Counting Exceptional Curves on Blowups

Takumi Murayama

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Let X be the blowup of $\mathbf{P}^2$ at r sufficiently general points $p_1, \ldots, p_r$. Repeated applications of [Har77, V, Prop. 3.2] give that $\operatorname{Pic} X \cong \mathbf{Z}^{r+1}$, and is generated by the classes $L, E_1, \ldots, E_r$, where L is (the proper transform of) a line in $\mathbf{P}^2$, and the E_i are the exceptional divisors.

Any $C \neq E_i$ is of the form $C = dL - \sum a_i E_i$ by [Har77, V, Prop. 3.6], where $a_i \geq 0$ are the intersection multiplicities of $C \cap E_i$ and where d is the degree of the image of C in $\mathbf{P}^2$ [Har77, II, Prop.6.4].

Suppose C is a (-1)-curve, i.e., a smooth rational curve with $C^2 = -1$; note that since $C \cong \mathbf{P}^1$, it has genus 0. We first have

$$-1 = C^2 = \left(dL - \sum a_i E_i\right)^2 = d^2 - \sum a_i^2,$$

and so $\sum a_i^2 = d^2 + 1$, by the fact that $L.E_i = 0$ [Har77, V, Prop. 3.2]. By the adjunction formula [Har77, V, Prop. 1.5], we also have

$$-2 = C^2 + C \cdot K_X = -1 + \left(dL - \sum a_i E_i\right) . \left(-3L + \sum E_i\right) = -1 - 3d + \sum a_i,$$

and so $\sum a_i = 3d - 1$.

A priori, we can apply the Cauchy-Schwarz inequality to get

$$\left(\sum a_i\right)^2 \leq r \sum a_i^2 \implies 9d^2 - 6d + 1 \leq rd^2 + r \implies (r-9)d^2 + 6d + (r-1) \geq 0.$$

For $r = 1$, then, there are no (-1)-curves other than E_1.

For $r = 2, 3, 4$, $d = 1$ is the only possibility for (-1)-curves, i.e., all (-1)-curves are lines. Note we have $\sum a_i = \sum a_i^2 = 2$, and so two of the a_i can equal 1.

For $r = 2$, $a_1 = a_2 = 1$, i.e., the only (-1)-curve not equal to E_1, E_2 is the proper transform of the line that goes through p_1, p_2. The dual graph then looks like $\cdot - \cdot - \cdot$, which has automorphism group $\mathbf{Z}/2\mathbf{Z}$.

For $r = 3$, there are $\binom{3}{2}$ possible (-1)-curves that are not the E_i. The dual graph then is a hexagon, which has automorphism group D_{12}.

For $r = 4$, there are $\binom{4}{2}$ possible (-1)-curves that are not the E_i. To draw the dual graph, we note

$$(L - E_i - E_j).(L - E_k - E_\ell) = 1 - \delta_i^k - \delta_i^\ell - \delta_j^k - \delta_j^\ell,$$

which is 1 if and only if $i \neq j \neq k \neq \ell$. Drawing the dual graph, we end up getting the Petersen graph (which has ten vertices), which has automorphism group S_5.

For $r = 5, 6$, we have that $d = 1$ or $d = 2$ are acceptable solutions for (-1)-curves, so any (-1)-curve that is not an exceptional divisor is either a line or a conic.

For $r = 5$, the $d = 1$ case gives us $\binom{5}{2} = 10$ (-1)-curves not equal to the exceptional divisors as above; by the same argument as above, we see that they have intersection 1 if they arise from lines that go through different blown up points. For the $d = 2$ case, we have that $\sum a_i = 5$ and $\sum a_i^2 = 5$, i.e., $a_i = 1$ for all i. This conic exists and is unique: it is the unique conic determined by the five points $p_1, \ldots, p_5$. Moreover, we see that the conic has zero intersection with the other (-1)-curves that are not the exceptional divisors:

$$\left(2L - \sum E_i\right).(L - E_j - E_k) = 2 - 1 - 1 = 0,$$

We therefore have 16 (-1)-curves, and the dual graph is a 5-regular graph with 16 vertices, which I don't know the automorphism group of...

For $r = 6$, the $d = 1$ case gives us $\binom{6}{2} = 15$ (-1)-curves that are lines. The $d = 2$ case gives us $\binom{6}{5} = 6$ (-1)-curves that are conics. This gives 27 (-1)-curves in all. Intersections between two linear (-1)-curves are as above. Between a linear and conic (-1)-curve, we have

$$\left(2L - \sum_{i \neq j} E_i\right).(L - E_k - E_\ell) = \delta_j^k + \delta_j^\ell,$$

and so the intersection number is 1 if the blown up point p_j missed by the conic is one of the points which the line goes through. Between two conic (-1)-curves,

$$\left(2L - \sum_{i \neq j} E_i\right).\left(2L - \sum_{i \neq k} E_i\right) = -\delta_j^k.$$

References

[Har77] R. Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics 52. New York: Springer-Verlag, 1977.