = 0.$$

We can therefore define an R-module map

$$\begin{aligned} \tilde{h}\colon \mathfrak{a}^{t+c} + \mathfrak{b} &\longrightarrow \Gamma_{\mathfrak{a}}(I) \\ s + r &\longmapsto rw \end{aligned}$$

which is well-defined since if $r_1 - r_2 \in \mathfrak{a}^{t+c}$, then

$$r_1 w - r_2 w = (r_1 - r_2)w \in \mathfrak{a}^{t+c}(Rw) \cap h(\mathfrak{b}) = 0.$$

This map $\tilde{h}$ fits into the commutative diagram

$$\begin{tikzcd} \mathfrak{b} \arrow[r, hook] \arrow[dr, "h"'] & \mathfrak{a}^{t+c} + \mathfrak{b} \arrow[r, hook] \arrow[d, "\tilde{h}"] & R \arrow[dl, dashed] \arrow[d, "1\mapsto m"] \\ & \Gamma_{\mathfrak{a}}(I) \arrow[r, hook] & I \end{tikzcd}$$

where we use the injectivity of I again to extend $\tilde{h}$ to $1 \mapsto m$. It remains to show that $m \in \Gamma_{\mathfrak{a}}(I)$. For all $s \in \mathfrak{a}^{t+c}$, we have

$$sm = \tilde{h}(s) = \tilde{h}(s+0) = 0 \cdot w = 0,$$

and hence $m \in \Gamma_{\mathfrak{a}}(I)$. □

As a consequence, we have:

[BS2013, Cor. 2.1.6, Cor. 2.1.7(i)]

Corollary 2.5.34. *Let R be a Noetherian ring. Consider an ideal $\mathfrak{a} \subseteq R$ and let M be an $\mathfrak{a}$-power-torsion R-module. Then, M has an injective resolution where each term is an $\mathfrak{a}$-power-torsion R-module. Moreover, $H^i_{\mathfrak{a}}(M) = 0$ for all $i > 0$.*

Proof. Since $\mathsf{Mod}(R)$ has enough injectives, there exists an injective module I'^0 and an inclusion $M \hookrightarrow I'^0$. Since $\Gamma_{\mathfrak{a}}(\,\cdot\,)$ is left exact, applying $\Gamma_{\mathfrak{a}}$ yields an inclusion $M \hookrightarrow I^0$ where $I^0 \coloneqq \Gamma_{\mathfrak{a}}(I'^0)$ is injective by Proposition 2.5.33.

Now suppose that we have an exact sequence

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^n$$

of $\mathfrak{a}$-power-torsion R-modules such that I^j is injective for all j. Applying the previous paragraph to $\operatorname{coker}(I^{n-1} \to I^n)$, we can find an $\mathfrak{a}$-power-torsion R-module I^{n+1} such that

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^n \longrightarrow I^{n+1}$$

is exact.

It remains to show that $H^i_{\mathfrak{a}}(M) = 0$ for all $i > 0$. Let $M \to I^\bullet$ be an injective resolution where I^j is $\mathfrak{a}$-power-torsion for all j as constructed above. Then, the functor $\Gamma_{\mathfrak{a}}(\,\cdot\,)$ has no effect on the resolution $I^\bullet$. Computing cohomology objects, we see that $H^i_{\mathfrak{a}}(M) = 0$ for all $i > 0$. □

We can now prove Theorem 2.5.32.

Proof of Theorem 2.5.32. We induce on $i \geq -1$. If $i = -1$, then $H^{-1}_{S_+}(M) = 0$ and there is nothing to show. Now suppose that $i \geq 0$. We proceed by dévissage for finitely generated graded modules over Noetherian graded rings (see [MurAGI, §1.7.3]). Since M is finitely generated, there exists a filtration

$$0 = M^0 \subseteq M^1 \subseteq \cdots \subseteq M^\ell = M$$

by graded S-modules such that for every i, we have

$$\frac{M^i}{M^{i-1}} \cong \left(\frac{S}{\mathfrak{p}_i}\right)(d_i)$$

where $\mathfrak{p}_i$ is a homogeneous prime ideal of S and $d_i \in \mathbf{Z}$ [MurAGI, Proposition 1.7.7]. If $\ell = 0$, then $M = 0$ and there is nothing to show. For the inductive case, the long exact sequence associated to the short exact sequence

$$0 \longrightarrow M^{\ell-1} \longrightarrow M^\ell \longrightarrow \left(\frac{S}{\mathfrak{p}_\ell}\right)(d_\ell) \longrightarrow 0$$

together with the inductive hypothesis implies it suffices to show the claim for $(S/\mathfrak{p}_\ell)(d_\ell)$. Set $\mathfrak{p} \coloneqq \mathfrak{p}_\ell$ and $d \coloneqq d_\ell$.

Suppose first that $S/\mathfrak{p}$ is concentrated in degree 0. Then, $S/\mathfrak{p}$ is S_+-torsion, and hence $H^i_{S_+}(S/\mathfrak{p}) = 0$ for all $i > 0$ by Corollary 2.5.34. It remains to show (i) and (ii) for $i = 0$. For (ii), we have that $H^0_{S_+}(S/\mathfrak{p}) = S/\mathfrak{p}$ is concentrated in degree 0, and hence $H^0_{S_+}(S/\mathfrak{p})_n = 0$ for all $n \geq 1$. For (i), we have $H^0_{S_+}(S/\mathfrak{p}) = S/\mathfrak{p}$, and the composition

$$S_0 \hookrightarrow S \longrightarrow S/\mathfrak{p}$$

is surjective since it is the composition $S_0 = S_0 \to (S/\mathfrak{p})_0$ on degree 0. We therefore see that $H^0_{S_+}(S/\mathfrak{p}) = S/\mathfrak{p}$ is finitely generated over S_0.

It remains to consider the case when $S/\mathfrak{p}$ has elements of positive degree. Choose a nonzero homogeneous element $s \in S/\mathfrak{p}$ of degree $t > 0$. The exact sequence

$$0 \longrightarrow \frac{S}{\mathfrak{p}} \xrightarrow{s} \left(\frac{S}{\mathfrak{p}}\right)(t) \longrightarrow \frac{S/\mathfrak{p}}{s(S/\mathfrak{p})}(t) \longrightarrow 0$$

and the corresponding long exact sequence of (graded) local cohomology modules yields the exact sequence

$$H^{i-1}_{S_+}\left(\frac{S/\mathfrak{p}}{s(S/\mathfrak{p})}\right)_{n+t} \longrightarrow H^i_{S_+}\left(\frac{S}{\mathfrak{p}}\right)_n \xrightarrow{s} H^i_{S_+}\left(\frac{S}{\mathfrak{p}}\right)_{n+t}$$

of S_0-modules for all $n \in \mathbf{Z}$. By the inductive hypothesis, the module on the left is 0 for $n + t \geq r$, and hence multiplication by s is injective for all $n \geq r - t$. However, $H^i_{S_+}(S/\mathfrak{p})$ is S_+-power-torsion by construction, and hence we must have $H^i_{S_+}(S/\mathfrak{p})_n = 0$ for all $n \geq r - t$.

The inductive hypothesis also says that the module on the left has finitely generated graded components. Fix n and choose $k \geq 0$ such that $n + kt \geq r - t$, in which case

$$H^i_{S_+}(S/\mathfrak{p})_{n+kt} = 0$$

by the previous paragraph. For each $j \in \{0, 1, \dots, k-1\}$, there is an exact sequence

$$H^{i-1}_{S_+}\left(\frac{(S/\mathfrak{p})}{s(S/\mathfrak{p})}\right)_{n+(j+1)t} \longrightarrow H^i_{S_+}(S/\mathfrak{p})_{n+jt} \xrightarrow{s} H^i_{S_+}(S/\mathfrak{p})_{n+(j+1)t}.$$

By descending induction on j, we see that $H^i_{S_+}(S/\mathfrak{p})_n$ is finitely generated over S_0. □

Remark 2.5.35. An alternative method to prove Theorem 2.5.32 is to use the short exact sequence

$$0 \longrightarrow \Gamma_{S_+}(M) \longrightarrow M \longrightarrow \frac{M}{\Gamma_{S_+}(M)} \longrightarrow 0,$$

the Γ_{S_+}-acyclicity of S_+-power-torsion modules, and homogeneous prime avoidance (Homework 2, Problem 6(c) in MA557) on $M/\Gamma_{S_+}(M)$. This replaces the role of dévissage for graded modules and is the approach taken in [BS2013].

2.5.9 Serre's equivalence for coherent sheaves on Proj

We are now ready to show:

[Har1977, Exer. II.5.9(b)] **Theorem 2.5.36.** ([FAC, n° 65, Proposition 5; EGAIII$_1$, Théorème 2.3.1]) *Let S be a Noetherian $\mathbf{N}$-graded ring that is generated by S_1 as an S_0-algebra and let $X = \operatorname{Proj}(S)$. Let M be a graded S-module satisfying* (TF). *Then, there exists an integer N such that*

$$\alpha_n \colon M_n \longrightarrow \Gamma\Big(X, \tilde{M}(n)\Big)$$

is an isomorphism for all $n \geq N$. In other words, the kernel and cokernel of the morphism

$$\alpha \colon M \longrightarrow \Gamma_*(M)$$

satisfy (TN).

Proof. Let $M' = \bigoplus_{k \geq n} M_k$ be finitely generated such that $\widetilde{M'} = \tilde{M}$. We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r] \arrow[d, hook] & M' \arrow[r, "\alpha"] \arrow[d, hook] & \Gamma_*\Big(\widetilde{M'}\Big) \arrow[r] \arrow[d, equal] & Q' \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & K \arrow[r] & M \arrow[r, "\alpha"] & \Gamma_*(\tilde{M}) \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

with exact rows. By construction, we have $Q' \twoheadrightarrow Q$. The left hand side of the diagram yields the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r] \arrow[d, hook] & M' \arrow[r, "\alpha"] \arrow[d, hook] & \operatorname{im}(\alpha) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & K \arrow[r] & M \arrow[r, "\alpha"] & \Gamma_*(\tilde{M})
\end{tikzcd}$$

with exact rows. The snake lemma [KS2006, Lemma 12.1.1] implies that $K/K' \hookrightarrow M/M'$, and the latter satisfies (TN). Since $K/K' \hookrightarrow M/M'$ and $Q' \twoheadrightarrow Q$, we may replace M by M' to assume that M itself is finitely generated. By Theorem 2.5.19, it suffices to show that

$$H^0_{S_+}(M)_n = H^1_{S_+}(M)_n = 0$$

for sufficiently large n. This holds by Theorem 2.5.32! □

Corollary 2.5.37. ([EGAII, Corollaire 2.3.2]) *Let S_0 be a Noetherian ring and let S be a $\mathbf{N}$-graded ring finitely generated by S_1 as an S_0-algebra. Let $X = \operatorname{Proj}(S)$. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type. Then, $\Gamma_*(\mathscr{F})$ satisfies* (TF).

Proof. By Corollary 2.5.31, we know that $\mathscr{F} \cong \tilde{M}$ for some finitely generated graded S-module M. The morphism $M \to \Gamma_*(\mathscr{F})$ is an isomorphism in sufficiently large degree by Theorem 2.5.36. $\square$

We therefore obtain:

Theorem 2.5.38. (Serre's equivalence for $\mathsf{Coh}(\mathrm{Proj}(S))$**)** [FAC, n$^{\mathrm{o}}$ 65, Propositions 5 and 6; EGAIII$_1$, Scholie 2.
Let S_0 be a Noetherian ring. Let S be a $\mathbf{N}$-graded ring finitely generated by S_1 an S_0-algebra. Let $X = \mathrm{Proj}(S)$. We have an equivalence of categories [Har1977, Exer. II.5.9(c)]

$$\frac{\left\{\begin{matrix}\textit{graded } S\textit{-modules}\\ \textit{satisfying } (\mathrm{TF})\end{matrix}\right\}}{\left\{\begin{matrix}\textit{graded } S\textit{-modules}\\ \textit{satisfying } (\mathrm{TN})\end{matrix}\right\}} \rightleftarrows \mathsf{Coh}(\mathscr{O}_X)$$

$$\begin{aligned} M &\longmapsto \tilde{M} \\ \Gamma_*(\mathscr{F}) &\longleftarrow\!\shortmid \mathscr{F}. \end{aligned}$$

To give a careful proof of this, we would need to discuss more preliminaries from homological algebra and category theory, namely the notion of a quotient category of an Abelian category by a Serre subcategory. Instead, we give an indication of the proof with some references.

Idea of proof. The subcategory of ${}^*\mathsf{Mod}(S)$ consisting of graded modules satisfying (TN) forms what is now called a *Serre subcategory* (or *thick* subcategory [TohokuI, p. 138]), which was first defined in [Ser1953, (I) on p. 259]. The quotient category on the left-hand side is defined as in [TohokuI, §1.11]: The objects are graded S-modules satisfying (TF) and the quotient category is obtained by formally inverting inclusions (resp. quotient maps) whose cokernel (resp. kernel) satisfy (TN). We can now show that $\mathsf{Coh}(\mathscr{O}_X)$ together with the functor $M \mapsto \tilde{M}$ from graded S-modules satisfying (TF) satisfies the universal property of quotient categories of Abelian categories as stated in [Gab1962, Chapitre III, §1, Proposition 1, Corollaire 2]. $\square$

Example 2.5.39. ([Smi2004, Example 4.6]**)** Serre's equivalence does not hold without "generated by S_1" hypotheses. Consider the weighted polynomial ring $S = k[x, y]$ over a field k where x has degree 1 and y has degree 2. Let $M = S/(x)$ and consider the graded module $M(1)$. Note that $M(1)$ is not isomorphic to 0 in the category on the left-hand side of Theorem 2.5.38. $\boxed{3/10}$

We claim $(M(1))^{\sim} = 0$, and hence the functor $N \mapsto \widetilde{N}$ is not faithful. It suffices to show that

$$M(1)_{(x)} = M(1)_{(y)} = 0.$$

First, $M(1)_{(x)} = 0$ since $x \in \mathrm{Ann}_S(M)$. Next, to show $M(1)_{(y)} = 0$, we note that

$$M(1)_{(y)} = \left\{ \frac{f(y)}{y^n} \;\middle|\; f(y) \in M(1)_{2n} = M_{2n+1},\ n \in \mathbf{Z}_{\geq 0} \right\} = 0$$

since the numerators $f(y)$ have odd degree, while the denominators y^n have even degree. Thus, $M(1)_{(y)} = 0$.

2.5.10 Serre's finiteness, vanishing, and global generation theorems for Proj

Three more important consequences of Theorem 2.5.32 and Serre's equivalence Theorem 2.5.38 are the following finiteness and vanishing theorems for Proj. For projective varieties (i.e., the case $S_0 = k$ for an algebraically closed field k), these results are due to Serre [FAC].

Theorem 2.5.40. ([FAC, n° 66, Théorème 1 and Théorème 2; EGAII, Corollaire 2.7.9; EGAIII$_1$, Théo
Let S_0 be a Noetherian ring and let S be an $\mathbf{N}$-graded ring finitely generated by S_1 as an S_0-algebra. Let $X = \operatorname{Proj}(S)$ and let $\mathscr{F}$ be a coherent $\mathcal{O}_X$-module.

[Har1977, Thm. II.5.19, Thm. III.5.2]

(*i*) (Finiteness) *For all $i \geq 0$ and all $n \in \mathbf{Z}$, the S_0-module $H^i(X, \mathscr{F}(n))$ is finitely generated.*

(*ii*) (Vanishing) *There exists $r \in \mathbf{Z}$ such that $H^i(X, \mathscr{F}(n)) = 0$ for all $i > 0$ and all $n \geq r$.*

[Har1977, Thm. II.5.17]

(*iii*) (Global generation) *There exists $r \in \mathbf{Z}$ such that $\mathscr{F}(n)$ is globally generated for all $n \geq r$.*

Proof. By Corollary 2.5.37, there exists a finitely generated graded S-module M such that $\tilde{M} \cong \mathscr{F}$. We then have the exact sequence

$$0 \longrightarrow H^0_{S_+}(M) \longrightarrow M \longrightarrow \bigoplus_{n \in \mathbf{Z}} H^0\big(X, \mathscr{F}(n)\big) \longrightarrow H^1_{S_+}(M) \longrightarrow 0 \tag{2.5.41}$$

and the isomorphisms

$$\bigoplus_{n \in \mathbf{Z}} H^i\big(X, \mathscr{F}(n)\big) \xrightarrow{\sim} H^{i+1}_{S_+}(M)$$

for all $i \geq 1$ by Theorem 2.5.19. The isomorphisms for $i \geq 1$ imply (*i*) and (*ii*) for $i \geq 1$ by Theorem 2.5.32.

We show (*i*) for $i = 0$. The exact sequence (2.5.41) restricts to the exact sequence

$$0 \longrightarrow H^0_{S_+}(M)_n \longrightarrow M_n \longrightarrow H^0\big(X, \mathscr{F}(n)\big) \longrightarrow H^1_{S_+}(M)_n \longrightarrow 0$$

in each degree n. Since all of the terms are finitely generated S_0-modules by Theorem 2.5.32 and the fact that M is finitely generated over S, we see that $H^0(X, \mathscr{F}(n))$ is finitely generated as an S_0-module.

[EGAII, Lem. 2.7.9.1]

It remains to show (*iii*). Since M is finitely generated as a graded S-module, we can find a surjective map

$$\bigoplus_i S(n_i) \longrightarrow\!\!\!\!\!\rightarrow M$$

of graded S-modules for finitely many direct summands $S(n_i)$ and integers n_i. Setting $r = -\min\{n_i\}$, we see that

$$\bigoplus_i S(n + n_i) \longrightarrow\!\!\!\!\!\rightarrow M(n)$$

is surjective and each $n+n_i \geq r+n_i \geq 0$ for all i. Since S is generated in degree 1, by choosing a set of generators for $S(n+n_i)_0 = S_{n+n_i}$ as an S_0-module, we obtain a surjection $S_0^{(I_i)} \twoheadrightarrow S(n+n_i)_0$ for finite indexing sets I_i. We therefore obtain the surjective composition

$$\bigoplus_i S^{(I_i)} \longrightarrow\!\!\!\!\rightarrow \bigoplus_i S(n+n_i) \longrightarrow\!\!\!\!\rightarrow M(n).$$

Taking associated sheaves, we are done by the discussion in Definition 2.2.2. □

2.6 Fiber products, separation axioms, and Čech cohomology

So far, we have developed enough tools to write down some interesting examples of Spec and Proj schemes and to say something about their sheaf cohomology, at least in theory. However, when we work with concrete geometric objects, at some point we will need to *compute* something. Our definition of sheaf cohomology via derived functors is not well-suited for computations since injective (or even flasque) resolutions are hard to get your hands on.

Our goal will be to define and study the necessary separation axioms for schemes that will allow us to compute sheaf cohomology easily using what is known as the *Čech complex* associated to a chosen affine open cover.

2.6.1 Fiber products of schemes

We start with the general notion of a fiber product.

Definition 2.6.1. ([EGAInew, Chapitre 0, (1.2.2)]) Let $\mathscr{C}$ be a category. Let S be an object of $\mathscr{C}$ and let X and Y be objects equipped with morphisms to S. The *fiber product* of X and Y over S, if it exists, is an object $X \times_S Y$ of $\mathscr{C}$ together with *projection morphisms* $p_1 \colon X \times_S Y \to X$ and $p_2 \colon X \times_S Y \to Y$ over S satisfying the following universal property: For every object Z of $\mathscr{C}$ and every solid commutative diagram

$$\begin{tikzcd} Z \arrow[dr, dashed, "\theta"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\ & X \times_S Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d] \\ & X \arrow[r] & S \end{tikzcd}$$

there exists a unique dashed morphism $\theta \colon Z \to X \times_S Y$ making the diagram commute. We call a square of the form above (where the object in the top left is the fiber product of the three objects in the rest of the square) a *Cartesian square* or a *pullback square*.

Example 2.6.2. Consider the category of quasi-projective varieties Var_k over an algebraically closed field k from last semester. Last semester, we showed that Var_k has products [MurAGI, Theorem 1.3.37]. Products in Var_k are special cases of fiber products where $S = \{*\}$. More generally, if $\mathscr{C}$ is a category with a terminal object $\{*\}$, then $X \times_{\{*\}} Y$ is the product of X and Y.

We want to show that fiber products exist in the category Sch of schemes. We will also show that the functor $\mathsf{Sch} \hookrightarrow \mathsf{LRS}$ preserves fiber products.

[Har1977, Thm. II.3.3]

Theorem 2.6.3. ([EGAI, Théorème 3.2.6; EGAInew, Théorème 3.2.1]) *Let S be a scheme and let X and Y be schemes over S. Then, the fiber product $X \times_S Y$ exists in Sch. Moreover, $X \times_S Y$ is a fiber product for X and Y over S in LRS.*

Remark 2.6.4. It turns out that fiber products exist in LRS. This is a fairly recent result attributed to Becker, and published in [Gil2011, Corollary 5].

We construct fiber products "by hand" following [EGAI; Har1977] instead of following [EGAInew] or [Gil2011]. As a preliminary result (used multiple times throughout the proof), we show the following.

Lemma 2.6.5. ([Stacks, Tag 01HI]) *Let $f\colon X \to Y$ be a morphism of ringed spaces. Let $U \hookrightarrow X$ and $V \hookrightarrow Y$ be open ringed subspaces such that $f(\mathrm{sp}(U)) \subseteq \mathrm{sp}(V)$. Then, there exists a unique morphism $f|_U\colon U \to V$ making the diagram*

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, dashed, "f|_U"'] & X \arrow[d, "f"] \\ V \arrow[r, hook] & Y \end{tikzcd}$$

commute. If f is a morphism of locally ringed spaces, then $f|_U$ is also a morphism of locally ringed spaces.

Proof. The map $\mathrm{sp}(f|_U)$ on underlying topological spaces exists and is unique. Since any open subset $W \subseteq V$ has open inverse image $f|_U^{-1}(W) = f^{-1}(W) \cap U \subseteq X$, we see that $\mathrm{sp}(f|_U)$ is continuous.

It remains to define the action of $f|_U$ on structure sheaves. Let $W \subseteq V$ be an open subset. We have the commutative diagram

$$\begin{tikzcd} \Gamma(W, \mathcal{O}_Y) \arrow[r] \arrow[d, equal] & \Gamma\bigl(f^{-1}(W), \mathcal{O}_X\bigr) \arrow[d] \\ \Gamma(W, \mathcal{O}_V) \arrow[r, dashed] & \Gamma\Bigl(\bigl(f|_U\bigr)^{-1}(W), \mathcal{O}_U\Bigr). \end{tikzcd}$$

This diagram determines the behavior of $f|_U$ on structure sheaves uniquely since the behavior must be compatible with restriction maps for f. $\square$

We now show Theorem 2.6.3.

Proof of Theorem 2.6.3. The idea is to first construct the fiber product for affine schemes and then to glue using the universal property on overlaps. We proceed in five steps.

Step 1. ([EGAI, Proposition 3.2.2]) *The case when* $X = \operatorname{Spec}(A)$*,* $Y = \operatorname{Spec}(B)$*, and* $S = \operatorname{Spec}(R)$ *are all affine.*

By the universal property of tensor products of algebras [MurCA, Theorem 7.11.2], we have the commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\ A \arrow[r] \arrow[drr, bend right] & A \otimes_R B \arrow[dr, dashed, "\exists!"] & \\ & & \Gamma(Z, \mathcal{O}_Z) \end{tikzcd}$$

By Proposition 2.1.21, giving a morphism $Z \to \operatorname{Spec}(A \otimes_R B)$ is equivalent to giving a morphism $A \otimes_R B \to \Gamma(Z, \mathcal{O}_Z)$, and similarly for $\operatorname{Spec}(A)$ and $\operatorname{Spec}(B)$. Applying Spec in the top left square and applying Proposition 2.1.21 three times to each of the morphisms involving Z, we are done.

Step 2. ([EGAI, Lemme 3.2.6.1]) *If* X *and* Y *are schemes over a scheme* S*, if* $U \subseteq X$ *is an open subset, and if the product* $X \times_S Y$ *exists, then* $p_1^{-1}(U) \subseteq X \times_S Y$ *(with the induced open subscheme structure) is a product for* U *and* Y *over* S*.*

Consider the commutative diagram

$$\begin{tikzcd} Z \arrow[drrr, bend left, "g"] \arrow[drr, dashed, "\theta", "\exists!"'] \arrow[ddr, bend right, "f"'] & & & \\ & p_1^{-1}(U) \arrow[r, hook] \arrow[d] & X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\ & U \arrow[r, hook] & X \arrow[r] & S \end{tikzcd}$$

where the restricted morphism $p_1^{-1}(U) \to U$ is constructed using Lemma 2.6.5. Since $X \times_S Y$ is a fiber product, the dashed map $Z \dashrightarrow X \times_S Y$ exists. We need to show that θ factors uniquely through $p_1^{-1}(U)$. This follows from Lemma 2.6.5 applied to the diagram

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d, dashed, "\exists!"'] & Z \arrow[d, "\theta"] \\ p_1^{-1}(U) \arrow[r, hook] & X \times_S Y. \end{tikzcd}$$

Step 3. ([EGAI, Lemme 3.2.6.2, (3.2.6.3)]) *Let* X, Y *be schemes over* S*. Suppose that* $\{X_i\}$ *is an open cover of* X *and suppose that* $X_i \times_S Y$ *exists for every* i*. Then,* $X \times_S Y$ *exists.*

For each i, j, let
$$U_{ij} \coloneqq p_1^{-1}(X_{ij}) \subseteq X_i \times_S Y.$$
By Step 2, U_{ij} is a product for X_{ij} and Y over S. By the uniqueness of fiber products, there are unique isomorphisms
$$\varphi_{ij} \colon U_{ij} \xrightarrow{\sim} U_{ji}$$
for every i, j compatible with the projection morphisms. Moreover, these isomorphisms are compatible with each other for each i, j, k in the sense of the gluing lemma (Lemma 2.1.30) by the uniqueness part of the universal property of fiber products. By the gluing lemma (Lemma 2.1.30), we therefore obtain a scheme $X \times_S Y$ which we claim is the fiber product for X and Y over S. The projection morphisms p_1, p_2 are defined by gluing the projection maps from the pieces $X_i \times_S Y$. For each i, set $Z_i = f^{-1}(X_i)$, and consider the commutative diagram
$$\begin{tikzcd}
Z_i \arrow[r, hook] \arrow[dr, "\theta_i", "\exists!"'] \arrow[ddr, bend right, "f_i"'] & Z \arrow[dr, dashed, "\theta", "\exists!"'] \arrow[drr, bend left, "g"] \arrow[ddr, "f"'] & & \\
& X_i \times_S Y \arrow[r] \arrow[d] & X \times_S Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d] \\
& X_i \arrow[r] & X \arrow[r] & S.
\end{tikzcd}$$
Then, the two compositions
$$\begin{aligned} Z_i \cap Z_j &\xrightarrow{\theta_i} X_i \times_S Y \longrightarrow X \times_S Y \\ Z_i \cap Z_j &\xrightarrow{\theta_j} X_j \times_S Y \longrightarrow X \times_S Y \end{aligned} \tag{2.6.6}$$
both factor through $X_{ij} \times_S Y$ by Lemma 2.6.5, and hence the two compositions (2.6.6) are equal by the universal property for $X_{ij} \times_S Y$. Thus, the morphisms (2.6.6) glue to give a map θ, which is unique by post-composing with the inclusion into $X_i \times_S Y$ and applying the universal property for each $X_i \times_S Y$.

Step 4. ([EGAI, (3.2.6.5)]) *The case when S is affine.*

By Step 1, $X \times_S Y$ exists when X, Y, S are all affine. By Step 3, $X \times_S Y$ exists when X is arbitrary but Y, S are affine. By interchanging the roles of X and Y and applying Step 3 again, $X \times_S Y$ exists when X, Y are arbitrary but S is affine.

Step 5. ([EGAI, (3.2.6.4), (3.2.6.5)]) *Conclusion of proof.*

Given arbitrary X, Y, S, let $q \colon X \to S$ and $r \colon Y \to S$ be the given structure morphisms. Let $\{S_i\}$ be an affine open cover of S. Let $X_i = q^{-1}(S_i)$ and

$Y_i = r^{-1}(S_i)$. Then, by Step 4, $X_i \times_{S_i} Y_i$ exists. The same scheme is a fiber product for X_i and Y over S because in the commutative diagram

$$\begin{tikzcd}
Z \arrow[dr, dashed] \arrow[drrr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & & \\
& X_i \times_{S_i} Y_i \arrow[r] \arrow[d] & Y_i \arrow[r, hook] \arrow[d] & Y \arrow[d] \\
& X_i \arrow[r] & S_i \arrow[r, hook] & S
\end{tikzcd}$$

the morphism g factors through Y_i by Lemma 2.6.5. Thus, $X_i \times_S Y$ exists for every i. Applying Step 3 again, $X \times_S Y$ exists.

Note that we did not use anywhere that Z is a scheme, only that it is a locally ringed space. Thus, $X \times_S Y$ is a fiber product for X and Y over S in the category LRS. In other words, the inclusion functor $\mathsf{Sch} \hookrightarrow \mathsf{LRS}$ preserves fiber products. $\square$

Caution 2.6.7. Let k be a field. The underlying topological space of $X \times_{\mathrm{Spec}(k)}$ [Har1977, Exer. II.3.9(a)]
Y is not the product of X and Y, even if k is algebraically closed! This is because

$$\mathbf{A}^1_k \times_{\mathrm{Spec}(k)} \mathbf{A}^1_k = \mathrm{Spec}\Big(k[s] \otimes_k k[t]\Big) \cong \mathrm{Spec}\Big(k[s,t]\Big) = \mathbf{A}^2_k,$$

which has many non-closed points corresponding to prime ideals of height 1 that are not of the form $((0),(t-b))$ or $((s-a),(0))$ by Theorem 2.1.7.

As one example of fiber products, we can define a scheme structure on fibers of morphisms.

Definition 2.6.8. **(**[EGAInew, (3.4.7)]**)** Let $f\colon X \to Y$ be a morphism of schemes [Har1977, p. 89]
and let $y \in Y$ be a point. Let $k(y) \coloneqq \mathcal{O}_{Y,y}/\mathfrak{m}_y$ be the residue field of Y at y and let $\mathrm{Spec}(k(y)) \to Y$ be the morphism obtained as the composition

$$\mathrm{Spec}\Big(k(y)\Big) \longrightarrow \mathrm{Spec}(\mathcal{O}_{Y,y}) \longrightarrow Y.$$

See [Har1977, Exercise II.2.7] (Homework 6, Problem 4(a)). The *fiber* of the morphism f over the point y is

$$X_y \coloneqq X \times_Y \mathrm{Spec}\Big(k(y)\Big).$$

The fiber X_y is a scheme over $k(y)$. The underlying topological space $\mathrm{sp}(X_y)$ [EGAInew, Prop. 3.4.6]
homeomorphic to $f^{-1}(y)$ with the subspace topology: The construction of fiber products (Theorem 2.6.3) says that fiber products can be constructed affine-locally, and for a ring map $\varphi\colon A \to B$, the spectrum

$$\mathrm{Spec}(B \otimes_A A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})$$

is homeomorphic to the fiber $\mathrm{Spec}(\varphi)^{-1}(\mathfrak{p})$ by [MurCA, Propositions 1.3.12 and 3.2.10(ii)].

This construction occurs throughout algebraic geometry: You can regard a morphism as a family of schemes, in which case the morphism encodes different deformations or degenerations of a scheme. The case when $Y = \operatorname{Spec}(\mathbf{Z})$ gives rise to the technique of *reduction modulo* p.

Example 2.6.9. ([Har1977, Examples II.3.3.1 and II.3.3.2]) Let k be an algebraically closed field.

(i) Consider

$$X = \operatorname{Spec}\left(\frac{k[x,y,t]}{(ty - x^2)}\right) \longrightarrow \operatorname{Spec}\big(k[t]\big).$$

This is a family of parabolas $V(ty - x^2) \subseteq \mathbf{A}^2_k$ parametrized by t. The fiber over $t \neq 0$ is an honest parabola, but over $t = 0$ we get the "double line" $\operatorname{Spec}(k[x,y]/(x^2))$, which is a non-reduced scheme. In this case, we say that the family of parabolas *degenerates* to a double line or that the double line *deforms* to a parabola.

Note that the generic fiber is

$$X_\eta = \operatorname{Spec}\left(\frac{k(t)[x,y]}{(ty - x^2)}\right) \cong \operatorname{Spec}\left(\frac{k(t)[x,y]}{(y - x^2)}\right),$$

which is a parabola over $k(t)$. Thus, the generic fiber describes the "generic behavior" of the family.

3/24 (ii) Consider

$$X = \operatorname{Spec}\left(\frac{k[x,y,t]}{(xy - t)}\right) \longrightarrow \operatorname{Spec}\big(k[t]\big).$$

This is a family of hyperbolas $V(xy - t) \subseteq \mathbf{A}^2_k$ parametrized by t. The fiber over $t \neq 0$ is an honest hyperbola, but over $t = 0$ we get the union of two axes $\operatorname{Spec}(k[x,y]/(xy))$, which is a reducible scheme. In this case, we say that the family of hyperbolas *degenerates* to the union of two lines or that the union of two lines *deforms* to a parabola.

The generic fiber in this case is

$$X_\eta = \operatorname{Spec}\left(\frac{k(t)[x,y]}{(xy - t)}\right) \cong \operatorname{Spec}\left(\frac{k(t)[x,y]}{(xy - 1)}\right),$$

which is a hyperbola over $k(t)$.

2.6.2 Base change

The construction of fiber products gives rise to the notion of base change, a special case of which includes the operation of taking fibers. Base change is a very useful way of replacing the base scheme that you are working over: If

$f\colon X \to X'$ is a morphism over S and $g\colon S' \to S$ is a morphism, taking fiber products yields a morphism

$$\begin{tikzcd}[column sep=tiny] X \times_S S' \arrow[rr, "f\times \mathrm{id}_{S'}"] \arrow[dr] & & X' \times_S S' \arrow[dl] \\ & S' & \end{tikzcd}$$

over S' by looking at the commutative diagram

$$\begin{tikzcd} X \times_S S' \arrow[dr, dashed, "f\times\mathrm{id}_{S'}", "\exists!"'] \arrow[drr, bend left, "p_2"] \arrow[dd, "p_1"'] & & \\ & X' \times_S S' \arrow[r] \arrow[d] & S' \arrow[d] \\ X \arrow[r, "f"] & X' \arrow[r] & S \end{tikzcd}$$

where the square is Cartesian. A common example of this is when $S = \operatorname{Spec}(k)$ and $S' = \operatorname{Spec}(k')$ for a field extension $k \subseteq k'$. We can even take the product of two morphisms by looking at the commutative diagram

$$\begin{tikzcd} X \times_S Y \arrow[rr, "p_2"] \arrow[dr, dashed, "f\times g", "\exists!"'] \arrow[dd, "p_1"'] & & Y \arrow[d, "g"] \\ & X' \times_S Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"] & X' \arrow[r] & S. \end{tikzcd}$$

Many properties of morphisms we have seen so far are “stable under base change.” Let us define two more, important classes of morphisms that we have not yet seen in this class.

Definition 2.6.10. ([EGAII, Définition 6.1.1]) Let $f\colon X \to Y$ be a morphism [Har1977, p. 84] of schemes. We say that f is *integral* (resp. *finite*) if there exists an affine open cover $Y = \bigcup_i \operatorname{Spec}(B_i)$ such that

$$f^{-1}\big(\operatorname{Spec}(B_i)\big) = \operatorname{Spec}(A_i)$$

is an affine open subset of X with A_i an integral (resp. module-finite) B_i-algebra. Note that we do not assume that $B_i \to A_i$ is injective, and hence this is not the same as saying that A_i is an integral or module-finite *extension* of B_i.

We have the implications

$$\text{finite} \implies \text{integral} \implies \text{affine}.$$

However, morphisms (locally) of finite type are not necessarily affine.

By [MurCA, Theorem 4.2.6] and [Har1977, Exercise II.3.3(c)] (Homework 8, Problem 1(b)), we know that

$$\text{integral} + \text{locally of finite type} \iff \text{finite}.$$

As usual, by Nike's trick [Vak2025, Proposition 5.3.1] (Homework 6, Problem 3), you can show that f is integral (resp. finite) if and only if the inverse image of *every* affine open $\operatorname{Spec}(B)$ is of the form $\operatorname{Spec}(A)$ where A is module-finite over B.

Proposition 2.6.12 below says that many properties of morphisms are stable under base change. Here are two examples of properties of morphisms that are *not* stable under base change.

[Har1977, Exer. II.3.15(c)] **Example 2.6.11.** Let $f\colon X \to \operatorname{Spec}(k)$ for a field k. Following [EGAInew, (3.2.2)], we denote $X \otimes_k k' \coloneqq X \times_{\operatorname{Spec}(k)} \operatorname{Spec}(k')$ for a field extension $k \subseteq k'$.

(i) The property "the fibers of f are irreducible" is not stable under base change. Let $k = \mathbf{R}$ and $X = \operatorname{Spec}(\mathbf{R}[x]/(x^2+1))$. Since x^2+1 is irreducible over $\mathbf{R}$, the scheme X is integral, and in particular, irreducible. However,
$$X \otimes_{\mathbf{R}} \mathbf{C} = \operatorname{Spec}\left(\frac{\mathbf{C}[x]}{(x+i)(x-i)}\right)$$
is a disjoint union of two points, which is reducible.

(ii) The property "the fibers of f are reduced" is not stable under base change. Let $k = \mathbf{F}_p(t)$ and $X = \operatorname{Spec}(\mathbf{F}_p(t)[x]/(x^p - t))$. Since $x^p - t$ is irreducible over $\mathbf{F}_p(t)$, the scheme X is integral, and in particular, reduced. However,
$$X \otimes_{\mathbf{F}_p(t)} \mathbf{F}_p(t^{1/p}) = \operatorname{Spec}\left(\frac{\mathbf{F}_p(t^{1/p})[x]}{(x - t^{1/p})^p}\right)$$
is non-reduced.

On the homework, you will investigate "geometric" versions of these properties. We say that a k-scheme X is *geometrically irreducible* (resp. *geometrically reduced*, *geometrically integral*) if $X \otimes_k k'$ is irreducible (resp. reduced, integral) for every finite field extension $k \subseteq k'$.

We want to show that various properties of morphisms are stable under base change.

Proposition 2.6.12. *Let S be a scheme. Let $f\colon X \to X'$ and $g\colon Y \to Y'$ be morphisms of schemes over S satisfying one of the following properties $\mathscr{P}$:*

(i) [EGAInew, Proposition 6.1.5(iv)] *quasi-compact.*

(ii) [EGAInew, Proposition 6.2.3(iv)] *locally of finite type.*

(*iii*) [EGAInew, Proposition 6.3.4(*iv*)] *finite type.* [Har1977, Exer. II.3.13(d)]

(*iv*) [EGAInew, Proposition 4.3.6(*iii*)] *closed immersion.* [Har1977, Exer. II.3.11(a)] [EGAI, Prop. 4.3.1]

(*v*) [EGAInew, Proposition 4.3.6(*iii*)] *open immersion.*

(*vi*) [EGAInew, Proposition 4.3.6(*iii*)] *immersion.*

(*vii*) [EGAII, Proposition 1.6.2(*iv*)] *affine.*

(*viii*) [EGAII, Proposition 6.1.5(*iv*)] *integral.*

(*ix*) [EGAII, Proposition 6.1.5(*iv*)] *finite.*

Then, the product

$$f \times g\colon X \times_S Y \longrightarrow X' \times_S Y'$$

has the same property $\mathscr{P}$*. In particular, setting* $g = \mathrm{id}_Y$*, the properties* $\mathscr{P}$ *are stable under base change.*

Proof. Let $\operatorname{Spec}(A' \otimes_R B') \subseteq X' \times_S Y'$ be an affine open subset. The inverse image of this affine open subset is

$$(f \times g)^{-1}\Big(\operatorname{Spec}(A' \otimes_R B')\Big) \xrightarrow{\sim} f^{-1}\Big(\operatorname{Spec}(A')\Big) \times_{\operatorname{Spec}(R)} g^{-1}\Big(\operatorname{Spec}(B')\Big)$$

by applying Step 2 of Theorem 2.6.3 to each factor, and then using the argument in Step 5 of Theorem 2.6.3 to replace the base scheme S in the fiber product with $\operatorname{Spec}(R)$.

For affine, integral, and finite morphisms, write

$$\begin{aligned} f^{-1}\Big(\operatorname{Spec}(A')\Big) &= \operatorname{Spec}(A) \\ g^{-1}\Big(\operatorname{Spec}(B')\Big) &= \operatorname{Spec}(B) \end{aligned}$$

for some R-algebras A and B. We then have

$$f^{-1}\Big(\operatorname{Spec}(A')\Big) \times_{\operatorname{Spec}(R)} g^{-1}\Big(\operatorname{Spec}(B')\Big) \xleftarrow{\sim} \operatorname{Spec}(A \otimes_R B),$$

and hence we are done in the affine case. The integral (resp. finite) cases follow from the fact that

$$A' \otimes_R B' \longrightarrow A' \otimes_R B \longrightarrow A \otimes_R B$$

is integral (resp. finite) if $A' \to A$ and $B' \to B$ are. For the finite case, module-finiteness is preserved under base change [MurCA, Corollary 7.5.4] and composition [MurCA, Lemma 4.2.8]. For the integral case, write the maps $A' \to A$ and $B' \to B$ as a direct limit of module-finite maps. The finite case implies that the base changes of these maps are direct limits of module-finite maps, and hence are integral.

See [BouCA, V.1.1, Props. 5, 6] for a textbook reference.

For quasi-compactness (resp. locally of finite type), write

$$f^{-1}\Big(\mathrm{Spec}(A')\Big) = \bigcup_i \mathrm{Spec}(A_i)$$
$$g^{-1}\Big(\mathrm{Spec}(B')\Big) = \bigcup_j \mathrm{Spec}(B_j)$$

as finite unions (resp. unions) of affine open subsets. Then,

$$(f \times g)^{-1}\Big(\mathrm{Spec}(A' \otimes_R B')\Big) = \bigcup_{i,j} \mathrm{Spec}(A_i \otimes_R B_j)$$

is a finite union (resp. union) of affine open subsets (resp. affine open subsets of finite type over $A' \otimes_R B'$). The statement for finite type follows from combining the statements for quasi-compactness and locally of finite type.

For open immersions, we decompose $f \times g$ as

$$X \times_S Y \xrightarrow{f \times \mathrm{id}_Y} X' \times_S Y \xrightarrow{\mathrm{id}_{X'} \times g} X' \times_S Y'.$$

For each individual morphism, the statement for open immersions follows from Step 2 of Theorem 2.6.3. Thus, the composition is also an open immersion by definition of an open immersion.

For closed immersions, replace the closed immersions with the canonical inclusions of closed subschemes. We can therefore consider the quasi-coherent ideal sheaves $\mathscr{I}_1 \subseteq \mathscr{O}_{X'}$ and $\mathscr{I}_2 \subseteq \mathscr{O}_{Y'}$ defining the closed subschemes $X \subseteq X'$ and $Y \subseteq Y'$ as in (2.4.13). We claim that

$$p_1^{-1}\mathscr{I}_1 \cdot \mathscr{O}_{X' \times_S Y'} + p_2^{-1}\mathscr{I}_2 \cdot \mathscr{O}_{X' \times_S Y'}$$

is quasi-coherent and defines a closed subscheme W that is the fiber product $X \times_S Y$. This sheaf is quasi-coherent since pullback preserve quasi-coherence, and since this sheaf is the image of the morphism

$$p_1^*\mathscr{I}_1 \oplus p_2^*\mathscr{I}_2 \xrightarrow{[1\ \ 1]} \mathscr{O}_{X' \times_S Y'}.$$

Here, we recall that images of quasi-coherent sheaves are quasi-coherent by Corollary 2.3.17. On affine open subsets $U = \mathrm{Spec}(A' \otimes_R B')$, the closed subscheme $W \cap U$ is defined by

$$I_1(A' \otimes_R B') + I_2(A' \otimes_R B')$$

where $I_1 = \Gamma(\mathrm{Spec}(A'), \mathscr{I}_1)$ and $I_2 = \Gamma(\mathrm{Spec}(B'), \mathscr{I}_2)$. We are now done by using the isomorphism

$$\frac{A' \otimes_R B'}{I_1(A' \otimes_R B') + I_2(A' \otimes_R B')} \xleftarrow{\sim} (A'/I_1) \otimes_R (B'/I_2)$$

from [MurCA, Corollary 7.2.4].

The statement for immersions follows by decomposing the immersions f, g as closed immersions followed by open immersions. $\square$

Index

Page numbers in bold refer to definitions. An "f", "n", or "t" following a page number refers to a figure, footnote, or table on that page, respectively.

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