

MAXIMAL TORI AND THE WEYL INTEGRATION FORMULA

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1. INTRODUCTION

Consider G a compact, connected Lie group. A maximal torus is a subgroup of G that is also a torus $\mathbb{T}^k \simeq \mathbb{R}^k/\mathbb{Z}^k$, and is a maximal such subgroup with respect to inclusion. The Weyl group of G , defined as $W = N(T)/T$ for a maximal torus T , depends definitionally on the choice of a maximal torus T , but by showing that any two tori are conjugate, we can see that W is unique up to isomorphism. This also shows that every conjugacy class in G meets T , and so to compute the integral of a class function, i.e., a function invariant under conjugation (such as the inner product of two characters), it should suffice to integrate only over the torus. The formula that allows this is the important Weyl Integration Formula (Theorem 4.1), which is fundamental in representation theory and in other areas like random matrix theory.

What is in fact remarkable about the conjugacy theorem and the Weyl Integration formula is the variety of approaches one can take in proving them, although the conjugacy theorem was first found by Élie Cartan and the Weyl Integration formula by Hermann Weyl. Some are based on the Hopf-Rinow theorem from differential geometry [4], others are based on André Weil's proof using the Lefschetz fixed-point formula from algebraic topology [1], and still others are based on a (rather tedious, yet enlightening) move back and forth between G and its associated Lie algebra, where there is a similar theorem on the conjugacy of Cartan subalgebras, which has an elegant proof due to Hunt [7], [8]. In previous versions of this paper, I have actually written up (most of) the Hopf-Rinow and Lie algebraic proofs, which I can provide for those interested. Here we take yet another approach, mostly following [3], which relies on the notion of mapping degree (see Theorem 3.3).

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2. DEFINITIONS AND PRELIMINARIES

In this paper, “Lie group” will mean “finite dimensional real Lie group,” and G will refer to a compact connected Lie group, unless stated otherwise.

Definition. A Lie subgroup $T \leq G$ is a *maximal torus* if T is a *torus*, i.e., a compact, connected, abelian subgroup, and is a maximal such subgroup with respect to inclusion.

$\{e\}$ is a torus. Since tori are compact and connected, if $T \subsetneq T'$, then $\dim T < \dim T'$. Since $\dim G < \infty$, this shows maximal tori exist. We also note that a torus is a maximal abelian subgroup, for if A is larger, then \overline{A} is a closed connected abelian subgroup, and hence a torus, contradicting maximality. The converse does not hold in general.

Our notion of tori is not the “intuitive” one of $\mathbb{T}^k \simeq \mathbb{R}^k/\mathbb{Z}^k$, the torus of dimension k . We would like to show they are equivalent. But first, we give two examples in Lie groups we have already studied closely:

Example 2.1 ($SU(n)$). Let

$$T = \{ \text{diag}(\exp(i\theta_1), \dots, \exp(i\theta_n)) \mid \sum \theta_j = 0 \} \leq SU(n) = \{ X \in SL(n, \mathbb{C}) \mid X^*X = I \}.$$

T is compact, connected, and abelian since it is isomorphic to \mathbb{T}^{n-1} , and maximal since any non-diagonal matrix would not commute with elements in T ; thus, T is a maximal torus of $SU(n)$.

Example 2.2 ($Sp(n)$). Let

$$T = \{ \text{diag}(\exp(i\theta_1), \dots, \exp(i\theta_n)) \} \leq Sp(n) = \{ X \in GL(n, \mathbb{H}) \mid X^*X = I \}$$

T is compact, connected, and abelian since it is isomorphic to \mathbb{T}^n , and maximal since any non-diagonal matrix would not commute with elements in T , and moreover, any diagonal matrix with entries in $\mathbb{H} \setminus \mathbb{C}$ would not commute either; thus, T is a maximal torus of $\mathrm{Sp}(n)$.

We now will show that any maximal torus is isomorphic to \mathbb{T}^k for some k , but we first need a few lemmas.

Lemma 2.3. *Let G be a connected Lie group and U a neighborhood of e . Then U generates G , i.e., $G = \bigcup_{n=1}^{\infty} U^n$ where U^n consists of all n -fold products of elements of U .*

Proof. Let $V = U \cap U^{-1}$, where U^{-1} is the set of inverses of elements in U ; V is open since the inverse map is continuous. Let $H = \bigcup_{n=1}^{\infty} V^n$, which is an open subgroup containing e . For $g \in G$, $gH = \{gh \mid h \in H\}$ contains g and is open for any $g \in G$, since left multiplication by g^{-1} is continuous. If we pick a representative $g_{\alpha}H$ for each coset in G/H , then $G = \coprod_{\alpha} g_{\alpha}H$. Connectedness of G implies $G = eH$. \square

We recall our definitions for the various “adjoint” actions. Recall from 10/12 that for any $g \in G$, we have the conjugation action:

$$\mathbf{Ad}(g) : G \rightarrow G, \quad h \mapsto ghg^{-1}.$$

The differential of \mathbf{Ad} at $e \in G$ provides a linear map

$$\mathrm{Ad}(g) : \mathfrak{g} \rightarrow \mathfrak{g}.$$

From the chain rule, $\mathrm{Ad}(gh) = \mathrm{Ad}(g) \circ \mathrm{Ad}(h)$, and so we have a homomorphism of Lie groups

$$\mathrm{Ad} : G \rightarrow \mathrm{GL}(\mathfrak{g}).$$

Differentiating this homomorphism we obtain

$$\mathrm{ad} : \mathfrak{g} \rightarrow \mathrm{End}(\mathfrak{g}).$$

We then have the following theorem from 10/12 concerning the exponential map and its relationship with these adjoint actions:

Theorem 2.4. *There exists a smooth map $\exp_G : \mathfrak{g} \rightarrow G$ with the following properties:*

- (a) $d\exp_G(e) = \mathrm{id}_{\mathfrak{g}}$.
- (b) $\mathrm{Ad} \circ \exp_G = \exp_{\mathrm{GL}(\mathfrak{g})} \circ \mathrm{ad}$.
- (c) $\mathbf{Ad}(x) \circ \exp_G = \exp_G \circ \mathrm{Ad}(x)$, $x \in G$.

from which we have the easy consequence

Lemma 2.5. *Let G be a compact Lie group, and $\mathfrak{g} = \mathrm{Lie}(G)$.*

- (a) *The map $\exp : \mathfrak{g} \rightarrow G$ is a local diffeomorphism near 0.*
- (b) *When G is connected, $\exp \mathfrak{g}$ generates G .*

Proof. (b) follows from (a) and Lemma 2.3, and so it suffices to show (a). Since $d\exp(e) = \mathrm{id}_{\mathfrak{g}}$ from Theorem 2.4, we are done by the Inverse Mapping theorem. \square

Note that this does *not* prove that \exp is onto; for this to hold, it is sufficient to have \exp be a homomorphism, which we prove in the lemma below. The lemma gives a relationship between the commutativity of a Lie group and its Lie algebra. Given the examples above, it is not too surprising that, as a result, we can establish a one-to-one correspondence between Cartan subalgebras and maximal tori; see [8], Theorem 5.4, or one of my previous drafts.

Lemma 2.6. *Let G be a compact Lie group, and $\mathfrak{g} = \mathrm{Lie}(G)$. If $X, Y \in \mathfrak{g}$, $[X, Y] = 0$, then e^X and e^Y commute. Moreover, \mathfrak{g} is abelian if and only if G^0 is abelian, and in this case $e^{X+Y} = e^X e^Y$.*

Proof. Suppose $[X, Y] = 0$. From Theorem 2.4 we have

$$\mathbf{Ad}(\exp X)(\exp Y) = \exp(\mathrm{Ad}(e^X)Y) = \exp(e^{\mathrm{ad} X}Y) = \exp(Y),$$

since $\mathrm{ad}(X)Y = 0 \implies e^{\mathrm{ad} X}(Y) = Y$. It follows that if $\mathrm{ad} = 0$, then the group generated by $\exp \mathfrak{g}$, which equals G^0 by Lemmas 2.3 and 2.5, is abelian. Conversely, if G^0 is abelian, then $\mathbf{Ad}(x)|_{G^0} = \mathrm{id}_{G^0}$, and so its differential $\mathrm{Ad}(x) = I$ on \mathfrak{g} . Differentiating with respect to x at $x = 1$ we get $\mathrm{ad} = 0$.

The last statement is a consequence of the Campbell-Baker-Hausdorff formula:

$$e^X e^Y = \exp \left(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [Y, X]] + \cdots \right),$$

where the remaining terms involve high-order iterations of bracket operations; see [6], Theorem I.5.5. \square

Finally, we have the result desired:

Theorem 2.7. *If T is a torus with $\dim T = k$, then $T \simeq \mathbb{R}^k/\mathbb{Z}^k \simeq \mathbb{T}^k$, as Lie groups.*

Proof. Let $\mathfrak{t} = \text{Lie}(T)$. \exp is a homomorphism $\mathfrak{t} \rightarrow T$ by Lemma 2.6. Since \exp is surjective by Lemma 2.5 and Lemma 2.6, this implies $T \simeq \mathfrak{t}/\ker(\exp)$. Since $\mathfrak{t} \simeq \mathbb{R}^k$ as a vector space and since $\ker(\exp)$ is discrete by Lemma 2.5(a) (it is a local bijection at the origin), $\ker(\exp) \simeq \mathbb{Z}^r$ for $r \leq k$. But since $T \simeq \mathbb{R}^k/\mathbb{Z}^r \simeq \mathbb{T}^k \times \mathbb{R}^{k-r}$ must be compact, $k = r$, and so $T \simeq \mathbb{R}^k/\mathbb{Z}^k \simeq \mathbb{T}^k$. \square

Now let $\varphi \in \text{Aut}(T)$. This induces the following commutative diagram of homomorphisms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^k & \xrightarrow{\subseteq} & \mathbb{R}^k & \xrightarrow{\exp} & \mathbb{T}^k & \longrightarrow & 0 \\ & & \downarrow \text{L}\varphi & & \downarrow \text{L}\varphi & & \downarrow \varphi & & \\ 0 & \longrightarrow & \mathbb{Z}^k & \xrightarrow{\subseteq} & \mathbb{R}^k & \xrightarrow{\exp} & \mathbb{T}^k & \longrightarrow & 0 \end{array}$$

where we identify T with $\mathbb{T}^k \simeq \mathbb{R}^k/\mathbb{Z}^k$ and $\text{Lie}(T)$ with \mathbb{R}^k . The pullback of φ is $\text{L}\varphi \in \text{Aut}(\mathbb{R}^k)$, so is an invertible matrix with integral entries by the fact that it preserves \mathbb{Z} . Thus, we can define

$$\text{L} : \text{Aut}(\mathbb{T}^k) \simeq \text{Aut}(\mathbb{Z}^k) = \text{GL}(k, \mathbb{Z}), \quad \varphi \mapsto \text{L}\varphi|_{\mathbb{Z}^k}.$$

In a similar fashion, we can analyze a function $f : T \rightarrow S^1$; this gives us a way to represent T as a subgroup generated by some element. This is useful later in making our analysis of the torus easier, and so we make it precise:

Definition. An element $t \in \mathbb{T}^k$ is a *generator* if the subgroup $\langle t \rangle$ generated by t is dense in \mathbb{T}^k .

Theorem 2.8 (Kronecker). *Let $(t_1, \dots, t_k) \in \mathbb{R}^k$, and let t be its image in $\mathbb{T}^k = (\mathbb{R}/\mathbb{Z})^k$. Then t is a generator of \mathbb{T}^k if and only if $1, t_1, \dots, t_k$ are linearly independent over \mathbb{Q} .*

Proof. A homomorphism $f : \mathbb{T}^k \rightarrow S^1$ induces the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^k & \xrightarrow{\subseteq} & \mathbb{R}^k & \xrightarrow{\exp} & \mathbb{T}^k & \longrightarrow & 0 \\ & & \downarrow \text{L}f & & \downarrow \text{L}f & & \downarrow f & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\subseteq} & \mathbb{R} & \xrightarrow{\exp} & S^1 & \longrightarrow & 0 \end{array}$$

and so $\text{L}f(t_1, \dots, t_k) = \alpha_1 t_1 + \dots + \alpha_k t_k$ for $\alpha_i \in \mathbb{Z}$. Now the following are equivalent:

- i) $1, t_1, \dots, t_k$ are linearly dependent over \mathbb{Q} .
- ii) $\sum q_i t_i \in \mathbb{Q}$ for some n -tuple $0 \neq (q_1, \dots, q_k) \in \mathbb{Q}^k$.
- iii) $\sum \alpha_i t_i \in \mathbb{Z}$ for some n -tuple $0 \neq (\alpha_1, \dots, \alpha_k) \in \mathbb{Z}^k$.
- iv) $t \bmod \mathbb{Z}^n$ is in the kernel of a nontrivial homomorphism $f : \mathbb{T}^k \rightarrow S^1$.
- v) $t \bmod \mathbb{Z}^n$ is not a generator.

$i \Leftrightarrow ii \Leftrightarrow iii$ are trivial, and $iii \Leftrightarrow iv$ follows from what was said.

It remains to show $iv \Leftrightarrow v$. Let $\bar{t} = t \bmod \mathbb{Z}^k \in \ker f$. If f is nontrivial, this kernel is not all of \mathbb{T}^k and hence is a proper closed subgroup of \mathbb{T}^k , so \bar{t} is not a generator. Conversely, a nongenerator \bar{t} is contained in a proper closed subgroup $H \leq \mathbb{T}^k$, and the quotient group \mathbb{T}^k/H is a nontrivial compact connected abelian Lie group. Thus \mathbb{T}^k/H is a torus \mathbb{T}^r , $r > 0$, and \bar{t} is in the kernel of

$$\mathbb{T}^k \rightarrow \mathbb{T}^k/H \simeq \mathbb{T}^r = S^1 \times \dots \times S^1 \xrightarrow{\text{proj}} S^1. \quad \square$$

Corollary 2.9. *Every torus T has a generator; indeed, they are dense in T .*

Proof. By Theorem 2.8, it suffices to show that k -tuples (t_1, \dots, t_k) such that $1, t_1, \dots, t_k$ are linearly independent over \mathbb{Q} are dense in \mathbb{R}^k . If $1, t_1, \dots, t_{i-1}$ are linearly independent, then linear independence of $1, t_1, \dots, t_i$ excludes only countably many t_i , and the result follows since \mathbb{R} is uncountable. \square

We now define the Weyl group mentioned in the introduction.

Definition. Let T be a maximal torus in G and $N = N(T) := \{g \in G \mid gTg^{-1} = T\}$, the normalizer of T in G . Then, the group $W = N/T$ is called the *Weyl group* of G .

Although it appears that the Weyl group depends on choice of T , we will show all maximal tori are conjugate, implying that different T yield isomorphic Weyl groups. Note the normalizer N is compact since if $t \in T$ is a generator, N is the preimage of T under $g \mapsto gtg^{-1}$.

The normalizer N of T operates on T by conjugation

$$N \times T \rightarrow T, \quad (n, t) \mapsto ntn^{-1},$$

but since the operation of T on T is trivial, we have the induced action of the Weyl group

$$W \times T \rightarrow T, \quad (nT, t) \mapsto ntn^{-1},$$

which is well-defined since if n, n' are in the same coset, say nT , then $n' = nt'$ for some $t' \in T$, and so t maps to $n'tn'^{-1} = nt'tt'^{-1}n^{-1} = ntn^{-1}$ since T is abelian. This gives us the following result:

Theorem 2.10. *The Weyl group is finite.*

Proof. Using the isomorphism above, we view the action of N above on T as the continuous map

$$N \rightarrow \text{Aut}(T) \xrightarrow{\text{L}} \text{GL}(k, \mathbb{Z}), \quad n \mapsto \text{Ad}(n)|_{\text{Lie}(T)}.$$

Then, the identity map $\text{id} \in \text{Im}(N_0)$ by definition, where N_0 is the connected component of the identity in N , and moreover since $\text{Im}(N_0)$ is a connected subset of the discrete space $\text{GL}(k, \mathbb{Z}) \subseteq \text{Aut}(\text{Lie}(T))$, $\text{Im}(N_0) = \{\text{id}\}$, so N_0 acts trivially on T .

Now let α be a one-parameter subgroup, i.e., a homomorphism $\mathbb{R} \rightarrow N_0$. Then, since $\alpha(\mathbb{R}) \subseteq N_0$, and conjugation by N_0 acts trivially on T , conjugation by $\alpha(\mathbb{R})$ also acts trivially on T . Thus, $\alpha(\mathbb{R})T$ is a connected abelian group containing T ($\alpha(\mathbb{R})$ is path connected by definition and is connected to e , which is in T), but since T is a maximal connected abelian subgroup of G , then $\alpha(\mathbb{R})T = T$. Thus, $\alpha(\mathbb{R}) \subseteq T$. Now, since the exponential maps $\exp(tX)$ are one-parameter subgroups which are locally bijective by Lemma 2.5, the groups $\alpha(\mathbb{R})$ cover an open neighborhood of the identity in N_0 and hence generate N_0 . Therefore, $T = N_0$.

This implies $W = N/N_0$, and so W is compact and discrete since N is compact, i.e., W is finite. \square

This is important since otherwise, the Weyl Integration Formula (Theorem 4.1) would not make sense. We also remark that given $w \in W$, $H \in \text{Lie}(T)$, $\lambda \in \text{Lie}(T)^\vee$, the dual space of $\text{Lie}(T)$, we can define the action

$$w(H) = \text{Ad}(w)H, \quad [w(\lambda)](H) = \lambda(w^{-1}(H)) = \lambda(\text{Ad}(w^{-1})H),$$

and this action is faithful on $\text{Lie}(T), \text{Lie}(T)^\vee$, making our definition of the Weyl group similar to that from 11/30; note that this is not absolutely precise, see [8], Theorem 6.36, or one of my previous drafts, for details.

3. CONJUGACY OF MAXIMAL TORI

The goal of this section is to prove the following theorem:

Theorem 3.1 (Maximal Torus Theorem). *Any two maximal tori in a compact connected Lie group G are conjugate, and every element of G is contained in a maximal torus.*

The proof of this theorem relies on the notion of mapping degree, which relies first on orientations on G and G/T (recall the latter has a natural smooth structure by requiring $G \times G/T \rightarrow G/T$, $(g, g'T) \mapsto gg'T$ is smooth). To begin, we need to know how the canonical volume form dg is transformed by

$$\psi : G/T \times T \rightarrow G, \quad (gT, t) \mapsto gtg^{-1},$$

which is well-defined since if g, g' are in the same coset, say gT , then $g' = gt'$ for some $t' \in T$, and so t maps to $g'tg'^{-1} = gt'tt'^{-1}g^{-1} = gtg^{-1}$ since T is abelian.

Choose an inner product (\cdot, \cdot) on $\text{Lie}(G)$ which is unitary relative to Ad , which exists from 11/9 since the representation $(\text{Ad}, \text{Lie}(G))$ is finite-dimensional. Relative to (\cdot, \cdot) , we can decompose $\text{Lie}(G)$ into

$$\text{Lie}(G) = \text{Lie}(G/T) \oplus \text{Lie}(T), \quad \text{Lie}(G/T) := \text{Lie}(T)^\perp,$$

where we are justified in identifying $\text{Lie}(T)^\perp$ with $\text{Lie}(G/T)$ since the projection $\pi : G \rightarrow G/T$ induces a map between $\text{Lie}(T)^\perp \oplus \text{Lie}(T)$ and $T_{eT}(G/T)$ (the tangent space to G/T at eT), which is an isomorphism

$\text{Lie}(T)^\perp \simeq T_e T(G/T) = \text{Lie}(G/T)$. This decomposition is invariant under $\text{Ad}|_T$, since the torus acts trivially on $\text{Lie}(T)$ and since T acts nontrivially on every nonzero vector of $\text{Lie}(G/T)$ by the fact that T is a maximal abelian subgroup. The induced action on $\text{Lie}(G/T)$ is

$$\text{Ad}_{G/T} : T \rightarrow \text{Aut}(\text{Lie}(G/T)).$$

Now recall from 11/9 that there exist left-invariant volume forms $d(gT), dt, dg$ on $G/T, T, G$ respectively with volume 1. Let $n = \dim G, k = \dim T$, and $\pi : G \rightarrow G/T$ the projection map. We obtain a left-invariant differential form $\pi^* d(gT) \in \Omega^{n-k}(G)$ from the volume form $d(gT)$ on G/T , and the alternating form $dt_e \in \text{Alt}^k(\text{Lie}(T))$ gives rise to the alternating form $\text{pr}_2^* dt_e \in \text{Alt}^k(\text{Lie}(G))$, where $\text{pr}_2 : \text{Lie}(G) = \text{Lie}(G/T) \oplus \text{Lie}(T) \rightarrow \text{Lie}(T)$ is projection onto the second summand. But $\text{pr}_2^* dt_e$ determines a left-invariant differential form $d\tau \in \Omega^k G$ such that $d\tau|_T = dt$ by its group structure, and $\pi^* d(gT) \wedge d\tau$ is a left-invariant volume form on G . We may choose our signs such that $\pi^* d(gT) \wedge d\tau = c \cdot dg$ with $c > 0$.

We want to show $c = 1$. We first give a version of Fubini's theorem which is useful in our context:

Theorem 3.2 ("Fubini"). *Let G be a compact Lie group, H a closed subgroup, and $d(gH)$ a left-invariant normalized volume form on G/H . For any continuous real-valued function f on G ,*

$$\int_G f(g) dg = \int_{G/H} \left(\int_H f(gh) dh \right) d(gH).$$

Proof. The right-hand side defines a normalized left-invariant integral on G , so the theorem follows from the uniqueness of such an integral from 11/9. \square

Now consider a bundle chart $\varphi : \pi^{-1}U \rightarrow U \times T$ of the T -principal bundle $\pi : G \rightarrow G/T$ and let f be a nonzero nonnegative continuous real-valued function on G with support in $\pi^{-1}(U)$. Then,

$$0 \neq \int_G \psi dg = \int_{G/T} \left(\int_T f(gt) dt \right) d(gT) = \int_U \left(\int_T f(gt) dt \right) d(gT),$$

using Fubini's theorem (Theorem 3.2). We now change coordinates based on the following diagram associated to the bundle chart φ :

$$\begin{array}{ccccc} G & \xleftarrow{\cong} & \pi^{-1}(U) & \xrightarrow{\varphi} & U \times T & \xrightarrow{\text{pr}_2} & T \\ \downarrow \pi & & \downarrow \pi & \swarrow & & & \\ G/T & \xleftarrow{\cong} & U & & & & \end{array}$$

Note that, for fixed $g \in \pi^{-1}(U)$, $\varphi(g) = (u, s)$ with $u = \pi(g)$ and some $s \in T$, and $\varphi(gt) = (u, ts)$, since φ is a right T -map.

We can arrange that the decomposition $\text{Lie}(G) = \text{Lie}(G/T) \oplus \text{Lie}(T)$ coincides with that induced by the bundle chart φ , so that $\text{pr}_2^* dt = (\varphi^{-1})^* d\tau$. Then,

$$\begin{aligned} \int_U \left(\int_T f(gt) dt \right) d(gT) &= \int_U \left(\int_T f \varphi^{-1}(u, t) \text{pr}_2^* dt \right) \text{pr}_1^* d(gT) = \int_{U \times T} f \varphi^{-1}(u, t) \text{pr}_1^* d(gT) \wedge \text{pr}_2^* dt \\ &= \int_{U \times T} f \varphi^{-1} (\varphi^{-1})^* (\pi^* d(gT) \wedge d\tau) = c \cdot \int_{U \times T} f \varphi^{-1} \cdot (\varphi^{-1})^* dg = c \int f dg, \end{aligned}$$

so $c = 1$.

We now have the volume form

$$dg = \pi^* d(gT) \wedge d\tau, \quad d\tau|_T = dt,$$

on G and the volume form

$$\alpha = \text{pr}_1^* d(gT) \wedge \text{pr}_2^* dt$$

on $G/T \times T$.

Definition. The determinant $\det(\psi) : G/T \times T \rightarrow \mathbb{R}$ of ψ is defined by the equation $\psi^* dg = \det(\psi) \cdot \alpha$.

Definition. $p \in G/T$ is a regular point of ψ if $d\psi(p) : T_p(G \times G/T) \rightarrow T_{\psi(p)}G$ is bijective. $q \in G$ is a regular value of ψ if $d\psi(p_i) : T_{p_i}(G \times G/T) \rightarrow T_q G$ is bijective for each $p_i \in \psi^{-1}(q)$, i.e., if each $p_i \in \psi^{-1}(q)$ is a regular point.

Theorem 3.3 (Theorem on Mapping Degrees). *There exists an integer $\deg(\psi)$ assigned to the homotopy class of ψ , called the mapping degree of ψ , such that, for every form $\alpha \in \Omega^n(G)$,*

$$\int_{G/T \times T} \psi^* \alpha = \deg(\psi) \int_G \alpha.$$

If $q \in G$ is a regular value, then the mapping degree is given by

$$\deg(\psi) = \sum_{p \in \psi^{-1}(q)} \operatorname{sgn}(\det(\psi)(p)),$$

In particular, if $\deg(f) \neq 0$, f is surjective.

Proof. For clarity, we omit the proof for now. See section A. □

The proof of Theorem 3.1, then, is easily proved by the following:

Lemma 3.4 (Weyl Covering Theorem). *Let T be a maximal torus in G . Then the map*

$$\psi : G/T \times T \rightarrow G, \quad (gT, t) \mapsto gtg^{-1},$$

has mapping degree $\deg(\psi) = |W|$, where $|W|$ is the order of the Weyl group associated to T . In particular, ψ is surjective.

Proof of Theorem 3.1 assuming Lemma 3.4. Let T, T' be maximal tori and t' a generator of T' by Corollary 2.9. By Lemma 3.4, there is a $g \in G$ with $t' \in gTg^{-1}$. Hence $T' \subseteq gTg^{-1}$, and by maximality $T' = gTg^{-1}$. Since ψ is surjective every element of G is contained in some conjugate of T . □

The statement for Lemma 3.4 is slightly different from Weyl Covering Theorem found in, say, [5], Theorem 3.7.2, which chooses to consider ψ through the factorization

$$\begin{array}{ccc} G/T \times T & \longrightarrow & G/T \times_W T \\ & \searrow \psi & \downarrow \tilde{\psi} \\ & & G \end{array}$$

where $G/T \times_W T$ denotes the orbit space of the action of W on $G/T \times T$ given by

$$w* : G/T \times T \rightarrow G/T \times T, \quad (gT, t) \mapsto (gw^{-1}T, wt w^{-1}),$$

for $wT \in W$, which is well-defined as in Section 2. The statement is then that the map $\tilde{\psi}$ has mapping degree 1, making the restriction of $\tilde{\psi}$ to regular points/values a diffeomorphism.

To prove Lemma 3.4, we proceed in steps. The first step gives a formula for $\det(\psi)$ in terms of the determinant on the Lie algebras.

Proposition 3.5. *The determinant of ψ is given by*

$$\det(\psi)(gT, t) = \det(\operatorname{Ad}_{G/T}(t^{-1}) - I_{G/T}),$$

where $I_{G/T}$ is the identity on $\operatorname{Lie}(G/T)$.

Proof. The forms $dg, d(gT)$ are left-invariant under the action of G , and the form dt is left-invariant under the action of T . This allows us to consider the following composition:

$$\begin{array}{ccccccc} G \times T & \xrightarrow{(g, t) \cdot} & G \times T & \xrightarrow{\tilde{\psi}} & G & \xrightarrow{gt^{-1}g^{-1}} & G \\ \downarrow \psi & & \downarrow \psi & & \downarrow \psi & & \downarrow \psi \\ (x, y) & \xrightarrow{(g, t) \cdot} & (gx, ty) & \xrightarrow{\tilde{\psi}} & (gx)(ty)(gx)^{-1} & \xrightarrow{gt^{-1}g^{-1}} & \mathbf{Ad}(g)(\mathbf{Ad}(t^{-1})(x) \cdot y \cdot x^{-1}) \end{array}$$

where $\tilde{\psi} : G \times T \rightarrow T : (g, t) \mapsto gtg^{-1}$. In particular, $(e, e) \mapsto e$. Thus, the determinant we want to compute is the determinant of the differential of this map at (e, e) , restricted to the subspace $\operatorname{Lie}(G/T) \oplus \operatorname{Lie}(T) \subseteq \operatorname{Lie}(G) \oplus \operatorname{Lie}(T) = \operatorname{Lie}(G \times T)$.

Now $L \mathbf{Ad}(g) = \operatorname{Ad}(g)$ has determinant 1, since $\operatorname{Ad}(g)$ is orthogonal and G is connected. Moreover, the differential of a product is the sum of the differentials, and so the determinant of q is the determinant of the linear endomorphism

$$(X, Y) \mapsto \operatorname{Ad}_{G/T}(t^{-1})X + Y - X$$

of $\text{Lie}(G/T) \oplus \text{Lie}(T)$. In matrix form, this is

$$\left[\begin{array}{c|c} \text{Ad}_{G/T}(t^{-1}) - I_{G/T} & \\ \hline & I_T \end{array} \right],$$

whose determinant is $\det(\text{Ad}_{G/T}(t^{-1}) - I_{G/T})$ as claimed. \square

To determine the mapping degree, we must count the number of elements in the preimage of a generator $t \in T$, which are clearly regular values of ψ , and keep track of orientation.

Lemma 3.6. *Let $t \in T$ be a generator as in Corollary 2.9. Then,*

- (a) $\psi^{-1}(t)$ consists of $|W|$ points,
- (b) $\det(\psi) > 0$ at each of these points

Proof. (a). Let $N = N(T)$. Then,

$$\psi(gT, s) = t \iff gsg^{-1} = t \iff s = g^{-1}tg \iff g^{-1}tg \in T \iff g^{-1}Tg \subseteq T \iff g \in N.$$

Thus, $\psi^{-1}(t) = \{(gT, g^{-1}tg) \mid g \in N\}$. Given $g_1, g_2 \in N$ such that $g_2 \in g_1T$, there exists $s \in T$ such that $g_2 = sg_1$, so

$$g_2^{-1}tg_2 = g_1^{-1}s^{-1}ts g_1 = g_1^{-1}ts^{-1}sg_1 = g_1^{-1}tg_1.$$

Otherwise, $g_1T \neq g_2T$, and so $(g_1T, g_1^{-1}tg_1) \neq (g_2T, g_2^{-1}tg_2)$, and hence $\psi^{-1}(t)$ has $[N : T] = |W|$ elements.

(b). The determinant we want is $\det(\text{Ad}_{G/T}(t^{-1}) - I_{G/T})$. Since this determinant is real, if we show that $\text{Ad}_{G/T}(t^{-1}) - I_{G/T}$ has no real eigenvalues, then eigenvalues will come in complex conjugate pairs, and hence, the determinant will be positive.

Suppose $\text{Ad}_{G/T}(t^{-1}) - I_{G/T}$ has a real eigenvalue. Then $\text{Ad}_{G/T}(t^{-1})$ does also, and would be ± 1 since $\text{Ad}_{G/T}(t^{-1})$ is orthogonal. In this case, $\text{Ad}_{G/T}(t^{-2})$ has 1 as eigenvalue. Since t^{-2} is also a generator of T , it suffices to show $\text{Ad}_{G/T}(t)$ cannot have 1 as an eigenvalue.

If $\text{Ad}_{G/T}(t)X = X$ for some $X \in \text{Lie}(G/T)$, then X is fixed by the action of t and hence by all of T . But then the one-parameter subgroup $H = \{\exp(sX) \mid s \in \mathbb{R}\}$ is left pointwise fixed under $\mathbf{Ad}(T)$. Thus HT is abelian and connected, implying $H \subseteq T$ and $X \in \text{Lie}(T) \cap \text{Lie}(G/T) = 0$. \square

We record a Corollary for later:

Corollary 3.7. *If t is a generating element of T , then $\text{Ad}_{G/T}(t)$ has no real eigenvalues. Hence $\dim G/T$ is even.*

Finally, we can prove Lemma 3.4:

Proof of Lemma 3.4. Let t be a generator of T as in Corollary 2.9. The mapping degree of ψ is $|W|$ since there are $|W|$ points in $\psi^{-1}(t)$, and $\det(\psi) > 0$ implies ψ preserves orientation for all elements in $\psi^{-1}(t)$. \square

4. WEYL INTEGRATION FORMULA

We finally get the main theorem of this paper. What is amazing is that this formula is a trivial consequence of the Weyl Covering Theorem (Lemma 3.4) above, despite being unexpected at first glance.

Theorem 4.1 (Weyl Integration Formula). *Let T be a maximal torus of G , and f a continuous function on G . Then,*

$$|W| \cdot \int_G f(g) dg = \int_T \left[\det(I_{G/T} - \text{Ad}_{G/T}(t^{-1})) \int_G f(gtg^{-1}) dg \right] dt.$$

Proof. Let $f_t : G/T \rightarrow \mathbb{R}$ be given by $g \mapsto f(gtg^{-1})$, so in the last integral $f(gtg^{-1}) = f_t \circ \pi(g)$, where $\pi : G \rightarrow G/T$ is the canonical projection. Then,

$$\begin{aligned} & \int_T \left[\det(I_{G/T} - \text{Ad}_{G/T}(t^{-1})) \int_G f(gtg^{-1}) dg \right] dt \\ &= \int_T \left[\det(I_{G/T} - \text{Ad}_{G/T}(t^{-1})) \int_G f_t \circ \pi(g) dg \right] dt = \int_T \left[\det(I_{G/T} - \text{Ad}_{G/T}(t^{-1})) \int_{G/T} f_t(gT) dgT \right] dt, \end{aligned}$$

by definition of dgT . We then use Fubini's theorem (Theorem 3.2), Lemma 3.4, and Proposition 3.5 to get

$$= \int_{G/T} \left[\int_T f \circ \psi(gT, t) \cdot \det(\psi)(gT, t) dgT \right] dt = \int_{G/T \times T} f \circ \psi(gT, t) \cdot \det(\psi)(gT, t) \operatorname{pr}_2^* dt \wedge \operatorname{pr}_1^* dgT.$$

Since $\dim G/T$ is even by Corollary 3.7, $\operatorname{pr}_1^* dgT$ is an even form, and so we have

$$\begin{aligned} &= \int_{G/T \times T} f \circ \psi(gT, t) \cdot \det(\psi)(gT, t) \operatorname{pr}_1^* dgT \wedge \operatorname{pr}_2^* dt = \int_{G/T \times T} f \circ \psi(gT, t) \cdot \det(\psi)(gT, t) \alpha \\ &= \int_{G/T \times T} f \circ \psi(gT, t) \cdot \psi^* dg = \int_{G/T \times T} \psi^*(f \cdot dg) = \deg(\psi) \cdot \int_G f(g) dg = |W| \cdot \int_G f(g) dg. \end{aligned} \quad \square$$

A. THEOREM ON MAPPING DEGREES

We prove here Theorem 3.3, which we omitted the proof for before, in a more general setting. We recall that if $\varphi : N \rightarrow M$ is a diffeomorphism, then for $\alpha \in \Omega_c^n(M)$ (the space of n -forms with compact support)

$$(A.1) \quad \int_M \alpha = \int_N \varepsilon \cdot \varphi^* \alpha,$$

where ε is locally constant with value 1 or -1 according to whether φ locally preserves or reverses orientation (see [3], I, (5.8)). We note that our proof of the Theorem on Mapping Degrees relies on the non-trivial Sard's theorem from differential topology (see [2], (6.1)).

Theorem A.1 (Theorem on Mapping Degrees). *Let M, N be compact, connected, oriented, n -dimensional manifolds. There is an integer $\deg(f)$ assigned to each homotopy class of (differentiable) maps $f : M \rightarrow N$, called the mapping degree of f , such that, for every form $\alpha \in \Omega^n(N)$,*

$$\int_M f^* \alpha = \deg(f) \cdot \int_N \alpha.$$

If $q \in N$ is a regular value with $f^{-1}\{q\}$ consisting of $k + \ell$ points $p_1, \dots, p_{k+\ell}$ such that f preserves orientation at p_1, \dots, p_k but reverses orientation at $p_{k+1}, \dots, p_{k+\ell}$, then $\deg(f) = k - \ell$. In particular, if $\deg(f) \neq 0$, f is surjective.

Proof. We first show the left-hand side depends on the homotopy class of f . If we have a homotopy $f : M \times [0, 1] \rightarrow N$, where $f = f_0$ on $M \times \{0\}$ and $f = f_1$ on $M \times \{1\}$, then since $d\alpha \in \Omega^{n+1}N = 0$, Stokes' theorem (see [3], I, (5.17)) and (A.1) gives

$$0 = \int_{M \times [0, 1]} f^* d\alpha = \int_{M \times [0, 1]} df^* \alpha = \int_{\partial(M \times [0, 1])} f^* \alpha = \int_M f_1^* \alpha + \int_{-M} f_0^* \alpha = \int_M f_1^* \alpha - \int_M f_0^* \alpha.$$

By Sard's theorem (see [2], (6.1)), for a smooth map $f : M \rightarrow N$, almost every point of N is a regular value. So let q be a regular value and $f^{-1}\{q\} = \{p_1, \dots, p_{k+\ell}\}$ as in the statement of the theorem. Then f is a local diffeomorphism around each p_i , and the complement of an open neighborhood of $f^{-1}\{q\}$ is mapped by f to a compact set not containing q . Thus we can choose small neighborhoods B around q and $B_1, \dots, B_{k+\ell}$ around $p_1, \dots, p_{k+\ell}$ such that

$$f|_{B_i} : B_i \xrightarrow{\sim} B$$

and $f^{-1}(B) = \coprod_i B_i$. The orientation of f is constant on each B_i and hence equal to its value at p_i . If $\operatorname{Supp}(\alpha) \subseteq B$, the theorem would hold for this particular q by (A.1). Now if we could find a diffeomorphism $\varphi : N \rightarrow N$ homotopic to the identity on N with $\operatorname{Supp}(\alpha) \subseteq \varphi(B)$, the theorem would be proved, since $\operatorname{Supp}(\varphi^* \alpha) \subseteq B$, and the integrals are homotopy invariant. We will show below that the sets $\varphi(B)$ cover N , where φ runs through diffeomorphisms $N \rightarrow N$ homotopic to the identity. Then we can choose a partition of unity $\{\psi_j \mid j \in \mathbb{N}\}$ subordinate to this covering, and the theorem follows since it is valid for each summand of the splitting $\alpha = \sum_j \psi_j \cdot \alpha$.

To show that the $\varphi(B)$ cover N we show that, given $x \in N$, there is a φ as above with $\varphi(q) = x$. If x, q are both contained in a compact ball of a chart domain, it is fairly easy to construct such a diffeomorphism φ , e.g., by integrating an appropriate vector field which vanishes outside the ball. From this case we obtain the general case by joining q and x with a chain $q = x_0, x_1, \dots, x_r = x$ such that x_j, x_{j+1} are always contained in a compact ball in some chart domain. \square

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