Étale Cohomology

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Abstract

These lecture notes accompanied a minicourse on étale cohomology offered by the author at the University of Michigan in the summer of 2017. They are only a preliminary draft and should not be used as a reference.

Contents

1 Overview **1**
1.1 The Weil Conjectures . . . 1
1.2 Weil Cohomology Theory . . . 2
1.3 A Sketch of ℓ-adic Cohomology . . . 3
1.4 Goals . . . 4

2 Sites **4**
2.1 The Yoneda Embedding . . . 4
2.2 Grothendieck Topologies . . . 5
2.3 Sheaves on a Site . . . 6
2.4 A Toy Model . . . 7
2.5 Sheafification . . . 8
2.6 Sheaves of Abelian Groups . . . 9

3 The Étale Topos **9**
3.1 Étale Morphisms . . . 9
3.2 Building Étale Sheaves . . . 11
3.3 The Étale Fundamental Group . . . 13
3.4 Further Operations on Étale Sheaves . . . 14

1 Overview

The machinery of étale cohomology and its relative, ℓ-adic cohomology, is formidable. This preliminary section explains the origins of such a theory, its general shape, and our approach to understanding it.

1.1 The Weil Conjectures

Historically, the Weil conjectures motivated the search for a robust cohomology theory of varieties over finite fields. These conjectures concern the zeta function of a variety over a finite field, a device for counting the number of points of the variety over finite extensions of the base field.

Definition 1. Let X be a variety over a finite field $\mathbb{F}_q$. The *zeta function* of X is the formal power series defined by the equation

$$\zeta_X(t) = \exp\left(\sum_{d\geq 1} \#X(\mathbb{F}_{q^d})\frac{t^d}{d}\right) \qquad (t = q^{-s}),$$

where $\#X(\mathbb{F}_{q^d})$ is the number of $\mathbb{F}_{q^d}$-points of X.

The zeta function $\zeta(X,t)$ is an object of the power series ring $\mathbb{Q}[\![t]\!]$; the exponential function $\exp : t\mathbb{Q}[\![t]\!] \to \mathbb{Q}[\![t]\!]^\times$ is defined only for power series with constant term zero.

The simplest examples to compute are projective and affine space. Since $\#\mathbb{A}^n(\mathbb{F}_{q^d}) = q^{dn}$,

$$\zeta_{\mathbb{A}^n}(t) = \exp\left(\sum_{d\geq 1} \frac{(q^n t)^d}{d}\right) = (1-q^n t)^{-1}.$$

Since $\mathbb{P}^n$ is the disjoint union $\bigsqcup_{i=0}^n \mathbb{A}^i$ of locally closed affine spaces,

$$\zeta_{\mathbb{P}^n}(t) = \prod_{i=0}^n \zeta_{\mathbb{A}^i}(t) = \prod_{i=0}^n (1-q^i t)^{-1}.$$

These zeta functions are also related to the more classical zeta functions of algebraic number fields: you can verify the identity

$$\zeta_{\mathcal{O}_K}(s) = \prod_{\substack{x\in\operatorname{Spec}\mathcal{O}_K \\ \text{closed}}} \zeta_{\kappa(x)}(s).$$

I have nothing more to say in this connection, but hopefully it motivates the definition.

Exercise 2: The zeta function of a finite disjoint union of subvarieties is the product of the zeta functions of the subvarieties.

The Weil conjectures state that for a smooth, n-dimensional, projective variety X over a finite field $\mathbb{F}_q$,

(1) (Rationality) $\zeta_X(t)$ is a rational function of t. More precisely, there are polynomials $P_0(t), \ldots, P_{2n}(t)$ such that

$$\zeta_X(t) = \prod_{i=0}^{2n} P_i(t)^{(-1)^i},$$

where $P_0(t) = 1-t$, $P_{2n}(t) = 1-q^n t$, and for each i, $P_i(t)$ has integer coefficients with $P_i(0) = 1$.

Factor $P_i(t)$ over $\mathbb{C}$ as $\prod_j (1-\alpha_{ij} t)$, where $\alpha_{ij} \in \mathbb{C}$.

(2) (Functional Equation) $\zeta\big(X, (q^n t)^{-1}\big) = \pm q^{nE/2} t^E \zeta(X,t)$ where E is the self-intersection number of the diagonal $\Delta \subseteq X \times X$.

(3) (Riemann Hypothesis) $|\alpha_{ij}| = q^{i/2}$.

(4) (Betti Numbers) If X is a (good) reduction mod p of a variety Y defined over the ring of integers of a number field embedded in $\mathbb{C}$, then $\deg P_i$ is the ith Betti number[1] of the space of complex points of Y.

Exercise 3: What does the Riemann Hypothesis part of the Weil conjectures say about the location of roots of $\zeta_X(s)$?

For more information about the Weil conjectures, see Appendix C of Hartshorne's *Algebraic Geometry* [6].

1.2 Weil Cohomology Theory

Grothendieck suggested that the Weil conjectures could be proved by constructing a cohomology theory for varieties over finite fields. Specifically, since the $\mathbb{F}_{q^d}$-points of a variety X are precisely the fixed points of the d-th power Φ^d of the Frobenius morphism $\Phi : \overline{X} \to \overline{X}$, where $\overline{X} = X_{\overline{\mathbb{F}}_q}$, a cohomology theory permitting a generalization of the classical Lefschetz fixed-point theorem, such as

$$\#X(\mathbb{F}_{q^d}) = \sum_{i\geq 0} (-1)^i \operatorname{tr}\left(\Phi^{d*}; H^i(\overline{X}, \mathbb{Q}_\ell)\right),$$

would give detailed information about $\#X(\mathbb{F}_{q^d})$.

In order to deduce the Lefschetz fixed-point theorem, the cohomology theory would need to satisfy certain formal properties. Kleiman [9] gave an axiomatic description of these properties.

[1] The ith Betti number of a topological space X is $\dim_{\mathbb{Q}} H^i(X, \mathbb{Q})$.

Definition 4 (See [9, Section 1.2]). Fix a coefficient field K of characteristic zero and a base field k. A *Weil cohomology theory* is a contravariant functor $X \mapsto H^\bullet(X)$ from the category of smooth projective varieties over k to the category of $\mathbb{N}$-graded K-algebras satisfying the following properties. Let $n = \dim X$.

(0) (Finiteness) $H^i(X)$ is finite dimensional for all i.

(1) (Dimensionality) $H^i(X) = 0$ for $i < 0$ or $i > 2n$.

(2) (Orientation Map) There is a canonical isomorphism $H^{2n}(X) \cong K$.

(3) (Poincaré Duality) There is a canonical non-degenerate pairing $H^i(X) \times H^{2n-i}(X) \to H^{2n}(X)$.

(4) (Künneth Isomorphism) There is a canonical isomorphism $H^\bullet(X) \otimes H^\bullet(Y) \cong H^\bullet(X \times Y)$.

(5) (Cycle Map) There is a cycle map $Z^i(X) \to H^{2i}(X)$ (satisfying certain compatibility conditions).

(6) Hard and weak Lefschetz axioms

There are two classical examples of Weil cohomology theories.

- **Singular cohomology** ($k = \mathbb{C}$, $K = \mathbb{Q}$). Smooth varieties over $\mathbb{C}$ are complex manifolds (using their analytic topology) with singular cohomology.
- **Algebraic de Rham cohomology** [5] ($\operatorname{char} k = 0$, $K = k$). The sheaf $\Omega_{X/k}$ of differentials of a variety X over k can be used to define an algebraic analogue of the classical de Rham complex of smooth manifolds, and the (hyper)cohomology of this complex is the *algebraic de Rham cohomology* of X.

Neither cohomology theory adequately captures varieties of positive characteristic. To address this deficiency, we will develop (pieces of) a Weil cohomology theory called *ℓ-adic cohomology.* Here ℓ is a prime not dividing q, the base field is $k = \mathbb{F}_q$, and the coefficient field is $K = \mathbb{Q}_\ell$.[2]

1.3 A Sketch of ℓ-adic Cohomology

For sufficiently nice topological spaces X, the singular cohomology $H^i_{\text{sing}}(X, \mathbb{Z})$ is isomorphic to the derived-functor cohomology $H^i(X, \mathbb{Z})$ of the constant sheaf $\mathbb{Z}$. By analogy, we would hope to define ℓ-adic cohomology as the derived-functor cohomology of $\mathbb{Z}$. Unfortunately, this definition is inadequate: on an irreducible variety, any constant sheaf is flabby and therefore has trivial derived-functor cohomology.

Exercise 5: Every constant sheaf is flabby.

Exercise 6: Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be an exact sequence of presheaves of abelian groups on a topological space X. Suppose that $\mathcal{F}$ is flabby.

(a) $0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{H}) \to 0$ is exact.

(b) $\mathcal{G}$ is flabby if and only if $\mathcal{H}$ is flabby.

Exercise 7: $\Gamma(X, -)$ maps bounded-from-the-left acyclic complexes of flabby sheaves to bounded-from-the-left acyclic complexes of abelian groups. Conclude that $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

The obstruction to defining ℓ-adic cohomology using sheaves is a shortage of open subsets in the Zariski topology, manifested in the topological irreducibility of varieties. In contrast, the analytic topology on a complex variety has enough open sets to ensure that constant sheaves are not flabby, since small analytic-open sets don't intersect.

To bypass the obstruction, we will generalize the analytic topology to an arbitrary scheme. The resulting "topology" is not a topology in the usual sense, but rather a Grothendieck topology, a mild generalization of a topology (or more precisely, the notion of an open covering). This process produces a space $X_{\text{ét}}$, the "etale topos" of X, which has enough "open subsets" (and the sheaves defined on them) for our purposes. A sheaf

[2] When ℓ divides q, the resulting cohomology theory behaves poorly. *Crystalline cohomology*, the fourth and final example of a Weil cohomology theory, provides a substitute in this case.

on $X_{\text{ét}}$ is called an étale sheaf, and these objects support a cohomology theory similar to the cohomology of sheaves on schemes. For instance, the cohomology $H^i(X_{\text{ét}}, \mathcal{F})$ of an étale sheaf $\mathcal{F}$ is just the right derived functor of the global sections functor $\text{Ab}(X_{\text{ét}}) \to \text{Ab}$.

Even after all of this work, it will turn out that $H^i(X_{\text{ét}}, \mathbb{Z})$, the cohomology of the constant sheaf $\mathbb{Z}$, is still not the right object (for instance, it is always trivial when $i = 1$). This time, though, the problem is not the topology, but the sheaf: étale cohomology is well-behaved only for torsion sheaves. We will therefore define

$$H^i(X, \mathbb{Z}_\ell) := \varprojlim_n H^i(X_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}).$$

(Warning: this is not the same as the étale cohomology of the constant sheaf $\mathbb{Z}_\ell$.) To get a cohomology theory over a field, we tensor with $\mathbb{Q}_\ell$ to remove torsion:

$$H^i(X, \mathbb{Q}_\ell) := H^i(X, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

1.4 Goals

Unfortunately for us, both étale cohomology and the passage to ℓ-adic cohomology are rather difficult: the first is the topic of SGA 4 [1] and the second is the topic of SGA 5 [8].[3] Treatments of the subject usually fall at one end of the following two extremes: systematically developing the necessary notions in great generality and abstraction, or giving only the main definitions and results of the theory with incomplete proofs. The first approach is impossible in a week-long lecture series, and I find the second approach pedagogically ineffective. Instead, I intend to focus on key definitions of the theory – such as sites, étale morphisms, and torsors – and leave proofs of their properties as guided exercises.

2 Sites

The first step in our construction of ℓ-adic cohomology was the formulation of a scheme-theoretic analogue of the complex-analytic topology, called the *étale topology.* I said that it was not a topology in the regular sense, but rather a structure called a *Grothendieck topology* which axiomatizes the notion of an open cover. Using Grothendieck topologies we can recover a great deal of classical sheaf theory, which we will ultimately specialize to the case of the étale topology.

2.1 The Yoneda Embedding

Let X be a topological space and let $\text{Op}(X)$ be the category of open subsets of X: the objects are open subsets of X and morphism $U \to V$ is an inclusion $U \subseteq V$. Recall that a *presheaf* on X is a set-valued contravariant functor on $\text{Op}(X)$, that is, a functor $\mathcal{F} : \text{Op}(X)^{\text{op}} \to \text{Sets}$. This definition makes no special use of the category $\text{Op}(X)$, and so we can (and do) define a presheaf on any category $\mathcal{C}$ to be a functor $\mathcal{F} : \mathcal{C}^{\text{op}} \to \text{Sets}$.

Exercise 8 (Review)**:** What is a morphism of presheaves on $\mathcal{C}$?

The category of presheaves on $\mathcal{C}$ is denoted by $\widehat{\mathcal{C}}$. The hat notation is standard here $\widehat{\mathcal{C}}$ can be thought of as a completion of $\mathcal{C}$, for the following reasons. First, the category $\mathcal{C}$ has all limits and colimits. This is because the same is true of the category of sets, and limits and colimits in functor categories can be computed component-by-component.

Exercise 9: Give a precise formulation of the following statement, and prove it: "limits and colimits in functor categories can be computed component-by-component."

Second, every object X of $\mathcal{C}$ gives rise to a presheaf h_X defined by $h_X(Y) = \text{Hom}_{\mathcal{C}}(Y, X)$. The assignment $X \mapsto h_X$ is functorial in X and defines an embedding of $\mathcal{C}$ in $\widehat{\mathcal{C}}$, known as the Yoneda embedding.

[3] See SGA 4½ [3] for a good summary of both topics.

Exercise 10: Work through the definition of the functor $\mathcal{C} \to \widehat{\mathcal{C}}$ and check that it is an embedding: for every two objects X and Y of $\mathcal{C}$, the map

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\widehat{\mathcal{C}}}(h_X, h_Y)$$

induced by the Yoneda embedding is an isomorphism. More generally, construct for any presheaf $\mathcal{F}$ an isomorphism $\mathrm{Hom}(h_X, \mathcal{F}) \to \mathcal{F}(X)$, natural in $\mathcal{F}$ and X.

In total, we get a canonical embedding of an arbitrary category $\mathcal{C}$ in a complete and cocomplete category, the category $\widehat{\mathcal{C}}$ of presheaves on $\mathcal{C}$. This observation isn't strictly relevant to the discussion that follows, but it should motivate the importance and naturality of the presheaf concept.[4]

We have seen how to define a presheaf on a general category, but how do we define a sheaf? Recall that a presheaf $\mathcal{F}$ on $\mathrm{Op}(X)$ is a *sheaf* if for every $U \in \mathrm{Op}(X)$ and every open cover $(U_i \subseteq U \mid i \in I)$ of U, the following property holds: for every family $(s_i \in \mathcal{F}(U_i) \mid i \in I)$ of sections such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $(i, j) \in I^2$, there is a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$. We can phrase this condition more concisely, and in a way that generalizes to presheaves taking values in other categories, as the statement that the sequence below is exact, that is, that the first map is an equalizer of the stacked arrows:

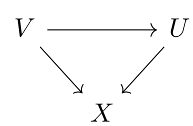
$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j) \in I^2} \mathcal{F}(U_i \cap U_j).$$

Unlike the notion of a presheaf, the notion of a sheaf *does not* make sense in any category, but only in those of the form $\mathrm{Op}(X)$. We would like to remedy the situation so that arbitrary categories can support a notion of sheaf. There are two parts of the definition that will need to be replaced.

- The intersection $U_i \cap U_j$ will be replaced by the fiber product $U_i \times_U U_j$.
- The open cover of U will be replaced by an axiomatization of the notion of an open cover, called a Grothendieck topology.

2.2 Grothendieck Topologies

Let $\mathcal{C}$ be a category and let X be an object of $\mathcal{C}$. Recall that a *morphism over* X is a morphism with target X, which we picture as an arrow $U \to X$. A morphism of objects over X is a commutative triangle as depicted below:



If you like, the objects of $\mathcal{C}$ over X form a category themselves, denoted by $\mathcal{C}_X$ and called the *slice category.*

Definition 11. Let $\mathcal{C}$ be a category with pullbacks. A *Grothendieck topology*[5] on $\mathcal{C}$ assigns to each object U of X a collection $\mathrm{Cov}(U)$ of morphisms over U, called the *coverings* of U, such that the following axioms are satisfied:

(a) (Trivial coverings) Every isomorphism $U' \to U$ is a covering.

(b) (Pullback) Let $(U_i \to U \mid i \in I)$ be a covering of U and let $V \to U$ be a morphism, and let $V_i \to V$ be the pullback of $U_i \to U$ along $V \to U$. Then $(V_i \to V \mid i \in I)$ is a covering of V.

(c) (Transitivity) Let $(U_i \to U \mid i \in I)$ be a covering of U, and for each $i \in I$, let $(U_{ij} \to U_i \mid j \in J_i)$ be a covering of U_i. Then $(U_{ij} \to U \mid i \in I, j \in J_i)$ is a covering of U.

A *site* is a category together with a Grothendieck topology.

Exercise 12: Show that if $(U_i \to X \mid i \in I)$ and $(V_j \to X \mid j \in J)$ are open covers then so is $(U_i \times V_j \mid (i, j) \in I \times J)$.

At this point, it would be most logical to define a sheaf on a site, the notion that motivated us to define sites in the first place. But before we define sheaves, here are a few examples of sites.

[4] The "Categorical Preliminaries" section of Mac Lane and Moerdijk's *Sheaves in Geometry and Logic* has a good discussion of this and other facts about categories. [10]

[5] There are several (equivalent) definitions of a Grothendieck topology. SGA 4 [1] uses the notion of a *sieve*, which more properly axiomatizes the notion of an open cover together with all its open subsets. There is also a notion of a *coverage* [12], a generalization for the case where $\mathcal{C}$ lacks pullbacks. These notes follow the conventions of the Stacks Project [14, Tag 00VH].

Topological Spaces In the category $\mathrm{Op}(X)$, declare $(U_i \to U \mid i \in I)$ to be a covering if $\bigcup_{i\in I} U_i = U$. This is the standard covering used to define sheaves on topological spaces. The Zariski topology is a particular example, and the one that we would like to modify for the purposes of étale cohomology. When X is a scheme, we will write X_{zar} for the site just defined and call it the *Zariski site*.

The Affine Site Let X be a scheme and let X_{aff} denote the category of affine open subsets of X with morphisms the inclusion relations between an affine scheme $\operatorname{Spec} A$ and its distinguished opens $\operatorname{Spec} A_f$. (In other words, $\operatorname{Hom}(U, V)$ has a single element if $U \subseteq V$ is a distinguished open and it is empty otherwise.) Declare $(U_i \to U \mid i \in I)$ to be a covering if $\bigcup_{i\in I} U_i = U$. The resulting site, which I'll call the *affine site*, is useful for constructing sheaves on schemes: it is the easiest site on which to define the structure sheaf $\mathcal{O}_X$ and quasicoherent sheaves of $\mathcal{O}_X$-modules.

***G*-Sets** Let G be a group. In the category of G-sets, declare $(\varphi_i : X_i \to X \mid i \in I)$ to be a covering if $\bigcup_{i\in I} \varphi_i(X_i) = X$.

One of the most important differences between general sites and the standard site on $\mathrm{Op}(X)$ is that general sites can have one-element covers that are not isomorphisms. The example of G-sets illustrates this phenomenon, as do most of the following examples.

Topological Étale Site In the category of topological spaces, declare $(\varphi_i : X_i \to X \mid i \in I)$ to be a covering if each φ_i is a local (on the source) homeomorphism (in other words, an *étale* map of topological spaces) and $\bigcup_{i\in I} \varphi_i(X_i) = X$.

We write $X_{\text{top-ét}}$ for the site consisting of étale topological spaces over X, together with the Grothendieck topology just defined.

The Smooth Site In the category of schemes, declare $(\varphi_i : X_i \to X \mid i \in I)$ to be a covering if each φ_i is smooth and $\bigcup_{i\in I} \varphi_i(X_i) = X$.

There is really nothing special about the word "smooth" here: we could have replaced it with any property of scheme morphisms closed under composition and pullbacks.

A Toy Model of the Étale Site Fix a field k. Let $(\operatorname{Spec} k)_{\text{ét}}$ denote the category whose objects are products of spectra of finite separable field extensions of k (over $\operatorname{Spec} k$), and whose morphisms are scheme morphisms (over $\operatorname{Spec} k$). Declare $(\varphi_i : X_i \to X \mid i \in I)$ to be a covering if $\bigcup_{i\in I} \varphi_i(X_i) = X$.

In a moment, we will study this example in detail. Later, we will define $X_{\text{ét}}$ for *any* scheme X, and this definition will generalize the definition of $(\operatorname{Spec} k)_{\text{ét}}$.

Exercise 13: Verify that each of our examples satisfies the axioms of a Grothendieck topology.

As these examples suggest, in many (most?) Grothendieck topologies appearing in algebraic geometry, the covering families are jointly surjective morphisms satisfying some additional property.

2.3 Sheaves on a Site

We can now define sheaves on a site by mimicking sheaves on a topological space.

Definition 14. Let $\mathcal{C}$ be a site. A presheaf $\mathcal{F}$ on $\mathcal{C}$ is a *sheaf* (resp. a *separated presheaf*) if for each object U of $\mathcal{C}$, each open cover $(U_i \to U \mid i \in I)$ of U_i, and each collection $(s_i \in \mathcal{F}(U_i) \mid i \in I)$ of sections such that

$$\mathcal{F}(U_i \times_U U_j \to U_i)(s_i) = \mathcal{F}(U_i \times_U U_j \to U_j)(s_j) \qquad \text{for all} \quad (i,j) \in I^2,$$

there is exactly one (resp. at most one) section $s \in \mathcal{F}(U)$ such that $\mathcal{F}(U_i \to U)(s) = s_i$ for all $i \in I$.

A (Grothendieck[6]) *topos* is a category (equivalent to a category) of sheaves on a site.

The notation used above for the restriction function may seem unusually pedantic, but it is necessary (at least in the beginning) because in general, when $i = j$ the two possible projection maps $U_i \times_U U_i \to U_i$ are different. The notation $s_i|_{U_i \times_U U_i}$ obscures this difference.

[6] In logic there is a notion of an *elementary topos*, which is strictly more general than a topos in algebraic geometry.

Exercise 15: Prove that $U_i \times_U U_i \to U_i$ is an isomorphism if and only if $U_i \to U$ is monic.

Given a site $\mathcal{C}$, write

- $\mathrm{PSh}(\mathcal{C})$ for the category of presheaves of sets on $\mathcal{C}$,
- $\mathrm{Sh}(\mathcal{C})$ for the category of sheaves of sets on $\mathcal{C}$,
- $\mathrm{PAb}(\mathcal{C})$ for the category of presheaves of abelian groups on $\mathcal{C}$, and
- $\mathrm{Ab}(\mathcal{C})$ for the category of sheaves of abelian groups on $\mathcal{C}$.

Eventually, when we work with sheaf cohomology, we will only be interested in (pre)sheaves of abelian groups. But there is no need to restrict to this case from the beginning.

Exercise 16: Phrase the definition of a sheaf (and a separated presheaf) using an equalizer diagram.

Giving examples of a sheaf on a site is harder than giving examples of sites, since the axioms to be verified are much more strenuous. Besides the unspoken example of sheaves on topological spaces, we will discuss sheaves on our toy model of the étale site.

Exercise 17: Let X be a scheme, let X_{aff} be the affine site on X, and let X_{zar} be the (usual) Zariski site on X. Show that the restriction map $\mathrm{Sh}(X_{\mathrm{zar}}) \to \mathrm{Sh}(X_{\mathrm{aff}})$ induced by inclusion $X_{\mathrm{aff}} \to X_{\mathrm{zar}}$ is an equivalence of categories.

Exercise 18: Let X be a topological space. Let X_{top} denote the usual topological site of X; that is, the underlying category of X_{top} is $\mathrm{Op}(X)$ and the covering families are the usual open covers. Let $X_{\text{top-ét}}$ denote the étale topological site of X, defined in the examples earlier. Show that the restriction map $\mathrm{Sh}(X_{\text{top-ét}}) \to \mathrm{Sh}(X_{\mathrm{top}})$ induced by inclusion $X_{\mathrm{top}} \to X_{\text{top-ét}}$ is an equivalence of categories.

2.4 A Toy Model

In this section, we analyze in detail the site $(\operatorname{Spec} k)_{\text{ét}}$ and its sheaves.

As a first step, let's determine what the sheaf condition says for covers of the form $\operatorname{Spec} \ell \to \operatorname{Spec} k$ with $k \subseteq \ell$ a finite Galois extension. Let $\mathcal{F}$ be a presheaf on $(\operatorname{Spec} k)_{\text{ét}}$ and let $G_{\ell/k}$ be the Galois group of ℓ over k. Since $G_{\ell/k}$ is the automorphism group $\operatorname{Aut}_{\operatorname{Spec} k}(\operatorname{Spec} \ell)$, the functoriality of $\mathcal{F}$ makes $\mathcal{F}(\operatorname{Spec} \ell)$ into a $G_{\ell/k}$-set.

For this particular cover, the sheaf equalizer sequence is

$$\mathcal{F}(\operatorname{Spec} k) \to \mathcal{F}(\operatorname{Spec} \ell) \rightrightarrows \mathcal{F}\big(\operatorname{Spec}(\ell \otimes_k \ell)\big).$$

Consequently, the restriction map $\mathcal{F}(\operatorname{Spec} k) \to \mathcal{F}(\operatorname{Spec} \ell)$ is injective, so we can think of $\mathcal{F}(\operatorname{Spec} k)$ as a subset of $\mathcal{F}(\operatorname{Spec} \ell)$. As for the second part of the sequence, it turns out that the coproduct $\ell \otimes_k \ell$ is isomorphic to $\prod_{\sigma \in G_{\ell/k}} \ell$, and the coproduct maps $\ell \rightrightarrows \prod_{\sigma \in G_{\ell/k}} \ell$ are $a \mapsto (a \mid \sigma \in G_{\ell/k})$ and $a \mapsto (\sigma a \mid \sigma \in G_{\ell/k})$. (This is an exercise below.) It follows that the equalizer sequence above is isomorphic to

$$\mathcal{F}(\operatorname{Spec} k) \to \mathcal{F}(\operatorname{Spec} \ell) \rightrightarrows \prod_{\sigma \in G_{\ell/k}} \mathcal{F}(\operatorname{Spec} \ell),$$

where the first stacked arrow is the diagonal and the arrow beneath it is $x \mapsto (\sigma x \mid \sigma \in G_{\ell/k})$. Consequently, $\mathcal{F}(\operatorname{Spec} k)$ consists of the elements of $\mathcal{F}(\operatorname{Spec} \ell)$ fixed by $G_{\ell/k}$:

$$\mathcal{F}(\operatorname{Spec} k) = \mathcal{F}(\operatorname{Spec} \ell)^{G_{\ell/k}}.$$

Exercise 19: Let $k \subseteq \ell$ be a finite Galois extension with Galois group $G_{\ell/k}$. Show that the map $\ell \otimes_k \ell \to \prod_{\sigma \in G_{\ell/k}} \ell$ given by $a \otimes b \mapsto (a \cdot \sigma b \mid \sigma \in G_{\ell/k})$ is an isomorphism. (Hint: use the primitive element theorem.)

More generally, for every finite Galois extension $k \subseteq \ell$, the absolute Galois group G_k of k acts on $\mathcal{F}(\operatorname{Spec} \ell)$ via the homomorphism $G_k \to G_{\ell/k}$. We can assemble this information into a single G_k-set S by taking a colimit (essentially a union, since the maps of the system are injective):

$$S = \varinjlim_{k \subseteq \ell \text{ Galois}} \mathcal{F}(\operatorname{Spec} \ell).$$

The set S is equipped with an action of the profinite group G_k and this action is continuous when S is endowed with the discrete topology (and G_k is endowed with the Krull topology). Hence every sheaf on $(\operatorname{Spec} k)_{\text{ét}}$ gives rise to a discrete G_k-set.

Exercise 20: Let G be a topological group, let X be a topological space, and let $G \times X \to X$ be a topological action of G on X. Show that the action of G on X is continuous if and only if the stabilizer subgroup of every element of G is open. Conclude that the action of G_k on S is continuous.

Notice that if we had allowed *all* Galois extensions in the category $(\operatorname{Spec} k)_{\text{ét}}$ then there would be no need to take a colimit; instead, we could just evaluate $\mathcal{F}$ on the separable closure k^{sep} of k. But since k^{sep} might be missing from $(\operatorname{Spec} k)_{\text{ét}}$, we have to approximate it.

Conversely, any discrete G_k-set S gives rise to a sheaf $\mathcal{F}$ on G_k defined by

$$\mathcal{F}(\operatorname{Spec} \ell) = S^{G_\ell}$$

for $k \subseteq \ell$ finite separable and extended to all objects of $(\operatorname{Spec} k)_{\text{ét}}$ using the sheaf axioms.

Theorem 21. *Let G_k be the absolute Galois group of the field k. The category $\operatorname{Sh}\big((\operatorname{Spec} k)_{\text{ét}}\big)$ is equivalent to the category of discrete G_k-sets.*

Exercise 22: As best you can, fill in the remaining details of the proof of the theorem.

As a corollary of the theorem, a sheaf of abelian groups on $(\operatorname{Spec} k)_{\text{ét}}$ can be thought of as a G_k-module, that is, an abelian group equipped with a continuous action of G_k by group homomorphisms. Under this identification, the "global sections" functor $\Gamma : \operatorname{Ab}\big((\operatorname{Spec} k)_{\text{ét}}\big) \to \operatorname{Ab}$ defined by $\mathcal{F} \mapsto \mathcal{F}(\operatorname{Spec} k)$ can be thought of as the functor $M \mapsto M^{G_k}$ which sends each G_k-module M to its G_k-invariant submodule. The derived functors of the functor of G_k-invariants are known as the *Galois cohomology* of M, denoted by $H^i(G_k, M)$ and studied extensively in number theory. Once we set up enough formalism of étale cohomology, we will see that these are precisely the *étale cohomology* groups of the sheaf $\mathcal{F}_M$ of abelian groups corresponding to M. Symbolically,

$$H^i((\operatorname{Spec} k)_{\text{ét}}, \mathcal{F}_M) := R^i\Gamma((\operatorname{Spec} k)_{\text{ét}}, \mathcal{F}_M) \cong H^i(G_k, M).$$

2.5 Sheafification

The usual method for sheafifying a presheaf on a topological space uses the stalks of the sheaf at the points of the space. This method does not immediately apply to a presheaf on an arbitrary site because sites lack an underlying topological space. Although there is a well-behaved notion of the "points of a topos," and although we will eventually describe the points of the étale topos (which can be used for sheafification), in this section we'll instead sketch an alternative method of sheafification that works for all toposes and avoids any mention of points, following the Stacks Project [14, Tag 00W1]. In the process, we'll introduce Čech cohomology and give a flavor of general sites.

Let $\mathcal{F}$ be a presheaf on a site $\mathcal{C}$. Given a cover $\mathcal{U} = (U_i \to U \mid i \in I)$, let $\check{H}^0(\mathcal{U}, \mathcal{F})$ denote the collection of $\mathcal{U}$-compatible families $(s_i \in \mathcal{F}(U_i) \mid i \in I)$, where "$\mathcal{U}$-compatible" means that $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$ for every $(i,j) \in I^2$. Now define $\check{H}^0(U, \mathcal{F}) := \varinjlim_{\mathcal{U}} \check{H}^0(\mathcal{U}, \mathcal{F})$, and let $\mathcal{F}^+$ denote the presheaf $U \mapsto \check{H}^0(U, \mathcal{F})$.

The next exercise checks that the definition of $\mathcal{F}^+$ makes sense, defines a presheaf, and is functorial.

Exercise 23: Let $\mathcal{V} = (V_j \to V \mid j \in J)$ and $\mathcal{U} = (U_i \to U \mid i \in I)$ be covers. A *morphism* $\mathcal{V} \to \mathcal{U}$ consists of a morphism $V \to U$ (called the *underlying morphism*), a set map $\alpha : J \to I$, and a family of maps $V_j \to U_{\alpha(j)}$ such that for each $j \in J$, the square at right commutes. We also call a morphism $\mathcal{V} \to \mathcal{U}$ a *refinement* of $\mathcal{U}$, and we say that $\mathcal{V}$ *refines* $\mathcal{U}$.

$$\begin{array}{ccc} V_j & \longrightarrow & U_{\alpha(j)} \\ \downarrow & & \downarrow \\ V & \longrightarrow & U \end{array}$$

(a) Show that any two covers of U have a common refinement. Conclude that the colimit defining $\check{H}^0(U, \mathcal{F})$ is directed (or really, cofiltered).

(b) Show how a morphism $\mathcal{V} \to \mathcal{U}$ of covers induces a set map $\check{H}^0(U, \mathcal{F}) \to \check{H}^0(V, \mathcal{F})$

(c) Check that the induced morphism from part (b) depends only on the underlying morphism $V \to U$. Check that this association is functorial. Conclude that $\mathcal{F}^+$ is a sheaf.

(d) Show that $\mathcal{F} \mapsto \mathcal{F}^+$ is functorial in $\mathcal{F}$.

It turns out that $\mathcal{F}^+$ is not necessarily a sheaf, but it is a separated presheaf. It also turns out that the functor $\mathcal{F} \mapsto \mathcal{F}^+$ transforms separated presheaves into sheaves. We therefore define the sheafification $\mathcal{F}^{\text{sh}}$ to be $\mathcal{F}^{\text{sh}} := (\mathcal{F}^+)^+$.

Exercise 24: Verify these two claims. That is, show that

(a) $\mathcal{F}^+$ is a separated presheaf, and

(b) if $\mathcal{F}$ is separated then $\mathcal{F}^+$ is a sheaf.

Exercise 25: Show that the sheafification functor $\text{PSh}(\mathcal{C}) \to \text{Sh}(\mathcal{C})$ is left adjoint to the inclusion functor $\text{Sh}(\mathcal{C}) \to \text{PSh}(\mathcal{C})$.

This concludes our discussion of sheafification. The suggestive notation $\check{H}^0$ is not accidental; our construction is a special case of Čech cohomology.

2.6 Sheaves of Abelian Groups

It turns out that for every site $\mathcal{C}$, the category $\text{Ab}(\mathcal{C})$ of sheaves of abelian groups on $\mathcal{C}$ is an abelian category with enough injectives, allowing us the formalism of derived functors. Proving these facts for the étale site is easier than for a general site, since the étale topos has "enough points;" we'll explain how this works later. Therefore, we're not going to devote any energy to proving that $\text{Ab}(\mathcal{C})$ is abelian with enough injectives. Instead, here's an exercise and references.

Exercise 26 (See [14, Tag 03A3])**:** This exercise shows that $\text{Ab}(\mathcal{C})$ is an abelian category.

(a) Show that $\text{Ab}(\mathcal{C})$ is an additive category.

(b) Let $a : \mathcal{A} \to \mathcal{B}$ and $b : \mathcal{B} \to \mathcal{A}$ be functors. Assuming that

  (ii) $\mathcal{A}$ and $\mathcal{B}$ are additive categories, a and b are additive functors, and a is left adjoint to b,

  (ii) $\mathcal{B}$ is abelian and b is left exact, and

  (ii) $ba \cong \text{id}_{\mathcal{A}}$,

  show that $\mathcal{A}$ is an abelian category.

(c) Assuming that sheafification $\text{PAb}(\mathcal{C}) \to \text{Ab}(\mathcal{C})$ is left exact (see [14, Tag 00WJ]), conclude that $\text{Ab}(\mathcal{C})$ is abelian with enough injectives.

The proof that $\text{Ab}(\mathcal{C})$ has enough injectives is more involved; for details, see the Stacks project page on this subject [14, Tag 01DL].

3 The Étale Topos

After the very general considerations of the previous section, we are almost ready to define the fundamental structure of étale cohomology, a certain category called the *étale topos*. All that remains is to define the notion of an étale morphism.

3.1 Étale Morphisms

In the study of étale morphisms, our guiding slogan is

> An étale scheme morphism is the analogue of a local isomorphism of smooth (complex-analytic) manifolds.

The simplest definition of an étale morphism of schemes would be a map that is a local isomorphism on the underlying topological spaces. However, this simple definition is wholly inadequate. To see why, consider the map $x \mapsto x^n$ on the Riemann surface $\mathbb{C}^\times$, a local isomorphism of manifolds. It corresponds to the scheme map $\pi : \operatorname{Spec} \mathbb{C}[x, x^{-1}] \to \operatorname{Spec} \mathbb{C}[x, x^{-1}]$ induced on rings by $x \mapsto x^n$. But since the morphism π induces on the residue field $\mathbb{C}(x)$ the map map $x \mapsto x^n$, which is not an isomorphism, no restriction of π to a Zariski open subset of $\operatorname{Spec} \mathbb{C}[x, x^{-1}]$ is a local isomorphism. Hence we cannot define étale maps using the Zariski topology alone.

Instead, we will take our inspiration from the implicit function theorem of differential geometry. Recall that a smooth map $\pi : X \to Y$ of manifolds is an isomorphism in a neighborhood of a point $x \in X$ if and only if the Jacobian determinant $J(F)$ of F does not vanish at x, or equivalently, if the smooth function $J(F) \in C^\infty(U)$ is a unit of the ring $C^\infty(U)$ for some neighborhood U of x. Although the Jacobian determinant is only defined locally and depends on a choice of coordinates, its vanishing or nonvanishing does not. Furthermore, given a notion of a derivative, the Jacobian determinant admits a purely algebraic description. We can use this description to define a Jacobian determinant for morphisms of schemes, since these morphisms are locally given by polynomials and polynomials admit a notion of a derivative.

As a first step, we could define a ring map $B \to A$ to be étale if A is isomorphic to a B-algebra of the form $B[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ such that the Jacobian determinant $J(f) = \det(\partial f_j/\partial x_i)$ is a unit. With this definition, the map $x \mapsto x^n$ from earlier is étale: its Jacobian determinant is nx^{n-1}, a unit of $\mathbb{C}[x, x^{-1}]$. We could then go on to define a scheme morphism to be étale if it locally looks like our affine model. This preliminary definition of étale is essentially right, except that we need to modify it slightly so that it will globalize.

Unfortunately for someone trying to quickly learn étale cohomology, there are many equivalent (and useful) definitions of étale, whose relationship to our definition is generally not immediately clear. I am going to hedge a little bit and give these various definitions without much of an indication as to why they might be equivalent.

Definition 27. A morphism $\pi : X \to Y$ of schemes is *étale* if any of the following (equivalent) conditions are satisfied:

(a) π is flat (for each $x \in X$ the $\mathcal{O}_{Y,\pi(x)}$-algebra $O_{X,x}$ is flat) and unramified (π is locally finitely presented and $\Omega_{X/Y} = 0$).

(b) π is formally étale (for every affine scheme Z and closed subscheme $Z_0 \subseteq Z$ defined by a nilpotent ideal, $\mathrm{Hom}_Y(Z_0, X) \to \mathrm{Hom}_Y(Z, X)$ is a bijection) and locally finitely presented.

(c) π is flat and locally finitely presented, and for each $y \in Y$, the fiber $\pi^{-1}(y)$ is a disjoint union of spectra of finite separable extensions of $\kappa(y)$.

(d) There is an affine open cover $(V_i \mid i \in I)$ of Y and for each i, an affine open cover $(U_{ij} \mid j \in J_i)$ of $\pi^{-1}(V_j)$ such that for each j, the map $U_{ij} \to V_i$ is Spec of $B \to B[x_1, \ldots, x_n]/(f_1, \ldots, f_n) = A$, where $\det(\partial f_j/\partial x_i) \in A^\times$.

(e) There is an affine open cover $(V_i \mid i \in I)$ of Y and for each i, an affine open cover $(U_{ij} \mid j \in J_i)$ of $\pi^{-1}(V_j)$ such that for each j, the map $U_{ij} \to V_i$ is Spec of $B \to B[x]_g/(f) = A$, where $f' \in A^\times$ (called a *standard étale* map).

The commutative algebra behind étale morphisms is daunting. Bosch's book on Néron Models [2] has a good summary, and portions of Mel Hochster's notes from a second-semester commutative algebra course [7] go into more detail.

Exercise 28: Prove as many implications as you can between the various definitions of étale. (Warning: proving implications for the definition using standard étale morphisms requires Zariski's main theorem.)

Here are several key examples of étale morphisms. (Later, when we discuss the algebraic fundamental group, we will classify the *finite* étale morphisms.)

- Open embeddings are étale.
- $\mathrm{Spec}\, \ell \to \mathrm{Spec}\, k$ is étale for $k \subseteq \ell$ finite and separable.
- The purely inseparable extension $\mathrm{Spec}\, k(x) \to \mathrm{Spec}\, k(x^{p^n})$ is *not* étale when k has characteristic p.
- If $A \subseteq B$ is an unramified extension of Dedekind domains then $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is étale.
- The normalization of a nodal cubic is étale, but the normalization of a cuspidal cubic is not.

Exercise 29: Verify these examples.

Exercise 30: Give an example of an étale morphism that is neither separated nor quasicompact.

Theorem 31 (Properties of Étale Morphisms)**.**

(a) The class of étale morphisms is closed under composition and base change (by arbitrary morphisms).

(b) The property of being an étale morphism is local on the target: $\pi : X \to Y$ *is étale if and only if for every open cover* $(V_i \mid i \in I)$ *of* Y *and every* $i \in I$*, the restriction* $\pi^{-1}(V_i) \to V_i$ *is étale.*

(c) Morphisms between étale morphisms are étale: if $U \to X$ *and* $V \to X$ *are étale then any morphism* $V \to U$ *over* X *is étale.*

(d) Étale morphisms are open.

Exercise 32: Using any of the equivalent definitions of étale, prove the theorem.

We can finally define the étale topos.

Definition 33. Given a scheme X, let $X_{\text{ét}}$, called the *étale site of* X, denote the category of étale X-schemes, together with the following Grothendieck topology: $(\varphi_i : U_i \to U \mid i \in I)$ is a covering if $U_i \to U$ is étale for each $i \in I$ and $\bigcup_{i\in I} \varphi_i(U_i) = U$. The *étale topos* of X is the category $\text{Sh}(X_{\text{ét}})$ of sheaves on $X_{\text{ét}}$.

The *big étale site*, denoted by $\text{Sch}_{\text{ét}}$, is the category of all schemes together with the topology given above. The site $X_{\text{ét}}$, sometimes called the *small* étale site for emphasis, is a slice category of $\text{Sch}_{\text{ét}}$ (consisting of the étale X-schemes). We often focus on $X_{\text{ét}}$ when proving properties about a specific scheme X, and focus on $\text{Sch}_{\text{ét}}$ when proving properties about all étale morphisms at once.

3.2 Building Étale Sheaves

After all this theory-building, we still lack a single nontrivial example of an étale sheaf. From our earlier discussion of the Yoneda embedding, we should expect one natural source of sheaves to be the representable presheaves. For example, in the category of topological spaces endowed with the usual Grothendieck topology, the sheaf exact sequence for the functor h_X and the covering $(U_i \to U \mid i \in I)$ is

$$\text{Hom}(U, X) \to \prod_{i\in I} \text{Hom}(U_i, X) \rightrightarrows \prod_{(i,j)\in I^2} \text{Hom}(U_i \cap U_j, X).$$

The assertion that this sequence is exact amounts to the usual assertion that continuous maps defined on an open cover of a topological space can be glued together, provided that they agree where jointly defined. Hence every representable presheaf on this site is a sheaf.

Exercise 34: Check that the same is true for the Zariski site: if X is a scheme, $(U_i \to U \mid i \in I)$ is an open covering (of schemes), and $(f_i : U_i \to X \mid i \in I)$ is a family of morphisms such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for every $(i, j) \in I^2$, then there is a unique morphism $f : U \to X$ such that $f|_{U_i} = f$ for each $i \in I$.

Exercise 35: The *canonical* topology on a site $\mathcal{C}$ is the finest topology with the property that each representable presheaf is a sheaf. In this exercise, we'll give an explicit description of the canonical topology.

A family $(U_i \to U \mid i \in I)$ is an *effective epimorphism* if for every object X, the following sequence is an equalizer:

$$\text{Hom}_{\mathcal{C}}(U, X) \to \prod_{i\in I} \text{Hom}_{\mathcal{C}}(U_i, X) \rightrightarrows \prod_{(i,j)\in I^2} \text{Hom}_{\mathcal{C}}(U_i \times U_j, X).$$

The family is a *universally effective epimorphism* [14, Tag 00WP] if for every $V \to U$, the family $(V \times_U U_i \to V \mid i \in I)$ is an effective epimorphism.

Show that there *is* a canonical topology on $\mathcal{C}$ (the unique finest topology in which all representable presheaves are sheaves), and that its covering families are the universally effective epimorphisms.

A Grothendieck topology is called *subcanonical* if it has the property that every representable sheaf is a presheaf. Subcanonicality is an extremely useful property because it gives us a ready-made store of sheaves. In the case of schemes, many of the sheaves that we know and love are representable.

- The structure sheaf $\mathcal{O}_X : U \mapsto \mathcal{O}_U(U)$ is represented by the affine line $\mathbb{A}^1$ (over $\mathbb{Z}$, this is $\operatorname{Spec}(\mathbb{Z}[x])$) and it inherits its ring structure from the ring structure of $\mathbb{A}^1$.
- The multiplicative group $\mathbb{G}_m$ (also denoted by $\mathcal{O}_X^\times$, or GL_1), defined by $\mathbb{G}_m(U) = \mathcal{O}_X(U)^\times$, is represented by the punctured affine line $\mathbb{A}^1 - \{0\}$ (over $\mathbb{Z}$, this is $\operatorname{Spec}(\mathbb{Z}[x, x^{-1}])$), and inherits its multiplicative structure from this scheme.
- The group μ_n of nth roots of unity, defined by $\mu_n(U) = \{a \in \mathcal{O}_X(U) \mid a^n = 1\}$, is represented (over $\mathbb{Z}$) by the scheme $\operatorname{Spec}(\mathbb{Z}[x]/(x^n - 1))$.

Theorem 36. *The étale topology on the category of schemes is subcanonical.*

There is a subtlety here: we are *not* asserting that the weaker statement that that the étale topology on $X_{\text{ét}}$ is subcanonical. This statement would not let us conclude that the structure sheaf is a sheaf, since $\mathbb{A}^1_X$ is never an étale X-scheme (unless X is empty). However, the theorem does imply that the topology on a small étale site is subcanonical.

To prove the subcanonicality theorem, we first show that it suffices to check the sheaf axiom on a singleton étale cover of one affine scheme by another. This will reduce our calculations to the affine case.

Lemma 37. *Let $\mathcal{F}$ be a presheaf on $\mathrm{Sch}_{\text{ét}}$. Suppose that $\mathcal{F}$ satisfies the sheaf axiom for both of the following classes of covers:*

(a) Zariski open covers, that is, covers $(\varphi_i : U_i \to U \mid i \in I)$ with each φ_i an open embedding and with $\bigcup_{i \in I} \varphi_i(U_i) = U$, and

(b) singleton étale covers $\operatorname{Spec} A \to \operatorname{Spec} B$ of one affine scheme by another.

Then $\mathcal{F}$ is a sheaf on $\mathrm{Sch}_{\text{ét}}$.[7]

Exercise 38: This exercise outlines a proof of Lemma 37. Assume that $\mathcal{F}$ satisfies the hypotheses of that lemma. In sequence, show that

(a) $\mathcal{F}$ is a sheaf with respect to any étale cover of the form $(U_i \to U : i \in I)$ with U and each U_i affine and with I finite;

(b) $\mathcal{F}$ is a separated sheaf with respect to every étale cover;

(c) $\mathcal{F}$ is a sheaf with respect to every étale cover of an affine scheme (you'll need the fact that étale maps are open);

(d) $\mathcal{F}$ is a sheaf on $X_{\text{ét}}$.

Next, we'll use the criterion of the lemma to verify that the structure sheaf is an étale sheaf. This amounts to verifying that for every étale map $A \to B$, the sequence

$$A \to B \to B \otimes_A B \qquad (*)$$

(where the second map is $b \mapsto b \otimes 1 - 1 \otimes b$) is exact. In fact, we can prove this result in more generality assuming only that $A \to B$ is faithfully flat. This method of proof is known as *faithfully flat descent*, and arises in other settings (namely, descent theory).

Exercise 39: This exercise recalls the definition of faithful flatness. Suppose that M is a flat A-module. Show that the following are equivalent:

(a) For all A-modules N, $M \otimes_A N = 0$ if and only if $N = 0$.

(b) For all chain complexes $C_\bullet$ of A-modules, $M \otimes_A C_\bullet$ is exact if and only if $C_\bullet$ is exact.

If in addition M is an A-algebra B, show that the two properties above are equivalent to the following:

(c) the set map $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.

An A-module (or A-algebra) satisfying any of these equivalent properties is called *faithfully flat.* A morphism $X \to Y$ of schemes is *faithfully flat* if it is flat and surjective.

[7] A similar lemma holds for the fpqc topology [14, Tag 021M]. The quasicompactness hypothesis in the fpqc topology is needed at the same step in the proof here where we use that étale maps are open.

Exercise 40: This exercise outlines a proof of the fact that $(*)$ is exact when $A \to B$ is faithfully flat.

(a) Show that $(*)$ is exact if $A \to B$ has a left inverse (as a map of rings).[8]

(b) Let $A \to A'$ be a faithfully flat ring map. Show that $(*)$ is exact if and only if $A' \to B' \to B' \otimes_{A'} B'$ is exact, where $B' = A' \otimes_A B$.

(c) Show that the map $B \to B \otimes_A B$ given by $b \mapsto b \otimes 1$ has a left inverse.

(d) Conclude that $(*)$ is exact if $A \to B$ is faithfully flat.

We can now conclude the proof of the theorem that the étale topology is subcanonical. This amounts to showing that for every étale cover $V \to U$ between affine X-schemes and every X-scheme Z, the following sequence is an equalizer:

$$\mathrm{Hom}_X(U, Z) \to \mathrm{Hom}_X(V, Z) \rightrightarrows \mathrm{Hom}_X(V \times_U V, Z). \tag{\dagger}$$

Exercise 41: This exercise verifies exactness of the above sequence, finishing the proof of the subcanonicality theorem. In sequence, show that

(a) $(\dagger)$ is an equalizer if X and Z are affine;

(b) $(\dagger)$ is an equalizer if X is affine;

(c) $(\dagger)$ is an equalizer.

Sheaves of Modules A similar strategy shows that quasicoherent sheaves on the Zariski site give rise to quasicoherent sheaves on $X_{\text{ét}}$.

Exercise 42: Let X be a scheme and let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module. Define the $\mathcal{O}_{X_{\text{ét}}}$-module $\mathcal{F}_{\text{ét}}$ by $\Gamma(\varphi : U \to X, \mathcal{F}_{\text{ét}}) = \Gamma(U, \varphi^*\mathcal{F})$. Show that $\mathcal{F}_{\text{ét}}$ is an étale sheaf.

The methods of this section fall under the general heading of *descent theory* and specifically *fpqc descent*, foundational material for studying stacks. Vistoli's notes on descent theory, which are freely available online (and also in *FGA Explained* [15, Chapter 2.3]), discuss fpqc descent in detail and prove that the fpqc topology in subcanonical.

3.3 The Étale Fundamental Group

In this section, we classify finite étale schemes over a base scheme X. This classification simultaneously generalizes the fundamental theorem of Galois theory for arbitrary Galois field extensions and the structure theory of (finite-sheeted) covering spaces. Let $X_{\text{fét}}$ denote the category of finite étale schemes over X. Recall that a topological group is *profinite* if it is a limit of a projective system of finite groups, or equivalently (this takes proof), if it is compact, Hausdorff, and totally disconnected.

Theorem 43 ([11, Theorem 5.24])**.** *Let X be a connected scheme. There is a unique profinite group $\pi_1(X)$ such that $X_{\text{fét}}$ is equivalent to the category of finite (discrete) $\pi_1(X)$-sets.*

The group $\pi_1(X)$ is called the (*algebraic* or *étale*) fundamental group of X. This group admits a concrete characterization, as follows.

We say that a finite étale scheme U over X is *Galois* if the categorical quotient $U/\operatorname{Aut}_X(U)$ is isomorphic to X. (Equivalently, $U \to X$ is Galois if the cardinality of $\operatorname{Aut}(X)$ is the degree of the cover $U \to X$.) The following exercise explains why $X_{\text{fét}}$ has categorical quotients by group actions, and the exercise after that motivates the definition of Galois by considering the case $X = \operatorname{Spec} k$.

Exercise 44: This exercise shows that $X_{\text{fét}}$ has categorical quotients by group actions.

(a) Let X be an object of a category $\mathcal{C}$ and let $G \to \operatorname{Aut}(X)$ be a group homomorphism. Define the categorical quotient $X \to X/G$ as a map satisfying a certain universal property.

(b) Let A be a ring and G a group of automorphisms of A, and let $A^G = \{a \in A \mid \forall g \in G,\ ga = a\}$ be the subring of elements of A fixed by G. Show that in the category of affine schemes, $\operatorname{Spec} A \to \operatorname{Spec}(A^G)$ is the categorical quotient of $\operatorname{Spec} A$ by G.

[8] This statement is often phrased as "$A \to B$ has a section," but the correct statement is "$\operatorname{Spec} B \to \operatorname{Spec} A$ has a section" (since a section is a *right* inverse, by definition.)

(c) Using the equivalence of affine schemes over X with quasicoherent sheaves of $\mathcal{O}_X$-algebras, show that the category of affine schemes over X admits quotients by group actions.

(d) Conclude that $X_{\text{fét}}$ admits quotients by group actions.

Exercise 45: Classically, a field extension $k \subseteq \ell$ is called *Galois* if it is normal, separable, and algebraic. Let G denote the group of automorphisms of ℓ fixing k. Show that $k \subseteq \ell$ is Galois if and only if $\ell^G = k$.

Although the theorem gives no indication of it, the étale fundamental group, like the topological fundamental group, relies on a choice of basepoint for its construction. In our setting, the proper analogue of a basepoint is a *geometric point* of the base scheme, a morphism from the spectrum of a separably closed field. Given a geometric point $\overline{x}$ of X, we define $\pi_1(X, \overline{x})$ as the inverse limit

$$\pi_1(X, \overline{x}) := \varprojlim_{(U, \overline{u})} \operatorname{Aut}_X(U),$$

taken over the partially ordered set of all pairs $(U, \overline{u})$, where $U \in X_{\text{fét}}$ is connected and Galois, $\overline{u}$ is a geometric point of U, and $(V, \overline{v}) \leq (U, \overline{u})$ if there is an X-morphism $V \to U$ sending $\overline{v}$ to $\overline{u}$. (If such a morphism exists, then it is unique.) When X is connected, $\pi_1(X, \overline{x})$ does not depend on the choice of $\overline{x}$ (up to an inner isomorphism of π_1, as in the topological setting) and we will often just write $\pi_1(X)$.

Unfortunately, there isn't enough space in these notes to adequately explain this definition and its relation to the structure theorem for $X_{\text{fét}}$. The idea is to axiomatize the categories equivalent to the category of finite G-sets, then verify that $X_{\text{fét}}$ satisfies the axioms. We'll content ourselves with providing several key examples.

- $\pi_1(\operatorname{Spec} k)$ is the absolute Galois group $\operatorname{Gal}(k^{\text{sep}}/k)$ of the separable closure k^{sep} of k.
- For A a Dedekind domain with fraction field K, $\pi_1(\operatorname{Spec} A)$ is the Galois group $\operatorname{Gal}(K^{\text{unr}}/K)$ of the maximal unramified extension K^{unr} of K.
- $\pi_1(\mathbb{P}^1_k) = \pi_1(\operatorname{Spec} k)$.
- For X a smooth projective variety over $\mathbb{C}$, $\pi_1(X)$ is isomorphic to the profinite completion of the topological fundamental group of the analytification X_{an} of X. (This is really a hard theorem, not an example.)

Exercise 46: Calculations of fundamental groups (for those who know some algebraic number theory).

(a) What is $\pi_1(\mathbb{F}_q)$?

(b) Using Minkowski's bound, show that $\pi_1(\operatorname{Spec} \mathbb{Z}) = 1$.

(c) What is $\pi_1(\mathbb{Z}_p)$ (explicitly)?

(d) Show that for each prime p the group $\pi_1(\mathbb{Z}[p^{-1}])$ is infinite.

Many of these examples are subsumed by the following theorem. For a normal integral scheme X with function field K, a finite field extension $K \subseteq L$ is said to be *unramified* if the normalization $Y \to X$ of X in L is an unramified morphism. The *maximal unramified extension* of K is the compositum of all unramified extensions of K (in some fixed algebraic closure).

Theorem 47 ([11, Theorem 6.17])**.** *Let X be a normal integral scheme with function field K. Then $\pi_1(X)$ is the Galois group of the maximal unramified extension of K.*

3.4 Further Operations on Étale Sheaves

In classical scheme theory, a morphism $\pi : X \to Y$ of schemes gives rise to a pair of adjoint functors (π^*, π_*), the *direct image* functor π_* and the *inverse image* functor π^*. In particular, when π is a morphism from a point the inverse image takes the stalk at that point, an important tool for studying sheaves. In much the same way, a morphism of schemes gives rise to a pair of adjoint functors between the étale toposes, and stalks of étale sheaves help us to study étale sheaves.

Direct Image The direct image functor $\pi_* : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(Y_{\text{ét}})$ on étale sheaves is defined in essentially the same way as the direct image functor on Zariski sheaves. Given an étale sheaf $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$, define the sheaf $\pi_*\mathcal{F} \in \mathrm{Sh}(Y_{\text{ét}})$ by

$$(\pi_*\mathcal{F})(V) = \mathcal{F}(X \times_Y V),$$

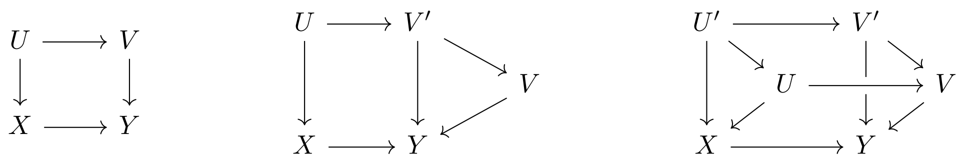
with restriction maps inherited from $\mathcal{F}$ in the obvious way.

Exercise 48: Verify that $\pi_*\mathcal{F}$ is indeed a sheaf.

Inverse Image The definition of the inverse image sheaf in the étale case mirrors the definition in the Zariski case: take a certain directed limit, then sheafify. Specifically, given a (pre)sheaf $\mathcal{G} \in \mathrm{Sh}(Y_{\text{ét}})$, define the presheaf $\pi^*_{\text{pre}}\mathcal{G}$ by

$$(\pi^*_{\text{pre}}\mathcal{G})(U) := \varinjlim_{V} \mathcal{G}(V),$$

where the directed limit is taken over all commutative squares as at left with $U \to X$ and $V \to Y$ étale, and where a morphism of such squares is a diagram as in the center, below.



Exercise 49: Show that the colimit defining π^*_{pre} is directed.

Exercise 50: Construct a bijection between the Hom sets below, natural in the presheaves $\mathcal{F} \in \mathrm{PSh}(X_{\text{ét}})$ and $\mathcal{G} \in \mathrm{PSh}(Y_{\text{ét}})$:

$$\mathrm{Hom}_{\mathrm{PSh}(X_{\text{ét}})}(\pi^*_{\text{pre}}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathrm{PSh}(Y_{\text{ét}})}(\mathcal{G}, \pi_*\mathcal{F}).$$

(Hint: show that both sets are in bijection with morphisms of squares as depicted above at right.)

We then define $\pi^*\mathcal{G}$ as the sheafification of $\pi^*_{\text{pre}}\mathcal{G}$:

$$\pi^*\mathcal{G} := \left(\pi^*_{\text{pre}}\mathcal{G}\right)^{\text{sh}}.$$

Exercise 51: Construct a bijection between the Hom sets below, natural in the sheaves $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$ and $\mathcal{G} \in \mathrm{Sh}(Y_{\text{ét}})$:

$$\mathrm{Hom}_{\mathrm{Sh}(X_{\text{ét}})}(\pi^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathrm{Sh}(Y_{\text{ét}})}(\mathcal{G}, \pi_*\mathcal{F}).$$

Stalks Traditionally, the stalks of an étale sheaf are defined only for the *geometric points* of the underlying scheme X, that is, the morphisms $\operatorname{Spec} k \to X$ with k a separably closed field. The reason for this restriction, instead of taking stalks at the points of X, is that points should have trivial étale cohomology. But $\operatorname{Spec} k$ is étale-cohomologically trivial if and only if k is separably closed.

Given a geometric point $i_{\overline{x}} : \overline{x} \to X$ and an étale sheaf $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$, we define the stalk $\mathcal{F}_{\overline{x}}$ of $\mathcal{F}$ at $\overline{x}$ to be the inverse image $i^*_{\overline{x}}\mathcal{F}$, a $\kappa(\overline{x})$-vector space. Concretely, $\mathcal{F}$ is the colimit of the sets $\mathcal{F}(U)$, taken over all étale neighborhoods $(U, \overline{u})$ of $\overline{x}$:

$$\mathcal{F}_{\overline{x}} = \varinjlim_{(U,\overline{u})} \mathcal{F}(U).$$

Here an *étale neighborhood* of $\overline{x}$ is a pair $(U, \overline{u})$ with $U \to X$ étale and $\overline{u}$ a geometric point of U mapping to $\overline{x}$ via $U \to X$. In this case, there is no need to sheafify because presheaves on $(\operatorname{Spec} k)_{\text{ét}}$ automatically satisfy the sheaf axiom when k is separably closed. (Avoiding sheafification is another reason to take stalks only at geometric points.)

Sheafification Now that we have a good notion of the "points of the étale topos," we could define sheafification by using stalks. But since we already discussed sheafification in the more general setting of sheaves on sites, I leave it to the interested reader to work out the details.

Exercise 52: Describe a sheafification functor $\mathrm{PSh}(X_{\text{ét}}) \to \mathrm{Sh}(X_{\text{ét}})$, constructed using stalks at geometric points.

Local Ring for the Étale Topology Just as a scheme carries a local ring at every point, defined as an inverse limit over Zariski-open neighborhoods, the étale topos carries a local ring at every geometric point, defined as an inverse limit over étale-open neighborhoods. The étale-local rings that arise in this way are known as *strict Henselian* rings, and have a concrete description using only commutative algebra: they are the local rings that satisfy a version of Hensel's lemma ("Henselian") and that have a separably closed residue field ("strict.") We can think of strict Henselian rings as intermediate between the localization $A_\mathfrak{p}$ and the completion $\widehat{A_\mathfrak{p}}$. I have nothing more to say about these local rings, although their study is important for gaining a finer understanding of étale cohomology.

For a nice discussion of local rings in various topologies on the category of schemes, see Gabber and Kelly's short paper "Points in Algebraic Geometry" [4]. The standard source for Henselian rings is Raynaud's *Anneaux local Henséliens* [13].

Monomorphisms and Epimorphisms For a general site $\mathcal{C}$, there is no difficulty in describing monomorphisms in the sheaf category $\mathrm{Sh}(\mathcal{C})$: since the inclusion functor $\mathrm{Sh}(\mathcal{C}) \to \mathrm{PSh}(\mathcal{C})$ admits a left adjoint (sheafification), inclusion preserves all limits. By the following exercise, it follows that inclusion preserves all monomorphisms. In other words, a map of sheaves is monic if and only if it is monic as a map of presheaves if and only if each of its components is injective.

Exercise 53: Let $f : X \to Y$ be a morphism in a category $\mathcal{C}$. Show that f is monic if and only if the square at right is a pullback square. Conclude that any functor that preserves all limits preserves all monomorphisms.

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

Epimorphisms are more difficult, since the inclusion functor from sheaves to presheaves does not usually preserve colimits.

Theorem 54. *Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on $X_{\text{ét}}$ and let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism. The following are equivalent.*

(a) ϕ is an epimorphism.

(b) For every geometric point $\overline{x}$ of X, $\phi_{\overline{x}} : \mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}}$ is surjective.

(c) ϕ is locally surjective: for every étale X-scheme $U \in X_{\text{ét}}$ and every section $s \in \mathcal{G}(U)$, there is an open cover $(U_i \to U \mid i \in I)$ such that for every $i \in I$, the section $s|_{U_i}$ lies in the image of $\mathcal{F}(U_i) \to \mathcal{G}(U_i)$.

In fact, it is true for a sheaf on *any* site that a morphism is epi if and only if it is locally surjective.[9]

Exercise 55: Prove the theorem. If you like, add in the following weakened version of (b): for every point x of X, there is a geometric point $\overline{x}$ of X mapping to x such that $\phi_{\overline{x}}$ is surjective.

In other words, we don't have to check the condition on all geometric points, just on "enough" of them. This is handy because the collection of geometric points is a proper class, not a set!

Exercise 56: Let $\overline{x}$ be a geometric point. Prove that the "taking stalks" functor $\mathrm{Ab}(X_{\text{ét}}) \to \mathrm{Ab}$ given by $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ is an exact functor of additive categories. (Hint: taking stalks has a right adjoint (what is it?) and thus is right exact.)

Putting this information together, we conclude that isomorphisms of sheaves can be tested on their stalks.

Theorem 57 ("The étale topos has enough points")**.** *Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on $X_{\text{ét}}$ and let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then ϕ is an isomorphism if and only if $\phi_{\overline{x}}$ is an isomorphism for every*[10] *geometric point of X.*

We can now prove that $\mathrm{Ab}(X_{\text{ét}})$ is an abelian category, along the same lines of the proof that $\mathrm{Ab}(X_{\text{zar}})$ is an abelian category: check that the necessary maps are isomorphisms by passing to stalks.

Exercise 58: Show that $\mathrm{Ab}(X_{\text{ét}})$ is an additive category, then show that $\mathrm{Ab}(X_{\text{ét}})$ is an abelian category.

[9] Stacks mentions this [14, Tag 00WN] but doesn't prove it.

[10] This condition can be weakened to the statement that $\phi_{\overline{x}}$ is an isomorphism for "enough points," à la Exercise 55.

Enough Injectives In order for the machinery of derived-functor cohomology to work, we need to show that the category of sheaves of abelian groups on $X_{\text{ét}}$ has enough injectives. The proof is the same as the one on the Zariski site.

Exercise 59: Let X be a scheme. For each geometric point $\overline{x}$ of X, let $i(\overline{x}) : \overline{x} \to X$ denote its inclusion.

Let $\mathcal{F}$ be an étale sheaf of abelian groups. Let $\mathcal{F} \to i(\overline{x})_* \mathcal{F}_{\overline{x}}$ be the morphism corresponding to $\mathrm{id}_{\mathcal{F}_{\overline{x}}}$ under the $(i(\overline{x})^*, i(\overline{x})_*)$ adjunction (in other words, the unit of the adjunction.) Consider the sheaf morphism

$$\mathcal{F} \to \prod_{\overline{x}} i(\overline{x})_* \mathcal{F}_{\overline{x}},$$

the product of these morphisms.

(a) Show that the morphism above is monic.

(b) Using the fact that for every ring A, the category of A-modules has enough injectives, show that $\mathrm{Ab}(X_{\text{ét}})$ has enough injectives.

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