Vector Bundles on Projective Space

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December 1, 2013

1 Preliminaries on vector bundles

Let X be a (quasi-projective) variety over k. We follow [Sha13, Chap. 6, §1.2].

Definition. A *family of vector spaces* over X is a morphism of varieties $\pi\colon E \to X$ such that for each $x \in X$, the fiber $E_x \coloneqq \pi^{-1}(x)$ is isomorphic to a vector space $\mathbf{A}^r_{k(x)}$. A *morphism* of a family of vector spaces $\pi\colon E \to X$ and $\pi'\colon E' \to X$ is a morphism $f\colon E \to E'$ such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & X & \end{array}$$

and the map $f_x\colon E_x \to E'_x$ is linear over $k(x)$. f is an isomorphism if f_x is an isomorphism for all x.

A *vector bundle* is a family of vector spaces that is locally trivial, i.e., for each $x \in X$, there exists a neighborhood $U \ni x$ such that there is an isomorphism $\varphi\colon \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbf{A}^r$ that is an isomorphism of families of vector spaces by the following diagram:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow[\varphi]{\sim} & U \times \mathbf{A}^r \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\mathrm{pr}_1} & \\ & U & \end{array} \tag{1.1}$$

where pr_1 denotes the first projection. We call $\pi^{-1}(U) \to U$ the *restriction* of the vector bundle $\pi\colon E \to X$ onto U, denoted by $E|_U$.

r is locally constant, hence is constant on every irreducible component of X. If it is constant everywhere on X, we call r the *rank* of the vector bundle.

The following lemma tells us how local trivializations of a vector bundle glue together on the entire space X.

Lemma 1.1. *For a vector bundle $\pi\colon E \to X$ and any nondisjoint open subsets $U_\alpha, U_\beta \subset X$ with isomorphisms $\varphi_\alpha, \varphi_\beta$ satisfying the diagram* (1.1) *above with the same value for r, the map $\varphi_\beta \circ \varphi_\alpha^{-1}\colon U_{\alpha\beta} \times \mathbf{A}^r \to U_{\alpha\beta} \times \mathbf{A}^r$ is given by an $r \times r$ invertible matrix in $\mathcal{O}_X(U_{\alpha\beta})$, where $U_{\alpha\beta} \coloneqq U_\alpha \cap U_\beta$.*

Proof. Note that $\varphi_\beta \circ \varphi_\alpha^{-1}$ defines an automorphism of the vector bundle $\mathrm{pr}_1\colon U_{\alpha\beta} \times \mathbf{A}^r \to U_{\alpha\beta}$. It therefore suffices to show that an automorphism of a trivial vector bundle, i.e., the vector bundle for which (1.1) commutes for $U = X$, is given by an $r \times r$ invertible matrix in $\mathcal{O}(X)$.

Let f denote our automorphism of the trivial bundle $\mathrm{pr}_1\colon X \times \mathbf{A}^r \to X$, and let $e_1, \ldots, e_r$ be a basis for $\mathbf{A}^r$, with corresponding coordinates $\xi_1, \ldots, \xi_r$ in $\mathcal{O}(\mathbf{A^r})$. The second projection $\mathrm{pr}_2\colon X \times \mathbf{A}^r \to \mathbf{A}^r$ give elements $x_i = \mathrm{pr}_2^*(\xi_i) \in \mathcal{O}(X \times \mathbf{A}^r)$. A point $\alpha = \{x\} \times \mathbf{A}^r$ for fixed $x \in X$ is then uniquely determined by the values $x_i(\alpha)$. So f is uniquely determined by specifying $f^*(x_i) \in X \times \mathbf{A}^r$.

Defining φ_i as the composite $X \xrightarrow{\sim} X \times \{e_i\} \hookrightarrow X \times \mathbf{A}^r$, and setting $a_{ij} = \varphi_i^*(f^*(x_j)) \in \mathcal{O}(X)$, we have

$$f^*(x_j) = \sum a_{ij} x_i, \tag{1.2}$$

since $f_x\colon \{x\} \times \mathbf{A}^r \to \{x\} \times \mathbf{A}^r$ must be linear, and (a_{ij}) is an $r \times r$ invertible matrix in $\mathcal{O}(X)$ by repeating the process above for the inverse morphism f^{-1}. Conversely, any such matrix defines an automorphism of the trivial bundle by (1.2). □

Since a vector bundle is locally trivial, we can construct one by gluing together trivial bundles over an open cover of V. Let $V = \bigcup U_\alpha$ be an open cover such that $\pi\colon E \to V$ is trivial on each U_α. For each U_α, first fix an isomorphism $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbf{A}^r$. Over $U_{\alpha\beta} \coloneqq U_\alpha \cap U_\beta$, then, we have that $\varphi_\alpha \circ \varphi_\beta^{-1}$ gives an automorphism of the trivial vector bundle $U_{\alpha\beta} \times \mathbf{A}^r$, and is therefore given by an invertible $r \times r$ matrix $C_{\alpha\beta}$ in $\mathcal{O}_X(U_{\alpha\beta})$ by Lemma 1.1. These matrices satisfy the conditions

$$C_{\alpha\beta} = \mathrm{id} \text{ and } C_{\alpha\gamma} = C_{\alpha\beta} \circ C_{\beta\gamma} \text{ on } U_{\alpha\beta\gamma} \coloneqq U_\alpha \cap U_\beta \cap U_\gamma. \tag{1.3}$$

We see conversely that specifying such matrices with entries in $\mathcal{O}_X(U_{\alpha\beta})$ define a vector bundle provided they satisfy (1.3) by gluing together the varieties $U_\alpha \times \mathbf{A}^r$ along these isomorphisms.

We now show how different choices of φ_α give different matrices. Any other isomorphism φ'_α is of the form $\varphi'_\alpha = f_\alpha \varphi_\alpha$ where f_α is an automorphism of $U_\alpha \times \mathbf{A}^r \to$

U_α. By Lemma 1.1, f_α can be expressed as a matrix B_α with entries in $\mathcal{O}_X(U_\alpha)$ with an inverse matrix of the same form; thus, we have new transition matrices $C'_{\alpha\beta} = B_\alpha C_{\alpha\beta} B_\beta^{-1}$. Conversely, any such change of the matrices $C_{\alpha\beta}$ leads to different isomorphisms φ_α. Any such change, however, leads to isomorphic vector bundles via the morphism of varieties defined by the $\varphi_\alpha^{-1} \circ \varphi'_\alpha$.

2 Vector bundles on $\mathbf{A}^n$

We would like to prove the following theorem:

Theorem 2.1. *Every vector bundle over $\mathbf{A}^n$ is trivial.*

There are two ways to go about this proof: if $k = \mathbf{C}$ we can use the analytic topology on $\mathbf{A}^n$ and E and prove this using topological methods. On the other hand, we can also use purely algebraic methods, and so the lemma will hold for algebraic vector bundles over any field k. We will present the latter. We will not prove the most general case.

2.1 Vector bundles and locally free sheaves

Vector bundles over X, in fact, correspond to locally free sheaves of $\mathcal{O}_X$-modules of finite rank.

Definition. A *section* of a vector bundle $\pi\colon E \to X$ is a morphism $\sigma\colon X \to E$ such that $\pi \circ \sigma = \mathrm{id}$ on X. Denote $\Gamma(U, E)$ to be the set of sections of the restriction $E|_U$.

Proposition 2.2 ([Ser55, n° 4]). *There is a one-to-one correspondence between vector bundles over a variety X and locally free $\mathcal{O}_X$-modules of finite rank, and between trivial bundles and free $\mathcal{O}_X$-modules of finite rank.*

Proof. We assume X is irreducible, for if it is not, it suffices to show the proposition on each irreducible component. Note this means that both the rank of a vector bundle and the rank of a locally free $\mathcal{O}_X$-module are constant.

If $\sigma(x) \in \Gamma(U, E)$ is a section on U, and $f(x) \in \mathcal{O}_X(U)$, then $f(x) \cdot \sigma(x)$ is another section in $\Gamma(U, E)$, using the vector space structure on E_x. Similarly, if $\sigma'(x) \in \Gamma(U, E)$, then $\sigma(x) + \sigma'(x)$ is another section. Thus, $\Gamma(U, E)$ is an $\mathcal{O}_X(U)$-module for all U, and $\mathscr{S}(E)\colon U \mapsto \Gamma(U, E)$ is a presheaf of $\mathcal{O}_X(U)$-modules, and is a sheaf since morphisms of varieties are defined locally. Since $\pi\colon E \to X$ is locally isomorphic to the trivial bundle $U \times \mathbf{A}^r$, we see that the sheaf $\mathscr{S}$ is locally isomorphic to $\mathcal{O}_X^r$.

Conversely, let $\mathscr{F}$ be a locally free $\mathcal{O}_X$-module of finite rank. Then, there exists an open cover $\{U_\alpha\}$ of X such that there are isomorphisms $\varphi_\alpha \colon \mathscr{F}|_{U_\alpha} \xrightarrow{\sim} \mathcal{O}_X|_{U_\alpha}^r$. Then,

$$\varphi_\alpha \circ \varphi_\beta^{-1} \colon \mathcal{O}_X|_{U_{\alpha\beta}}^r \xrightarrow{\sim} \mathcal{O}_X|_{U_{\alpha\beta}}^r$$

is an isomorphism of $\mathcal{O}_X|_{U_{\alpha\beta}}$-modules, given by a matrix $C_{\alpha\beta} = (c_{ij})$ with $c_{ij} \in \mathcal{O}_X(U_{\alpha\beta})$. These obviously satisfy (1.3), and so we get a vector bundle $E(\mathscr{F})$.

Since these two operations $\mathscr{S}(-)$ and $E(-)$ clearly define a correspondence between trivial bundles and free $\mathcal{O}_X$-modules of finite rank, and since vector bundles (resp. sheaves) can be uniquely glued together using the matrices $C_{\alpha\beta}$, we see that we have a correspondence between vector bundles and locally free $\mathcal{O}_X$-modules. □

This reduces questions about vector bundles to questions about locally free sheaves of finite rank.

2.2 Serre's correspondence

We can reduce this question about locally free sheaves of finite rank further for affine spaces to questions about modules using Serre's correspondence [FAC, Ch. II, §4].

Let $X = \operatorname{Spec} A$ be the affine variety of a noetherian k-algebra A, and $\mathcal{O}_X$ its structure sheaf.

Definition (Sheaf associated to a module)**.** Let M be an A-module. A defines a constant sheaf of rings on X denoted by $\mathscr{A}$, and M defines a constant sheaf of $\mathscr{A}$-modules denoted by $\mathscr{M}$. Put $\mathscr{A}(M) = \mathcal{O}_X \otimes_{\mathscr{A}} \mathscr{M}$, the tensor product sheaf. We call $\mathscr{A}(M)$ the *sheaf associated* to M. If $\varphi \colon M \to M'$ is an A-homomorphism, then we get a morphism $\mathscr{A}(\varphi) \coloneqq \mathrm{id} \otimes \varphi \colon \mathscr{A}(M) \to \mathscr{A}(M')$, and so $\mathscr{A}(-)$ is a functor.

Definition (Module associated to a sheaf)**.** Let $\mathscr{F}$ be an $\mathcal{O}_X$-module. Define the *module associated* to $\mathscr{F}$ as $\Gamma(\mathscr{F}) \coloneqq \Gamma(X, \mathscr{F})$; $\Gamma(\mathscr{F})$ is then an A-module. If $\psi \colon \mathscr{F} \to \mathscr{G}$ is a morphism of $\mathcal{O}_X$-modules, we get a morphism $\Gamma(\psi)$ by taking global sections.

If M is finitely generated, $\mathscr{A}(M)$ is a coherent $\mathcal{O}_X$-module; conversely, if $\mathscr{F}$ is a coherent $\mathcal{O}_X$-module, $\Gamma(\mathscr{F})$ is finitely generated [FAC, n°s 48, 49]. The two operations $\mathscr{A}(-)$ and $\Gamma(-)$ are then "inverse" to each other:

Theorem 2.3 ([FAC, n° 49, Thm. 1])**.**

(*a*) *If M is a finitely generated A-module, $\Gamma(\mathscr{A}(M))$ is canonically isomorphic to M.*

(*b*) *If $\mathscr{F}$ is a coherent $\mathcal{O}_X$-module, $\mathscr{A}(\Gamma(\mathscr{F}))$ is canonically isomorphic to $\mathscr{F}$.*

Using this correspondence, we find that vector bundles have a characterization based on their global sections. Recall that an A-module M is *projective* if it is the direct summand of a free A-module, or equivalently if M_p is isomorphic to A_p^r for some r for all $p \in X$ [FAC, n° 50, Prop. 4].

Now if $\mathscr{F}$ is a coherent $\mathcal{O}_X$-module, and if $\mathscr{F}_p$ is isomorphic to $\mathcal{O}_{X,p}^r$, then it is isomorphic to $\mathcal{O}_X|_U^r$ in a neighborhood U of p. If this holds for all $p \in X$, then $\mathscr{F}$ is locally free of rank r, where r is constant on every irreducible component of X. Applying the functor $\mathscr{A}(-)$ and using Prop. 2.2 gives:

Corollary. *Let $\mathscr{F}$ be a coherent $\mathcal{O}_X$-module over an irreducible affine variety X. Then, the following are equivalent:*
(*i*) $\Gamma(\mathscr{F})$ *is a projective A-module.*
(*ii*) $\mathscr{F}$ *is locally free of rank r.*
(*iii*) $\mathscr{F}$ *is of the form $\mathscr{S}(E)$ for some vector bundle $E \to X$.*

Moreover, if E is a vector bundle over $X = \operatorname{Spec} A$, $E \mapsto \Gamma(\mathscr{S}(E))$ is then a one-to-one correspondence between vector bundles and finitely generated projective A-modules, in which trivial bundles correspond to free modules.

Theorem 2.1 then takes the form:

Theorem 2.1*. *Every finitely generated projective A-module P for $A = k[x_1, \ldots, x_n]$ is free.*

Note that the proof of this theorem in full generality is fairly involved; we will only show a couple special cases.

2.3 The case $n = 1$

The case $n = 1$ is rather simple. Recall that $k[x]$ is a PID.

Proof of Theorem 2.1.* Suppose P is a finitely generated projective module over $A = k[x]$. Since $k[x]$ is a PID, by the classification of finitely generated modules over a PID, it can be decomposed into into $A^r \oplus T$, where T is the torsion submodule of P. But since P is projective, it is the direct summand of a free A-module, and so cannot contain torsion elements. Hence $T = 0$ and $P = A^r$, i.e., P is free. □

2.4 The case of rank 1 projectives

Let K be the fraction field of $A = k[x_1, \ldots, x_n]$. If P is a projective A-module, let $\operatorname{rk} P = \dim_K P \otimes_A K$ be the *rank* of P. We consider the case when $\operatorname{rk} P = 1$. Recall that $k[x_1, \ldots, x_n]$ is a UFD. We follow [CE56, VII.3].

Proof of Theorem 2.1.* Since P is torsion free, $P \subset K \otimes_R P \cong K$ since P has rank 1. Let $\psi\colon F \to P$ be a homomorphism of a free module with base $\{e_\alpha\}$ onto P; then, the $a_\alpha = \psi(e_\alpha)$ generate P. Since P is finitely generated, we can take F to be of finite rank. Since P is a finitely generated A-submodule of K, by clearing denominators of its generators, we can identify P with some ideal of A, and assume $a_\alpha \in A$ for all α.

Now P being projective is equivalent to the existence of a section $\varphi\colon P \to F$ such that $\psi \circ \varphi = \mathrm{id}_P$. Writing $\varphi(p) = \sum_\alpha \varphi_\alpha(p) e_\alpha$, we obtain morphisms $\varphi_\alpha\colon P \to A$. The condition $\psi \circ \varphi = \mathrm{id}_P$ is then equivalent with

$$p = \sum_\alpha \varphi_\alpha(p) a_\alpha. \tag{2.1}$$

Now for each α, we have $x\varphi_\alpha(y) = \varphi_\alpha(xy) = y\varphi_\alpha(x)$. Letting $q_\alpha = \varphi_\alpha(x)/x \in K$ for some $x \neq 0$ in P gives $q_\alpha y = y\varphi_\alpha(x)/x = \varphi_\alpha(y)$ for all $y \in P$. Thus $q_\alpha \cdot P \subset A$ for all q_α, and so $Q \cdot P \subset A$ where $Q \coloneqq \sum_\alpha A \cdot q_\alpha$. (2.1) then becomes

$$p = \sum_\alpha \varphi_\alpha(p) a_\alpha = \sum_\alpha (q_\alpha p) a_\alpha = p \sum_\alpha q_\alpha a_\alpha,$$

and so $\sum_\alpha q_\alpha a_\alpha = 1$.

Now write $q_\alpha = \frac{b_\alpha}{d_\alpha}$ for $b_\alpha, d_\alpha \in A$ such that they share no common non-unit factors. Since $\frac{b_\beta}{d_\beta} a_\alpha \in A$ for all α, β, it follows from unique factorization that d_β divides a_α for all pairs α, β. So, letting $d = \ell\mathrm{cm}\{d_\beta\}$, we have $a_\alpha \in A \cdot d \implies P \subset A \cdot d$. We also have the relation $1 = \sum_\alpha \frac{b_\alpha}{d_\alpha} a_\alpha$ from above; multiplying by d gives $d = \sum_\alpha b_\alpha \frac{d}{d_\alpha} a_\alpha \in P$. Thus, $P = A \cdot d \cong A$. □

3 Vector bundles on $\mathbf{P}^1$

We can now classify all vector bundles on $\mathbf{P}^1$ following [HM82].

Let $U_0 = \operatorname{Spec} k[s]$ and $U_1 = \operatorname{Spec} k[t]$, $U_{01} = \operatorname{Spec} k[s, s^{-1}] = U_0 \setminus \{0\}$ and $U_{10} = \operatorname{Spec} k[t, t^{-1}] = U_1 \setminus \{0\}$. $\mathbf{P}^1$ is then obtained by gluing together U_0, U_1 along the isomorphism identifying U_{01} and U_{10} by the isomorphism induced by $k[s, s^{-1}] \xrightarrow{\sim} k[t, t^{-1}]$ defined by $s \mapsto t^{-1}$.

Let $\pi\colon E \to \mathbf{P}^1$ be a rank r vector bundle. By Theorem 2.1, $E|_{U_i}$ is trivial for $i = 0, 1$, i.e., $E|_{U_i} \cong U_i \times \mathbf{A}^r$. Thus, up to isomorphism, E can be considered as the result of gluing together $U_0 \times \mathbf{A}^r$ and $U_1 \times \mathbf{A}^r$ along an isomorphism $U_0 \setminus \{0\} \times \mathbf{A}^r \xrightarrow{\sim} U_1 \setminus \{0\}\mathbf{A}^r$ which is given by $(s, v) \mapsto (s^{-1}, A(s, s^{-1})v)$, where $A(s, s^{-1}) \in$

$\mathbf{GL}(r, k[s, s^{-1}])$, i.e., it has nonzero determinant for all $s \neq 0, s^{-1} \neq 0$, and so

$$\det A(s, s^{-1}) = c \cdot s^n, \quad n \in \mathbf{Z},\ c \in k^\times. \tag{3.1}$$

By the discussion at the end of §1, we note that depending on our choice of isomorphisms $\varphi_i \colon \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbf{A}^r$, we can have different matrices $A(s, s^{-1})$ that give isomorphic vector bundles. But isomorphisms of $U_0 \times \mathbf{A}^r$ are of the form $(s, v) \mapsto (s, U(s)v)$ for $U(s) \in \mathbf{GL}(r, k[s])$, so $\det U(s) \in k^\times$; similarly, isomorphisms of $U_1 \times \mathbf{A}^r$ correspond to $V(s^{-1}) \in \mathbf{GL}(r, k[s^{-1}])$, so $\det V(s^{-1}) \in k^\times$. This gives us the following:

Proposition 3.1. *Isomorphism classes of rank r vector bundles over $\mathbf{P}^1$ are in one-to-one correspondence to the set*

$$\left\{ \mathbf{GL}(r, k[s, s^{-1}]) \Big/ A(s, s^{-1}) \sim A'(s, s^{-1}) \iff A'(s, s^{-1}) = V(s^{-1}) A(s, s^{-1}) U(s) \right\}$$

where $U(s) \in \mathbf{GL}(r, k[s])$, $V(s^{-1}) \in \mathbf{GL}(r, k[s^{-1}])$.

In particular, this means that we can assume that in (3.1), the constant $c = 1$.

3.1 A canonical form for matrices in $\mathbf{GL}(r, k[s, s^{-1}])/\sim$

We now want to find a canonical form for matrices in this set $\mathbf{GL}(r, k[s, s^{-1}])/\sim$ defined in Proposition 3.1 above.

Proposition 3.2. *Let $A(s, s^{-1}) \in \mathbf{GL}(r, k[s, s^{-1}])$ with $\det A(s, s^{-1}) = s^n$ for $n \in \mathbf{Z}$. Then, there exist $U(s) \in \mathbf{GL}(r, k[s])$, $V(s^{-1}) \in \mathbf{GL}(r, k[s^{-1}])$ such that*

$$V(s^{-1}) A(s, s^{-1}) U(s) = \begin{pmatrix} s^{d_1} & & & 0 \\ & s^{d_2} & & \\ & & \ddots & \\ 0 & & & s^{d_r} \end{pmatrix} \tag{3.2}$$

with $d_1 \geq d_2 \geq \cdots \geq d_r$, $d_i \in \mathbf{Z}$. The d_i are uniquely determined by $A(s, s^{-1})$.

Proof. We first prove uniqueness. Write $D(d_1, \ldots, d_r)$ for the matrix on the right side of (3.2). If there are two matrices $D(d_1, \ldots, d_r)$ and $D(d'_1, \ldots, d'_r)$ equivalent to $A(s, s^{-1})$, then there are $U(s) \in \mathbf{GL}(r, k[s])$, $V(s^{-1}) \in \mathbf{GL}(r, k[s^{-1}])$ such that

$$C = V(s^{-1}) D(d_1, \ldots, d_r) = D(d'_1, \ldots, d'_r) U(s). \tag{3.3}$$

We now recall the Cauchy-Binet formula [Gan59, I, §2.5]. If A is an $m \times n$ matrix and B is an $n \times q$ matrix, denoting

$$A \begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}$$

to be the minor of A obtained by taking the determinant of A after removing all rows with index in $\{1, \dots, r\} \setminus \{i_1, \dots, i_k\}$ and removing all columns with index in $\{1, \dots, r\} \setminus \{j_1, \dots, j_k\}$, then

$$(AB) \begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix} = \sum_{\ell_1 < \ell_2 < \cdots < \ell_k} A \begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ \ell_1 & \ell_2 & \cdots & \ell_k \end{pmatrix} B \begin{pmatrix} \ell_1 & \ell_2 & \cdots & \ell_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}.$$

We want to apply this to (3.3). We first note

$$D(d_1, \dots, d_r) \begin{pmatrix} \ell_1 & \ell_2 & \cdots & \ell_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix} \neq 0 \iff r_i = j_i \forall i.$$

Thus, we have

$$\begin{aligned} C \begin{pmatrix} 1 & 2 & \cdots & k \\ i_1 & i_2 & \cdots & i_k \end{pmatrix} &= V(s^{-1}) \begin{pmatrix} 1 & 2 & \cdots & k \\ i_1 & i_2 & \cdots & i_k \end{pmatrix} s^{d_{i_1} + d_{i_2} + \cdots + d_{i_k}} \\ &= s^{d'_1 + d'_2 + \cdots + d'_k} U(s) \begin{pmatrix} 1 & 2 & \cdots & k \\ i_1 & i_2 & \cdots & i_k \end{pmatrix} \end{aligned}$$

for all k and sequences $i_1 < i_2 < \cdots < i_k$. Since $\det U(s) \neq 0$, for all k, there exists at least one sequence $i_1 < i_2 < \cdots < i_k$ such that

$$U(s) \begin{pmatrix} 1 & 2 & \cdots & k \\ i_1 & i_2 & \cdots & i_k \end{pmatrix} \neq 0.$$

Thus, $d'_1 + d'_2 + \cdots d'_k \leq d_{i_1} + d_{i_2} + \cdots + d_{i_k} \leq d_1 + d_2 + \cdots d_k$ for all k. Multiplying on the right by $U(s)^{-1}$ and on the left by $V(s^{-1})^{-1}$ in (3.3) and applying the same argument, we get $d_1 + d_2 + \cdots d_k \leq d'_1 + d'_2 + \cdots d'_k$ for all k. Thus, $d_i = d'_i$ for all $i = 1, \dots, r$.

We now prove existence. We proceed by induction. For $r = 1$, the proposition clearly holds. Now for arbitrary r, we assume it works for $(r-1) \times (r-1)$ matrices. First multiply $A(s, s^{-1})$ by s^n for some $n \in \mathbf{Z}_{\geq 0}$ so that we obtain a polynomial matrix $B(s)$. Now by multiplying $B(s)$ by suitable $U(s)$ on the right, i.e., by performing elementary column operations, we can find a $B = (b'_{ij})$ with $b'_{11} \neq 0$ and $b'_{1i} = 0$ for all $i = 2, \dots, r$ (then, $b'_{11} = \gcd\{b_{1i}\}$). Then, $b'_{11} = s^{k_1}$ for some k_1 since $\det B(s)$ is some power of s. Denoting $B_2(s)$ to be the lower-right $(r-1) \times (r-1)$ submatrix of B, by induction there exist $U_2(s), V_2(s^{-1})$ such that $V_2(s^{-1})B_2(s)U_2(s)$ is of the form on the right hand side of (3.2). Then, we have

$$C(s) = \begin{pmatrix} 1 & 0 \\ 0 & V_2(s^{-1}) \end{pmatrix} B \begin{pmatrix} 1 & 0 \\ 0 & U_2(s) \end{pmatrix} = \begin{pmatrix} s^{k_1} & 0 & \cdots & 0 \\ c_2 & s^{k_2} & & 0 \\ \vdots & & \ddots & \\ c_r & 0 & & s^{k_r} \end{pmatrix} \tag{3.4}$$

for some $k_1, \dots, k_r \in \mathbf{Z}_{\geq 0}$ and $c_2, \dots, c_r \in k[s, s^{-1}]$. By multiplying $C(s)$ by suitable $V(s^{-1})$ on the left, i.e., by performing elementary row operations, we can assume that $c_i \in k[s]$ for all i.

Now consider all matrices equivalent to $B(s)$ of the form (3.4). There is one representative with k_1 maximal, for $k_2, \dots, k_r \geq 0$ implies $k_1 \leq \deg(\det B(s))$. We claim that $k_1 \geq k_i$ for all $i = 2, \dots, r$. So suppose $k_1 < k_i$. Subtracting a suitable $k[s^{-1}]$-multiple of the first row, (3.4) then has $c_i = s^{k_1+1}c'(s)$ for some $c'(s) \in k[s]$. Interchanging the first and i-th rows, we get a polynomial matrix $B'(s)$ such that the greatest common divisor of the first row is $s^{k'_1}$ with $k'_1 \geq k_1 + 1$. But applying to $B'(s)$ the same process as above for $B(s)$, we get a $C'(s)$ of the form (3.4) with $k'_1 > k_1$, contradicting maximality of k_1.

We can therefore assume in (3.4) that $k_1 \geq k_i$ and $c_i \in k[s]$ for $i = 2, \dots, r$. Now subtracting suitable $k[s]$-multiples of the ith column for $i = 2, \dots, r$ from the first column (i.e., multiplying by suitable $U(s)$ on the right) we get a matrix (3.4) with $\deg c_i \leq k_i$ for all i. Then, $\deg c_i < k_1$, and so subtracting suitable $k[s^{-1}]$-multiples of the first row from the ith row (i.e., multiplying by suitable $V(s^{-1})$ on the left), we get $c_2 = c_3 = \cdots = c_r = 0$ in (3.4). This shows that there are $k_1, \dots, k_r \in \mathbf{Z}_{\geq 0}$, $k_1 \geq k_2 \geq \cdots \geq k_r$ and $U(s) \in \mathbf{GL}(r, k[s])$, $V(s^{-1}) \in \mathbf{GL}(r, k[s^{-1}])$ such that

$$V(s^{-1})s^n A(s, s^{-1})U(s) = V(s^{-1})B(s)U(s) = D(k_1, \dots, k_r).$$

Multiplying by s^{-n} gives

$$V(s^{-1})A(s, s^{-1})U(s) = D(d_1, \dots, d_r), \quad d_i = k_i - n. \qquad \square$$

3.2 Classification of vector bundles over $\mathbf{P}^1$

Let $\mathcal{O}(d)$, $d \in \mathbf{Z}$ be the line bundle over $\mathbf{P}^1$ defined by the gluing matrix $A(s, s^{-1}) = s^{-d}$. Then, the bundle defined by the gluing matrix $A(s, s^{-1}) = D(d_1, \dots, d_r)$ is equal to the direct sum $\mathcal{O}(-d_1), \dots, \mathcal{O}(-d_r)$, for the direct sum of two vector bundles defined by gluing matrices $C_{\alpha\beta}$ and $D_{\alpha\beta}$ on the same cover is defined as the vector bundle defined by gluing matrices $C_{\alpha\beta} \oplus D_{\alpha\beta}$. We then have

Theorem 3.3. *Let E be a rank r vector bundle over $\mathbf{P}^1$ which is defined over k. Then E is isomorphic over k to a direct sum of line bundles*

$$E \cong \mathcal{O}(\kappa_1) \oplus \cdots \oplus \mathcal{O}(\kappa_r), \quad \kappa_1 \geq \cdots \geq \kappa_r, \quad \kappa_i \in \mathbf{Z},\ i = 1, \dots, r,$$

and the κ_i are uniquely determined by the isomorphism class of E.

4 Line bundles on $\mathbf{P}^n$

Recall that $\operatorname{Pic}(X)$ (as a set) is the isomorphism classes of line bundles (vector bundles of rank 1) on X. Theorem 3.3 in the case $r = 1$ then gives

Corollary. $\operatorname{Pic}(\mathbf{P}^1) = \mathbf{Z} = \{\mathcal{O}(d) \mid d \in \mathbf{Z}\}$.

We would like to generalize this to $\mathbf{P^n} = \operatorname{Proj} k[s_0, \dots, s_n]$. Since we have shown Theorem 2.1 for arbitrary n in the case $r = 1$, we see that as before, a line bundle is trivial on each affine chart $U_0, \dots, U_n$, where $U_i = \{s_i \neq 0\}$.

In this case, we see that the transition matrices C_{ij} are given by

$$C_{ij} = \left(\frac{x_i}{x_j}\right)^d \tag{4.1}$$

for some d, by the same reasoning as in §3. The cocycle condition (1.3) gives us

$$\left(\frac{x_i}{x_k}\right)^{d_{ik}} = \left(\frac{x_i}{x_j}\right)^{d_{ij}} \left(\frac{x_j}{x_k}\right)^{d_{jk}},$$

and so $d_{ij} = d_{jk}$ for all i, j, k. Defining $\mathcal{O}(d)$ as the line bundle defined by gluing matrices $C_{ij} = (x_i/x_j)^d$ (note that this matches what we had for $\mathbf{P}^1$ since we defined $s = x_1/x_0$), we see that then, all line bundles on $\mathbf{P}^n$ are isomorphic to $\mathcal{O}(d)$ for some $d \in \mathbf{Z}$, and so

Theorem 4.1. $\operatorname{Pic}(\mathbf{P}^n) = \mathbf{Z} = \{\mathcal{O}(d) \mid d \in \mathbf{Z}\}$.

Finally, we prove the following:

Theorem 4.2. $\Gamma(X, \mathcal{O}(d)) = k[x_0, \dots, x_n]_d$*, the d-graded piece of $k[x_0, \dots, x_n]$ consisting of homogeneous polynomials of degree d in $k[x_0, \dots, x_n]_d$.*

Proof. Recall that sections are morphisms $\sigma\colon \mathbf{P}^n \to E$ such that $\pi \circ \sigma = \mathrm{id}_{\mathbf{P}^n}$. Since $E|_{U_i} \cong U_i \times \mathbf{A}^1$ for each open affine $U_i = \{x_i \neq 0\}$, and these U_i are glued via transition matrices (4.1) above, we see that a morphism σ consists of all $n+1$-tuples of regular functions $(\sigma_0, \dots, \sigma_n)$ such that $\sigma_i = \sigma_j$ on U_{ij} for all i, j via the gluing defined in (4.1), i.e., $C_{ij}\sigma_i = \sigma_j$. Thus, we have

$$\left(\frac{x_i}{x_j}\right)^d \sigma_i = \sigma_j,$$

where $\sigma_i \in k[\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}]$, and similarly for σ_j, for all i, j. But this is possible if and only if $\deg \sigma_i \leq d$ for all i; we also note that one choice of σ_i uniquely determines all other σ_j. But the polynomials in $k[\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}]$ of degree less than d are in bijection with the homogeneous polynomials of degree d in $k[x_0, \dots, x_n]$ by multiplying σ_i by x_i. Thus, we are done. □

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